On the Confidence Interval for the parameter of Poisson Distribution

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Abstract

In present paper the possibility of construction of continuous analogue of Poisson distribution with the search of bounds of confidence intervals for parameter of Poisson distribution is discussed and the results of numerical construction of confidence intervals are presented.

PACS number(s): 02.70.Lq, 06.20.Dk
Keywords: Statistics, Confidence Intervals, Poisson Distribution.

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Introduction

In paper [1] the unified approach to the construction of confidence intervals and confidence limits for a signal with a background presence, in particular for Poisson distributions, is proposed. The method is widely used for the presentation of physical results [2] though a number of investigators criticize this approach [3] (in particular, this approach avoids a violation of the coverage principle). Series of Workshops on Confidence Limits has been held in CERN and Fermilab. At these meetings demands for properties of constructed confidence intervals and confidence limits have been formulated [4]. On the other hand, the results of experiments often give noninteger values of a number of observed events (for example, after fitting [5]) when Poisson distribution take place. That is why there is a necessity to search a continuous analogue of Poisson distribution. The present work offers some generalization of Poisson distribution for continuous case. The generalization given here allows to construct confidence intervals and confidence limits for Poisson distribution parameter both for integer and real values of a number of observed events, using conventional methods. More than, the supposition about continuous of some function $f(x, \lambda)$ described below allows to use Gamma distribution for construction of confidence intervals and confidence limits of Poisson distribution parameter. In present paper we consider only the construction of confidence intervals.

In the Section 1 the generalization of Poisson distribution for the continuous case is introduced. An example of confidence intervals construction for the parameter of analogue of Poisson distribution is given in the Section 2. In the Section 3 the results of construction of confidence intervals having the minimal length for the parameter of Poisson distribution using Gamma distribution are discussed. The main results of the paper are formulated in the Conclusion.

1 The Generalization of Discrete Poisson Distribution for the Continuous Case

Let us have a random value ξ, taking values from the set of numbers $x \in X$. Let us consider two-dimensional function $f(x, \lambda) = \dfrac{\lambda^x}{x!}e^{-\lambda}$, where $x \geq 0 \quad \lambda > 0$.

Assume, that set X includes only integer numbers, then discrete function $f(x, \lambda)$ describes distribution of probabilities for Poisson distribution with the parameter λ and random variable x

Let us rewrite the density of Gamma distribution using unconventional notation

$f(x, a, \lambda) = \dfrac{a^{x+1}}{\Gamma(x+1)} e^{-a\lambda} \lambda^x$, where a is a scale parameter, $x > -1$ is a shape parameter and $\lambda > 0$ is a random variable. Here the quantities of x and λ take values from the set of real numbers. Let $a = 1$ and as is the convention $x! = \Gamma(x+1)$, then a continuous function

$f(x, \lambda) = \dfrac{\lambda^x}{x!} e^{-\lambda}$, $\lambda > 0$, $x > -1$ is the density of Gamma distribution with the scale parameter $a = 1$.

To anticipate a little, it is indicative of the Gamma distribution of parameter λ for the Poisson distribution in case of observed value $x = \hat{x}$.

Figure 1 shows the surface described by the function $f(x, \lambda)$. Smooth behaviour of this function along x and λ (see Fig.2) allows to assume that there is such a function $-1 < l(\lambda)$, that $\int_{l(\lambda)}^{\infty} f(x, \lambda) dx = 1$ for given value of λ. It means that in this way we introduce continued analogue of Poisson distribution with the probability density $f(x, \lambda) = \dfrac{\lambda^x}{x!} e^{-\lambda}$ over the area of function definition, i.e. for $x \geq l(\lambda)$ and $\lambda > 0$. The values of the function $f(x, \lambda)$ for integer x coincide with corresponding magnitudes in the probabilities distribution of discrete Poisson distribution. Dependences of the values of function $l(\lambda)$, the means and the variances for the suggested distribution on λ were calculated by using programme DGQUAD from the library CERNLIB [6] and the results are presented in Table 1. This Table shows that series of properties of Poisson distribution ($E\xi = \lambda, D\xi = \lambda$) take place only if the value of the parameter $\lambda > 3$.

It is appropriate at this point to say that

$\int_0^{\infty} f(x, \lambda) dx = \int_0^{\infty} \dfrac{\lambda^x e^{-\lambda}}{\Gamma(x+1)} dx = e^{-\lambda} \nu(\lambda)$.

The function $\nu(\lambda) = \int_0^{\infty} \dfrac{\lambda^x}{\Gamma(x+1)} dx$ is well known and, according to ref. [7],

$\nu(\lambda) = \sum_{n=-N}^{\infty} \dfrac{\lambda^n}{\Gamma(n+1)} + O(|\lambda|^{-N-0.5}) = e^{\lambda} + O(|\lambda|^{-N})$

if $\lambda \to \infty$, $|arg\lambda| \le \frac{\pi}{2}$ for any integer N. Nevertheless we have to use the function $l(\lambda)$ in our calculations in Section 2. We consider it as a mathematical trick for easy construction of confidence intervals by numerically.

In principle, we can numerically to transform the function $f(x,\lambda)$ in the interval $x \in (0,1)$ so that
$\int_0^\infty f(x,\lambda)dx = 1$, $E\xi = \int_0^\infty xf(x,\lambda)dx = \lambda$ and
$D\xi = \int_0^\infty (x - E\xi)^2 f(x,\lambda)dx = \lambda$ for any λ. In this case we can construct confidence intervals without introducing of $l(\lambda)$.

In Section 3 only assumption about continuous of the function $f(x,\lambda)$ along the variable x are used for construction of confidence intervals of parameter λ for any observed $\hat{x}$.

Let us construct a central confidence intervals for the continued analogue of Poisson distribution using function $l(\lambda)$.

2 The Construction of the Confidence Intervals for Continued Analogue of Poisson Distribution.

Assume that in the experiment with the fixed integral luminosity the $\hat{x}$ events ($\hat{x}$ is not necessity integer) of some Poisson process were observed. It means that we have an experimental estimation $\hat{\lambda}(\hat{x})$ of the parameter λ of Poisson distribution. We have to construct a confidence interval $(\hat{\lambda}_1(\hat{x}), \hat{\lambda}_2(\hat{x}))$, covering the true value of the parameter λ of the distribution under study with confidence level $1-\alpha$, where α is a significance level. It is known from the theory of statistics [8], that the value of mean of selected data is an unbiassed estimation of mean of distribution under study. In our case the sample consists of one observation $\hat{x}$. For the discrete Poisson distribution the mean coincides with the estimation of parameter value, i.e. $\hat{\lambda} = \hat{x}$. This is not true for small value of λ in considered case (see Table 1). That is why in order to find the estimation of $\hat{\lambda}(\hat{x})$ for small value $\hat{x}$ there is necessary to introduce correction in accordance with Table 1. Let us construct the central confidence intervals using conventional method assuming that

$\int_{\hat{x}}^\infty f(x,\hat{\lambda}_1)dx = \dfrac{\alpha}{2}$ for the lower bound $\hat{\lambda}_1$ and

$\int_{l(\hat{\lambda}_2)}^{\hat{x}} f(x, \hat{\lambda}_2)dx = \frac{\alpha}{2}$ for the upper bound $\hat{\lambda}_2$ of confidence interval.

Figure 3 shows the introduced in the Section 1 distributions with parameters defined by the bounds of confidence interval ($\hat{\lambda}_1 = 1.638, \hat{\lambda}_2 = 8.498$) for the case $\hat{x} = \hat{\lambda} = 4$ and the Gamma distribution with parameters $a = 1$, $x = \hat{x} = 4$. The association between the confidence interval and the Gamma distribution is seen from this Figure. The bounds of confidence interval with 90% confidence level for parameter of continued analogue of Poisson distribution for different observed values $\hat{x}$ (first column) were calculated and are given in second column of the Table 2. It is necessary to notice that the confidence level of the constructed confidence intervals always coincides exactly with the required confidence level. As it results from Table 2 that the suggested approach allows to construct confidence intervals for any real and integer values of the observed number of events in the case of the values of parameter $\lambda > 3$. The Table 2 shows that the left bound of central confidence intervals is not equal to zero for small $\hat{x}$. It is not suitable.

Also note that 90% of the area of Gamma distributions with parameter $x = \hat{x}$ are contained inside the constructed 90% confidence intervals for observed value $\hat{x}$ (for small values of $\lambda < 0.3$ we have got 88%). It points out the possibility of Gamma distribution usage for confidence intervals construction for parameter of Poisson distribution.

3 Shortest Confidence Intervals for Parameter of Poisson Distribution.

As is follow from formulae for $f(x, \lambda)$ (see Fig.3) we may suppose that the parameter λ of Poisson distribution for the observed value $\hat{x}$ has Gamma distribution[1] with the parameters $a = 1$ and $x = \hat{x}$. This supposition allows to choose confidence interval of minimum length from all possible confidence intervals of given confidence level without violation of the coverage principle. The bounds of minimum length area, containing 90% of the corresponding Gamma distribution square, were found by numerically both for integer value of $\hat{x}$ and for real value of $\hat{x}$. Here we took into account that $\hat{\lambda} = \hat{x}$, constructed the central 90% confidence interval and, then, found the shortest 90% con-

[1]The similar supposition is discussed in ref. [9]

fidence interval for the parameter of Poisson distribution. The results are presented in third column of Table 2. For comparison with the results of conventional procedure [2] of finding confidence intervals, the results of calculations of confidence intervals for integer value of $\hat{x}$ [1] are adduced in the Table 2. By this means confidence intervals, got using Gamma distribution, may be used for real values of $\hat{x}$, even though the $\hat{x}$ is negative ($\hat{x} > -1$).

Conclusion

In the paper the attempt of introducing of continued analogue of Poisson distribution for the construction of classical confidence intervals for the parameter λ of Poisson distribution is described. Two approaches (with using of function $l(\lambda)$ and with using of Gamma distribution) are considered. Confidence intervals for different integer and real values of number of observed events for Poisson process in the experiment with given integral luminosity are constructed. As seems the approach with the use of Gamma distribution for construction of confidence intervals more preferable than approach with using of function $l(\lambda)$.

Acknowledgments

We are grateful to V.A. Matveev, V.F. Obraztsov and Fred James for the interest to this work and for valuable comments. We are thankful to S.S. Bityukov and V.A. Litvine for useful discussions. We would like to thank E.A.Medvedeva for the help in preparing the paper. This work has been supported by RFFI grant 99-02-16956 and grant INTAS-CERN 377.

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Table 1: The function $l(\lambda)$, mean and variance versus λ.

λ	$l(\lambda)$	mean ($E\xi$)	variance ($D\xi$)
0.001	-0.297	-0.138	0.024
0.002	-0.314	-0.137	0.029
0.005	-0.340	-0.130	0.040
0.010	-0.363	-0.120	0.052
0.020	-0.388	-0.100	0.071
0.050	-0.427	-0.051	0.113
0.100	-0.461	0.018	0.170
0.200	-0.498	0.142	0.272
0.300	-0.522	0.256	0.369
0.400	-0.539	0.365	0.464
0.500	-0.553	0.472	0.559
0.600	-0.564	0.577	0.653
0.700	-0.574	0.681	0.748
0.800	-0.582	0.785	0.844
0.900	-0.590	0.887	0.939
1.00	-0.597	0.989	1.035
1.50	-0.622	1.495	1.521
2.00	-0.639	1.998	2.012
2.50	-0.650	2.499	2.506
3.00	-0.656	3.000	3.003
3.50	-0.656	3.500	3.501
4.00	-0.647	4.000	3.999
4.50	-0.628	4.500	4.498
5.00	-0.593	5.000	4.997
5.50	-0.539	5.500	5.497
6.00	-0.466	6.000	5.996
6.50	-0.373	6.500	6.495
7.00	-0.262	7.000	6.995
7.50	-0.135	7.500	7.494
8.00	0.000	8.000	7.993
8.50	0.000	8.500	8.496
9.00	0.000	9.000	8.997
9.50	0.000	9.500	9.498
10.0	0.000	10.00	9.999

Table 2: 90% C.L. intervals for the Poisson signal mean λ for total events observed $\hat{x}$.

	bounds	(Section 2)	bounds	(Section 3)	bounds	(ref[1])
$\hat{x}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_1$	$\hat{\lambda}_2$
0.000	0.121E-08	2.052	0.0	2.302	0.00	2.44
0.001	0.205E-08	2.054	0.0	2.304		
0.002	0.292E-08	2.056	0.0	2.306		
0.005	0.666E-08	2.061	0.0	2.311		
0.01	0.307E-07	2.076	0.0	2.320		
0.02	0.218E-06	2.098	0.0	2.337		
0.05	0.765E-05	2.166	1.66E-05	2.389		
0.10	0.137E-03	2.275	2.23E-05	2.474		
0.20	0.186E-02	2.490	6.65E-05	2.642		
0.30	0.696E-02	2.692	1.49E-04	2.806		
0.40	0.161E-01	2.891	2.60E-03	2.969		
0.50	0.295E-01	3.084	5.44E-03	3.129		
0.60	0.466E-01	3.269	1.35E-02	3.290		
0.70	0.673E-01	3.450	2.63E-02	3.452		
0.80	0.911E-01	3.629	4.04E-02	3.611		
0.90	0.1179	3.804	6.12E-02	3.773		
1.0	0.1473	3.977	8.49E-02	3.933	0.11	4.36
1.5	0.3257	4.800	0.2391	4.718		
2.0	0.5429	5.582	0.4410	5.479	0.53	5.91
2.5	0.7896	6.340	0.6760	6.220		
3.0	1.056	7.076	0.9284	6.937	1.10	7.42
3.5	1.340	7.792	1.219	7.660		
4.0	1.638	8.493	1.511	8.358	1.47	8.60
4.5	1.946	9.188	1.820	9.050		
5.0	2.264	9.869	2.120	9.714	1.84	9.99
5.5	2.590	10.55	2.453	10.39		
6.0	2.924	11.21	2.775	11.05	2.21	11.47
6.5	3.264	11.87	3.126	11.72		
7.0	3.609	12.53	3.473	12.38	3.56	12.53
7.5	3.961	13.18	3.808	13.01		
8.0	4.316	13.82	4.160	13.65	3.96	13.99
8.5	4.677	14.46	4.532	14.30		
9.0	5.041	15.10	4.905	14.95	4.36	15.30
9.5	5.406	15.73	5.252	15.56		
10.	5.779	16.36	5.640	16.21	5.50	16.50
20.	13.65	28.49	13.50	28.33	13.55	28.52

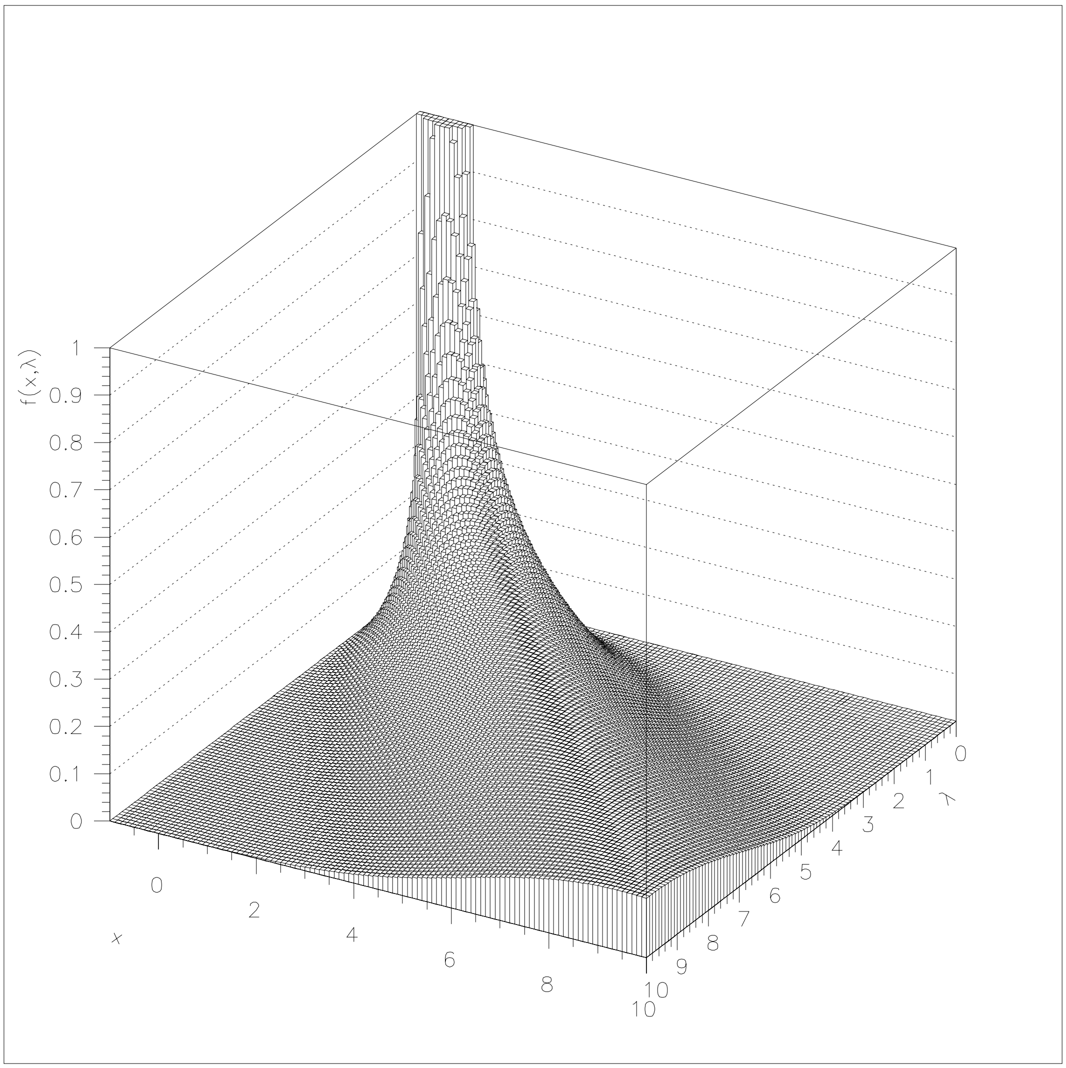

Figure 1: The behaviour of the function $f(x, \lambda)$ versus λ and x if $f(x, \lambda) < 1$.

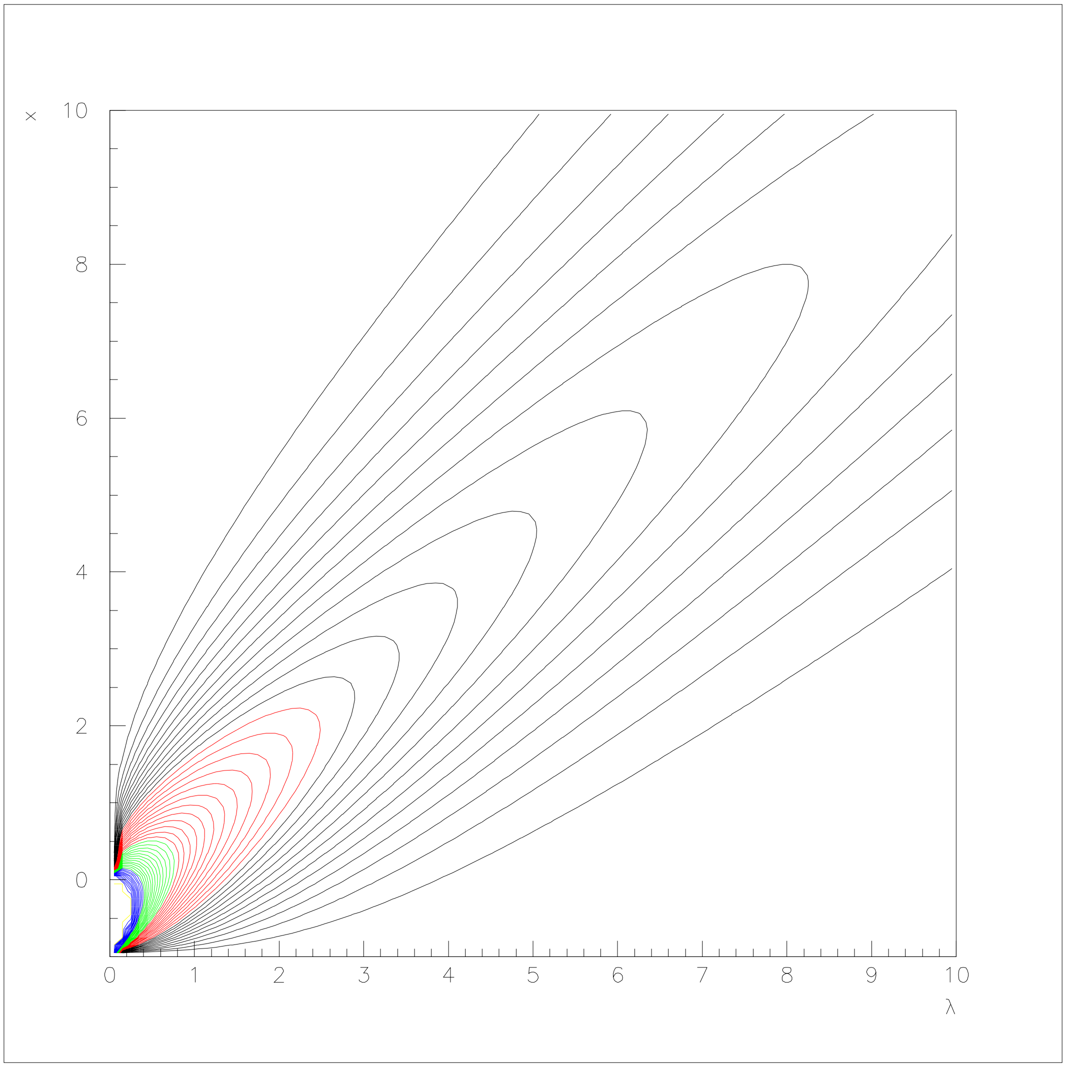

Figure 2: Two-dimensional representation of the function $f(x, \lambda)$ versus λ and x for values $f(x, \lambda) < 1$.

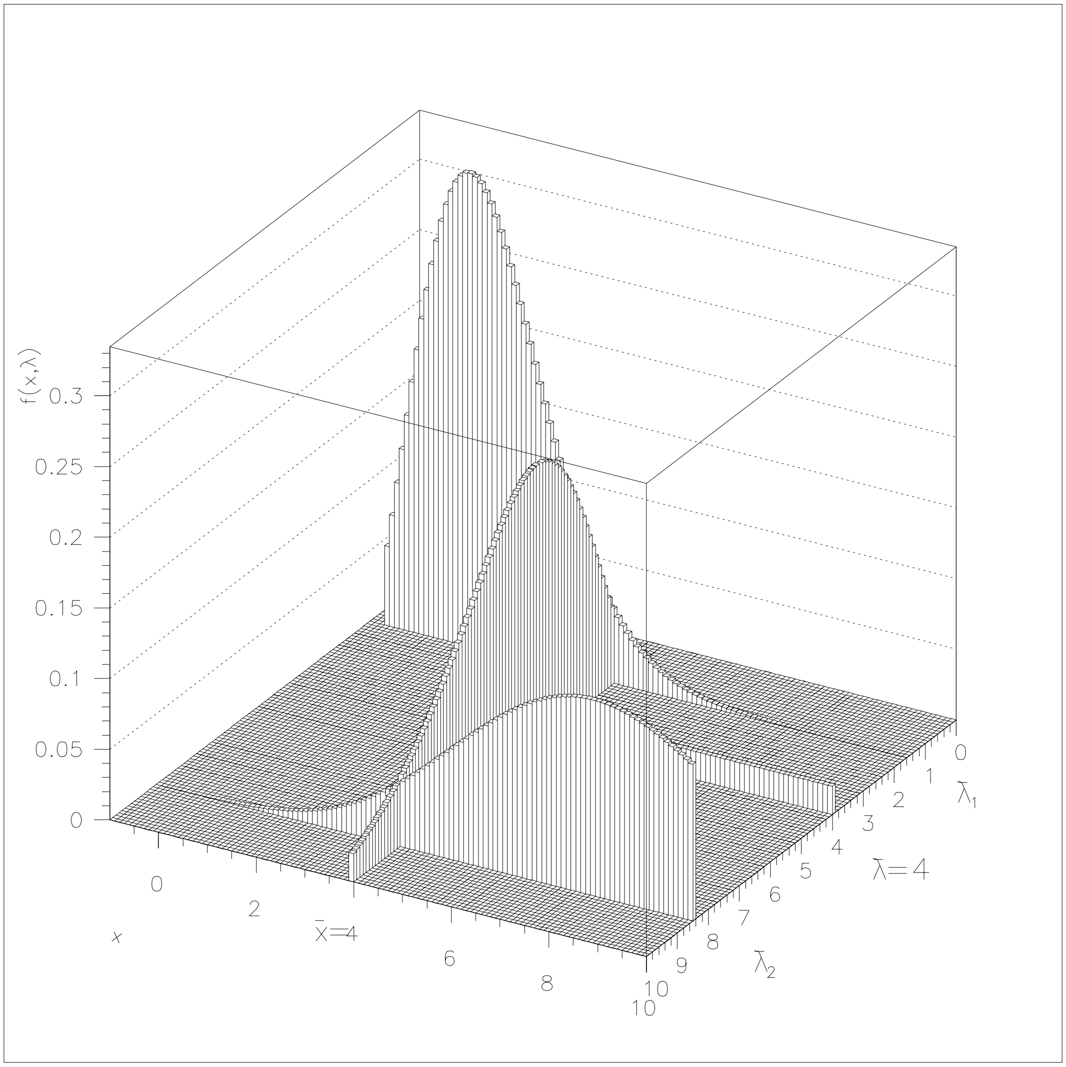

Figure 3: The probability densities $f(x, \lambda)$ of continuous analogous Poisson distribution for λ's determined by the confidence limits $\hat{\lambda}_1$ and $\hat{\lambda}_2$ in case of observed number of events $\hat{x} = 4$ and the probability density of Gamma distribution with parameters $a = 1$ and $x = \hat{x} = 4$.