FIELD QUALITY IN ACCELERATOR MAGNETS

S. Russenschuck
CERN, 1211 Geneva 23, Switzerland

Abstract
The field quality in the superconducting magnets is expressed in terms of the coefficients of the Fourier series expansion of the field in the aperture, at a reference radius. Scaling laws for different coefficients and reference radii are presented and the field generated by line currents in 2 and 3 dimensions is derived from basic principles.

1 Fourier series expansion of B_r

The magnetic field errors in the aperture of the superconducting accelerator magnets are expressed as the coefficients of the Fourier series expansion of the radial field component at a given reference radius (in the 2-dimensional case). In the 3-dimensional case, the transverse field components are given at a longitudinal position z_0 or integrated over the entire length of the magnet. For beam tracking it is sufficient to consider the transverse field components, since the effect of the z-component of the field, which is present in the magnet ends, on the particle motions can be neglected.

Assuming that the radial component of the magnetic flux density B_r at a given reference radius $r = r_0$ inside the aperture of a magnet is measured or calculated as a function of the angular position φ (if nothing else is stated, the local coordinate system r_2, φ_2 of aperture 2 [right one seen from the conncetion side] is used and the index is omitted), we get for the Fourier series expansion of the field

$$B_r(r_0, \varphi) = \sum_{n=1}^{\infty} (B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi). \tag{1}$$

If the field components are related to the main field component B_N we get for N=1 dipole, N=2 quadrupole etc.:

$$B_r(r_0, \varphi) = B_N(r_0) \sum_{n=1}^{\infty} (b_n(r_0) \sin n\varphi + a_n(r_0) \cos n\varphi). \tag{2}$$

The B_n are called the normal and the A_n the skew components of the field given in T, b_n are the normal relative, and a_n the skew relative field components. They are dimensionless and are usually given in units of 10^{-4} at a 17 mm reference radius.

2 The field components

Let us now consider a single coil centered in an iron yoke with circular inner aperture and a uniform high permeability. The coil can be accurately described by a set of line currents at the position of the superconducting strands. It will be shown below, that for a set of ns of these line currents at the position (r_i, Θ_i) carrying current I_i the coefficients are given by

$$B_n(r_0) = -\sum_{i=1}^{ns} \frac{\mu_0 I_i r_0^{n-1}}{2\pi} \left(\frac{1}{r_i^n} + \frac{\mu_\mathrm{r} - 1}{\mu_\mathrm{r} + 1} (\frac{r_i}{R_\mathrm{yoke}^2})^n \right) \cos n\Theta_i \tag{3}$$

$$A_n(r_0) = \sum_{i=1}^{ns} \frac{\mu_0 I_i r_0^{n-1}}{2\pi} \left(\frac{1}{r_i^n} + \frac{\mu_r - 1}{\mu_r + 1} (\frac{r_i}{R_{\text{yoke}}^2})^n \right) \sin n\Theta_i \tag{4}$$

where R_{yoke} is the inner radius of the iron yoke with the relative permeability μ_r. It is reasonable to focus on the fields generated by line currents since the field of any current distribution over an arbitrary cross-section can be approximated by summing the fields of a number of line currents distributed within the cross-section. As superconducting cables are made of strands with a diameter of about 1 mm, a good computational accuracy can be obtained by representing each cable by two layers of equally spaced line currents (same number as strands). Thus the grading of the current density in the cable due to the different compaction on its narrow and wide side is automatically considered.

With equation (3) and (4), a semi-analytical method for calculating the fields in superconducting magnets is given. The iron yoke is represented by image currents (second term in the parentheses). At low field level, the saturation of the iron yoke is low and this method is sufficient for optimizing the coil cross-section. Under that assumption some important conclusions can be drawn:

- For a coil without iron yoke the field errors scale with $1/r^n$ where n is the order of the multipole and r is the mid radius of the coil. It is clear, however, that the consequence of an increase of coil aperture is a linear drop in dipole field. Other limitations of the coil size are the beam distance, the electromagnetic forces, the yoke size, and the stored energy which results in an increase of the hot-spot temperature during a quench.
- The relative contribution of the iron yoke to the total field (coil field plus iron magnetization) is for a non saturated yoke approximately $(1 + (\frac{R_{\text{yoke}}}{r})^{2n})^{-1}$. For the main dipoles with $r = 43.5$ mm and $R_{\text{yoke}} = 89$ mm we get for the B_1 component about 19 % of contribution from the yoke, whereas for the B_5 component the influence of the yoke is only about 0.07 %.
- For certain symmetry conditions in the magnet, some of the multipole components vanish i.e. for an up-down symmetry in a dipole magnet (positive current I_0 at (r_0, Θ_0) and at $(r_0, -\Theta_0)$) no A_n terms occur. If there is in addition a left-right symmetry, only the odd $B_1, B_3, B_5, B_7, ..$ components appear.

It is therefore appropriate to optimize for higher harmonics first, using the analytical approach, and only at a later state calculate the effect of iron saturation on the lower-order multipoles. When the LHC magnets are ramped to their nominal field of 8.4 T in the aperture, the yoke is highly saturated, and numerical methods have to be used to replace the imaging method. Then it is advantageous to use numerical methods that allow a distinction between the coil-field and the iron magnetization effects, to confine both modelling problems on the coils and FEM-related numerical errors to the 20 % of field contribution from the iron magnetization. Collaborative efforts with the University of Graz, Austria, and the University of Stuttgart, Germany, have been undertaken for this task. Using the methods of reduced vector-potentials or the BEM-FEM coupling method yields the reduced field in the aperture caused by the magnetization of the iron yoke and avoids the representation of the coil in the FE-meshes, see companion papers in this yellow report.

In order to avoid field approximations by differential quotients, it is useful to use the vector-potentials A_z instead of the B_r components in the Fourier series expansion. In order to do so, transformation laws are derived.

3 The solution of the Laplace equation

With Maxwell's equations

$$\mathrm{curl}\vec{H} = \vec{J} \tag{5}$$

$$\mathrm{div}\,\vec{B} = 0 \tag{6}$$

for magnetostatic problems and the constitutive equation

$$\vec{B} = \mu(\vec{H}) \cdot \vec{H} = \mu_0(\vec{H} + \vec{M}) \tag{7}$$

and together with the vector-potential formulation $\vec{B} = \mathrm{curl}\,\vec{A}$ we get

$$\mathrm{curl}\vec{A} = \mu_0(\vec{H} + \vec{M}) \tag{8}$$

$$\vec{H} = \frac{1}{\mu_0}\mathrm{curl}\vec{A} - \vec{M} \tag{9}$$

$$\frac{1}{\mu_0}\mathrm{curlcurl}\vec{A} = \vec{J} + \mathrm{curl}\vec{M} \tag{10}$$

In the two-dimensional case with $\frac{\partial}{\partial z} = 0$, $\vec{A}$ has only a z-component. In the absence of iron magnetization, we get the scalar Poisson differential equation

$$\nabla^2 A_z = -\mu_0 J_z \tag{11}$$

and for current free regions Eq. (11) reduces to the Laplace equation which reads in cylindrical coordinates

$$r^2\frac{\partial^2 A_z}{\partial r^2} + r\frac{\partial A_z}{\partial r} + \frac{\partial^2 A_z}{\partial \varphi^2} = 0. \tag{12}$$

The general solution of this homogeneous differential equation (which is valid only inside the aperture of the magnet containing neither iron nor currents) is derived using the method of separation and reads

$$A_z(r,\varphi) = \sum_{n=1}^{\infty}(C1_n r^n + C2_n r^{-n})(D1_n \sin n\varphi + D2_n \cos n\varphi). \tag{13}$$

Considering that A_z is finite at r=0 the $C2_n$ have to be zero for the vector potential inside the aperture of the magnet. For the solution in the area outside the coil all $C1_n$ are zero. Rearranging Eq. (13) yields:

$$A_z(r,\varphi) = \sum_{n=1}^{\infty} r^n(\mathcal{C}_n \sin n\varphi + \mathcal{D}_n \cos n\varphi). \tag{14}$$

At a reference radius r_0 we get:

$$A_z(r_0,\varphi) = \sum_{n=1}^{\infty}(C_n \sin n\varphi + D_n \cos n\varphi). \tag{15}$$

With $B_r = \frac{1}{r}(\frac{\partial A_z}{\partial \varphi})$ the radial field component can be expressed as

$$B_r(r,\varphi) = \sum_{n=1}^{\infty} r^{(n-1)}(\mathcal{B}_n \sin n\varphi + \mathcal{A}_n \cos n\varphi), \tag{16}$$

$$B_r(r_0,\varphi) = \sum_{n=1}^{\infty}(B_n \sin n\varphi + A_n \cos n\varphi) = B_N \sum_{n=1}^{\infty}(b_n \sin n\varphi + a_n \cos n\varphi). \tag{17}$$

The small b_n, a_n are the multipoles related to the main field B_N which is B_1 for the dipole, B_2 for the quadrupole etc. $\mathcal{B}_n$ are given in T, T/m, T/m^2 etc., B_n are given in T, and b_n are dimensionless and usually given in units of 10^{-4} at a reference radius of 17 mm. For the B_φ component we get $B_\varphi(r,\varphi) = -\frac{\partial A_z}{\partial r} = \sum_{n=1}^{\infty} r^{(n-1)}(\mathcal{A}_n \sin n\varphi - \mathcal{B}_n \cos n\varphi)$ and therefore

$$|B_n| = \sqrt{B_{r,n}^2 + B_{\varphi,n}^2} = r^{(n-1)}\sqrt{\mathcal{A}_n^2 + \mathcal{B}_n^2}. \tag{18}$$

4 Some scaling laws

From equation (18) it can be seen that the magnitude of a $2n$-pole field component does not depend on φ and scales with r^{n-1}. A $\mathcal{B}_3$ component produces on the x-axis a B_y field that rises with x^2. The relation between the coefficients in Eq. (15) and (17) is as follows:

$$A_n(r_0) = \frac{n}{r_0}C_n(r_0), \qquad B_n(r_0) = \frac{-n}{r_0}D_n(r_0). \tag{19}$$

These scaling laws can be used to calculate the field components from the Fourier series expansion of the vector-potential which is more accurate when numerical field computation methods are applied. For the scaling of different reference radii we get:

$$A_n(r_1) = (\frac{r_1}{r_0})^{n-1}A_n(r_0), \qquad B_n(r_1) = (\frac{r_1}{r_0})^{n-1}B_n(r_0), \tag{20}$$

$$a_n(r_1) = (\frac{r_1}{r_0})^{n-N}a_n(r_0), \qquad b_n(r_1) = (\frac{r_1}{r_0})^{n-N}b_n(r_0). \tag{21}$$

Of course the problem still remains how to calculate the field harmonics from a given current distribution.

5 The field of a line current

As previously explained, it is reasonable to focus on the fields generated by line currents as the field of any current distribution over an arbitrary cross-section can be approximated by summing the fields of a number of line currents distributed within the cross-section. In the 3-dimensional case, Eq. (11) can be separated and we get for the A_x, A_y and the A_z component the solution applying Greens theorem,

$$A_i = \frac{\mu_0}{4\pi}\int_V \frac{J_i}{R}\mathrm{d}V \tag{22}$$

where $R = |\vec{R}| = |\vec{r} - \vec{r_0}|$ with the source point $\vec{r}$ and the field point $\vec{r_0}$. Assembling the components we get

$$\vec{A} = A_x\vec{e_x} + A_y\vec{e_y} + A_z\vec{e_z} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}}{R}\mathrm{d}V \tag{23}$$

and therefore

$$\vec{B} = \mathrm{curl}\vec{A} = \frac{\mu_0}{4\pi}\int_V \mathrm{curl}\frac{\vec{J}}{R}\mathrm{d}V = -\frac{\mu_0}{4\pi}\int_V \vec{J}\times\mathrm{grad}(\frac{1}{R})\mathrm{d}V = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}\times\vec{R}}{R^3}\mathrm{d}V \tag{24}$$

which is called Biot-Savart's law. The integral can be approximated by the integration over segments of line currents of finite length which are used to approximate the current distribution in the magnet. In 2-d the required particular solution for A_z is

$$A_z = \int_A -\frac{\mu_0 J_z}{2\pi}\ln R dA = -\frac{\mu_0 I}{2\pi}\ln R \tag{25}$$

with the source point $\vec{r} = (r, \Theta)$ and the field point $\vec{r_0} = (r_0, \varphi)$ and $R = |\vec{R}| = |\vec{r} - \vec{r_0}|$. The cosine law

$$R^2 = r^2 + r_0^2 - 2rr_0 \cos(\varphi - \Theta) \tag{26}$$

can be rewritten as

$$R^2 = r^2 (1 - \frac{r_o}{r} e^{i(\varphi - \Theta)}) \cdot (1 - \frac{r_o}{r} e^{-i(\varphi - \Theta)}) \tag{27}$$

and therefore

$$\ln R = \ln r + \frac{1}{2} \ln(1 - \frac{r_o}{r} e^{i(\varphi - \Theta)}) + \frac{1}{2} \ln(1 - \frac{r_o}{r} e^{-i(\varphi - \Theta)}). \tag{28}$$

With the Taylor series expansion of $\ln(1 - x)$ which gives for $|x| < 1$ (or $r_0 < r$ inside the aperture of the magnet),

$$\ln(1 - x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n, \tag{29}$$

Eq. (25) can be transformed to

$$A_z = -\frac{\mu_0 I}{2\pi} \ln r + \frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} (\frac{r_0}{r})^n \cos\left(n(\varphi - \Theta)\right). \tag{30}$$

The r component of the magnetic field is then

$$B_r = -\frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} (\frac{r_0^{n-1}}{r^n}) \sin\left(n(\varphi - \Theta)\right) \tag{31}$$

$$B_r = -\frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} (\frac{r_0^{n-1}}{r^n}) (\sin n\varphi \, \cos n\Theta - \cos n\varphi \, \sin n\Theta). \tag{32}$$

Comparison of the coefficients with equation (17) yields

$$B_n(r_0) = -\frac{\mu_0 I}{2\pi} \frac{r_0^{n-1}}{r^n} \cos n\Theta, \tag{33}$$

$$A_n(r_0) = \frac{\mu_0 I}{2\pi} \frac{r_0^{n-1}}{r^n} \sin n\Theta. \tag{34}$$

The effect of an iron yoke with constant permeability and perfect circular inner shape with radius R_{yoke} is taken into account by means of the imaging method. The image current of the strength $\frac{\mu_r - 1}{\mu_r + 1} I$ is located at the same angular position and the radius $r' = R_{\text{yoke}}^2 / r$. Thus

$$B_n(r_0) = -\frac{\mu_0 I r_0^{n-1}}{2\pi} \left(\frac{1}{r^n} + \frac{\mu_r - 1}{\mu_r + 1} (\frac{r}{R_{\text{yoke}}^2})^n \right) \cos n\Theta, \tag{35}$$

$$A_n(r_0) = \frac{\mu_0 I r_0^{n-1}}{2\pi} \left(\frac{1}{r^n} + \frac{\mu_r - 1}{\mu_r + 1} (\frac{r}{R_{\text{yoke}}^2})^n \right) \sin n\Theta. \tag{36}$$

For the calculation of the integrated multipole content in the coil-end region, no analytical equation exists. The coefficients $B_n(r_0, z_0)$, $A_n(r_0, z_0)$ can be estimated by means of the Fourier series expansion of the field $B_r(r_0, \varphi, z_0)$ which is calculated with the Biot-Savart integrals for the coil contribution and with numerical methods (BEM-FEM coupling method) for the field generated by the iron magnetization. The integration of the transverse field components is sufficient as the effect of the The magnetic length of the coil-end is given by

$$l_{\mathrm{mag}} = \frac{1}{B_{\mathrm{N,xsec}}} \int_{zs}^{ze} B_N(z) dz \tag{37}$$

where $B_{\mathrm{N,xsec}}$ is the main field component in the magnet cross-section. zs is the starting point and ze the end point of the integration path. The field harmonics produced by the coil-end can then be calculated by integrating the B_n and A_n components along the z-axis and dividing by $l_{\mathrm{mag}} \cdot B_{\mathrm{N,xsec}}$.

For the calculation of the field generated by a line current segment in 3 dimensions we now assume that the line current starts in the origin and ends at a point (x_2, y_2, z_2). The field point is (x_1, y_1, z_1). With

$$\vec{ds} = dx\vec{ex} + dy\vec{ey} + dz\vec{ez} = dx\vec{ex} + (\frac{y_2}{x_2}dx)\vec{ey} + (\frac{z_2}{x_2}dx)\vec{ez} \tag{38}$$

$$\vec{r} = (x_1 - x)\vec{ex} + (y_1 - y)\vec{ey} + (z_1 - z)\vec{ez} = (x_1 - x)\vec{ex} + (y_1 - \frac{y_2}{x_2}x)\vec{ey} + (z_1 - \frac{z_2}{x_2}x)\vec{ez} \tag{39}$$

we get:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \cdot \int_0^{x_2} \frac{dx}{\sqrt{d - \frac{2e}{x_2}x + \frac{f}{x_2^2}x^2}^3} \; (a\, e_x, -b\, e_y, c\, e_z) \tag{40}$$

where

$$a = \frac{y_2}{x_2}z_1 - y_1\frac{z_2}{x_2} \qquad b = z_1 - x_1\frac{z_2}{x_2} \qquad c = y_1 - x_1\frac{y_2}{x_2} \tag{41}$$

$$d = x_1^2 + y_1^2 + z_1^2 \qquad e = x_1x_2 + y_1y_2 + z_1z_2 \qquad f = x_2^2 + y_2^2 + z_2^2. \tag{42}$$

The integral in eq. (40) yields:

$$\int_0^{x_2} \frac{dx}{\sqrt{d - \frac{2e}{x_2}x + \frac{f}{x_2^2}x^2}^3} = -\frac{e\, x_2}{\sqrt{d}\, (e^2 - d)} + \frac{(-e + f)\; x_2}{\sqrt{d - 2\, e + f}\; (-e^2 + d\, f)} \tag{43}$$