A study of center vortices in $SU(2)$ and $SU(3)$ gauge theories [a]

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We show how center vortices and Abelian monopoles both appear as local gauge ambiguities in the Laplacian Center gauge. Numerical results, for $SU(2)$ and $SU(3)$, support the view that the string tension obtained in the center-projected theory matches the full string tension when the continuum limit is taken.

1 Introduction

There are several approaches addressing the problem of understanding the mechanism of color confinement in non-Abelian gauge theories. The most popular share the idea that only a subset of the degrees of freedom are relevant for confinement. In the Abelian projection approach[1,2], one takes into account the maximal Abelian subgroup $U(1)^{N-1}$ of the gauge group $SU(N)$ and monopoles are the effective degrees of freedom under study. In the Center projection approach[3,4], it is the center group Z_N which is considered and center vortices are the effective degrees of freedom under study.

These two schemes are commonly believed to give alternative descriptions of confinement. Many analytical and numerical studies have been and are being performed using these two approaches. They give clear evidence that both Abelian and center degrees of freedom play a relevant role.

The reduction of the gauge symmetry and the selection of the effective degrees of freedom is usually carried out by a partial fixing of the initial $SU(N)$ gauge freedom. We show that, in the Laplacian Center gauge, monopoles and center vortices are closely related in a unified description and that the latter are the effective degrees of freedom relevant for confinement. Indeed center vortices are related to a more reduced gauge symmetry than monopoles. A fundamental issue of this study is the use of the Laplacian Center gauge, which is free from the problem of the lattice Gribov copies affecting the widely used Maximal Center gauge.

[a] presented by M. Pepe.

2 The center degrees of freedom

Let us consider an $SU(2)$ gauge field defined in a plane and expressed in terms of the radial and angular components (A_r, A_φ). Suppose that the radial component is vanishing, $A_r = 0$, and the angular one is given by $A_\varphi = \frac{1}{2r}\sigma_3$. This gauge field configuration describes a magnetic flux tube crossing the plane at the origin. If we consider a Wilson loop encircling the origin, we see that it has a non–trivial value $\mathrm{e}^{i\pi\sigma_3} = -1$ with respect to the center group Z_2 of $SU(2)$. Conversely a Wilson loop non encircling the origin has a trivial value $+1$ with respect to Z_2. This is an example of what a center vortex is.

Consider now an $SU(2)$ gauge theory on the lattice and decompose the link variable $U_\mu(x)$ as the product of two parts

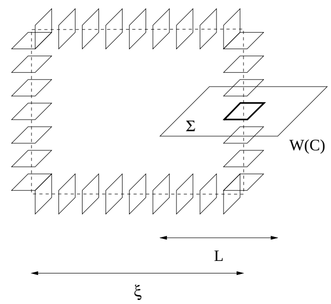
$$U_\mu(x) = Z_\mu(x)\, U'_\mu(x) \tag{1}$$

where $Z_\mu(x)$ lives in the center group Z_2 and $U'_\mu(x)$ is the coset part belonging to $SU(2)/Z_2$. For instance, this splitting can be carried out defining $U'_\mu(x)$ as having positive trace. If $W(C)$ is a Wilson loop along the closed path C, making use of the decomposition (1), we can write

$$W(C) = \sigma(C)\, W'(C) = \left[\prod_{p\in\Sigma} \sigma(p)\right] W'(C) \tag{2}$$

$W'(C)$ and $\sigma(C) \equiv \prod_{p\in\Sigma} \sigma(p)$ are respectively the Wilson loops evaluated with the coset links $U'_\mu(x)$ and with the center links $Z_\mu(x)$. $\prod_{p\in\Sigma}$ is the product over all the plaquettes p belonging to a surface Σ supported by C; the value of $\sigma(C)$ does not depend on the choice of Σ and, for simplicity, we can choose it as the planar surface bounded by C. If we fix a gauge where U'_μ is smooth, then $W(C)$ has a non–trivial value with respect to Z_2 if $\sigma(C)$ does. Moreover, considering (2) for a single plaquette, $\sigma(p) = -1$ is a "signal" for a center vortex. In 4–dimensional space–time, center vortices form closed surfaces in the dual lattice.

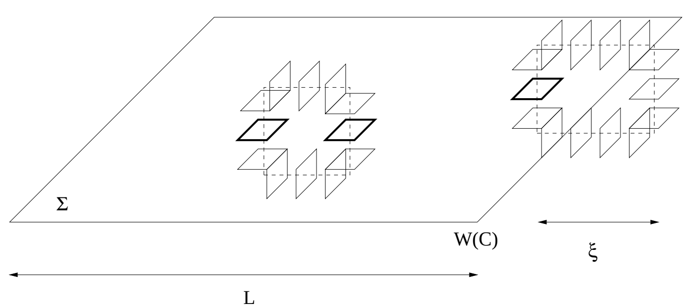
It is possible to give a handwaving and qualitative argument showing how the described center vortices can give rise to an area law behavior for the Wilson loops. According to the previous argument, we are interested in the plaquettes p having $\sigma(p) = -1$. Suppose that t is the probability for a plaquette to have $\sigma(p) = -1$, consequently $(1-t)$ is the probability to have $\sigma(p) = +1$. Consider a time–slice of a lattice $SU(2)$ gauge configuration, then center vortices form 1–dimensional closed strings in the dual lattice. The following figure displays an example of such a time–slice



The drawn plaquettes have $\sigma(p) = -1$ and the broken line is the closed string in the dual lattice representing the center vortex. Consider a Wilson loop $W(C)$ which, in the figure, is represented by the continuous line; the plaquette in bold line belongs to the planar surface Σ spanned by C. Let L be the linear extension of C and ξ the average linear size of the center vortex strings. If $\xi \gg L$, the value $\sigma(p)$ of each plaquette p in Σ is independent of the others; then, with respect to the center degrees of freedom, one can write

$$< W(C) > \sim < \sigma(p) >^{A} = \mathrm{e}^{A \log(1-2t)} \tag{3}$$

where A is the area of Σ. With this qualitative argument we have obtained on one hand that the center degrees of freedom can give rise to an area law behavior for the Wilson loop and, on the other hand, that the string tension is about twice the probability t for a plaquette to have $\sigma(p) = -1$. We have assumed that $\xi \gg L$: in order for this inequality to be satisfied for arbitrarily large Wilson loops, ξ must be divergent, that is center vortex strings must percolate through the lattice. Conversely, suppose that the center vortices do not percolate, so that ξ has a finite value, then for L sufficiently large, one must have $\xi \ll L$. The next figure displays such a case



The values $\sigma(p)$ of the plaquettes in Σ well inside the boundary C are strongly correlated by pairs; moreover these pairs do not give a net contribution to $\sigma(C)$. So only the plaquettes belonging to a ring of thickness ξ around the boundary of Σ can give a non–trivial contribution. According to this reasoning and, with respect to the center degrees of freedom, we can write

$$< W(C) > \sim < \sigma(p) >^{\xi P} = \mathrm{e}^{\xi P \log(1-2t)} \tag{4}$$

where P is the perimeter of Σ. Thus, we have obtained a perimeter law behavior for the Wilson loop. The conclusion from this handwaving argument is that the center degrees of freedom can give rise to an area law for the Wilson loop and that the area/perimeter behavior can be recast in terms of percolation/non-percolation of center vortices.

3 Laplacian gauge fixing

Many lattice simulations – starting with the initial results by Greensite and collaborators[5] – have been performed in order to investigate the role of the center degrees of freedom in the non perturbative features of the non-Abelian gauge theories. In these studies the selection of the center degrees of freedom is carried out by the numerical partial gauge fixing of an ensemble of configurations. The most widely used gauge is the Maximal Center gauge. In this gauge, it has been shown[6,7] that the removal of the center vortices leads to the loss of the area law for the Wilson loop, to the restoration of chiral symmetry and to the disappearance of non–trivial topological features. The numerical implementation of the Maximal Center gauge fixing is performed by a local iterative procedure and, as there are many local extrema, lattice Gribov copies are present. This is a serious problem[8,9] that can lead to a complete loss of meaningful information in the Z_2 projected model. Thus, it is important to study the role of the center degrees of freedom in the confinement mechanism considering a smooth gauge not affected by this problem of the lattice Gribov copies. In the Laplacian gauge the reduction of the gauge degrees of freedom is carried out in an unambiguous way. This gauge was proposed by Vink and Wiese[10]: they suggested to use the eigenvectors of the covariant Laplacian operator to fix the gauge. Since we are interested in reducing unambiguously the symmetry of the gauge group from $SU(N)$ to its center Z_N, it is useful to consider the Laplacian operator in the adjoint representation. In fact, as the adjoint representation is invariant under gauge transformations in Z_N, the adjoint Laplacian procedure fixes unambiguously the gauge up to the center symmetry.

Consider the 4–dimensional lattice $SU(N)$ gauge theory. The adjoint covariant Laplacian operator $\Delta^{ab}_{yx}(\dot{U})$ is given by

$$-\Delta^{ab}_{yx}(\dot{U}) = \sum_{\mu} \Big(2\,\delta_{y,x}\,\delta^{ab} - \dot{U}^{ab}_{\mu}(x-\hat{\mu})\,\delta_{y,x-\hat{\mu}} - \dot{U}^{ba}_{\mu}(x)\,\delta_{y,x+\hat{\mu}} \Big) \tag{5}$$

where a,$b = 1,\ldots,(N^2-1)$ are color indices and x,y are space–time lattice coordinates. The dotted $\dot{U}_{\mu}(x)$ are the link variables in the adjoint representation and are related to the links $U_{\mu}(x)$ in the fundamental by

$$\dot{U}^{ab}_{\mu}(x) = \frac{1}{2}\mathrm{Tr}\left(\lambda_a U_{\mu}(x)\lambda_b U^{\dagger}_{\mu}(x)\right) \tag{6}$$

λ_i, $i = 1,\ldots,(N^2-1)$ being the generators of $SU(N)$ with the normalization $\mathrm{Tr}(\lambda_a\lambda_b) = 2\delta_{ab}$. If V is the volume of the lattice, $\Delta(\dot{U})$ is a $[(N^2-1)V] \times [(N^2-1)V]$ real symmetric matrix which depends on the gauge field. The eigenvalues μ_j of Δ are real and the eigenvector equation is

$$\Delta^{ab}_{yx}(\dot{U})\phi^{(j)}_b(x) = \mu_j\,\phi^{(j)}_a(y) \tag{7}$$

where $\phi^{(j)}$, $j = 1,\ldots,[(N^2-1)V]$ are the (real) eigenvectors. So we can associate (N^2-1)–dimensional real vectors $\phi^{(j)}(x)$ to every lattice site x. Consider now a gauge transformation on the links in the fundamental representation $U'_{\mu}(x) = \Omega(x)U_{\mu}(x)\Omega^{\dagger}(x+\hat{\mu})$; the eigenvector equation (7) becomes

$$\dot{\Omega}^{\dagger\;ai}(y)\Delta^{ik}_{yx}(\dot{U}')\dot{\Omega}^{kb}(x)\phi^{(j)}_b(x) = \mu_j\,\phi^{(j)}_a(y) \tag{8}$$

where $\dot{\Omega}$ is the gauge transformation in the adjoint representation. This relation shows that the eigenvalues are gauge invariant and the eigenvectors transform according to $\dot{\Omega}^{ab}(x)\phi^{(j)}_b(x) = \phi^{(j)}_a(x)'$. This transformation law can be rewritten as follows

$$\Omega(x)\Phi^{(j)}(x)\Omega^{\dagger}(x) = \Phi^{(j)}(x)' \tag{9}$$

where we have defined the $su(N)$ matrices – i.e. in the $SU(N)$ algebra – $\Phi^{(j)}(x) = \sum_{a=1}^{N^2-1}\phi^{(j)}_a(x)\lambda_a$ and $\Phi^{(j)}(x)' = \sum_{a=1}^{N^2-1}\phi^{(j)}_a(x)'\lambda_a$. Gauge transformations rotate the vectors $\phi^{(j)}(x)$ in color space and we can fix the gauge by requiring a conventional arbitrary orientation for the $\phi^{(j)}(x)$. To perform the reduction of the gauge symmetry from $SU(N)$ to Z_N we only need to fix the orientation of two eigenvectors of the Laplacian operator. As we are interested in the non–perturbative features and in fixing a smooth gauge, we consider the two lowest-lying eigenmodes $\phi^{(1)}$ and $\phi^{(2)}$.

The gauge fixing procedure can be split into two steps. In the first, one rotates $\Phi^{(1)}(x)$ at every x so that $\Phi^{(1)}(x)'$ is diagonal. This leaves a residual symmetry corresponding to gauge transformations in the Cartan subgroup $U(1)^{N-1}$. To further reduce the gauge freedom we have to consider a second step where the second eigenvector $\phi^{(2)}$ is taken into account. The gauge transformation that has rotated $\Phi^{(1)}(x)$ to the Cartan subalgebra, maps $\Phi^{(2)}(x)$ to $\Phi^{(2)}(x)'$. The remaining $U(1)^{N-1}$ symmetry can be fixed to Z_N by requiring that some conventionally chosen color components of the twice rotated matrix $\Phi^{(2)}(x)$" vanish. Now we describe explicitly how to perform the presented two–step program for the $SU(2)$ gauge theory[7], where, at each x, we consider the 3–dimensional real vectors $\phi^{(1)}(x)$ and $\phi^{(2)}(x)$.

Step 1. We consider equation (9) for the first eigenvector and we conventionally define $\Omega(x)$ to be the gauge transformation that rotates $\phi^{(1)}(x)$ along the direction 3 in color space: $\dot{\Omega}(x)\phi^{(1)}(x) = \phi^{(1)}(x)' \propto (0,0,1)$. $\Omega(x)$ is unambiguously defined up to gauge transformations $V(x)$ in $U(1)$. So we have reduced the gauge symmetry to the Cartan subgroup of $SU(2)$. This is the Laplacian Abelian gauge[11,12].

Step 2. We apply the gauge transformation $\Omega(x)$ found in **Step 1** to $\phi^{(2)}(x)$: $\phi^{(2)}(x)' = \dot{\Omega}(x)\phi^{(2)}(x)$. $\phi^{(2)}(x)'$ is not invariant under the rotations $V(x) \in U(1)$ that leave $\phi^{(1)}(x)'$ unchanged; then we can fix this symmetry by requiring, for example, that $\phi^{(2)}(x)" = \dot{V}(x)\phi^{(2)}(x)'$ lie in the 1–3 color plane in the positive direction.

We have completely fixed the gauge symmetry in the adjoint representation and, as it is center–blind, we are left with the center symmetry Z_2. The described two–step procedure for $SU(2)$ can be extended[13] to $SU(N)$ with the same pattern of gauge symmetry fixing: $SU(N) \to U(1)^{N-1}$ in the first step and $U(1)^{N-1} \to Z_N$ in the second one.

Two last, important remarks concern the accidental degeneracy of μ_1 and μ_2 in (7) and the arbitrariness in the eigenvectors $\phi^{(j)}$. If it happens that μ_1 or μ_2 is degenerate, the gauge fixing can not be carried out unambiguously and one has a (global) Gribov ambiguity. This case is really exceptional and never occurs in the numerical simulations. The second point is about the scale and sign arbitrariness of the eigenvectors $\phi^{(j)}$. Rescaling can not give rise to any ambiguity in the procedure while the freedom in the choice of the sign does. This global freedom can be eliminated with a conventional choice on $\phi^{(j)}$.

4 Local gauge ambiguities

The adjoint Laplacian gauge fixing procedure has local defects. Now we discuss how these defects can show up and how they can be identified with monopoles and center vortices in $SU(2)$. This discussion can be generalized to $SU(N)$[13].

Step 2 ill–defined: the second step of the Laplacian gauge fixing is not defined at the points x where $\phi^{(1)}(x) \,//\, \phi^{(2)}(x)$. In such a case also $\phi^{(2)}(x)'$ is invariant under rotations $V(x) \in U(1)$. The condition $\phi^{(1)}(x) \,//\, \phi^{(2)}(x)$ sets two constraints and so these points x – where the gauge symmetry is promoted from Z_2 to $U(1)$ – form 2–dimensional surfaces in 4–dimensional space–time.

Step 1 ill–defined: the gauge fixing procedure can not be even started at the points x where $\phi^{(1)}(x) = (0, 0, 0)$. These defects constitute 1–dimensional strings in the 4–dimensional space–time since 3 constraints must be satisfied. At these points the symmetry is not fixed and the gauge freedom is $SU(2)$.

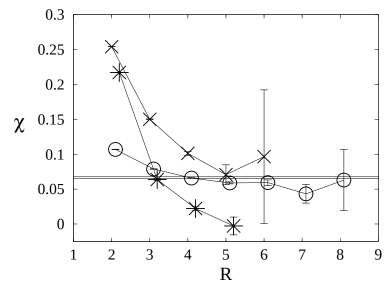
The 2–dimensional surfaces of **Step 2 ill–defined** can be identified with center vortices. Suppose that at a point x_0 it happens that $\phi^{(1)}(x_0) \,//\, \phi^{(2)}(x_0)$, then moving along a small loop around the singularity point x_0, $\phi^{(2)}$ describes a 2π rotation in color space. As a 2π phase in the adjoint representation of $SU(2)$ corresponds to a π phase in the fundamental one, it follows that a Wilson loop encircling x_0 has a non trivial value with respect to the center Z_2.

The 1–dimensional strings of **Step 1 ill–defined** can be identified with monopole world–lines. In fact, in analogy with the Georgi–Glashow model, the points x where the Higgs field vanishes and the gauge symmetry can not be reduced from $SU(2)$ to $U(1)$, correspond to the monopole world–lines.

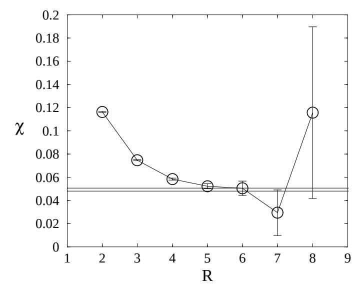
In the adjoint Laplacian gauge, monopoles and center vortices turn out to be closely related in a unified description. Consider the 2–dimensional surface $\mathcal{S}$ of the center vortices. At every $x \in \mathcal{S}$, $\phi^{(1)}(x)$ and $\phi^{(2)}(x)$ are parallel or anti–parallel. So $\mathcal{S}$ is divided in parts where $\phi^{(1)}(x)$ is parallel to $\phi^{(2)}(x)$ and parts where it is anti–parallel. By continuity, these patches must be separated by 1–dimensional strings where $\phi^{(1)}(x) = 0$ or $\phi^{(2)}(x) = 0$. Moreover, in the neighbourhood of a monopole, $\phi^{(1)}(x)$ has a hedgehog–like shape and there will be a direction where $\phi^{(1)} \,//\, \phi^{(2)}$. Thus monopole world–lines are embedded within the 2–dimensional surfaces of center vortices.

5 Numerical results and their interpretation

We have performed numerical simulations to investigate the role of the center degrees of freedom in the $SU(2)$ and $SU(3)$ lattice gauge theories. For $SU(2)$ we have collected 1000 configurations at $\beta = 2.3$, 2.4, 2.5 on a 16^4 lattice; for $SU(3)$ we have generated 500 configurations on a 16^4 lattice at $\beta = 6.0$. The following figure shows the measurement of the Creutz ratios $\chi(R) = -\ln(\langle W(R,R)\rangle\langle W(R-1,R-1)\rangle/\langle W(R,R-1)\rangle^2)$ for $SU(2)$ at $\beta = 2.4$ ($W(R,T)$ is the $R \times T$ Wilson loop). Crosses refer to $SU(2)$, circles to center projection after Laplacian gauge fixing and stars to the coset part. The contin-



uous band is the value in the literature[14,15] for the $SU(2)$ string tension at the considered set of parameters. The results show, on one hand, the flattening of the Creutz ratios in the Z_2 sector and, on the other hand, the vanishing of the Creutz ratios computed with the coset links. We have obtained a similar behaviour for $\beta = 2.3$ and 2.5. In the case of $SU(3)$ also, the following figure shows the Creutz ratios in the Z_3 sector after Laplacian gauge fixing. The



continuous band is the value in literature[16] for the $SU(3)$ string tension at the chosen set of parameters. Also in this case, one can clearly see flattening to a non vanishing value for the Creutz ratios evaluated with center projected links.

The good agreement with the values in the literature for the string tension in $SU(2)$ and $SU(3)$ should not be over–estimated. Numerical simulations are performed at finite lattice spacing and lattice artifacts can give non-negligible effects in the center projected theory. Our conjecture is that even if, at finite lattice spacing, the flattening value of the Creutz ratios in the center sector changes with the particular lattice Laplacian used to fix the gauge, this dependence vanishes in the continuum limit. The observation of such a behaviour would be a robust confirmation of the relevance of the center degrees of freedom in the confinement mechanism. To investigate this issue, we have considered three different lattice Laplacians differing by irrelevant operators: they have been built using smeared links in (6). Every set of 1000 configurations at $\beta = 2.3$, 2.4, 2.5, has been fixed in each one of the three gauges and the Creutz ratios have been measured after center projection. The table summarizes our results: $R_i = \sqrt{\sigma_i/\sigma_{SU(2)}}$ where σ_i is the string tension measured in the Z_2

	$\beta = 2.3$	$\beta = 2.4$	$\beta = 2.5$
R_0	0.813(23)	0.860(20)	0.978(18)
R_1	0.592(12)	0.720(11)	0.804(12)
R_2	0.547(8)	0.653(7)	0.739(11)

sector; $i = 0, 1, 2$ is an index for the three lattice Laplacians and $\sigma_{SU(2)}$ is the value in literature for the $SU(2)$ string tension at the three values of β. Thus, it is in the continuum limit ($\beta \to \infty$) that the string tension measured after center projection correctly reproduces the value of the full gauge theory.

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