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Non-Relativistic Effective Field Theory for Perturbative Heavy Quark–Antiquark Systems

A. H. Hoang

Theory Division, CERN,
CH-1211 Geneva 23, Switzerland

Abstract

Recent developments for an effective theory for non-relativistic perturbative quark–antiquark systems are reviewed and some applications are discussed.

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Non-Relativistic Effective Field Theory for Perturbative Heavy Quark–Antiquark Systems*

A. H. HOANG
Theory Division, CERN
CH-1211 Geneva 23, Switzerland
E-mail: andre.hoang@cern.ch

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1 Introduction

The study of non-relativistic two-body systems consisting of a heavy quark–antiquark ($Q\bar{Q}$) pair has a long tradition since the discovery of the J/ψ and the work of Appelquist and Politzer [1], which had shown that, owing to asymptotic freedom, non-relativistic quantum mechanics should apply to heavy $Q\bar{Q}$ systems. Thus, heavy $Q\bar{Q}$ systems share many features of the positronium and are an ideal tool for studying the interplay of perturbative and non-perturbative physics. It is clear that non-perturbative effects become suppressed more efficiently for increasing heavy quark mass. To get a more quantitative feeling about this, it is helpful to consider the scales that are relevant to the dynamics of a positronium-like $Q\bar{Q}$ pair with relative velocity $v \ll 1$, neglecting for now the influence of non-perturbative effects. There are three: the mass, m_Q (relevant to processes with high virtuality such as annihilation), $m_Q v$ (which sets the size of the system and the relative momentum), and $m_Q v^2$ (which is of order of the binding energy). To assess the importance of non-perturbative effects, one can compare these scales with the typical hadronic scale $\Lambda_{\rm QCD}$. The tools needed to describe the $Q\bar{Q}$ pair depend on the relation of $\Lambda_{\rm QCD}$ to the three scales just mentioned.

In this talk I consider heavy $Q\bar{Q}$ systems where $m_Q \gg m_Q v \gg m_Q v^2 \gg \Lambda_{\rm QCD}$, and I call such systems "perturbative"[2]. In fact, only a top–antitop ($t\bar{t}$) pair can clearly satisfy this condition, and it satisfies it even for arbitrarily small velocities because of the large top decay rate. (I will come back to this point in Sec. 6.) For a bottom–antibottom ($b\bar{b}$) pair, the condition is, if at all, only valid for the bottomonium ground state, whereas for a charm–anticharm pair it can certainly never be satisfied. For $b\bar{b}$ pairs, on the other hand, one can construct quantities, such as moments of the total cross section of hadrons

*This write-up was updated for some relevant publications that have appeared after the time of the conference.

containing a bottom and and antibottom quark in e^+e^- annihilation where the condition is fairly well satisfied. Perturbative $Q\bar{Q}$ systems are the ones that are best understood. They allow for a rather precise quantitative description from first principles QCD, i.e. without a sizeable model-dependence. This makes perturbative $Q\bar{Q}$ systems in principle an ideal tool to determine QCD parameters such as the heavy quark mass m_Q and the strong coupling α_s.

In this talk I review the recent achievements in the description of perturbative non-relativistic $Q\bar{Q}$ systems using the concept of effective field theories. These developments have allowed a systematic determination of corrections of order v^2 with respect to the non-relativistic limit for heavy quark production (or annihilation) in e^+e^- annihilation and opened up the access to the determination of even higher order corrections. In Sec. 2 I will briefly motivate why the concept of effective field theories is so helpful, and in fact necessary to understand perturbative $Q\bar{Q}$ systems quantitatively, and in Sec. 3 I discuss basics of the construction of an effective theory for non-relativistic perturbative $Q\bar{Q}$ systems ignoring non-perturbative effects altogether. In Sec. 4 the calculation of the total $Q\bar{Q}$ production cross section in e^+e^- annihilation close to threshold at NNLO is sketched, while in Sec. 5 the need to introduce heavy quark mass definitions that are adapted to the non-relativistic framework is explained. Section 6 contains the application of the NNLO cross section to the top mass determination at a future lepton pair collider. In Sec. 7 I briefly mention further applications and give the conclusions.

2 Why do we need an Effective Theory?

The lowest order description of the perturbative $Q\bar{Q}$ pair in the non-relativistic expansion is provided by the well-known Schrödinger equation

$$\bigg(-\frac{\boldsymbol{\nabla}^2}{m_Q} - \frac{4}{3}\frac{\alpha_s}{|\boldsymbol{r}|} - E \bigg)\, G(\boldsymbol{r},\boldsymbol{r}',E) = \delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}')\,. \tag{1}$$

Because all relevant scales are larger than $\Lambda_{\rm QCD}$, the coupling is weak and the potential can be calculated perturbatively (similar to QED). Equation (1) provides a resummation of the most singular terms $\propto (\alpha_s/v)^n$ in the elastic $Q\bar{Q} \to Q\bar{Q}$ scattering amplitude for $v \to 0$ in full QCD. There is nothing more we have to do in the non-relativistic limit. However, a problem arises when we try to calculate relativistic corrections. As an example, let us consider the Darwin term $\delta H_D = \frac{4}{3}\frac{\alpha_s \pi}{m_Q^2}\delta^{(3)}(\boldsymbol{r})$. The first order corrections to the energy levels can be determined trivially as $\delta E_n = \langle n|H_D|n\rangle \sim |\psi_n(0)|^2$, using Rayleigh–Schrödinger time-independent perturbation theory, where n stands for the appropriate set of quantum numbers. For the description of production and decay rates, however, we also need the corrections to the wave function at the origin. Formally the answer reads

$$\delta\psi_n(0) \,=\, \langle\,\mathbf{0}\,|\sum_{i\neq n}\!\!\!\!\!\!\int \frac{|\,i\,\rangle\,\langle\,i\,|}{E_n - E_i}\,H_D\,|\,n\,\rangle \sim \lim_{E\to E_n}\left[G(0,0,E) - \frac{|\psi_n(0)|^2}{E_n - E}\right]\psi_n(0)\,. \tag{2}$$

Interestingly the expression is UV-divergent because $G(0,r,E)$ contains terms proportional to $1/r$ and $\ln r$ for $r \to 0$. (The power divergence can already be found in the free Green function $G_{\rm free}(0,r,E) = \frac{m_Q}{4\pi r}e^{ipr}$, $p = \sqrt{Em_Q}$.) The same divergence also exists in the energy levels, but in the next order of perturbation theory.

Obviously, the UV-divergence reflects the fact that the non-relativistic Schrödinger equation is only an approximation, which is valid for small momenta of order $m_Q v$ and smaller. This means that the summation over the intermediate states with very large energies of order m_Q and larger is incorrect. The traditional approach to avoid this problem has been promoted by Bethe and Salpeter [3] and simply avoids the non-relativistic approximation completely. This is achieved by formulating the problem in terms of fully relativistic functional equations of Green functions. The Bethe–Salpeter approach, however, has two disadvantages: first, the corresponding equations are quite difficult to solve (because it is up to the person who solves them to find out what exactly has to be calculated at a specific order in the non-relativistic expansion) and, second, there is no systematic way to separate perturbative and non-perturbative effects.

3 The Effective Field Theory Approach

The effective field theory (EFT) approach avoids this dilemma. The EFT approach for non-relativistic perturbative $Q\bar{Q}$ pairs had already been initiated some time ago by Caswell and Lepage [4], but only recently has it been fully elaborated. To construct the EFT one has to separate those physical degrees of freedom (d.o.f.'s) that can become on-shell for the non-relativistic $Q\bar{Q}$ pair from those that can only be off-shell. This separation is, of course, not unambiguous. So, the separation between what we count as on-shell and off-shell is dependent on our choice of the regularization scheme. The idea is to integrate out the off-shell d.o.f.'s, in much the same way as particles with masses that are much larger then the available energies can be integrated out in a given process. In order to describe production and annihilation processes, it is also necessary to carry out the same program for external currents. Once the EFT Lagrangian is obtained one can derive equations of motion for Green functions involving the $Q\bar{Q}$ pair. Here, the UV-divergence mentioned before does not arise from the fact that the EFT is defined within a regularization scheme. The resulting dependences on the regularization parameter are cancelled once the Wilson coefficients of interactions and the external currents in the EFT (which contain the contributions from the off-shell d.o.f.'s) are taken into account.

For the case of the non-relativistic $Q\bar{Q}$ pair this program is non-trivial because we have to distinguish between on- and off-shell components of the single heavy quark d.o.f. in full QCD. The same it true for the gluon† d.o.f. in full QCD. A very useful way to identify the relevant d.o.f.'s for the non-relativistic $Q\bar{Q}$ system is to identify the momentum regimes needed to set up an asymptotic expansion of Feynman diagrams (in full QCD) that describe elastic $Q\bar{Q} \to Q\bar{Q}$ scattering or $Q\bar{Q}$ production and annihilation for centre-of-mass energies very close to $2m_Q$. It is obvious that, in general, one cannot naively carry out the expansion *before* integration, because the resulting terms might not lead to finite results. As a simple mathematical example we might consider that we want to determine the expansion of the parameter integral $f(a) = \int_0^1 \arctan(1-x)/(x^2+a^2)dx$ for $a \to 0$. Naive expansion before integration will lead to divergent expressions. Here we have to distinguish between the regions where $x \sim a$ and $x \sim 1$. We might introduce a cutoff Λ that separates the regions such that $a \ll \Lambda \ll 1$, but we could also use an analytic regularization scheme. In the region $x \sim a$ we can now conveniently expand the arctan for small x, and for $x \sim 1$ we can expand the whole integral in a. The integrations can now be done rather easily. Adding both contributions and expanding properly in Λ and a, the cutoff dependence cancels and leaves us with the correct expansion in a.

The corresponding momentum regions relevant to a non-relativistic $Q\bar{Q}$ pair are more involved, because there are three widely separated scales one has to take into account: m_Q, $m_Q v$ and $m_Q v^2$. The regions were first fully identified by Beneke and Smirnov [5]: $(k^0, \boldsymbol{k}) \sim (m_Q, m_Q)$ ("hard"), $(k^0, \boldsymbol{k}) \sim (m_Q v, m_Q v)$ ("soft"), $(k^0, \boldsymbol{k}) \sim (m_Q v^2, m_Q v)$ ("potential") and $(k^0, \boldsymbol{k}) \sim (m_Q v^2, m_Q v^2)$ ("ultrasoft"). The energy and momentum components can scale differently with v because the choice of a c.m. system for the $Q\bar{Q}$ pair breaks Lorentz invariance. All these regions can be populated by quark and gluonic d.o.f.'s. For the $Q\bar{Q}$ system, where the available energy is of order $m_Q v^2$, only quarks in the potential regime have a chance to be on-shell. For gluons this can only happen in the ultrasoft regime. In order to get to the EFT we have to integrate out all the other d.o.f.'s. There have been different attempts and philosophies to achieve this. Pineda and Soto [6] have proposed a two-step procedure, where first the d.o.f.'s in the hard regime are integrated out at

† We assume that all light quarks are massless, and we count them as gluons.

the scale m_Q, leading to "non-relativistic QCD" (NRQCD), the theory formulated in [4]. In a second step, the other off-shell d.o.f.'s are integrated out at a scale of order $m_Q v$, leading to "potential NRQCD" (pNRQCD). This two-step procedure has the feature that it contains two separate matching scales. Manohar and collaborators [7,8] propose a single-step procedure where the off-shell d.o.f.'s are integrated out at the scale m_Q, except for soft gluons. The resulting EFT is called "velocity NRQCD" (vNRQCD). The crucial issue of vNRQCD is that it is regulated by a regularization velocity ν in dimensional regularization. This regularization allows the scaling of the boundaries of the different momentum regions with v at the same time. A common feature of both EFTs is that their Lagrangians contain four-quark $(Q\bar{Q})(Q\bar{Q})$ interactions that depend non-locally on the quark three momenta, but locally on the quark energies. These interactions represent a generalization of what is called "potential" in the Schrödinger equation (1). We note that, in the EFT for perturbative $Q\bar{Q}$ pairs, these potentials are purely short-distance objects that can be calculated perturbatively and do, in contrast to potential models, not contain any confining contributions.

Setting aside the problem of properly resumming logarithms of the velocity [9] (which is acceptable because the corresponding logarithms are not dominant numerically for the practical applications involving $t\bar{t}$ and $b\bar{b}$ pairs) both formulations of the EFT agree on how to calculate the relativistic corrections to $Q\bar{Q}$ production. From now on, I will therefore only talk about "the EFT for perturbative non-relativistic $Q\bar{Q}$ pairs". Based on the velocity scaling of the different momentum regimes, it is possible to unambiguously determine to which power in v an individual interaction in the Lagrangian of the EFT scales. This "power counting" allows an unambiguous identification of the interactions (and the contributions in their Wilson coefficients) that have to be taken into account to describe a process involving the non-relativistic $Q\bar{Q}$ pair at a certain order in v (i.e. in the non-relativistic expansion). One well-known fact that can be derived from formal power counting is that the Coulomb interaction scales with the same power in v as the kinetic energy, i.e. the Coulomb interaction cannot be treated as a perturbation. As a particularly important application of the power counting, one can also show that the interaction of the quark with ultrasoft gluons is suppressed by v^3 with respect to the non-relativistic limit described in the Schrödinger equation (1). This means that up to NNLO[‡] we can simply neglect the effects of ultrasoft gluons—a great simplification because ultrasoft gluons cause Lamb-shift-type retardation cor-

[‡]The non-relativistic limit described in the Schrödinger equation (1) is usually called "leading-order" (LO); corrections of order v^n with respect to the non-relativistic limit are called N^{n}LO.

rections that are quite difficult to calculate. For $Q\bar{Q}$ production they have never been fully calculated to date; only partial results exist so far [10]. Another important issue is that the EFT gives an exact definition for the energy E in Eq. (1), which is particularly important for the determination of heavy quark masses from experimental data.

4 The Cross Section $\sigma(e^+e^- \to Q\bar{Q})$ Close to Threshold at NNLO

To determine the total $Q\bar{Q}$ production cross section in e^+e^- annihilation in the non-relativistic region at NNLO in the EFT, we start from the corresponding expressions in full QCD ($R \equiv \sigma(e^+e^- \to Q\bar{Q})/\sigma(e^+e^- \to \mu^+\mu^-)$):[§]

$$\begin{aligned} R(q^2) &= \frac{4\,\pi\,e_Q^2}{q^2}\,\mathrm{Im}\bigg[\,-i\,\int d^4x\;e^{i\,q.x}\,\langle\,0\,|T\,j_\mu(x)\,j^\mu(0)\,|\,0\,\rangle\,\bigg] \\ &\equiv \frac{4\,\pi\,e_Q^2}{q^2}\,\mathrm{Im}\Big[\,-i\,\langle\,0\,|T\,\tilde{j}_\mu(q)\,\tilde{j}^\mu(-q)\,|\,0\,\rangle\,\Big]\,, \end{aligned} \tag{3}$$

where e_Q is the heavy-quark electric charge and $\sqrt{q^2}$ the c.m. energy; $\tilde{j}_\mu(\pm q)$ are the electromagnetic currents that produce and annihilate the $Q\bar{Q}$ pair with c.m. energy $\sqrt{q^2}$. In the EFT these (external) currents are replaced by a sum of ${}^3S_1$ EFT currents with increasing dimension. For NNLO we need currents up to dimension 8 ($i = 1, 2, 3$):

$$\begin{aligned} \tilde{j}_i(q) &= c_1\left(\tilde{\psi}^\dagger\sigma_i\tilde{\chi}\right)(q) - \frac{c_2}{6\,m_Q^2}\left(\tilde{\psi}^\dagger\sigma_i(-\tfrac{i}{2}\overleftrightarrow{\boldsymbol{D}})^2\tilde{\chi}\right)(q) + \ldots\,, \\ \tilde{j}_i(-q) &= c_1\left(\tilde{\chi}^\dagger\sigma_i\tilde{\psi}\right)(-q) - \frac{c_2}{6\,m_Q^2}\left(\tilde{\chi}^\dagger\sigma_i(-\tfrac{i}{2}\overleftrightarrow{\boldsymbol{D}})^2\tilde{\psi}\right)(-q) + \ldots\,, \end{aligned} \tag{4}$$

where the Pauli spinors ψ and χ describe the heavy quark and antiquark fields in the EFT, respectively. The constants c_1 and c_2 are Wilson coefficients that contain a perturbative series in α_s originating from integrating out the off-shell d.o.f.'s. At NNLO c_1 has to be calculated at order α_s^2 (Refs. [11,12]) and contains only contributions from hard d.o.f.'s. For c_2 the Born expression is sufficient. The dependence of c_1 on the regularization parameter is not written out. We note that only the spatial components of the currents contribute at the NNLO level. Using the expansions (4) we arrive at the following expression for the cross section at NNLO

$$R^{\mathrm{thr}}_{\mathrm{NNLO}}(q^2) = \frac{\pi\,e_Q^2}{m_Q^2}\,C_1\,\mathrm{Im}\Big[\,\mathcal{A}_1(q^2)\,\Big] - \frac{4\,\pi\,e_Q^2}{3\,m_Q^4}\,C_2\,\mathrm{Im}\Big[\,\mathcal{A}_2(q^2)\,\Big] + \ldots\,, \tag{5}$$

[§]For simplicity we consider only the photon-mediated cross section.

where

$$\mathcal{A}_1(q^2) = i\,\langle\,0\,|\,(\tilde{\psi}^\dagger\vec{\sigma}\,\tilde{\chi})(q)\;(\tilde{\chi}^\dagger\vec{\sigma}\,\tilde{\psi})(-q)\,|\,0\,\rangle\,, \tag{6}$$

$$\mathcal{A}_2(q^2) = \tfrac{1}{2}\,i\,\langle\,0\,|\,(\tilde{\psi}^\dagger\vec{\sigma}\,\tilde{\chi})(q)\;(\tilde{\chi}^\dagger\vec{\sigma}\,(-\tfrac{i}{2}\,\overleftrightarrow{\boldsymbol{D}})^2\tilde{\psi})(-q)\;+\mathrm{h.c.}\,|\,0\,\rangle\,, \tag{7}$$

are EFT current–current correlators and m_Q is the heavy quark pole mass. Using the EFT equations of motion for the heavy quark fields $\mathcal{A}_2$ can be related directly to $\mathcal{A}_1$:

$$\mathcal{A}_2 = m_Q\,E\,\mathcal{A}_1\,. \tag{8}$$

The current–current correlators are also regularization-parameter-dependent and, in fact cancel the regularization-parameter dependence of $C_{1/2}$. $\mathcal{A}_1$ could be calculated from Feynman diagrams in the EFT. However, even in LO we would have to resum an infinite number of them (and we would have to carry out this resummation at NNLO). It is more convenient to consider the four-point function ($p \equiv (\sqrt{q^2}, \mathbf{0})$)

$$\tilde{G}(k,k',q^2) \;=\; \Big\langle\,0\,\Big|\,\Big(\tilde{\psi}^\dagger(\frac{p}{2}+k)\tilde{\chi}(\frac{p}{2}-k)\Big)\;\Big(\tilde{\chi}^\dagger(-\frac{p}{2}+k')\tilde{\psi}(-\frac{p}{2}-k')\Big)\,\Big|\,0\,\Big\rangle\,, \tag{9}$$

that describes off-shell elastic scattering of a $Q\bar{Q}$ pair in the EFT with c.m. energy $\sqrt{q^2}$. Integrating $\tilde{G}$ over k and k' we arrive at $\mathcal{A}_1(q^2)$ (up to a trivial factor). Upon integration over k_0 and k'_0 the equation of motion of $\tilde{G}$ at NNLO is just the Schrödinger equation (1) modified by NNLO corrections.:

$$\begin{aligned}&\bigg[\,\frac{\boldsymbol{k}^2}{M_t}-\frac{\boldsymbol{k}^4}{4M_t^3}-\Big(\sqrt{q^2}-2\,m_Q-2\,\delta m_Q\Big)\bigg]\,\tilde{G}(\boldsymbol{k},\boldsymbol{k}';q^2)\\ &\quad+\int\frac{d^3\boldsymbol{p}}{(2\,\pi)^3}\,\tilde{V}(\boldsymbol{k},\boldsymbol{p})\,\tilde{G}(\boldsymbol{p},\boldsymbol{k}';q^2)\;=\;(2\,\pi)^3\,\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\,.\end{aligned} \tag{10}$$

All terms displayed in Eq. (10) are of order $v^2 \sim \alpha_s^2$ (LO) and higher. The potential $\tilde{V}$ contains the Coulomb potential shown in Eq. (1), its one- and two-loop corrections [13], the order α_s/m_Q^2 Breit–Fermi potential known from positronium, and an order α_s^2/m_Q non-Abelian potential. This equation can be solved either perturbatively, starting from the known LO Coulomb solution, or exactly, using numerical methods. It is important that this calculation is carried out in exactly the same regularization scheme as was used to determine the constants $C_{1/2}$. As mentioned before, we obtain $\mathcal{A}_1(q^2)$ by integrating $\tilde{G}(\boldsymbol{k},\boldsymbol{k}';q^2)$ over $\boldsymbol{k}$ and $\boldsymbol{k'}$. From the EFT we can also uniquely determine the exact definition of what is E in Eq. (1): in the pole mass scheme we find $\delta m_Q = 0$ in Eq. (10).

5 The Heavy Quark Mass

The pole mass definition seems to be the natural choice to formulate the non-relativistic effective theory that describes the $Q\bar{Q}$ dynamics close to threshold. After all, the heavy quark pole mass is gauge-invariant and IR-finite [14]. Moreover, in the pole mass scheme the equation of motion for the non-relativistic $Q\bar{Q}$ pair has the simple form of Eq. (1) (and also of (10) for $\delta m_Q = 0$), which is well known from the non-relativistic problems in QED. Intuition also seems to favour the pole mass definition, because close to threshold the heavy quarks only have a very small virtuality of order $m_Q^2 v^2$, i.e. they are very close to the mass shell. However, it is known that the use of the pole mass can lead to (artificially) large high order corrections, because of its strong sensitivity to small momenta [15]. Quantitatively, the pole mass is ambiguous to an amount of order $\Lambda_{\rm QCD}$. Thus the use of the pole mass is unfavourable if we intend to use the calculation for the $Q\bar{Q}$ cross section at NNLO for a precise determination of the heavy quark mass. [At this point I emphasize that the notion of the "heavy quark mass" is not a physical one, because we can only define quark masses as parameters in the QCD Lagrangian, not as physical quantities.] In fact, the first theoretical calculations of the NNLO cross section for $t\bar{t}$ production close to threshold [16,17,18] were carried out in the pole mass scheme; they seemed to indicate a rather large instability of the location where the $t\bar{t}$ cross section at threshold rises, because this rise did not seem to converge when comparing LO, NLO and NNLO calculations. (See Ref. [19] for a synopsis of all calculations that have been done for the total $t\bar{t}$ cross section at NNLO.) The situation was similar for the first determinations of the bottom quark mass from moments of the total $b\bar{b}$ cross section at NNLO, where the uncertainties were quite large [20,21] if the pole mass was employed and the latter were estimated conservatively.

The bottom line is that we should use a heavy quark mass definition that has an infrared sensitivity much smaller than the pole definition. In principle, the $\overline{\rm MS}$ mass definition might be an ideal candidate because its definition does not depend on infrared physics. However, using the $\overline{\rm MS}$ mass would mean that $\delta m_Q = m_Q^{\rm pole} - \overline{m}_Q$ is of order α_s and parametrically larger than any other term in the Schrödinger equation (10). This means that the $\overline{\rm MS}$ mass formally breaks the non-relativistic power counting. In order to find an adequate "threshold mass" definition, it helps to have a closer look at Eq. (10). Technically, the instability observed in the pole mass scheme originates from the potential $\tilde{V}$. It can be shown that the potential causes large corrections from momenta smaller than $m_Q\alpha_s$ at large orders of perturbation theory. This can be visualized by considering the small momentum contribution to the heavy quark potential in configuration space representation for distances of the order of the inverse Bohr radius $1/m_Q\alpha_s$:

$$\Big[V(r\approx 1/M_t\alpha_s)\Big]^{\rm IR} \sim \int\limits^{|\mathbf{q}|<\mu\ll M_t\alpha_s} \frac{d^3\boldsymbol{q}}{(2\pi)^3}\,\tilde V_c(\boldsymbol{q})\,\exp(i\,\boldsymbol{q}\boldsymbol{r}) \tag{11}$$

$$= \int\limits^{|\mathbf{q}|<\mu\ll M_t\alpha_s} \frac{d^3\boldsymbol{q}}{(2\pi)^3}\,\tilde V_c(\boldsymbol{q})\,+\,\dots\,, \tag{12}$$

where $\tilde V_c$ is the Coulomb potential in momentum space representation. At large orders of perturbation theory the RHS of Eq. (12) grows asymptotically like $-\mu\,\alpha_s^n n!$ (Ref. [22]), where μ is the scale of α_s. On the other hand, it can be shown that the total static energy $2m_Q^{\rm pole}+V_c(r)$ does not contain these large corrections [23,24,25]. Therefore, an adequate mass definition can be found by reabsorbing the r-independent small-momentum contribution of the Coulomb potential into the heavy quark mass. This procedure is not unique. Several different mass definitions have been suggested in the literature [23,26,27] that fulfil this task. Let me mention only the 1S mass [27] in some detail. The 1S mass is defined as one half of the mass of the perturbative contribution of a fictitious $n=1$, $^3S_1$ $Q\bar Q$ bound state, assuming that the quarks are stable:

$$m_Q^{\rm 1S}-m_Q^{\rm pole} = \frac{1}{2}\,\langle 1\,^3S_1|\mathcal{H}|1\,^3S_1\rangle \,=\, -\frac{2}{9}\,\alpha_s^2\,m_Q\,+\,\dots\,, \tag{13}$$

where $\mathcal{H}$ is the non-relativistic perturbative Hamiltonian. The NNLO expression for the RHS of Eq. (13), which is not displayed here, was first calculated in Ref. [28]. The 1S mass is less sensitive to infrared momenta than the pole mass (its ambiguity is only of order $\Lambda^2_{\rm QCD}/m_Q$) and, at the same time, does not break the non-relativistic power counting.

In order to relate the 1S mass to the $\overline{\rm MS}$ mass, some care has to be taken, because it is not a priori clear how to combine the corresponding perturbative series. The guiding principle is the proper cancellation of the asymptotically large perturbative corrections contained in the perturbative relations of the 1S and the $\overline{\rm MS}$ masses to the pole mass in each order of perturbation theory. It can be shown that the cancellation happens properly if a modified perturbative expansion, called the "upsilon expansion" [29] is used. In the upsilon expansion one has to consider terms of order α_s^n in Eq. (13) as of order α_s^{n-1}. Technically this unusual prescription comes from the fact that the coefficient of the order α_s^n term in Eq. (13) contains the contribution $\sum_{i=0}^{n-2}\ln(1/\alpha_s)/i!$, which leads asymptotically to an additional factor $1/\alpha_s$. The upsilon expansion should be applied whenever the 1S mass is used outside the framework of the non-relativistic problems. As an example, it has been applied for the description of inclusive semileptonic B decays [29], where it leads to a very well behaved perturbative expansion.

6 Top-Antitop Production Close to Threshold

Top–antitop pair production at future lepton pair colliders is one of the most important applications of the EFT for non-relativistic $Q\bar{Q}$ pairs discussed in this talk. The $t\bar{t}$ production close to threshold will provide a rich phenomenological environment where the top quark can be studied in quite a unique manner. Assuming the upcoming Linear Collider or a muon collider will spend around a $100\,\mathrm{fb}^{-1}$ at the top–antitop threshold, the top mass can be determined from the $t\bar{t}$ line shape with experimental uncertainties below 50 MeV. The fact that the $t\bar{t}$ pair is produced in a colour singlet state is crucial to make this possible. Even the top quark width might be extracted from the line shape with an uncertainty of around 20%, assuming that the theoretical uncertainties are small. Assuming that the Higgs mass is around 100 GeV also a fairly good determination of the top Yukawa coupling could be achieved. In addition, from studies of distributions one might get measurements of anomalous top couplings and further measurements of the top width. What makes the $t\bar{t}$ system so interesting theoretically is the fact that its total width ($\Gamma_t \approx 1.5$ GeV) is much larger than Λ_{QCD}. Thus, the top quark decays before it hadronizes, and the effects of non-perturbative effects should be small. One can show that the effective velocity of the top quarks close to threshold reads $v_{\mathrm{eff}} = |\sqrt{(E+i\Gamma_t)/m_Q}|$ rather than $\sqrt{E/m_Q}$. This means that the hierarchy $m_t \gg m_t v_{\mathrm{eff}} \gg m_t v_{\mathrm{eff}}^2 > \Lambda_{\mathrm{QCD}}$ holds even for c.m. energies that are extremely close to the threshold and that non-perturbative effects should only represent relatively small corrections.

In view of this prospect, it is obvious that a careful analysis and assessment of theoretical uncertainties in the prediction of the total cross section is mandatory, in order to determine whether the theoretical precision can meet the experimental one. Within the last two and a half years, considerable progress has been achieved in the calculation of NNLO corrections to the $t\bar{t}$ production process within the framework of the EFT for non-relativistic $Q\bar{Q}$ pairs. For the total cross section the complete set of NNLO QCD corrections have been calculated [16,17,18,30,31,32]. So far, for the total cross section the complete electroweak effects have only been implemented at NLO. The implementation up to NLO is quite easy: the only thing to do is to replace $(\sqrt{q^2} - 2\,m_Q - 2\,\delta m_Q)$ by $(\sqrt{q^2} - 2\,m_Q - 2\,\delta m_Q) + i\Gamma_t$ in Eq. (10). The width has to be counted

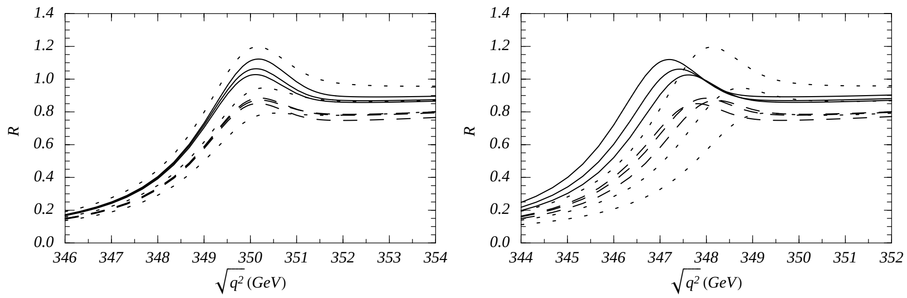

Figure 1: The total photon-induced $t\bar{t}$ cross section divided by the point cross section at the LC versus the c.m. energy in the threshold regime at LO (dotted curves), NLO (dashed) and NNLO (solid) in the 1S (left figure) and the pole (right figure) mass schemes for $\alpha_s(M_Z) = 0.118$ and $\mu = 15$, 30, 60 GeV. The effects of beamstrahlung and initial-state radiation have not been included. The plots are taken from Ref.[27].

as of order $\alpha_s^2 \sim v_{\rm eff}^2$ and cannot be treated as a perturbation. In the left picture of Fig. 1 the result for the total $t\bar{t}$ production cross section is shown for $m_t^{1S} = 175$ GeV, where dotted, dashed and solid curves represent LO, NLO and NNLO results, respectively. As a comparison, the corresponding results are shown in the pole mass scheme in the right picture of Fig. 1. It is conspicuous that the normalization uncertainties are rather large, at the level of 20%. However, realistic simulation studies [33] have shown that these uncertainties do not severely affect the determination of a threshold mass such as the 1S mass. A top mass determination with combined theoretical and experimental uncertainties of order or smaller than 200 MeV seems realistic.

7 Other Applications and Conclusions

There are a number of other applications of the EFT, which I will only discuss briefly. As mentioned in the introduction, moments of the total $b\bar{b}$ cross section in e^+e^- annihilation can be used to determine the bottom quark mass. These moments are defined as

$$P_n \equiv \int \frac{dq^2}{(q)^{2(n+1)}} \frac{\sigma_{e^+e^-\to b\bar{b}+X}(q^2)}{\sigma_{e^+e^-\to\mu^+\mu^-}(q^2)} \,. \tag{14}$$

It can be shown that the effective velocity of the bottom quarks in the n-th moments is $v_{\rm eff} = 1/\sqrt{n}$, which means that $m_b > m_b v_{\rm eff} > m_b v_{\rm eff}^2 > \Lambda_{\rm QCD}$ holds as long as n is not chosen larger than about 10. Determinations of the bottom quark mass based on threshold mass definitions have been carried out in Refs. [21,34,35]. The results are perfectly consistent and yield a value for the $\overline{\rm MS}$ bottom mass $\overline{m}_b(\overline{m}_b)$ of 4.2 GeV, with an uncertainty of 60–80 MeV.

It is clear that the non-relativistic EFT for heavy $Q\bar{Q}$ pairs can also be applied to two-body systems where the binding is caused by the electromagnetic forces. Apart from classic systems such as hydrogen, positronium or muonium, where the use of effective field theoretical methods has become increasingly popular within the last two years, hadronic atoms are a quite interesting field of application. Among the most interesting of these systems are the bound state of two charged pions ($\pi^+\pi^-$), called pionium, or muonic or kaonic hydrogen. A number of experiments such as DIRAC at CERN or DEAR at DAΦNE are already running or are under way, with the aim of measuring parameters of the chiral Lagrangian with unprecedented precision in order to provide tests of our understanding of chiral symmetry breaking.

To conclude, one can say that the advent of EFT methods has revived the interest of many theorists in non-relativistic two-body systems. The strength of the EFT approach is that it represents a "theory" and not just a "method". It explicitly uses the concepts of separation of scales, factorization and renormalization, and it simplifies systematic calculations through a set of power counting rules that can be applied before any calculation is actually started. This has made non-relativistic heavy $Q\bar{Q}$ systems accessible to a wider public. Today it is clear how to calculate NNNLO corrections to the total $Q\bar{Q}$ cross section. This is a tremendous technical project, but it is clear conceptually. The calculation of these corrections might be important in obtaining a more realistic estimate of theoretical uncertainties in the non-relativistic expansion for $t\bar{t}$ and $b\bar{b}$ systems. An important conceptual development, which still has to be achieved, is the proper implementation of the instability of a heavy quark such as the top into the EFT framework. To study instability, the non-relativistic framework seems to be a much better setting than the full theory, because the non-relativistic EFT provides a natural separation of on- and off-shell heavy quark d.o.f.'s.

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