SLOW MOTION AND METASTABILITY FOR A NON LOCAL EVOLUTION EQUATION

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Abstract. In this paper we continue the analysis, started in [2] and [3], of a non local mean field equation, proving the existence of an invariant, unstable, one dimensional manifold connecting the critical droplet with the stable and metastable phases. We prove that the points on the manifold are droplets longer or shorter than the critical one, and that their motion is very slow in agreement with the theory of metastable patterns.

1. Introduction

There are many examples in physics of metastable states like supercooled liquids, supersaturated vapours and solutions, or ferromagnets with magnetization opposite to the field. The phenomenon of metastability occurs in thermodynamic systems close to a first order phase transition. A system is initially prepared in an equilibrium pure phase, and then the thermodynamic parameters are changed to values for which there is a different equilibrium pure phase. Under suitable conditions, the system does not undergo a phase transition, but it remains in an apparent equilibrium, the *metastable state*, which is very similar to the initial one. This state persists until some (even slight) disturbance leads the system to the stable equilibrium. For instance we can prepare a glass of water in an environment at temperature slightly below zero degrees centigrade. The glass looks apparently in an equilibrium liquid phase, and this state may last for a long time; but eventually an irreversible process takes place and the glass of water suddenly freezes.

A first theoretical explanation of metastability goes back to the classical van der Waals Maxwell theory of liquid-vapour phase transition. In this theory the notion of metastable states emerges in a natural way, as describing homogeneous, almost equilibrium phases which however have higher free energy with respect to the corresponding stable equilibria.

As a matter of fact, the appearence of metastable states is a common feature of all the microscopic *mean field theories* of phase transition. In fact, due to the "mean field approximation", the range of the interaction between the particles coincides with the macroscopic size of the system, and this fact implies the non convexity of the free energy, thus allowing metastable branches in the phase diagram.

However, the mean field approximation is unrealistic, and in fact it produces unphysical features like termodynamical instabilities and the (already mentioned) non convexity of the free energy. As an example let us mention the famous *Maxwell equal area rule* introduced to

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correct the van der Waals diagram of phases in order to eliminate the (unphysical) branch corresponding to negative compressibility.

A more realistic mean field approximation, called the *local mean field limit*, has been introduced in the 60's by Kac, Uhlenbeck, and Hemmer, [8]. They consider interactions, usually called *Kac potentials*, that depend on a scaling parameter $\gamma > 0$, studying the limit $\gamma \downarrow 0$ where the range of the interaction becomes infinite. This program has been carried out by Kac, Uhlenbeck and Hemmer, [8], in some particular models, and then by Lebowitz and Penrose in a more general class of systems, [9]. These results prove that the phase diagram converges, for any temperature, to the van der Waals phase diagram, comprehensive of the Maxwell equal area rule.

Because of the dynamical nature of the phenomenon, a fully satisfactory description of metastabilty can be found only in the framework of non equilibrium Statistical Mechanics. In this setting the first rigorous approach to metastability goes back to Lebowitz and Penrose, [10], who give a general method for describing metastable states. Moreover, they apply this method to systems with Kac potentials, giving a rigorous justification (for what concerns the static properties) of the van der Waals (mean field) description of metastable states.

In more recent years, [4], Ising spin systems with Glauber dynamics and Kac potentials have been introduced in order to analyze non equilibrium phenomena like phase separation and interface dynamics; this will be the model considered in this paper. The Ising spin system is the most elementary microscopic caricature of ferromagnets. At each site of a lattice there is a *spin* variable with two possible values, ± 1 (up or down); the interaction between the spins is chosen so that aligned spins are favoured. In this case the mean field free energy density is given by

$$F(s) = -\frac{1}{2}s^2 - hs - \beta^{-1} i(s), \quad s \in [-1,1] \tag{1.1}$$

$$i(s) = -\frac{1+s}{2}\log\frac{1+s}{2} - \frac{1-s}{2}\log\frac{1-s}{2} \tag{1.2}$$

where β is the inverse temperature, s the average magnetization, and h an external positive magnetic field. The quadratic term $-s^2/2$ is the internal energy density, $-hs$ the energy density due to the field h, and $i(s)$ the entropy. The critical points of $F(s)$ are the solutions of the so called mean field equation:

$$s = \tanh\{\beta[s+h]\} \tag{1.3}$$

Given $\beta > 1$, there is an $h_\beta > 0$ such that for $h \in [0, h_\beta]$ (1.3) has three and only three different roots, denoted by

$$m^-_{\beta,h} < m^0_{\beta,h} \le 0 < m^+_{\beta,h} \tag{1.4}$$

For $h > 0$, $|m^-_{\beta,h}| < m^+_{\beta,h}$ and $m^0_{\beta,h} < 0$; for $h = 0$, $m^0_{\beta,h} = 0$ and $m^+_{\beta,h} = -m_{\beta,h} =: m_\beta$. The two phases $\pm m_\beta$ are thermodynamically stable at $h = 0$, while $m^0_{\beta,h} = 0$ is unstable. For $h > 0$, $m^+_{\beta,h}$ is the only stable phase, $m^0_{\beta,h}$ is still unstable, while $m^-_{\beta,h}$ becomes metastable.

However these are static considerations, the dynamics of persistence and decay of the metastable state can be investigated only going back to the stochastic evolution of the underlying spin system, characterizing the tunneling from the metastable to the stable phase.

According to general heuristic arguments we expect the transition occurs through the *nucleation* of a sufficiently large droplet of the stable phase, which will start to grow undergoing an irreversible process leading to the stable phase everywhere. On the contrary, small droplets will have a tendency to shrink. This arguing leads to believe that the transition occurs through the formation of a well defined *critical droplet*, which breaks the spatial homogeneity in the metastable state.

A specific feature of stochastic dynamics with Kac potentials is its almost deterministic behaviour for small values of the scaling paramater γ (i.e. when the range of the interaction is large). In fact in [4] it is shown that in the continuum limit the Ising spin system with Glauber dynamics and Kac potentials gives rise to a local magnetization density $m = m(t, x)$ which evolves according to the non local evolution equation

$$\frac{\partial m}{\partial t} = -m + \tanh\{\beta[J * m + h]\} \tag{1.5}$$

where J is a non negative, even function which is related to the (long range) coupling of the spin-spin interaction, and $J * m$ denotes the convolution between J and m. In agreement with (1.1) we assume also that J is normalized so that

$$\int dx\, J(x) = 1 \tag{1.6}$$

which implies that the homogeneous stationary solutions of (1.5) coincide with the solutions of (1.3).

According to the pathwise approach to metastability in the case of reversible dynamics, see [1] and [7], we expect the metastable behaviour of the deterministic evolution (1.5) will play a central role in determining the tunnelling transition for the underlying stochastic spin dynamics.

Since the mean field theory of phase transition makes sense also in one dimension, we restrict to this simpler case. In [3] it is proved (under additional hypothesis on J) the existence, for h small enough, of the critical droplet, also called the *bump*, i.e. a spatially non homogeneous, symmetric solution q of the non local, one dimensional equation

$$q(x) = \tanh\{\beta[(J * q)(x) + h]\}, \qquad x \in \mathbb{R} \tag{1.7}$$

with asymptotic conditions

$$\lim_{|x|\to\infty} q(x) = m^-_{\beta,h} \tag{1.8}$$

which is therefore a stationary solution of (1.5). All the translations of q are stationary solutions, and it is not known whether these are the only solutions of (1.7) with asymptotic conditions (1.8). In [3] it has been proven that the region where q is close to the stable phase is of order $|\log h|$ for h small. Here, in Section 3, we improve this result by showing that the critical droplet is a strictly decreasing function for $x > 0$, converging exponentially fast to the metastable phase as $|x| \to +\infty$.

The existence of an invariant, one dimensional, unstable manifold through q, follows in a standard way (see for instance [11]) from the existence, proven in [2], of an isolated, simple, positive eigenvalue of the operator obtained by linearizing (1.7) around the critical droplet q. Nevertheless we give, in Section 4, a proof useful for establishing the other results. The points on the manifold are symmetric functions, non increasing for $x > 0$. Thus they are droplets that we call sub-critical or super-critical droplets according to the length of the region where they are close to the stable phase.

Finally, in Section 5, we study the motion along the manifold. In the branch of the manifold where the length of the droplets is shorter of that of q, the evolution shrinks it further, while it grows if it is larger. The analysis is global in time, and we prove that the manifold connects the critical droplet with the stable and metastable phases. The motion along the manifold is very slow: the velocity of propagation of the stable phase vanishes exponentially fast with the length of the interval where it is present.

2. Definitions and results

We first state the assumptions on the interaction J appearing in (1.5). $J \in C^3(\mathbb{R})$ is a symmetric, non negative function satisfying (1.6). Moreover $\sup\{x \in \mathbb{R} : J(x) > 0\} = 1$ and $J'(x) < 0$ for $x \in (0,1)$.

In the whole paper we consider the evolution defined by (1.5) as an equation in the space $L_\infty(\mathbb{R}; [-1,1])$, which we rewrite as

$$\frac{dm}{dt} = f(m) \tag{2.1}$$

where

$$f(m) := -m + \tanh\{\beta[J * m + h]\} \tag{2.2}$$

We observe that the Cauchy problem has a unique solution in $L_\infty(\mathbb{R})$ because the map f is uniformly Lipschitz continuous. Moreover, since $|\tanh z| < 1$ for all $z \in \mathbb{R}$, the set $L_\infty(\mathbb{R}; [-1,1])$ is an invariant for the dynamics. Analogously there exists a unique solution of the Cauchy problem in $C_0(\mathbb{R})$, the space of continuous and bounded functions on $\mathbb{R}$, and the set $C(\mathbb{R}, [-1,1])$ is left invariant. We denote by $S_t(m)$ the flow solution with initial datum m, so that S_t defines a semigroup on $L_\infty(\mathbb{R}; [-1,1])$ for which $C(\mathbb{R}, [-1,1])$ is an invariant (closed) subspace.

Since J is a symmetric function, by uniqueness, the evolution preserves the parity of the initial datum. In particular the space $C^{\text{sym}}(\mathbb{R}, [-1,1])$ of symmetric, continuous functions with range $[-1,1]$ is an invariant set.

In [3] the existence of the bump is obtained by studying (1.7) perturbatively around $h = 0$. For $h = 0$ there is no critical droplet, however there exist many spatially non homogeneous solutions of the equation (1.7). The relevant one for proving the existence of the bump is the standing wave solution (also called the *instanton*). In the following theorem we collect the main properties of the instanton.

Theorem 2.1. ([2], [5], [6]) *Given $\beta > 1$, there exists a solution $\bar{m}(x)$ of (1.7) with $h = 0$,*

$$\bar{m}(x) = \tanh\{\beta(J * \bar{m})(x)\} \tag{2.3}$$

which is a C^∞, strictly increasing, antisymmetric function with asymptotes

$$\lim_{x \to \pm\infty} \bar{m}(x) = \pm m_\beta \tag{2.4}$$

$\bar m(x)$ is, modulo translations, the unique solution of (2.3) with asymptotes (2.4). Moreover, letting $\alpha > 0$ be such that

$$\beta(1 - m_\beta^2) \int dz\, J(z) e^{-\alpha z} = 1 \tag{2.5}$$

there are $a > 0$, $\delta > 0$, and $c > 0$ so that, for all $x \geq 0$,

$$\left|\bar m(x) - (m_\beta - a e^{-\alpha x})\right| + \left|\bar m'(x) - \alpha a e^{-\alpha x}\right| + \left|\bar m''(x) + \alpha^2 a e^{-\alpha x}\right| \leq c e^{-(\alpha+\delta)x} \tag{2.6}$$

where $\bar m'$ and $\bar m''$ are respectively the first and second derivatives of $\bar m$.

For each $z > 0$ we denote by $\bar m_z$ the symmetric function which is an instanton shifted by $-z$ on the negative half line and its mirror image on the positive half line, i.e. $\bar m_z(x) = \bar m(z - |x|)$, $x \in \mathbb{R}$.

Theorem 2.2. ([3]) *Given $\beta > 1$, there is $h_0 > 0$ and, for any $h \in (0, h_0]$, there is $q \in C^{\mathrm{sym}}(\mathbb{R}; [-1,1])$ which solves (1.7) with asymptotes as in (1.8). Moreover there are $\xi^* = \xi^*(h)$ and $c^* > 0$ such that*

$$\lim_{h \downarrow 0} \|q - \bar m_{\xi^*}\|_\infty = 0 \tag{2.7}$$

and

$$\lim_{h \downarrow 0} e^{-2\alpha \xi^*(h)} h = c^* \tag{2.8}$$

with α as in (2.5).

Our first result concerns the spatial structure of the critical droplet which is the content of the following Proposition, proved in Section 3.

Proposition 2.3. *Given $\beta > 1$, there is $h^* \in (0, h_0]$ (h_0 as in Theorem 2.2) such that for any $h \in (0, h^*]$ the bump $q(x)$ is a strictly decreasing function on $\mathbb{R}_+$ (actually $q'(x) < 0$ for all $x > 0$). Moreover, letting $\gamma > 0$ be such that*

$$\beta \left(1 - (m_{\beta,h}^-)^2\right) \int dz\, J(z) e^{-\gamma z} = 1 \tag{2.9}$$

and ξ be the (unique) positive zero of $q(x)$, there are $A > 0$, $\delta > 0$, and $C > 0$ so that, for all $x \geq \xi$,

$$\begin{aligned} &\left|q(x) - (m_{\beta,h}^- + A e^{-\gamma(x-\xi)})\right| + \left|q'(x) + \gamma A e^{-\gamma(x-\xi)}\right| \\ &\qquad + \left|q''(x) - \gamma^2 A e^{-\gamma(x-\xi)}\right| \leq C e^{-(\gamma+\delta)(x-\xi)} \end{aligned} \tag{2.10}$$

Finally, as $h \downarrow 0$, A, δ, and C remain strictly positive and bounded, while $\xi \to +\infty$.

The qualitative behavior of the dynamics around the bump follows from the spectral properties of the linear operator $L := Df|_q$, the derivative of $f(m)$ at $m = q$. Since q satisfies (1.7) we compute, for any $\psi \in L_\infty(\mathbb{R})$,

$$L\psi = -\psi + pJ * \psi \tag{2.11}$$

where

$$p(x) := \beta\left(1 - q(x)^2\right) \tag{2.12}$$

We denote by $L_\infty^{\rm sym}(\mathbb{R})$ the space of the symmetric functions in $L_\infty(\mathbb{R})$, $C_0^{\rm sym}(\mathbb{R})$ is defined analogously. Since q and J are even and continuous functions the above spaces are invariant under L.

Given $\zeta \in \mathbb{R}$, we introduce the normed spaces

$$X_\zeta := \{w : \mathbb{R} \to \mathbb{R} \text{ measurable and symmetric } : \|w\|_{\zeta,\infty} < \infty\} \tag{2.13}$$

where

$$\|w\|_{\zeta,\infty} := \sup_{x\in\mathbb{R}_+} e^{\zeta x}|w(x)|$$

In the following Proposition we collect results proven in [2, 3] and for the reader convenience at the end of Section 3 we give detailed references on where the proofs can be found.

Proposition 2.4. ([2, 3]) *Given $\beta > 1$ let h^* be as in Proposition 2.3. Then there are constants $C_0 > 1$ and $C_1 > 0$ such that for any $h \in (0, h^*]$ the following holds.*

1) There are $\lambda > 0$ and strictly positive functions $v, v^* \in C_0^{\rm sym}(\mathbb{R})$ so that

$$Lv = \lambda v, \qquad v^* L = \lambda v^* \tag{2.14}$$

$$v^*(x) = p(x)v(x) \qquad \forall\, x \in \mathbb{R} \tag{2.15}$$

and

$$\frac{h}{C_0} \le \lambda \le C_0 h \tag{2.16}$$

2) There is a unique $\gamma_v > \gamma > 0$, (γ as in (2.9)) such that

$$\beta\left(1 - (m_{\beta,h}^-)^2\right) \int dz\, J(z) e^{-\gamma_v z} = 1 + \lambda \tag{2.17}$$

and there is $M_v > 0$ so that

$$\lim_{x\to+\infty} e^{\gamma_v x} v(x) = M_v \tag{2.18}$$

Moreover for any $\zeta \le \gamma_v$ we have

$$\left\|e^{Lt}\right\|_{\zeta,\infty} \le C_1 e^{\lambda t} \qquad \forall\, t \ge 0 \tag{2.19}$$

3) Assume v and v^* are normalized in such a way that

$$\int_0^\infty dx\, \frac{v(x)^2}{p(x)} \equiv \int_0^\infty dx\, v^*(x)v(x) = 1 \tag{2.20}$$

and define the linear functional π on X_ζ, $\zeta < \gamma_v$, as

$$\pi(\psi) := \int_0^\infty dx\, v^*(x)\psi(x) \tag{2.21}$$

Then there is $\omega > 0$ so that, for any $w \in X_\zeta$, $\zeta < \gamma_v$, such that $\pi(w) = 0$ and $t \ge 0$,

$$\left\|e^{Lt} w\right\|_{\zeta,\infty} \le C_1 e^{-\omega t} \left\|w\right\|_{\zeta,\infty} \tag{2.22}$$

Moreover

$$\frac{1}{C_0} \le \omega \le C_0 \tag{2.23}$$

4) Finally, setting

$$\tilde{m}(x) := C_{\bar{m}}^{1/2}\bar{m}(x), \qquad C_{\bar{m}} := \left[\int dy \, \frac{\bar{m}'(y)^2}{\beta(1-\bar{m}(y)^2)}\right]^{-1} \tag{2.24}$$

and defining $\tilde{m}_z(x) := \tilde{m}(z-|x|)$*, we have*

$$\lim_{h\downarrow 0} \|v - \tilde{m}'_{\xi^*}\|_\infty = 0 \tag{2.25}$$

with ξ^* *as in* (2.7).

As we are going to see, from the fact that the linearized operator around q has only one positive eigenvalue, it follows the existence, for any h small enough, of two distinct, one dimensional manifolds $\mathcal{M}_\pm \subset C^{\text{sym}}(\mathbb{R};[-1,1])$. We give the precise statement in the next theorem.

Theorem 2.5. *Given* $\beta > 1$ *let* h^* *be as in Proposition 2.3. Then, for any* $h \in (0, h^*]$*, there are two distinct, one dimensional manifolds* $\mathcal{M}_\pm \subset C^{\text{sym}}(\mathbb{R};[-1,1])$*,*

$$\mathcal{M}_\pm = \left\{ m_s^\pm : \, s \in \mathbb{R} \right\} \tag{2.26}$$

For any $s \in \mathbb{R}$ *and* $t \geq 0$

$$S_t\left(m_s^\pm\right) = m_{s+t}^\pm \tag{2.27}$$

and

$$\lim_{s\to -\infty} \|m_s^\pm - q\|_\infty = 0 \tag{2.28}$$

Moreover, given $C_{\bar{m}}$ *as in* (2.24)*, we have*

$$\lim_{s\to-\infty} e^{-\lambda s} \left\| \frac{dm_s^\pm}{ds} \mp \lambda e^{\lambda s} C_{\bar{m}}^{-1/2} v \right\|_\infty = 0 \tag{2.29}$$

Finally, for any $s \in \mathbb{R}$*, the (symmetric) functions* $m_s^\pm$ *are non increasing on* $\mathbb{R}_+$ *and satisfy*

$$m_{\beta,h}^- \leq m_s^-(x) \leq q(x) \leq m_s^+(x) \leq m_{\beta,h}^+ \qquad \forall x \in \mathbb{R} \tag{2.30}$$

Thus the one dimensional manifolds $\mathcal{M}_\pm$ originates at $s = -\infty$ from q, (2.28), and are time invariant, (2.27). Each one of them is therefore described by a single orbit of S_t with time going from $-\infty$ to $+\infty$. The two orbits are denoted by $m_s^\pm$ and the parameter s is identified with time. Of course the origin of time is arbitrary and this can be exploited to fix up the constants in such a way that (2.29) holds, we refer to Section 4 for details on this point.

By integrating (2.29) from $-\infty$ to s we get

$$m_s^\pm \approx q \pm e^{\lambda s} C_{\bar{m}}^{-1/2} v \tag{2.31}$$

Therefore, since from (2.25) we have

$$\lim_{h\downarrow 0} \sup_{x \leq 0} \left| C_{\bar{m}}^{-1/2} v(x) - \bar{m}'(x+\xi^*) \right| = 0 \tag{2.32}$$

by (2.7) and (2.25) for h small and $x \leq 0$ we have

$$m_s^\pm(x) \approx \bar{m}(x+\xi^*) \pm e^{\lambda s}\bar{m}'(x+\xi^*) \approx \bar{m}(x+\xi^* \pm e^{\lambda s}) \tag{2.33}$$

By symmetry the result extends to $x \geq 0$. Thus the points in a neighborhood of q that are in $\mathcal{M}_+$ are "droplets longer" than q while those in $\mathcal{M}_-$ are shorter. Their length changes at the exponential rate λ, which is therefore the Lyapunov exponent at q with $\mathcal{M}_\pm$ the corresponding unstable manifolds. Since $\lambda \approx h$, for h small, there are a "dormant instability" and a "slow motion" in the sense that for small h (which is the case of interest in metastability) even though ultimately unstable, q seems in fact stable for very long times $\approx O(h^{-1})$. But the sub-critical and super-critical droplets will eventually go to the metastable and stable phase respectively: this is the content of the next theorem.

Theorem 2.6. *Given $\beta > 1$ there is $h^\dagger \in (0, h^*]$ (h^* as in Proposition 2.3) such that, for any $h \in (0, h^\dagger]$,*

$$\lim_{s\to+\infty} \left\| m_s^- - m_{\beta,h}^- \right\|_\infty = 0 \tag{2.34}$$

$$\lim_{s\to+\infty} m_s^+(x) = m_{\beta,h}^+ \qquad \forall\, x \in \mathbb{R} \tag{2.35}$$

The paper is organized as follows: in Section 3 we prove Proposition 2.3 and we give comments and references on the proof of Proposition 2.4; Theorem 2.5 is proved in Section 4; finally, in Section 5, we prove Theorem 2.6.

3. Spatial properties of the bump

In this section we prove Proposition 2.3. The critical droplet is uniquely determined by its restriction to the semispace $\mathbb{R}_+$, which solves

$$q(x) = \tanh\left\{\beta\left[(J_+ q)(x) + h\right]\right\}, \qquad x \in \mathbb{R}_+ \tag{3.1}$$

where, for any $f \in C(\mathbb{R}_+)$,

$$(J_\pm f)(x) := \int_0^{+\infty} dy\, J_\pm(x,y) f(y), \qquad J_\pm(x,y) := J(x-y) \pm J(x+y) \tag{3.2}$$

Observe that $J_\pm(x,y) \geq 0$ for all $x, y \geq 0$ and also that if $x > 1$ then $J_\pm(x,y) = J(x-y)$ for all $y \geq 0$.

To prove the proposition we will exploit the fact that the critical droplet is a solution of (3.1) which is close, for h small, to a suitable reflected instanton, see (2.7).

Let $\bar m_{\xi^*}$ be as in Theorem 2.2. Since $\beta(1 - m_\beta^2) < 1$, from (2.4) and (2.8) there are $\theta \in (0,1)$, $h_1 \in (0, h_0]$, and a positive integer $\ell^* \in (1, \xi^* - 1)$ such that

$$\beta\left(1 - \bar m_{\xi^*}(x)^2\right) \leq \theta \qquad \forall\, |x - \xi^*| \geq \ell^* - 1, \quad \forall\, h \in (0, h_1] \tag{3.3}$$

On the other hand, recalling the definition (2.12), (2.7) implies

$$\lim_{h\downarrow 0} \left\| p - \beta(1 - \bar m_{\xi^*}^2) \right\|_\infty = 0 \tag{3.4}$$

From (3.3) and (3.4) it follows there are $\delta \in (\theta, 1)$ and $h_2 \in (0, h_1]$ such that, for $\ell^* \in (1, \xi^* - 1)$ as before,

$$p(x) \leq \delta \qquad \forall\, |x - \xi^*| \geq \ell^* - 1, \quad \forall\, h \in (0, h_2] \tag{3.5}$$

Lemma 3.1. *Let $h \in (0, h_2]$, h_2 and ℓ^* be as in (3.5). Then, for each $k \in [0, \xi^* - \ell^*]$ and $s \geq \xi^* + \ell^*$, we have*

$$q'(x) = \int_k^{k+1} dy\, H_k(x,y) q'(y) \qquad \forall\, x \in [0,k) \tag{3.6}$$

$$q'(x) = \int_{s-1}^{s} dy\, K_s(x,y) q'(y) \qquad \forall\, x \in (s,+\infty) \tag{3.7}$$

where $H_k(x,y)$, $x \in (0,k)$, and $K_s(x,y)$, $x > s$, are non negative continuous functions of y, strictly positive for some $y \in [k,k+1]$, $y \in [s-1,s]$ respectively.

Proof. Let us prove (3.6). We differentiate (3.1) at $x \in [0,k)$, obtaining (recall (3.2))

$$q'(x) = p(x) \int_0^k dy\, J_-(x,y) q'(y) + p(x) \int_k^{k+1} dy\, J_-(x,y) q'(y)$$

After N iteration we get

$$q'(x) = \int_k^{k+1} dy\, H_k^{(N)}(x,y) q'(y) + \int_0^k dy\, D_k^{(N)}(x,y) q'(y) \tag{3.8}$$

where

$$H_k^{(N)}(x,y) := \sum_{n=1}^{N} D_k^{(n)}(x,y), \qquad D_k^{(1)}(x,y) := p(x) J_-(x,y)$$

and, for $n > 1$, setting $x = y_0$ and $y = y_n$,

$$D_k^{(n)}(y_0,y_n) = \int_0^k dy_1 \cdots \int_0^k dy_{n-1} \prod_{i=1}^{n} p(y_{i-1}) J_-(y_{i-1},y_i)$$

The assumptions on J imply

$$0 \leq J_-(x,y) \leq J(x-y), \quad J_-(0,y) \equiv 0 \qquad \forall\, x,y \in \mathbb{R}_+ \tag{3.9}$$

and

$$\sup\{|y-x| \in \mathbb{R}_+ \,:\, J_-(x,y) > 0\} > 0 \qquad \forall\, x > 0 \tag{3.10}$$

From (2.12), (3.5), and (3.9) we get

$$0 \leq D_k^{(n)}(y_0,y_n) \leq \delta^{n-1} J^n(y_0,y_n) \tag{3.11}$$

Since $J^n(y_0,y_n)$ is a probability density and $\|q'\|_\infty < \infty$, the second integral in the r.h.s. of (3.8) vanishes as $N \to +\infty$ and we obtain (3.6) with

$$H_k(x,y) = \sum_{n=1}^{\infty} D_k^{(n)}(x,y) \tag{3.12}$$

and the series converges exponentiall y fast. Clearly $H_k(x,\cdot)$ is non negative and continuous. Moreover, from (3.10), it is strictly positive for some $y \in [k,k+1]$.

The case $x > s$ can be treated in the same manner, getting

$$K_s(x,y) = \sum_{n=1}^{\infty} R_s^{(n)}(x,y), \qquad R_s^{(1)}(x,y) := p(x) J(x-y) \tag{3.13}$$

where, for $n > 1$, setting $x = y_0$ and $y = y_n$,

$$R_s^{(n)}(y_0, y_n) = \int_s^{+\infty} dy_1 \cdots \int_s^{+\infty} dy_{n-1} \prod_{i=1}^{n} p(y_{i-1}) J(y_{i-1} - y_i) \tag{3.14}$$

In (3.13) we used that $J_-(u, v) = J(u - v)$ for $u > s > 1$ and $v \geq 0$. □

Proof of the monotonicity property. We first prove that there is an $h_3 \in (0, h_2]$ such that

$$q'(x) < 0 \qquad \forall\, |x - \xi^*| \leq \ell^*, \quad \forall\, h \in (0, h_3] \tag{3.15}$$

To prove (3.15) we differentiate (3.1) for $|x - \xi^*| \leq \ell^*$. Recalling $J_+(x, y) = J(x - y)$ when $x + y > 1$, we get,

$$q'(x) = p(x)(J * q)'(x) = p(x)(J' * (q - \bar m_{\xi^*}))(x) + p(x)(J * \bar m'_{\xi^*})(x) \tag{3.16}$$

Since $\bar m_{\xi^*}$ is strictly decreasing on $\mathbb{R}_+$, from (2.7) we get (3.15).

From (3.15) and Lemma 3.1 it follows $q'(x) < 0$ for all $x > 0$ and $h \in (0, h_3]$, thus getting the monotonicity property of the bump. □

We will prove Proposition 2.3 with $h^* = h_3$. We are thus left with the proof of (2.10). We follow the same strategy used in [2, §3]. In fact large part of that strategy can be adapted to our context without modification. We first need a weaker result.

Lemma 3.2. *There are $\eta > 0$ and $c > 0$ such that*

$$|q'(x)| \leq c e^{-\eta(x - \xi)} \qquad \forall\, x \in \mathbb{R}_+, \quad \forall\, h \in (0, h^*] \tag{3.17}$$

where $\xi = \xi(h)$ is the (unique) positive zero of the bump.

Proof. By definition (3.14), $R_s^{(n)}(x, y) = 0$ if $x > n + s$ and $y \in [s - 1, s]$, and it satisfies a bound analogous to (3.11). Then, from (3.7), for any $x > s \geq \xi^* + \ell^*$, we have

$$|q'(x)| \leq \beta \|q'\|_\infty \sum_{n \geq x - s} \delta^{n-1} \leq \delta^{-1} \beta \|q'\|_\infty e^{-(x-s)|\log \delta|} \tag{3.18}$$

Let ξ be the (unique) zero of $q(x)$ in $\mathbb{R}_+$. By (2.7) $\bar m_{\xi^*}(\xi) = \bar m(\xi - \xi^*)$ vanishes as $h \downarrow 0$, hence

$$\lim_{h \downarrow 0} [\xi(h) - \xi^*(h)] = 0 \tag{3.19}$$

In particular (3.19) implies there is $\ell < \infty$ such that

$$\xi + \ell \geq \xi^* + \ell^* \qquad \forall\, h \in [0, h^*] \tag{3.20}$$

and (3.17) follows from (3.18) with $s = \xi + \ell$. □

From (3.7), we have, for each $s \geq \ell$,

$$q'(x + \xi) = \int_{s-1}^{s} dy\, G_s(x, y) q'(y + \xi), \qquad \forall\, h \in (0, h^*] \tag{3.21}$$

where, setting $p_\xi(x) := p(x+\xi)$,

$$G_s(x,y) := \sum_{n=1}^{\infty} \int_s^{+\infty} dy_1 \cdots \int_s^{+\infty} dy_{n-1} \prod_{i=1}^{n} p_\xi(y_{i-1}) J(y_{i-1}-y_i) \tag{3.22}$$

We observe that $p_\xi(x)$ is a strictly decreasing function of x for $x>0$,

$$p_\xi(x) > \inf_{x>0} p_\xi(x) = p_\infty = \beta\left(1-(m^-_{\beta,h})^2\right) < 1 \tag{3.23}$$

and, by Lemma 3.2, there is $c'>0$ such that

$$p_\xi(x) \le p_\infty + c' e^{-\eta x}, \qquad \forall\, h \in [0,h^*] \tag{3.24}$$

In [2, Thm. 3.1] the asymptotics of $\bar m'(x)$ follows from an analogous (to (3.21)) expression for $\bar m'(x)$, where $p_\xi(x)$ is replaced by $p_{\bar m}(x) := \beta(1-\bar m(x)^2)$ in the definition of $G_s(x,y)$. The proof does not depend on the specific form of the function $p_{\bar m}(x)$, but only on the monotonicity property and the analogous of (3.23) and (3.24). Then a result as [2, Thm. 3.1] holds in our case. We conclude that there exist $M>0$ and $\delta\in(0,\gamma)$, γ as in (2.9), such that

$$\lim_{x\to+\infty} e^{\gamma x} q'(x+\xi) = -M, \qquad \lim_{x\to+\infty} e^{\delta x}\left(e^{\gamma x} q'(x+\xi)+M\right) = 0 \tag{3.25}$$

As in [2] the constant M is non zero because of the monotonicity property of $q'(x)$. Moreover, since $0<p_\infty<1$ and (3.24) holds uniformly in h, the constant $M=M(h)$ appearing in (3.25) remains bounded away from 0 as $h\downarrow 0$.

Analogously we obtain (2.10) (with $A=M\gamma^{-1}$) from (3.25) by arguing exactly as in the proofs of [2, Thms. 3.2, 3.3] where (2.6) follows as a corollary of [2, Thm. 3.1]. We omit the details. □

We end this section with a few remarks on the proof of Proposition 2.4. First of all we observe that in [2] it has been considered the finite volume case, i.e. the interval $[0,\ell]$ with Neumann boundary conditions. But since the estimates given in that paper are uniform in ℓ they include the case $\ell=\infty$ treated here.

Proof of 1). The existence of $\lambda>0$ and (2.14) are proven in [2, Thm. 2.1]. The bounds (2.16) follow from (2.8) and [2, Eq. (2.17)].

Proof of 2). The existence and uniqueness of $\gamma_v>\gamma$ solving (2.17) follows from [2, Lemma 3.1]. The proof of (2.18) is not given in [2], but it can be done in the same way as the proof of (2.10) given above. In fact, recalling the definition (2.11) of the linear operator L, we can rewrite the equation for v in (2.14) as $v=(1+\lambda)^{-1} pJ * v$; then by arguing as before we have

$$v(x+\xi) = \int_{s-1}^{s} dy\, \tilde G_s(x,y) v(y+\xi), \qquad \forall\, x > s \ge \ell$$

with ξ, ℓ as in (3.20) and

$$\tilde G_s(x,y) := \sum_{n=1}^{\infty} \frac{1}{(1+\lambda)^n} \int_s^{+\infty} dy_1 \cdots \int_s^{+\infty} dy_{n-1} \prod_{i=1}^{n} p_\xi(y_{i-1}) J(y_{i-1}-y_i)$$

Finally (2.19) is exactly [2, Eq. (2.23)], the only difference here is that this estimate can be proven also for $\zeta=\gamma_v$.

Proof of 3). This is done in [2, Thm. 2.4].
Proof of 4). (2.25) follows from [2, Eqs. (2.19), (9.44)].

4. The invariant manifold $\mathcal{M}$

In this section we prove Theorem 2.5, i.e. the existence of a one dimensional, invariant, expanding manifold $\mathcal{M}$ in $C^{\rm sym}(\mathbb{R};[-1,1])$ consisting of two branches that originate from the bump q.

In the sequel we will often need to study the dependence of the flow solution of (1.5) $S_t(m)$ on the initial datum m. A first estimate is

$$\|S_t(m+u)-S_t(m)\|_\infty \le e^{k_1 t}\|u\|_\infty \tag{4.1}$$

where $k_1>0$ is the Lipschitz coefficient of f, i.e. for any $m,u\in L_\infty(\mathbb{R})$,

$$\|f(m+u)-f(m)\|_\infty \le k_1\|u\|_\infty \tag{4.2}$$

For a more refined bound we observe that there is $k_2>0$ so that

$$\|f(m+u)-f(m)-L_m u\|_\infty \le k_2\|u\|_\infty^2 \tag{4.3}$$

where

$$L_m u := Df|_m u = -u + \frac{\beta}{\cosh^2\{\beta[J*m+h]\}} J*u \tag{4.4}$$

For $h\in(0,h^*]$ (h^* as in Proposition 2.3) let L, λ, v be as in Proposition 2.4. We next derive some properties of the evolution $S_t(q+u_0)$ starting from an initial datum $q+u_0$ with u_0 small. We set $u_t := S_t(q+u_0)-q$. Since $S_t(q)=q$ and $f(q)=0$ we have

$$\frac{du_t}{dt} = Lu_t + [f(q+u_t)-f(q)-Lu_t] \tag{4.5}$$

which implies

$$u_t = e^{Lt}u_0 + \int_0^t ds\, e^{L(t-s)}[f(q+u_s)-f(q)-Lu_s] \tag{4.6}$$

Then by (4.3) and (2.19) (with $\zeta=0$)

$$\left\|u_t - e^{Lt}u_0\right\|_\infty \le C_2\int_0^t ds\, e^{\lambda(t-s)}\|u_s\|_\infty^2 \tag{4.7}$$

where

$$C_2 := C_1 k_2 \tag{4.8}$$

Lemma 4.1. *There is $N>0$ such that if $u_0\in L_\infty(\mathbb{R})$ satisfies*

$$\sigma(u_0) := \frac{1}{\lambda}\log\frac{1}{N\|u_0\|_\infty} > 0 \tag{4.9}$$

then, for all $t<\sigma(u_0)$,

$$\left\|u_t - e^{Lt}u_0\right\|_\infty \le N\left(e^{\lambda t}\|u_0\|_\infty\right)^2 \tag{4.10}$$

and

$$\|u_t\|_\infty \le (1+C_1)e^{\lambda t}\|u_0\|_\infty \tag{4.11}$$

with C_1 as in (2.19).

Proof. The lemma will follows with $N := 4C_2\lambda^{-1}$. We prove (4.10) by contradiction. Fix $\sigma < \sigma(u_0)$ and define $\rho_\tau := e^{\lambda\tau}\|u_0\|_\infty$. Let $T \le \tau$ be the first time when the inequality (4.10) becomes an equality. Then, by (4.7) with $t = T$ (and supposing without loss of generality that $\|u_0\|_\infty > 0$),

$$\begin{aligned} N\left(e^{\lambda T}\|u_0\|_\infty\right)^2 &\le C_2\int_0^T ds\, e^{\lambda(T-s)}\left[e^{\lambda s}\|u_0\|_\infty + N\left(e^{\lambda s}\|u_0\|_\infty\right)^2\right]^2 \\ &\le C_2\, e^{\lambda T}\|u_0\|_\infty \int_0^T ds\, e^{\lambda s}\|u_0\|_\infty\,(1+N\rho_\tau)^2 \\ &\le C_2\,(1+N\rho_\tau)^2\,\lambda^{-1}\left(e^{\lambda T}\|u_0\|_\infty\right)^2 < N\left(e^{\lambda T}\|u_0\|_\infty\right)^2 \end{aligned} \tag{4.12}$$

where in the last inequality we used $N\rho_\tau < 1$. We have thus reached a contradiction and (4.10) is proved for all $t \le \tau$. Hence, by (2.19),

$$\begin{aligned} \|u_t\|_\infty &\le C_1 e^{\lambda t}\|u_0\|_\infty + N\left(e^{\lambda t}\|u_0\|_\infty\right)^2 \\ &\le (C_1 + N\rho_\tau)\, e^{\lambda t}\|u_0\|_\infty \le (1+C_1)e^{\lambda t}\|u_0\|_\infty \end{aligned} \tag{4.13}$$

for all $t \le \tau$ and Lemma 4.1 is proved. □

We use in the sequel the following notation. For v, N as in Proposition 2.4 and Lemma 4.1 we denote by ρ any positive number such that

$$N\rho\|v\|_\infty < 1 \tag{4.14}$$

and define

$$\psi_{\pm\varepsilon} := q \pm \varepsilon v, \qquad \varepsilon \in [0,\rho] \tag{4.15}$$

$$\tau(\rho,\varepsilon) := \frac{1}{\lambda}\log\frac{\rho}{\varepsilon}, \qquad \text{i.e. } e^{\lambda\tau(\rho,\varepsilon)} = \frac{\rho}{\varepsilon} \tag{4.16}$$

We observe that $\pm\varepsilon v$, $\varepsilon \in [0,\rho]$ satisfy the hypothesis of Lemma 4.1 and that $\tau(\rho,\varepsilon) < \sigma(\pm\varepsilon v)$, $\sigma(\cdot)$ as in (4.9). Hence, for any $t \le \tau(\rho,\varepsilon)$,

$$\left\|S_t\left(\psi_{\pm\varepsilon}\right) - \left(q \pm e^{\lambda t}\varepsilon v\right)\right\|_\infty \le N\left(e^{\lambda t}\varepsilon\|v\|_\infty\right)^2 \tag{4.17}$$

and

$$\|S_t\left(\psi_{\pm\varepsilon}\right) - q\|_\infty \le (1+C_1)e^{\lambda t}\varepsilon\|v\|_\infty \tag{4.18}$$

Theorem 4.2. *For any $h \in (0,h^*]$ (h^* as in Proposition 2.3), there are $\rho > 0$ and $w_s^\pm \in C_0^{\mathrm{sym}}(\mathbb{R})$, $s \le 0$, such that, for any $s \le 0$,*

$$\lim_{\varepsilon\downarrow 0}\|S_{\tau(\rho,\varepsilon)+s}(\psi_{\pm\varepsilon}) - w_s^\pm\|_\infty = 0 \tag{4.19}$$

Moreover

$$\lim_{s\to-\infty}\|w_s^\pm - q\|_\infty = 0; \qquad S_t(w_s^\pm) = w_{s+t}^\pm \quad \text{if } s+t \le 0 \tag{4.20}$$

A uniformity in $s \le 0$ of the limit (4.19) is proven in Proposition 4.6 below to which we refer for a precise statement.

Proof of Theorem 4.2. We will next prove that if ρ is small enough then $\{S_{\tau(\rho,\varepsilon)}(\psi_{\pm\varepsilon}) : \varepsilon \in (0,\rho]\}$ is a Cauchy sequence as $\varepsilon \downarrow 0$. Without loss of generality we restrict to the case with the plus sign. Then we need to estimate

$$S_{\tau(\rho,\varepsilon')}(\psi_{\varepsilon'}) - S_{\tau(\rho,\varepsilon)}(\psi_\varepsilon) \qquad 0 < \varepsilon' < \varepsilon \tag{4.21}$$

Observing that

$$\psi_\varepsilon = q + e^{\lambda\tau(\varepsilon,\varepsilon')}\varepsilon' v$$

by (4.17),

$$\left\| S_{\tau(\varepsilon,\varepsilon')}(\psi_{\varepsilon'}) - \psi_\varepsilon \right\|_\infty \le N\|v\|_\infty^2 \varepsilon^2 \tag{4.22}$$

We thus need to compare $S_t(\psi_\varepsilon)$ and $S_t(\tilde m)$, $t \le \tau(\rho,\varepsilon)$, for all functions $\tilde m$ such that

$$\|\tilde m - \psi_\varepsilon\|_\infty \le N\|v\|_\infty^2 \varepsilon^2 \tag{4.23}$$

Let

$$\Delta_t := S_t(\psi_\varepsilon) - S_t(\tilde m) \tag{4.24}$$

By (4.3) we have

$$\frac{d\Delta_t}{dt} = L_{S_t(\psi_\varepsilon)}\Delta_t + R_t^{(1)}, \qquad \|R_t^{(1)}\|_\infty \le k_2\|\Delta_t\|_\infty^2 \tag{4.25}$$

Since $\|L_{m+u}\Delta - L_m\Delta\|_\infty \le c'\|u\|_\infty\|\Delta\|_\infty$ with c' a suitable constant, by (4.18) there is C_3 so that

$$R_t^{(2)} := L_{S_t(\psi_\varepsilon)}\Delta_t - L\Delta_t, \qquad \|R_t^{(2)}\|_\infty \le C_3\rho\|\Delta_t\|_\infty \qquad \forall\, t \le \tau(\rho,\varepsilon) \tag{4.26}$$

Thus

$$\frac{d\Delta_t}{dt} = L\Delta_t + R_t^{(2)} + R_t^{(1)} \tag{4.27}$$

and

$$\Delta_t = e^{Lt}\Delta_0 + \int_0^t ds\, e^{L(t-s)}[R_s^{(2)} + R_s^{(1)}] \tag{4.28}$$

Then by (2.19) and the bounds in (4.25)-(4.26), for any $t \le \tau(\rho,\varepsilon)$,

$$\|\Delta_t\|_\infty \le C_1 e^{\lambda t}\Delta_0 + C_1\int_0^t ds\, e^{\lambda(t-s)}[C_3\rho\|\Delta_s\|_\infty + k_2\|\Delta_s\|_\infty^2]$$

Calling

$$\lambda^* := \lambda + C_2C_3\rho \tag{4.29}$$

we have, by iteration and recalling (4.8),

$$\|\Delta_t\|_\infty \le C_1 e^{\lambda^* t}\Delta_0 + C_2\int_0^t ds\, e^{\lambda^*(t-s)}\|\Delta_s\|_\infty^2 \tag{4.30}$$

Setting $W_t := e^{-\lambda^* t}\|\Delta_t\|_\infty$ and using (4.23), from (4.30) we get, for all $t \le \tau(\rho,\varepsilon)$,

$$W_t \le C_1 N \|v\|_\infty^2 \varepsilon^2 + C_2 \int_0^t ds\, W_s^2 \tag{4.31}$$

which implies

$$W_t \le c\varepsilon^2 \sum_{n=0}^{\infty} \left(c\varepsilon^2 t\right)^n, \qquad c := C_1(1 \vee C_2) N \|v\|_\infty^2 \tag{4.32}$$

Since $\varepsilon\tau(\rho,\varepsilon) \to 0$ as $\varepsilon \downarrow 0$, we can choose $\varepsilon_1 \in (0,\rho]$ so that the series converges and $W_t \le 2c\varepsilon^2$ for all $\varepsilon \in (0,\varepsilon_1]$ and $t \le \tau(\rho,\varepsilon)$. Choosing ρ small enough so that

$$C_2 C_3 \rho \le \frac{\lambda}{2} \qquad \text{i.e.} \qquad e^{\lambda^* \tau(\rho,\varepsilon)} \le \left(\frac{\rho}{\varepsilon}\right)^{3/2} \tag{4.33}$$

and recalling (4.16), (4.29), and the definition of W_t, we get

$$\|\Delta_t\|_\infty \le C_4 \sqrt{\varepsilon}, \qquad \forall\, \varepsilon \in (0,\varepsilon_1] \quad \forall\, t \le \tau(\rho,\varepsilon) \tag{4.34}$$

with $C_4 := 2c\rho^{3/2}$.

By (4.22) and (4.34) we conclude that

$$\left\| S_{\tau(\rho,\varepsilon')}\left(\psi_{\varepsilon'}\right) - S_{\tau(\rho,\varepsilon)}\left(\psi_{\varepsilon}\right) \right\|_\infty \le C_4 \sqrt{\varepsilon} \qquad \text{if } 0 < \varepsilon' < \varepsilon \le \varepsilon_1 \tag{4.35}$$

which shows $\{S_{\tau(\rho,\varepsilon)}\left(\psi_\varepsilon\right)\}$ is a Cauchy sequence as $\varepsilon \downarrow 0$ for all ρ small enough. Analogously we argue for the case with the minus sign.

The same argument shows that also $S_{\tau(\rho,\varepsilon)+s}\left(\psi_{\pm\varepsilon}\right)$ is, for each $s \le 0$, a Cauchy sequence. Then $S_{\tau(\rho,\varepsilon)+s}\left(\psi_{\pm\varepsilon}\right)$ converges in sup norm as $\varepsilon \downarrow 0$ to a function $w_s^\pm$, hence (4.19). Moreover if $t+s \le 0$, $t \ge 0$, then $S_t\left(S_{\tau(\rho,\varepsilon)+s}\left(\psi_\varepsilon\right)\right) = S_{\tau(\rho,\varepsilon)+s+t}\left(\psi_\varepsilon\right)$. By (4.1) for each $t \ge 0$, $S_t(m)$ depends continuously on m, thus $S_t\left(S_{\tau(\rho,\varepsilon)+s}\left(\psi_\varepsilon\right)\right) \to S_t(w_s^\pm)$ as $\varepsilon \downarrow 0$. On the other hand $S_{\tau(\rho,\varepsilon)+s+t}\left(\psi_\varepsilon\right) \to w_{s+t}^\pm$ as $\varepsilon \downarrow 0$, hence $S_t\left(w_s^\pm\right) = w_{s+t}^\pm$, proving the second relation in (4.20). Finally, from (4.18),

$$\left\| S_{\tau(\rho,\varepsilon)+s}\left(\psi_{\pm\varepsilon}\right) - q \right\|_\infty \le C_5\, e^{\lambda s}, \qquad C_5 := (1+C_1)\rho\|v\|_\infty \tag{4.36}$$

from which, letting $\varepsilon \downarrow 0$,

$$\left\| w_s^\pm - q \right\|_\infty \le C_5\, e^{\lambda s} \tag{4.37}$$

proving the first statement in (4.20), Theorem 4.2 is proved. □

Proof of Theorem 2.5. The manifold

$$\mathcal{M} := \mathcal{M}^+ \cup \mathcal{M}^-, \quad \mathcal{M}^\pm := \left\{ S_t(w_s^\pm) \,:\, s \le 0 \le t \right\} \tag{4.38}$$

and both its branches $\mathcal{M}^\pm$ are invariant under S_t which on $\mathcal{M}^\pm$ is invertible. By (4.20) $\mathcal{M}^\pm$ originate at $s = -\infty$ from q.

Recalling (4.16), from (4.17)

$$\left\| S_{\tau(\rho,\varepsilon)+s}\left(\psi_{\pm\varepsilon}\right) - \left(q \pm e^{\lambda s}\rho v\right) \right\|_\infty \le C_6\, e^{2\lambda s}, \qquad C_6 := N\left(\rho\|v\|_\infty\right)^2 \tag{4.39}$$

from which, letting $\varepsilon \downarrow 0$,

$$\left\| w_s^\pm - \left(q \pm e^{\lambda s}\rho v\right) \right\|_\infty \le C_6\, e^{2\lambda s} \tag{4.40}$$

Next, by (4.5) and recalling that $f(q)=0$,

$$\begin{aligned}\frac{dw_s^\pm}{ds} &= L(w_s^\pm - q) + \left[f(w_s^\pm) - f(q) - L\left(w_s^\pm - q\right)\right] = L\left[w_s^\pm - \left(q \pm e^{\lambda s}\rho v\right)\right] \\ &\quad \pm \lambda e^{\lambda s}\rho v + \left[f(w_s^\pm) - f(q) - L\left(w_s^\pm - q\right)\right] \end{aligned} \tag{4.41}$$

Denoting by $\|L\|_\infty$ the norm of the operator L (which is finite), by (4.3), (4.37), and (4.40) we have

$$\left\|\frac{dw_s^\pm}{ds} \mp \lambda e^{\lambda s}\rho v\right\|_\infty \le C_7\, e^{2\lambda s}, \qquad C_7 := \|L\|_\infty C_6 + k_2 C_5^2 \tag{4.42}$$

Recalling that $v(x) \approx \tilde m'(\xi^* - x)$ in the sense of (2.25), we set

$$s_0 :\ \rho e^{\lambda s_0} = C_{\tilde m}^{-1/2}, \qquad m_s^\pm := w_{s+s_0}^\pm \tag{4.43}$$

Then, letting $\bar v(x) = C_{\tilde m}^{-1/2}\, v(x)$, (4.42) implies

$$\left\|\frac{dm_s^\pm}{ds} \mp \lambda e^{\lambda s}\bar v\right\|_\infty \le C_7\, e^{2\lambda s}$$

which gives (2.29).

The proofs of the monotonicity property of $m_s^\pm$ and of the bound (2.30) will be given in Proposition 4.3 below, Theorem 2.5 is then proved. ☐

Proposition 4.3. *For any $s \in \mathbb{R}$, the symmetric functions $m_s^\pm$ are non increasing on $\mathbb{R}_+$ and* (2.30) *holds.*

In order to prove Proposition 4.3 we need the following properties of the flow S_t.

Theorem 4.4. (The Comparison Theorem, [4]**)** *Let $m, \tilde m \in L_\infty(\mathbb{R})$ be such that $m(x) \le \tilde m(x)$ for all $x \in \mathbb{R}$. Then $S_t(m)(x) \le S_t(\tilde m)(x)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.*

Lemma 4.5. *Let $m \in L_\infty^{\rm sym}(\mathbb{R})$ be a non increasing function on $\mathbb{R}_+$. Then $S_t(m)$ has the same monotonicity property for all $t \in \mathbb{R}_+$.*

Proof. The flow solution $S_t(m)$ solves the integral equation

$$S_t(m) = e^{-t}m + \int_0^t ds\, \tanh\left\{\beta\left[J * S_s(m) + h\right]\right\}$$

Since J is smooth, the function $g_t(x) := S_t(m)(x) - e^{-t}m(x)$ is differentiable. Further its spatial derivative $g_t'(x)$ is an antisymmetric function which satisfies, for any $x \in \mathbb{R}_+$,

$$\begin{aligned} g_t'(x) &= \int_0^t ds\, p_s(x)\left(J_- g_s'\right)(x) + z_t(x) \\ p_s(x) &:= \frac{\beta}{\cosh^2\left\{\beta\left[\left(J * S_s(m)\right)(x) + h\right]\right\}}, \qquad z_t(x) := \int_0^t ds\, e^{-s}p_s(x)(J' * m)(x) \end{aligned} \tag{4.44}$$

where we used that, since g_s is differentiable,

$$\frac{d}{dx}\left(J * S_s(m)\right)(x) = \left(J * g_s'\right)(x) + e^{-s}(J' * m)(x) = \left(J_- g_s'\right)(x) + e^{-s}(J' * m)(x)$$

with $J_- g_s$ as in definition (3.2). By iteration of (4.44), calling $(t,x)=(s_0,x_0)$, we get

$$g_t'(x)=\sum_{n=1}^{\infty}\int_0^{s_0}ds_1\cdots\int_0^{s_{n-1}}ds_n\int_0^{+\infty}dx_1\cdots\int_0^{+\infty}dx_n\prod_{k=1}^{n-1}p_s(x_k)J_-(x_k,x_{k+1})z_{s_n}(x_n) \tag{4.45}$$

Using the fact that both J and m are symmetric and non increasing on $\mathbb{R}_+$, it is easy to check that $J'*m$ is a non positive function on $\mathbb{R}_+$. On the other hand both $J_-(x,y)$ and $p_s(x)$ are non negative for $x,y\ge 0$. We conclude from (4.45) that also g_t' is a non positive function on $\mathbb{R}_+$. Then $S_t(m)$ is non increasing on $\mathbb{R}_+$ because sum of two functions with this property. The Lemma is proved. □

Proof of Proposition 4.3. Since the difference between $m_s^\pm$ and $w_s^\pm$ is only a time shift, see (4.43), it is enough to prove the proposition for $w_s^\pm$.

We start with the monotonicity property. We use Theorem 4.2 and Lemma 4.5. Thus the first step is to show that for ε small the functions $\psi_{\pm\varepsilon}=q\pm\varepsilon v$ are non increasing on $\mathbb{R}_+$. To this purpose we first notice that, by definition (see (2.14)),

$$v'(x)=-\frac{2\beta}{1+\lambda}q(x)q'(x)(J*v)(x)+\frac{\beta}{1+\lambda}(J'*v)(x) \tag{4.46}$$

Then from (2.10) and (2.18), for a suitable constant C_8

$$\sup_{x>1}\left|e^{\gamma_v x}v'(x)\right|\le C_8\sup_{x>1}e^{\gamma_v x}v(x)<\infty \tag{4.47}$$

which implies that

$$\sup_{x>1}\left|\frac{v'(x)}{q'(x)}\right|<\infty \tag{4.48}$$

For $x\in[0,1]$, since $q'(0)=0$, we need to show that $q''(0)\neq 0$. This is easily seen by noticing that since $q'(0)=0$, and both J' and q' are antisymmetric functions,

$$q''(0)=-2\beta(1-q(0)^2)\int_0^1 dy\,J'(y)q'(y)<0 \tag{4.49}$$

In the last inequality we used that, by our assumptions on the function J and Proposition 2.3, $J'(x)q'(x)>0$ for $x\in(0,1)$. From (4.48) and (4.49) we then get

$$\sup_{x\in\mathbb{R}_+}\left|\frac{v'(x)}{q'(x)}\right|<\infty \tag{4.50}$$

Lemma 4.5 and (4.50) imply that for any $s\le 0$ there is $\varepsilon_s\in(0,\rho]$ such that $\{S_{\tau(\rho,\varepsilon)+s}(\psi_{\pm\varepsilon}) : \varepsilon\in(0,\varepsilon_s]\}$ is a sequence of non increasing functions on $\mathbb{R}_+$. Hence from (4.19) the same property holds for $w_s^\pm$, $s\le 0$. Then the monotonicity property of $w_s^\pm$ for all $s\in\mathbb{R}$ follows from Lemma 4.5.

We are left with the bound (2.30). Since q solves (1.7) and it is strictly decreasing on $\mathbb{R}^+$, it follows that $m_{\beta,h}^-<q(x)<m_{\beta,h}^+$ for all $x\in\mathbb{R}$. We also recall that q satisfies (2.10).

Since v is a positive function which satisfies (2.18) with $\gamma_v > \gamma$, we conclude that, for all ε small enough,

$$m^-_{\beta,h} \le \psi_{-\varepsilon}(x) < q(x) < \psi_\varepsilon(x) < m^+_{\beta,h} \tag{4.51}$$

Then (2.30) follows from Theorem 4.2 and the Comparison Theorem. □

We conclude this section by proving Proposition 4.6 below, which is a stronger version of Theorem 4.2, since we show that the curves $\{w^\pm_s\}$ are the limits, in sup norm, of the curves $S_{\tau(\rho,\delta)}\mathcal{C}_\delta$ where, for any $\delta > 0$,

$$\mathcal{C}_\delta := \{\psi_\varepsilon \,:\, 0 < \varepsilon < \delta\}$$

Proposition 4.6. *Let $\delta > 0$, $s \le 0$, and*

$$\delta(s) := e^{\lambda s}\delta \tag{4.52}$$

Then

$$\lim_{\delta\downarrow 0}\sup_{s\le 0}\left\| S_{\tau(\rho,\delta)}\left(\psi_{\pm\delta(s)}\right) - w^\pm_s\right\|_\infty = 0 \tag{4.53}$$

Proof. Without loss of generality we restrict to the case with the plus sign in (4.52) and (4.53). We need to show that for any $\eta > 0$ there is $\delta_\eta > 0$ so that for any $\delta < \delta_\eta$ and $s \le 0$

$$\left\| S_{\tau(\rho,\delta)}\left(\psi_{\delta(s)}\right) - w^+_s\right\|_\infty \le \eta \tag{4.54}$$

We approximate w^+_s by $S_t(\psi_\varepsilon)$ for suitable values of ε and t: given $s \le 0$ let ε_0 be such that for $\varepsilon \in (0,\varepsilon_0]$

$$\left\| S_{\tau(\rho,\varepsilon)+s}\left(\psi_\varepsilon\right) - w^+_s\right\|_\infty \le \frac{\eta}{2} \tag{4.55}$$

For $\delta < \rho$ we have

$$S_{\tau(\rho,\varepsilon)+s}\left(\psi_\varepsilon\right) = S_{\tau(\rho,\delta)}\left(S_{\tau(\delta(s),\varepsilon)}\left(\psi_\varepsilon\right)\right) \tag{4.56}$$

By (4.17),

$$\left\| S_{\tau(\delta(s),\varepsilon)}\left(\psi_\varepsilon\right) - \psi_{\delta(s)}\right\|_\infty = \left\| S_{\tau(\delta(s),\varepsilon)}\left(\psi_\varepsilon\right) - q - e^{\lambda\tau(\delta(s),\varepsilon)}\varepsilon v\right\|_\infty = N\|v\|^2_\infty\delta(s)^2 \tag{4.57}$$

We define

$$D_t := S_t\left(\psi_{\delta(s)}\right) - S_t\left(S_{\tau(\delta(s),\varepsilon)}\left(\psi_\varepsilon\right)\right) \tag{4.58}$$

so that

$$\left\| D_{\tau(\rho,\delta)}\right\|_\infty = \left\| S_{\tau(\rho,\varepsilon)+s}\left(\psi_\varepsilon\right) - S_{\tau(\rho,\delta)}\left(\psi_{\delta(s)}\right)\right\|_\infty \tag{4.59}$$

The analysis of D_t is identical to that of Δ_t in the proof of Theorem 4.2. In fact, by comparing (4.23)–(4.24) with (4.57)–(4.58), we see that D_t satisfies the conditions defining the function Δ_t when the parameter ε appearing in (4.23)–(4.24) is replaced by $\delta(s)$. Then the bound (4.34) applied to D_t becomes

$$\|D_t\|_\infty \le C_4\sqrt{\delta(s)} \qquad \forall\,\delta \in (0,\varepsilon_1] \quad \forall\, t \le \tau(\rho,\delta(s))$$

which implies

$$\left\| D_{\tau(\rho,\delta)}\right\|_\infty \le C_4\sqrt{\delta} \qquad \forall\,\delta \in (0,\varepsilon_1] \tag{4.60}$$

We then choose $\delta_\eta \in (0, \varepsilon_1]$ so small that

$$C_4 \sqrt{\delta_\eta} \le \frac{\eta}{2} \tag{4.61}$$

and we have by (4.55) and (4.59)-(4.61) that, for all $\delta < \delta_\eta$,

$$\begin{aligned}\left\| S_{\tau(\rho,\delta)}\left(\psi_{\delta(s)}\right) - w_s^+ \right\|_\infty &\le \left\| S_{\tau(\rho,\varepsilon)+s}\left(\psi_\varepsilon\right) - w_s^+ \right\|_\infty \\ &+ \left\| S_{\tau(\rho,\varepsilon)+s}\left(\psi_\varepsilon\right) - S_{\tau(\rho,\delta)}\left(\psi_{\delta(s)}\right) \right\|_\infty \le \eta \end{aligned} \tag{4.62}$$

Proposition 4.6 is then proved. □

5. Motion along the manifold and convergence to $m^\pm_{\beta,h}$

In this section we prove Theorem 2.6. To this purpose we will define suitable functions $Q_a^+ \le q \le Q_a^-$, a a small parameter, which are close to q, see (5.10) below. We shall prove that the functions m_s^+ (resp. m_s^-) at a certain time s are above Q_a^- (resp. below Q_a^+). Then, by the Comparison Theorem it is enough to study the evolution of $Q_a^\pm$. Using the spectral properties of the linear operator L, we show that, for a time interval $T_a \sim |\log a|$, the evolution $S_{T_a}(Q_a^+)$ (resp. $S_{T_a}(Q_a^-)$) can be bounded from above (resp. below) by the same functions Q_a^+ (resp. Q_a^-) suitably translated in space, see Theorem 5.2 below. By the Comparison Theorem we can iterate the argument, thus getting bounds at longer times which, combined with general properties of the flow S_t, lead to the desired result, Corollary 5.3 below, from which Theorem 2.6 will follow.

In the sequel we shall need a more refined *apriori* bound on the evolution around the critical droplet, which is the content of the following lemma.

Lemma 5.1. *There is $K > 0$ such that if $u_t := S_t(q + u_0) - q$, $u_0 \in L_\infty(\mathbb{R})$, then for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$*

$$\left| u_t(x) - e^{Lt} u_0(x) \right| \le K \int_0^t ds\, e^{L(t-s)} \left(J * e^{Ls} u_0 \right)^2 (x) + K \mathcal{R}_t[u.] \tag{5.1}$$

where

$$\mathcal{R}_t[u.] = e^{\lambda t} \sup_{s \in [0,t]} \left\{ \|u_s\|_\infty^3 + \|u_s\|_\infty \left\| u_s - e^{Ls} u_0 \right\|_\infty + \left\| u_s - e^{Ls} u_0 \right\|_\infty^2 \right\} \tag{5.2}$$

Proof. We recall u_t solves (4.6). Expanding the tanh appearing in (2.2) around $\beta(J * q + h)$ and using that q solves (1.7) we have

$$f\left(q + u_s\right) - f(q) - L u_s = \Phi \left(J * u_s \right)^2 + \frac{\beta^3}{3!} \tanh'''(\theta_s) \left(J * u_s \right)^3 \tag{5.3}$$

where

$$\Phi(x) := -\beta^2 q(x) \left(1 - q(x)^2 \right) \tag{5.4}$$

while θ_s is a number in the interval with endpoints $\beta[J*q+h]$ and $\beta[J*(q+u_s)+h]$. Then we rewrite (4.6) as

$$u_t = e^{Lt}u_0 + \int_0^t ds\, e^{L(t-s)}\left[\Phi\left(J*e^{Ls}u_0\right)^2 + R[u_s]\right] \tag{5.5}$$

where, using (5.3),

$$R[u_s] = \Phi\left[(J*u_s)^2 - \left(J*e^{Ls}u_0\right)^2\right] + \frac{\beta^3}{3!}\tanh'''(\theta_s)\,(J*u_s)^3 \tag{5.6}$$

Since Φ is a bounded function on $\mathbb{R}$, the first integral on the r.h.s. of (5.5) is bounded by the first term on the r.h.s. of (5.1) with any $K \ge \|\Phi\|_\infty$.

We next rewrite the square bracket on the r.h.s. of (5.6) as

$$(J*u_s)^2 - \left(J*e^{Ls}u_0\right)^2 = \left[J*(u_s - e^{Ls}u_0)\right]^2 + 2\left(J*e^{Ls}u_0\right)\left[J*(u_s-e^{Ls}u_0)\right]$$

Then using $\tanh'''$ is bounded and J has compact support, from (2.19) we have, for any K large enough,

$$\left\|\int_0^t ds\, e^{L(t-s)}R[u_s]\right\|_\infty \le K\mathcal{R}_t[u.]$$

The lemma is proved. □

Warning: In the sequel we shall denote by C a generic constant whose numerical value may change from line to line. From the statements it will appear clear on which parameters it depends on.

Let γ and λ be as in (2.9) and (2.16) respectively. We fix δ and R_0 such that

$$0 < \delta < \frac{1}{8}, \qquad \frac{3}{2} < \gamma R_0 < 2 - 4\delta \tag{5.7}$$

and we set, for any $a \in (0,1]$,

$$T_a := \frac{\delta}{\lambda}|\log a|, \qquad R_a := R_0|\log a|, \qquad \Delta_a := a^{1-\delta/2} \tag{5.8}$$

Recalling (2.16), (2.17), and (2.23), there exists an $\bar h \in (0,h^*]$, h^* as in Proposition (2.3), such that

$$(\gamma_v - \gamma)R_0 \le \frac{\delta}{4} \quad \text{ and } \quad \delta\lambda^{-1}\omega > 3 \qquad \forall\, h \in [0,\bar h] \tag{5.9}$$

We define the symmetric functions

$$Q_a^\pm(x) := q_a^\pm(x)\mathbf{1}_{|x|\le R_a} + \left[m_{\beta,h}^- \pm a^{3/2}\right]\mathbf{1}_{|x|>R_a} \tag{5.10}$$

where

$$q_a^+(x) := q(|x|+a), \qquad q_a^-(x) := q(0)\mathbf{1}_{|x|\le a} + q(|x|-a)\mathbf{1}_{|x|>a} \tag{5.11}$$

The main result of this section is the content of the following theorem.

Theorem 5.2. *Let $h \in [0, \bar{h}]$ with $\bar{h}$ (5.9). Then there is $a_0 \in (0,1]$ such that, for any $a \in (0, a_0]$,*

$$S_{T_a}\left(Q_a^+\right)(x) \le Q_a^+(x + \Delta_a) \tag{5.12}$$

and

$$S_{T_a}\left(Q_a^-\right)(x) \ge Q_a^-(x - \Delta_a) \tag{5.13}$$

with T_a and Δ_a as in (5.8).

Proof. From Proposition 2.3 there is a constant c such that $|q''(x)| \le c|q'(x)|$ for any $|x| \ge 1$. Then, by expanding to second order $q_a^{\pm}(x)$ around $q(x)$ for $|x| \ge 1$, and using $q_a^+(x) \le q(x) \le q_a^-(x)$ for all $x \in \mathbb{R}$ (see (5.11)) we have, for any a small enough,

$$q_a^+(x) \le q(x) + \frac{a}{2} q'(|x|) \mathbf{1}_{|x|\ge 1}, \qquad q_a^-(x) \ge q(x) - \frac{a}{2} q'(|x|) \mathbf{1}_{|x|\ge 1} \tag{5.14}$$

Observing $q'(|x|) = -|q'(x)|$ for any $x \in \mathbb{R}$, if we define

$$\varphi(x) := \frac{1}{2}|q'(x)| \mathbf{1}_{|x|\ge 1} \tag{5.15}$$

from (5.10) and (5.14) we obtain

$$Q_a^+(x) \le q(x) - a\varphi(x) + \left[m_{\beta,h}^- - q(x) + a^{3/2} + a\varphi(x)\right] \mathbf{1}_{|x|>R_a} \tag{5.16}$$

$$Q_a^-(x) \ge q(x) + a\varphi(x) + \left[m_{\beta,h}^- - q(x) - a^{3/2} - a\varphi(x)\right] \mathbf{1}_{|x|>R_a} \tag{5.17}$$

Moreover, from (2.10) and (5.7), for any a small enough,

$$|m_{\beta,h}^- - q(x)| + a|\varphi(x)| \le \frac{1}{2} a^{3/2} \qquad \forall\, |x| > R_a$$

so that, if we define

$$U_0^{\pm}(x) := \mp a\varphi(x) \pm \frac{3}{2} a^{3/2} \mathbf{1}_{|x|>R_a} \tag{5.18}$$

from (5.16) and (5.17) we get, for any a small enough,

$$Q_a^+(x) \le q(x) + U_0^+(x), \qquad Q_a^-(x) \ge q(x) + U_0^-(x) \tag{5.19}$$

We shall now obtain good bounds on $S_{T_a}\left(q + U_0^{\pm}\right)$. We can apply Lemma 5.1 to $U_t^{\pm} := S_t\left(q + U_0^{\pm}\right) - q$ so that

$$\left|U_t^{\pm} - e^{Lt} U_0^{\pm}\right| \le K \int_0^t ds\, e^{L(t-s)} \left(J * e^{Ls} U_0^{\pm}\right)^2 + K \mathcal{R}_t\left[U_\cdot^{\pm}\right] \tag{5.20}$$

We will use (5.53) to obtain good bounds on $U_{T_a}^{\pm}$. We analyze separately the various terms appearing.

Estimate on $e^{LT_a} U_0^{\pm}$. Since $e^{\lambda T_a} = a^{-\delta}$, see (5.8), we have

$$e^{LT_a} U_0^{\pm} = \mp a^{1-\delta} \pi(\varphi) v \mp a e^{LT_a}\left[\varphi - \pi(\varphi) v\right] \pm \frac{1}{2} a^{3/2} e^{LT_a} \mathbf{1}_{|x|>R_a} \tag{5.21}$$

where, recalling (2.21) and (5.15),

$$\pi(\varphi) = \int_1^{\infty} dx\, \frac{v(x)}{p(x)}\, |q'(x)| > 0 \tag{5.22}$$

From the spectral gap property (2.22) and (5.9),

$$\left\| e^{LT_a}\left[\varphi - \pi(\varphi)v\right]\right\|_\infty \le e^{-\omega T_a}\|\varphi - \pi(\varphi)v\|_\infty \le Ca^3 \tag{5.23}$$

Analogously we estimate

$$e^{LT_a}\mathbf{1}_{|x|>R_a} = a^{-\delta}\pi\left(\mathbf{1}_{|x|>R_a}\right)v + e^{LT_a}\left[\mathbf{1}_{|x|>R_a} - \pi\left(\mathbf{1}_{|x|>R_a}\right)v\right] \le Ca^{3/2-\delta} \tag{5.24}$$

where we used $\pi\left(\mathbf{1}_{|x|>R_a}\right) \le Ca^{\gamma_v R_0}$ with $\gamma_v R_0 > \gamma R_0 > 3/2$. From (5.21), (5.23) and (5.24) we obtain

$$\left|e^{LT_a}U_0^\pm \pm a^{1-\delta}\pi(\varphi)v\right| \le Ca^{3-\delta} \tag{5.25}$$

Estimate on $\int_0^{T_a} ds\, e^{L(T_a-s)}\left(J * e^{Ls}U_0^\pm\right)^2$. Using (2.19) with $\zeta = 0$ and (5.24), we get, for any a small enough,

$$\begin{aligned}\int_0^{T_a} ds e^{L(T_a-s)}\left(J * e^{Ls}U_0^\pm\right)^2 &\le Ca^2\int_0^{T_a} ds\, e^{L(T_a-s)}\Big[\left(J * e^{Ls}\varphi\right)^2 + a\left(J * e^{Ls}\mathbf{1}_{|x|\ge R_a}\right)^2 \\ &\qquad + \sqrt{a}\left(J * e^{Ls}\varphi\right)\left(J * e^{Ls}\mathbf{1}_{|x|\ge R_a}\right)\Big] \\ &\le Ca^{3-2\delta} + Ca^2\int_0^{T_a} ds\, e^{L(T_a-s)}(J * e^{Ls}\varphi)\left[(J * e^{Ls}\varphi) + a^{2-\delta}\right]\end{aligned} \tag{5.26}$$

Now, recalling the definitions (5.15) and (2.13), from the asymptotics (2.10) it follows $\varphi \in X_\gamma$. Since $\gamma < \gamma_v$ we can use (2.19) with $\zeta = \gamma$. Hence, since J has compact support,

$$\left|\left(J * e^{Ls}\varphi\right)(x)\right| \le Ce^{\lambda s - \gamma|x|}$$

Therefore, by applying again (2.19) with $\zeta = \gamma$,

$$\int_0^{T_a} ds\, e^{L(T_a-s)}\left(J * e^{Ls}\varphi\right)^2 \le Ca^{-2\delta}e^{-\gamma|x|}$$

so that from (5.26), for all a small enough,

$$\int_0^{T_a} ds\, e^{L(t-s)}\left(J * e^{Ls}U_0^\pm\right)^2 \le C\left(a^{2-2\delta}e^{-\gamma|x|} + a^{3-2\delta}\right) \tag{5.27}$$

Estimate on $\mathcal{R}_t[U_\cdot^\pm]$. We use Lemma 4.1 to obtain *apriori* bounds. Since $\|U_0^\pm\|_\infty \le Ca$, comparing the definitions (4.9) and (5.8) and using $\delta < 1$ we conclude that for all a small enough $\sigma(U_0^\pm) > T_a$. Therefore from (4.10) and (4.11)

$$\|U_t^\pm\|_\infty \le (1 + C_1)a^{1-\delta} \qquad \forall\, t \le T_a \tag{5.28}$$

and

$$\left\|U_t^\pm - e^{Lt}U_0^\pm\right\|_\infty \le Na^{2-2\delta} \qquad \forall\, t \le T_a \tag{5.29}$$

Recalling (5.2), from (5.28) and (5.29) we get

$$\mathcal{R}_{T_a}\left[U_\cdot^\pm\right] \le Ca^{3-4\delta} \tag{5.30}$$

Collecting (5.20), (5.25), (5.27), and (5.30), we conclude that, for any a small enough,

$$\left|U_{T_a}^\pm(x) \pm a^{1-\delta}\pi(\varphi)v(x)\right| \le C\left(a^{2-2\delta}e^{-\gamma|x|} + a^{3-4\delta}\right) \tag{5.31}$$

Therefore, from the Comparison Theorem and (5.19), recalling $U_t^\pm = S_t\left(q + U_0^\pm\right) - q$, we finally get

$$S_{T_a}\left(Q_a^+\right)(x) \le q(x) - C\left[a^{1-\delta}v(x) - a^{2-2\delta}e^{-\gamma|x|} - a^{3-4\delta}\right] \tag{5.32}$$

$$S_{T_a}\left(Q_a^-\right)(x) \ge q(x) + C\left[a^{1-\delta}v(x) - a^{2-2\delta}e^{-\gamma|x|} - a^{3-4\delta}\right] \tag{5.33}$$

Next we shall find good bounds on $Q_a^\pm(x \pm \Delta_a)$. We first observe that, since $\Delta_a \le 1$, $|x| \le R_a - 1$ implies $|x + \Delta_a| \le R_a$ while $|x| > R_a + 1$ implies $|x + \Delta_a| > R_a$. Moreover, from (2.10), (5.7), and (5.11), $|q_a^\pm(x \pm \Delta_a) - m_{\beta,h}^-| \le a^{3/2}$ if $R_a - 1 < |x| \le R_a + 1$ and a is small enough. Hence

$$Q_a^+(x + \Delta_a) \ge q_a^+(x + \Delta_a)\mathbf{1}_{|x|\le R_a+1} + \left[m_{\beta,h}^- + a^{3/2}\right]\mathbf{1}_{|x|>R_a+1}$$

$$Q_a^-(x - \Delta_a) \le q_a^-(x - \Delta_a)\mathbf{1}_{|x|\le R_a+1} + \left[m_{\beta,h}^- - a^{3/2}\right]\mathbf{1}_{|x|>R_a+1}$$

Now we notice $q_a^+(x + \Delta_a) \ge q_{a+\Delta_a}^+(x)$ and $q_a^-(x - \Delta_a) \le q_{a+\Delta_a}^-(x)$ for all $x \in \mathbb{R}$. Moreover, since $|q''(x)| \le c|q'(x)|$ for $|x| \ge 1$, by expanding to the second order for $|x| > 1$ and to the first one for $|x| \le 1$, we get, if a is small enough,

$$q_{a+\Delta_a}^+(x) \ge q(x) - (a + \Delta_a)\psi(x), \qquad q_{a+\Delta_a}^-(x) \le q(x) + (a + \Delta_a)\psi(x) \tag{5.34}$$

where

$$\psi(x) := 2\left[|q'(x)| + \|q'\|_\infty \mathbf{1}_{|x|\le 1}\right] \tag{5.35}$$

hence

$$Q_a^+(x + \Delta_a) \ge \left[q(x) - (a + \Delta_a)\psi(x)\right]\mathbf{1}_{|x|\le R_a+1} + \left[m_{\beta,h}^- + a^{3/2}\right]\mathbf{1}_{|x|>R_a+1} \tag{5.36}$$

$$Q_a^-(x - \Delta_a) \le \left[q(x) + (a + \Delta_a)\psi(x)\right]\mathbf{1}_{|x|\le R_a+1} + \left[m_{\beta,h}^- - a^{3/2}\right]\mathbf{1}_{|x|>R_a+1} \tag{5.37}$$

We can now conclude the proof of the theorem. We consider first the case $|x| \le R_a + 1$. Since v is strictly positive and obeys the aymptotics (2.18), and q satisfies (2.10), from (5.9) and (5.35) we have

$$v(x) \ge Ca^{\delta/4}\psi(x) \qquad \forall\, |x| \le R_a + 1 \tag{5.38}$$

On the other hand, using (5.7),

$$a^{2-2\delta}e^{-\gamma|x|} + a^{3-4\delta} \le Ca\psi(x) \qquad \forall\, |x| \le R_a + 1$$

Therefore, for any a small enough,

$$a^{1-\delta}v(x) - a^{2-2\delta}e^{-\gamma|x|} - a^{3-4\delta} \ge Ca^{1-3\delta/4}\psi(x) \qquad \forall\, |x| \le R_a + 1 \tag{5.39}$$

Since $(a + \Delta_a)a^{-1+3\delta/4}$ vanishes as $a \downarrow 0$, (5.12) and (5.13) for $|x| \le R_a + 1$ follow from (5.32), (5.33), (5.36), (5.37), and (5.39).

Finally we consider the case $|x| > R_a + 1$. Using $\gamma_v R_0 > \gamma R_0$ and (5.7), from (5.32) and (5.33) we get

$$\lim_{a\downarrow 0} a^{-3/2} \sup_{|x|>R_a+1} \left|S_{T_a}\left(Q_a^\pm\right)(x) - m_{\beta,h}^-\right| = 0 \tag{5.40}$$

Then (5.12) and (5.13) for $|x| > R_a + 1$ and a small enough follow from (5.36), (5.37), and (5.40). □

Corollary 5.3. *In the same hypothesis of Theorem 5.2, there is $a_1 \in (0, a_0]$ such that, for any $a \in (0, a_1]$,*

$$\lim_{t\to+\infty} \left\| S_t\left(Q_a^+\right) - m_{\beta,h}^- \right\|_\infty = 0 \tag{5.41}$$

$$\lim_{t\to+\infty} S_t\left(Q_a^-\right)(x) = m_{\beta,h}^+ \qquad \forall\, x \in \mathbb{R} \tag{5.42}$$

To prove the above Corollary we need the following Barrier Lemma.

Lemma 5.4. (The Barrier Lemma, [4]**)** *There are V and C^* positive so that if $m, \tilde m \in L_\infty(\mathbb{R}; [-1,1])$ and, for some $x_0 \in \mathbb{R}$ and $T > 0$, $m(x) = \tilde m(x)$ for all $|x - x_0| \le VT$, then*

$$|S_t(m)(x_0) - S_t(\tilde m)(x_0)| \le C^* e^{-T}$$

Proof of Corollary 5.3. We first prove (5.41). By (5.12) and the Comparison Theorem, for any integer n,

$$S_{nT_a}\left(Q_a^+\right)(x) \le Q_a^+(x + n\Delta_a) \qquad \forall\, x \in \mathbb{R}$$

From (5.10) the function on the r.h.s. of the above inequality is identically equal to $m_{\beta,h}^- + a^{3/2}$ for all $x > R_a - n\Delta_a$. On the other hand $S_{nT_a}\left(Q_a^+\right)$ is a symmetric function for all integer n, then

$$S_{nT_a}\left(Q_a^+\right)(x) \le m_{\beta,h}^- + a^{3/2} \qquad \forall\, x \in \mathbb{R} \quad \forall\, n > \frac{R_a}{\Delta_a}$$

Using again the Comparison Theorem and recalling $m_{\beta,h}^- \le Q_a^+$, we conclude that

$$m_{\beta,h}^- \le S_t\left(Q_a^+\right) \le S_t\big(m_{b,h}^- + a^{3/2}\big) \qquad \forall\, t > \left(1 + \frac{R_a}{\Delta_a}\right) T_a \tag{5.43}$$

We now observe that $S_t\big(m_{\beta,h}^- + a^{3/2}\big)$ solves the homogenous equation

$$\frac{d\rho(t)}{dt} = -\rho(t) + \tanh\{\beta[\rho(t) + h]\} \tag{5.44}$$

with initial datum $m_{\beta,h}^- + a^{3/2}$. Since the free energy density (1.1) is a Lyapunov functional for the flow evolution (5.44), it is easy to verify that the intervals $(-1, m_{\beta,h}^0)$ and $(m_{\beta,h}^0, 1)$ are basins of attraction of the stationary solutions $m_{\beta,h}^-$ and $m_{\beta,h}^+$. Then for any a small enough,

$$\lim_{t\to+\infty} S_t\big(m_{\beta,h}^- + a^{3/2}\big) = m_{\beta,h}^- \tag{5.45}$$

From (5.43) and (5.45) we get (5.41).

We shall next prove (5.42). We need to show that for any $x_0 \in \mathbb{R}$ and $\varepsilon > 0$ there is T_{ε,x_0} so that

$$\left| S_t\left(Q_a^-\right)(x_0) - m_{\beta,h}^+ \right| < \varepsilon \qquad \forall\, t > T_{\varepsilon,x_0} \tag{5.46}$$

By (5.13) and the Comparison Theorem, for any integer n,

$$S_{nT_a}\left(Q_a^-\right)(x) \ge Q_a^-(x - n\Delta_a) \qquad \forall\, x \in \mathbb{R}$$

Recalling the definition (5.10) we have

$$S_{nT_a}\left(Q_a^-\right)(x) \geq q(x - n\Delta_a - a) \qquad \forall\, x > n\Delta_a$$

Since Q_a^- is symmetric and non increasing for $x > 0$, from Lemma 4.5 we have

$$S_{nT_a}\left(Q_a^-\right)(x) \geq q(0) \qquad \forall\, x \in [0, n\Delta_a]$$

Hence

$$S_{nT_a}\left(Q_a^-\right)(x) \geq q(0)\mathbf{1}_{|x|\leq n\Delta_a} + q(|x| - n\Delta_a)\mathbf{1}_{|x|\geq n\Delta_a}$$

We conclude that for any $R > 0$ we can find a time T_R so that

$$S_t\left(Q_a^-\right)(x) \geq q(0) \qquad \forall\, |x| \leq R \quad \forall\, t > T_R \tag{5.47}$$

We can now prove (5.46). Given any $\varepsilon > 0$ we choose T_ε so large that $C^* e^{-T_\varepsilon} < \varepsilon/2$ and $R \geq |x_0| + VT_\varepsilon$, C^*, V as in the Barrier Lemma 5.4. Hence from the Comparison Theorem, (5.47), and the Barrier Lemma it follows that

$$S_t\left(Q_a^-\right)(x_0) > S_t\left(q(0)\right) - \frac{\varepsilon}{2} \qquad \forall\, t > T_R + T_\varepsilon \tag{5.48}$$

On the other hand since $q(0)$ belongs to the basin of attraction of $m^+_{\beta,h}$ w.r.t. the dynamics (5.44) (in fact $m^0_{\beta,h} < 0 < q(0) < m^+_{\beta,h}$), there is $\bar{T}$ such that

$$\left|S_t\left(q(0)\right) - m^+_{\beta,h}\right| < \frac{\varepsilon}{2} \qquad \forall\, t > \bar{T} \tag{5.49}$$

Recalling $Q_a^- \leq m^+_{\beta,h}$, from the Comparison Theorem, (5.48), and (5.49) we finally get

$$m^+_{\beta,h} - \varepsilon < S_t\left(Q_a^-\right) \leq m^+_{\beta,h} \qquad \forall\, t > \bar{T} \vee (T_\varepsilon + T_R)$$

which implies (5.46) with $T_{\varepsilon,x_0} = \bar{T} \vee (T_\varepsilon + T_R)$. □

Proof of Theorem 2.6. Let $w_s^\pm$, $s \leq 0$, be as in Theorem 4.2. We will next prove that for any $a > 0$ small enough there is $s_a < 0$ such that

$$w^-_{s_a}(x) \leq Q_a^+(x) \quad \text{and} \quad Q_a^-(x) \leq w^+_{s_a}(x) \qquad \forall\, x \in \mathbb{R} \tag{5.50}$$

Theorem 2.6 will then follows from the Comparison Theorem, Corollary 5.3, (2.30), and (5.50) (recall that the relation between $w_s^\pm$ and $m_s^\pm$ is only a time shift, see (4.43)).

To prove (5.50) we need a more accurated estimate on the difference $w_s^\pm - \left(q \pm e^{\lambda s}\rho v\right)$. This is the content of Proposition 5.5 below.

Proposition 5.5. *Let $w_s^\pm$ be as in Theorem 4.2 and γ as in* (2.9)*. Then there is a constant $\bar{C}$ so that, for all $x \in \mathbb{R}$ and $s \leq 0$,*

$$\left|w_s^\pm(x) - q \mp e^{\lambda s}\rho v(x)\right| \leq \bar{C}\left(e^{2\lambda s - \gamma|x|} + e^{3\lambda s}\right) \tag{5.51}$$

Proof. We apply Lemma 5.1 with $u_0 = \pm\varepsilon v$, getting

$$\left|S_t\left(\psi_{\pm\varepsilon}\right) - q \mp e^{\lambda s}\varepsilon v(x)\right| \leq K\varepsilon^2 \int_0^t dt'\, e^{2\lambda t'} e^{L(t-t')}(J * v)^2 + K\mathcal{R}_t\left[S_\cdot\left(\psi_{\pm\varepsilon}\right) - q\right] \tag{5.52}$$

We bound $(J*v)^2$ by $\|v\|_\infty J*v$; next we observe that since J has compact support and v satisfies (2.18) with $\gamma_v > \gamma$, hence $J*v \in X_\gamma$, see (2.13). Then, by applying (2.19) with $\zeta = \gamma$,

$$K\varepsilon^2 \int_0^t dt'\, e^{2\lambda t'} e^{L(t-t')}(J*v)^2 \le KC_1 \|J*v\|_{\gamma,\infty} \|v\|_\infty \lambda^{-1}\varepsilon^2 e^{2\lambda t - \gamma|x|} \tag{5.53}$$

We now observe that for $t = \tau(\rho,\varepsilon) + s$, $s \le 0$, the r.h.s. of (5.53) is bounded, uniformly as $\varepsilon \downarrow 0$, by const $e^{2\lambda s - \gamma|x|}$. Analogously, from (4.36), (4.39), and (5.2), we get that $K\mathcal{R}_{\tau(\rho,\varepsilon)+s}\,[S.\,(\psi_{\pm\varepsilon}) - q]$ is bounded by const $e^{3\lambda s}$. The Proposition is proved. □

Proof of (5.50). We first observe that, arguing as in getting (5.36) and (5.37), for all a small enough we have

$$Q_a^+(x) \ge [q(x) - a\psi(x)]\,\mathbf{1}_{|x|\le R_a} + \left[m_{\beta,h}^- + a^{3/2}\right]\mathbf{1}_{|x|>R_a} \tag{5.54}$$

$$Q_a^-(x) \le [q(x) + a\psi(x)]\,\mathbf{1}_{|x|\le R_a} + \left[m_{\beta,h}^- - a^{3/2}\right]\mathbf{1}_{|x|>R_a} \tag{5.55}$$

where ψ is defined in (5.35). We then set

$$s_a := \frac{r}{\lambda}\log a \qquad \text{with } \frac{1}{2} < r < 1 - \frac{\delta}{4} \text{ and } \delta \text{ as in (5.7)} \tag{5.56}$$

Recalling (2.18) and that $\gamma_v R_0 > \gamma R > 3/2$, from (5.51) it follows

$$\lim_{a\downarrow 0} a^{-3/2} \sup_{|x|>R_a} \left| w_{s_a}^{\pm}(x) - m_{\beta,h}^- \right| = 0 \tag{5.57}$$

On the other hand, by using (5.38), we also have, if a is small enough,

$$e^{\lambda s_a}\rho v(x) > a\psi(x) \qquad \forall\, x \in \mathbb{R} \tag{5.58}$$

Then (5.50) follows from (5.51), (5.54)–(5.58). Theorem 2.6 is proved. □

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