THE USE OF A REDUCED VECTOR POTENTIAL A_r FORMULATION FOR THE CALCULATION OF IRON INDUCED FIELD ERRORS

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Abstract
The application of the method of finite elements to computing the magnetostatic field due to a given current density distribution in the presence of ferromagnetic media is reviewed with the high precision needed for the analysis of LHC magnets taken into account. Various formulations in terms of either a magnetic scalar or vector potential are described. The basic concepts of the method of finite elements are presented using both node based and edge based elements. The formulation in terms of a reduced magnetic vector potential is shown to be the best choice. It can be realized with the aid of node based finite elements for two-dimensional models but edge based elements are necessary when analyzing three-dimensional arrangements.

1. INTRODUCTION

The magnetic field occurring in superconducting magnets including ferromagnetic iron parts can only be computed numerically. Several potential formulations have been proposed in the past to serve as the underlying boundary value problem [1-3]. These use either a reduced magnetic scalar potential or a magnetic vector potential. Their merits and shortcomings will be discussed in the paper. The conclusion arrived at is that, if highly permeable iron parts are present, the precision of formulations based on a vector potential is higher.

The most versatile numerical technique for computing magnetic fields is the method of finite elements (FEM). As pointed out below, node based or edge based finite elements can be employed depending upon the potential formulation used. The scalar potential is best approximated with the aid of nodal elements as is the single-component vector potential in two-dimensional problems. However, if the vector potential is used in three-dimensional arrangements, the question of gauging arises. The best method turns out to be to use an ungauged vector potential realized by edge elements [4].

Since the magnetic field in superconducting magnets such as the LHC dipoles has to be computed with extremely high precision, it is desirable that the part of the field due to the superconducting conductors be computed analytically using the Biot-Savart Law and only the part due to the iron be obtained numerically with the aid of the method of finite elements. This means that a reduced vector potential has to be used. This formulation, which ensures the high precision required, will be presented in detail.

2. MAGNETOSTATIC FIELD

The differential equations of magnetostatic fields are the following Maxwell equations:

$$curl\ \mathbf{H} = \mathbf{J} \tag{1}$$

$$div\ \mathbf{B} = 0 \tag{2}$$

where **H** is the magnetic field intensity, **B** is the magnetic flux density and **J** is the known current density. The field quantities satisfy the following constitutive equations:

$$\mathbf{B} = \mu(\mathbf{H})\,\mathbf{H} \quad \text{or} \quad \mathbf{H} = \nu(\mathbf{B})\,\mathbf{B} \tag{3}$$

where μ is the permeability and ν the reluctivity, the reciprocal of the permeability. Due to the saturation of iron, these material parameters depend on the magnetic field. For isotropic soft magnetic media, wherein hysteresis is negligible, they can be assumed to be scalar quantities and monovalued functions of the magnitude of the field. The closed domain in which the magnetic field is to be calculated will be denoted by Ω.

The field quantities **B** and **H** satisfy boundary conditions on the boundary of Ω. Two types of boundary conditions cover all practical cases. They are prescribed on two disjunct parts Γ_B and Γ_H of the boundary with the union of Γ_B and Γ_H forming the entire boundary.

On the part Γ_B of the boundary, the normal component of the magnetic flux density is known. In many cases this value is zero, as on symmetry planes parallel to the field. Since, in order to employ the method of finite elements, a closed domain Ω has to be assumed, artificial far boundaries are frequently introduced. These far boundaries may also be part of Γ_B with the normal component of **B** vanishing. In some special problems, the distribution of B_{normal} can be estimated along a physical surface. As an example, it can often be assumed that no flux leaves the outer boundary of an iron structure completely surrounded by air or that the flux distribution in the air gap of an electrical machine is sinusoidal. All these boundary conditions can be written in the form

$$\mathbf{B}\cdot\mathbf{n} = -b \quad \text{on } \Gamma_B \tag{4}$$

where **n** is the outer unit normal vector on Γ_B and b can interpreted as a fictitious magnetic surface charge density. (The negative sign in (4) implies that positive values of b correspond to positive surface charges.)

On the part Γ_H of the boundary, the tangential component of the magnetic field intensity is known. In many cases this value is zero, as on symmetry planes perpendicular to the field. Far boundaries may also be part of Γ_H with the tangential component of **H** vanishing. In some special problems, the distribution of $\mathbf{H}_{tangential}$ can be estimated along a physical surface. For example, it can often be assumed that the field enters highly (infinitely) permeable iron structures at right angle or that the tangential component of **H** is determined by a surface current flowing on the surface of an infinitely permeable magnetic pole. All these boundary conditions can be written in the form

$$\mathbf{H}\times\mathbf{n} = \mathbf{K} \quad \text{on } \Gamma_H \tag{5}$$

where **n** is the outer unit normal vector on Γ_H and **K** can interpreted as a real or fictitious electric surface current density.

The interface conditions on any surface between two regions with different magnetic properties are the continuity of B_{normal} and of $\mathbf{H}_{tangential}$. Denoting the outer unit normal vectors of the two abutting regions Ω_1 and Ω_2 by $\mathbf{n}_1$ and $\mathbf{n}_2$ and using the indices *1* and *2* to denote the field quantities in the two regions, the interface conditions on the interface Γ_{12} can be written as

$$\left.\begin{aligned} \mathbf{B}_1\cdot\mathbf{n}_1 + \mathbf{B}_2\cdot\mathbf{n}_2 &= 0 \quad (6a)\\ \mathbf{H}_1\times\mathbf{n}_1 + \mathbf{H}_2\times\mathbf{n}_2 &= \mathbf{0} \quad (6b)\end{aligned}\right\} \text{ on } \Gamma_{12}\,.$$

In case Γ_H is a connected surface, no further conditions are necessary to define the static magnetic field. If, however, Γ_H consists of n_H+1 disjunct parts Γ_{H0}, Γ_{H1}, ..., Γ_{HnH}, then either the n_H magnetic voltages of between Γ_{H1}, ..., Γ_{HnH} and Γ_{H0} must be defined as

$$\int_{C_{Hi}} \mathbf{H} \cdot d\mathbf{l} = U_{mi} \,, \;\; i = 1, 2, ..., n_H \tag{7}$$

where C_{Hi} is an arbitrary curve connecting Γ_{Hi} and Γ_{H0}, or the n_H magnetic fluxes of the surfaces Γ_{H1}, ..., Γ_{HnH} have to be given as

$$\int_{\Gamma_{Hi}} \mathbf{B} \cdot \mathbf{n} d\Gamma = \Psi_i \,, \;\; i = 1, 2, ..., n_H. \tag{8}$$

3. POTENTIAL FORMULATIONS

The solution of the differential equations (1) and (2) with the constitutive equation (3), the boundary conditions (4) and (5), the interface conditions (6) as well as the integral conditions (7) or (8) is unique [5]. Equivalent boundary value problems involving second order elliptic differential equations as well as Dirichlet and Neumann boundary conditions can be set up in terms of potential functions. These boundary value problems lend themselves well to numerical solution with the aid of the method of finite elements. The various potential formulations will be reviewed in the following.

3.1 Reduced scalar potential

Since the curl of the magnetic field intensity is, in general, nonzero, it cannot always be written as the gradient of a scalar potential function. If, however, a function **T** is found satisfying

$$curl\mathbf{T} = \mathbf{J} \quad \text{in } \Omega, \tag{9}$$

then **H-T** is curl free and the magnetic field intensity can be written as

$$\mathbf{H} = \mathbf{T} - grad\Phi \tag{10}$$

where Φ is the *reduced magnetic scalar potential*.

The choice (10) automatically satisfies Ampere's Law (1), so Maxwell's equation (2) remains to be solved. Taking account of the material relationship (3), it has the form

$$-div(\mu grad\Phi) = -div(\mu\mathbf{T}) \quad \text{in } \Omega. \tag{11}$$

This is a generalized Laplace-Poisson equation.

The boundary condition (4) becomes

$$\mu\mathbf{n} \cdot grad\Phi = \mu \frac{\partial \Phi}{\partial n} = \mu\mathbf{T} \cdot \mathbf{n} + b \quad \text{on } \Gamma_B, \tag{12}$$

a nonhomogeneous Neumann boundary condition. Expressing the condition (5) with the aid of the scalar potential,

$$grad\Phi \times \mathbf{n} = \mathbf{T} \times \mathbf{n} - \mathbf{K} \quad \text{on } \Gamma_H \tag{13}$$

is obtained. Let us assume that the current density **J** has no normal component on Γ_H and that the integral quantities specified if Γ_H is composed of several nonconnected parts are the magnetic

voltages as in Eq. (7). Then, choosing the value of Φ to be the magnetic voltage U_{mi} defined in (7) at an arbitrary point P_{0i} in Γ_{Hi}, $i = 1, 2, ..., n_H$, $U_{m0} = 0$, the value of Φ at any point P in Γ_{Hi} is obtained as the sum of U_{mi} and of the integral of the tangential component of $grad\Phi$ along some curve C_{Pi} connecting P_{0i} to P. The tangential component of $grad\Phi$ can be obtained from Eq. (13) as

$$\frac{\partial \Phi}{\partial t} = \mathbf{n} \times (grad\Phi \times \mathbf{n}) = \mathbf{n} \times (\mathbf{T} \times \mathbf{n}) + \mathbf{K} \times \mathbf{n} \quad \text{on } \Gamma_H. \tag{14}$$

Hence, the boundary condition on Γ_H is

$$\Phi(P) = U_{mi} + \int_{C_{Pi}} [\mathbf{n} \times (\mathbf{T} \times \mathbf{n}) + \mathbf{K} \times \mathbf{n}] \cdot d\mathbf{l} = \Phi_0(P), \quad P \in \Gamma_{Hi}, \tag{15}$$

a Dirichlet boundary condition. If fluxes of the form (8) are the specified as integral quantities, then the reduced scalar potential formulation cannot be employed directly.

The boundary value problem consisting of the differential equation (11), the Neumann boundary condition (12) and the Dirichlet boundary condition (15) can be cast in a *weak form* facilitating the application of the method of finite element to its numerical solution. Let us consider all functions satisfying the Dirichlet boundary condition (15). From among these, the solution of the boundary value problem is the function Φ fulfilling

$$\int_\Omega w[-div(\mu grad\Phi)]d\Omega + \int_{\Gamma_B} w(\mu \mathbf{n} \cdot grad\Phi)d\Gamma = \int_\Omega w[-div(\mu \mathbf{T})]d\Omega + \int_{\Gamma_B} w(\mu \mathbf{T} \cdot \mathbf{n} + b)d\Gamma \tag{16}$$

with w being an *arbitrary* weighting function that obeys the homogeneous counterpart of the Dirichlet boundary condition (15):

$$w = 0 \quad \text{on } \Gamma_H. \tag{17}$$

Using the identities

$$\int_\Omega w[-div(\mu grad\Phi)]d\Omega = \int_\Omega gradw \cdot \mu grad\Phi d\Omega - \oint_{\Gamma_B + \Gamma_H} w(\mu \mathbf{n} \cdot grad\Phi)d\Gamma, \tag{18}$$

$$\int_\Omega w[-div(\mu \mathbf{T})]d\Omega = \int_\Omega gradw \cdot \mu \mathbf{T} d\Omega - \oint_{\Gamma_B + \Gamma_H} w(\mu \mathbf{T} \cdot \mathbf{n})d\Gamma \tag{19}$$

as well as the boundary condition (17), the following can be stated:

The solution of the boundary value problem (11), (12), (15) is the function Φ satisfying the Dirichlet boundary condition (15) if the weak form

$$\int_\Omega gradw \cdot \mu grad\Phi d\Omega = \int_\Omega gradw \cdot \mu \mathbf{T} d\Omega + \int_{\Gamma_B} wbd\Gamma \tag{20}$$

holds for any function w satisfying Eq. (17).

In order to satisfy the interface condition (6a), it is sufficient that the scalar potential as well as the tangential component of **T** be continuous. The interface condition (6b) is included in the weak form provided w is continuous along Γ_{12}. This can be seen by adding the two integrals over Γ_{12} corresponding to (6b) to the left hand side of (16). Applying the identities (18) and (19) over Ω_1 and Ω_2, the form (20) is obtained.

Several options are open for the choice of the function **T** satisfying Eq. (9). The most straightforward one is the Biot-Savart field $\mathbf{H}_s$ computed from the current density as

$$\mathbf{H}_s(P) = \frac{1}{4\pi} \int_{\Omega_Q} \frac{\mathbf{J}(Q) \times \mathbf{e}_{QP}}{r_{QP}^2} d\Omega_Q \tag{21}$$

where $\mathbf{e}_{QP}$ is the unit vector pointing from the source point Q to the field point P, r_{QP} is the distance between Q and P and Ω_Q is the domain where **J** is nonzero. The curl of $\mathbf{H}_s$ is obviously **J**, so it satisfies Eq. (9) and is thus a valid choice as **T**. In highly permeable media, the magnitude of **H** is frequently much less than the magnitude of $\mathbf{H}_s$. This means that using $\mathbf{H}_s$ as **T** can result in large cancellation errors when computing **H** from Eq. (10). These cancellation errors are ruinous if Φ is computed numerically, e.g. approximated by piecewise continuous functions with discontinuous derivatives as in the method of finite elements, and $\mathbf{H}_s$ is simultaneously computed analytically as a smooth function. One method to avoid these errors is using a total scalar potential in ferromagnetic regions assumed to be current free [1]. An alternative is to interpolate $\mathbf{H}_s$ with the aid similar functions as those used for the approximation of *grad*Φ [3]. The choice of using $\mathbf{H}_s$ as **T** is, naturally, not the only possibility. Some useful options in the context of the method of finite elements can be found in [6] and [3].

The scalar potential formulation of magnetostatic fields is appealing since it offers the most economic description in terms of unknown functions. In view of the fact, however, that the quantity derived directly from the potential function is the magnetic field intensity, the error of the flux density is much higher in ferromagnetic iron regions than in nonferromagnetic air domains especially if the field runs mainly parallel to the iron/air interface and hence **H** has about the same value in both regions [7]. This fact suggest that formulations yielding the magnetic flux density directly from the potentials perform better in problems involving highly permeable parts.

3.2 Vector potential

Since the divergence of the magnetic flux density is zero, it can be written as the curl of a *magnetic vector potential* function **A**:

$$\mathbf{B} = curl\mathbf{A} \quad \text{in } \Omega. \tag{22}$$

The choice (22) automatically satisfies Maxwell's equation (2), so Ampere's Law (1) remains to be solved. Taking account of the material relationship (3), it has the form

$$curl(vcurl\mathbf{A}) = \mathbf{J} \quad \text{in } \Omega. \tag{23}$$

This is a second order partial differential equation.

The boundary condition (5) becomes

$$vcurl\mathbf{A} \times \mathbf{n} = \mathbf{K} \quad \text{on } \Gamma_H, \tag{24}$$

a nonhomogeneous Neumann boundary condition. Expressing the condition (4) with the aid of the vector potential,

$$\mathbf{n} \cdot curl\mathbf{A} = b \quad \text{on } \Gamma_B \tag{25}$$

is obtained. The normal component of the curl of **A** is completely determined by the tangential component of the vector potential. Therefore it is possible to choose a function **a** so that the Dirichlet boundary condition

$$\mathbf{n} \times \mathbf{A} = \mathbf{a} \quad \text{on } \Gamma_B \tag{26}$$

implies (25). Naturally, several possible functions **a** exist. All they have to fulfill are the two conditions

$$div\mathbf{a} = div(\mathbf{n} \times \mathbf{A}) = \mathbf{n} \cdot curl\mathbf{A} = b \quad \text{on } \Gamma_B \tag{27}$$

and

$$\oint_{C_{Hi}} (\mathbf{a} \times \mathbf{n}) \cdot d\mathbf{l} = \oint_{C_{Hi}} \mathbf{A}_{\text{tangential}} \cdot d\mathbf{l} = \oint_{C_{Hi}} \mathbf{A} \cdot d\mathbf{l} = \int_{\Gamma_{Hi}} \mathbf{n} \cdot curl\mathbf{A} d\Gamma = \Psi_i \ , \ i = 1, 2, ..., n_H \tag{28}$$

where C_{Hi} is the curve bounding the surface Γ_{Hi} and as such separates Γ_{Hi} from Γ_B. The satisfaction of Eq. (28) ensures the fulfillment of the integral conditions (8). If magnetic voltages of the form (7) are the specified integral quantities, then the vector potential formulation cannot be employed directly.

The solution of the boundary value problem consisting of the differential equation (23), the Neumann boundary condition (24) and the Dirichlet boundary condition (26) is not unique. The gradient of any scalar function can be added to any of its solutions if the scalar function is constant on Γ_B. This lack of uniqueness can be eliminated by modifying the boundary value problem to include the Coulomb gauge on the vector potential [2]. In this case, the differential equation (23) is replaced by

$$curl(\nu curl\mathbf{A}) - grad(\nu div\mathbf{A}) = \mathbf{J} \quad \text{in } \Omega, \tag{23a}$$

and the boundary conditions are supplemented by the following two conditions:

$$\mathbf{A} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_H, \tag{24a}$$

$$\nu div\mathbf{A} = 0 \quad \text{on } \Gamma_B. \tag{26a}$$

The boundary value problem (23a), (24), (24a), (26), (26a) has a unique solution satisfying the Coulomb gauge

$$\nu div\mathbf{A} = 0 \quad \text{in } \Omega \tag{29}$$

and hence also the differential equation (23).

To obtain the weak form of the ungauged, nonunique boundary value problem (23), (24), (26), the satisfaction of the Dirichlet boundary condition (26) is assumed and (23), (24) are written by requiring that

$$\int_{\Omega} \mathbf{w} \cdot [curl(\nu curl \mathbf{A})] d\Omega + \int_{\Gamma_H} \mathbf{w} \cdot (\nu curl \mathbf{A} \times \mathbf{n}) d\Gamma = \int_{\Omega} \mathbf{w} \cdot \mathbf{J} d\Omega + \int_{\Gamma_H} \mathbf{w} \cdot \mathbf{K} d\Gamma \tag{30}$$

is satisfied with **w** being an *arbitrary* vector weighting function that obeys the homogeneous counterpart of the Dirichlet boundary condition (26):

$$\mathbf{n} \times \mathbf{w} = \mathbf{0} \quad \text{on } \Gamma_B. \tag{31}$$

Using the identity

$$\int_{\Omega} \mathbf{w} \cdot [curl(\nu curl \mathbf{A})] d\Omega = \int_{\Omega} curl \mathbf{w} \cdot \nu curl \mathbf{A} d\Omega - \oint_{\Gamma_B + \Gamma_H} \mathbf{w} \cdot (\nu curl \mathbf{A} \times \mathbf{n}) d\Gamma \tag{32}$$

as well as the boundary condition (31), the following can be stated:

A solution of the boundary value problem (23), (24), (26) is any function **A** satisfying the Dirichlet boundary condition (26) if the weak form

$$\int_{\Omega} curl \mathbf{w} \cdot \nu curl \mathbf{A} d\Omega = \int_{\Omega} \mathbf{w} \cdot \mathbf{J} d\Omega + \int_{\Gamma_H} \mathbf{w} \cdot \mathbf{K} d\Gamma \tag{33}$$

holds for any function **w** satisfying Eq. (31). If the current density **J** and the surface current density **K** are described with the aid of a function **T** satisfying (9) and

$$\mathbf{T} \times \mathbf{n} = \mathbf{K} \quad \text{on } \Gamma_H, \tag{34}$$

then, using the identity

$$\int_{\Omega} \mathbf{w} \cdot curl \mathbf{T} d\Omega = \int_{\Omega} curl \mathbf{w} \cdot \mathbf{T} d\Omega - \oint_{\Gamma_B + \Gamma_H} \mathbf{w} \cdot (\mathbf{T} \times \mathbf{n}) d\Gamma \tag{35}$$

and the boundary condition (31), the weak form (33) can be rewritten as

$$\int_{\Omega} curl \mathbf{w} \cdot \nu curl \mathbf{A} d\Omega = \int_{\Omega} curl \mathbf{w} \cdot \mathbf{T} d\Omega . \tag{36}$$

In order to satisfy the interface condition (6b), it is sufficient that the tangential component of **A** be continuous. The interface condition (6a) is included in the weak form provided the tangential component of **w** is continuous along Γ_{12}. This can be seen by adding the two integrals over Γ_{12} corresponding to (6a) to the left hand side of (30). Applying the identity (32) over Ω_1 and Ω_2, the form (33) is obtained.

To obtain the weak form of the gauged, unique boundary value problem (23a), (24), (24a), (26), (26a), one has to assume the satisfaction of the Dirichlet boundary conditions (24a) and (26) whereas (23a), (24) and (26a) are written by requiring that

$$\int_{\Omega} \mathbf{w} \cdot [curl(\nu curl\mathbf{A}) - grad(\nu div\mathbf{A})] d\Omega + \int_{\Gamma_H} \mathbf{w} \cdot (\nu curl\mathbf{A} \times \mathbf{n}) d\Gamma + \int_{\Gamma_B} \mathbf{w} \cdot \mathbf{n} \nu div\mathbf{A} d\Gamma = \int_{\Omega} \mathbf{w} \cdot \mathbf{J} d\Omega + \int_{\Gamma_H} \mathbf{w} \cdot \mathbf{K} d\Gamma$$

(37)

is satisfied with **w** being an *arbitrary* vector weighting function that obeys the homogeneous counterparts of the Dirichlet boundary conditions (24a) and (26):

$$\mathbf{w} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_H, \tag{38}$$

$$\mathbf{n} \times \mathbf{w} = \mathbf{0} \quad \text{on } \Gamma_B. \tag{39}$$

Using the identities (32) and

$$\int_{\Omega} \mathbf{w} \cdot [-grad(\nu div\mathbf{A})] d\Omega = \int_{\Omega} div\mathbf{w} \, \nu div\mathbf{A} d\Omega - \oint_{\Gamma_B + \Gamma_H} \mathbf{w} \cdot \mathbf{n} \nu div\mathbf{A} d\Gamma \tag{40}$$

as well as the boundary conditions (38) and (39), the following can be stated:

The solution of the boundary value problem (23a), (24), (24a), (26), (26a) is the function **A** satisfying the Dirichlet boundary conditions (24a), (26) if the weak form

$$\int_{\Omega} (curl\mathbf{w} \cdot \nu curl\mathbf{A} + div\mathbf{w} \, \nu div\mathbf{A}) d\Omega = \int_{\Omega} \mathbf{w} \cdot \mathbf{J} d\Omega + \int_{\Gamma_H} \mathbf{w} \cdot \mathbf{K} d\Gamma \tag{41}$$

holds for any function **w** satisfying Eqs. (38) and (39).

It is easy to verify that, besides the interface condition (6a), this weak also form implies the condition

$$\mathbf{n}_1 \nu_1 div\mathbf{A}_1 + \mathbf{n}_2 \nu_2 div\mathbf{A}_2 = \mathbf{0} \quad \text{on } \Gamma_{12}. \tag{42}$$

It will be shown in section 4 that the application of the method of finite elements ensures the *approximate* satisfaction of the weak forms. This means that a certain error in the fulfillment of the Coulomb gauge (29) is invariably present. Along an iron/air interface, the condition (42) implies that this error is much higher in the ferromagnetic region (where ν is low) than in the air domain (where ν may be several thousand times higher). A large error in the satisfaction of the Coulomb gauge results in even larger errors in fulfilling Ampere's Law in iron, since the additional term $grad(\nu div\mathbf{A})$ in (23a) is far from zero [8]. All in all, this feature of the gauged formulation makes it unsuitable for solving problems involving ferromagnetic materials. Therefore, in the following, the ungauged boundary value problem (23), (24), (26) will be considered only.

4. THE METHOD OF FINITE ELEMENTS

The method of finite elements requires the problem region Ω to be discretized into elementary domains called finite elements forming a mesh.. The elements have simple geometrical forms such as triangles or quadrilaterals in two-dimensional problems or tetrahedra, hexahedra, prisms, etc. in three-dimensional arrangements. The elements are defined by nodes and scalar interpolating functions called shape functions are associated with each node in each element. The shape functions are low order polynomials. Scalar potentials can be interpolated with the aid of the shape functions providing approximations which are continuous on the interfaces between finite elements. Such nodal finite

elements will be discussed in section 4.1. Besides nodes, edges can also be defined within finite elements and vector shape functions are then associated with each edge in each element. These edge shape functions are low order vector polynomials. They can be used to interpolate vector functions. The approximations thus generated are vector functions whose tangential components are continuous across element interfaces but whose normal components are, in general, discontinuous. They are, therefore, suitable for approximating vector potentials. Edge finite elements will be treated in section 4.2. A detailed exposition of the method of finite elements can be found e.g. in [9].

4.1 Node based elements

An *element node shape function* $N_k^{(e_j)}$ (k = 1, 2, ..., $n_n^{(e)}$) is associated with each of the $n_n^{(e)}$ nodes of the j-th finite element. They are usually defined in a local coordinate system and are low order polynomials of the local coordinates ξ, η, ζ. They are constructed to obey the equations

$$N_k^{(e_j)}(\xi,\eta,\zeta) = \begin{cases} 1 \text{ at the local node } k, \\ 0 \text{ at all other local nodes,} \end{cases} \qquad k = 1, 2, ..., n_n^{(e)}. \tag{43}$$

The element node shape functions may also serve to establish a transformation between local and global coordinates. With x, y and z denoting the global coordinates, this transformation has the following form within the j-th element:

$$x(\xi,\eta,\zeta) = \sum_{k=1}^{n_n^{(e)}} x_k N_k^{(e_j)}(\xi,\eta,\zeta), \; y(\xi,\eta,\zeta) = \sum_{k=1}^{n_n^{(e)}} y_k N_k^{(e_j)}(\xi,\eta,\zeta), \; z(\xi,\eta,\zeta) = \sum_{k=1}^{n_n^{(e)}} z_k N_k^{(e_j)}(\xi,\eta,\zeta). \tag{44}$$

where x_k, y_k and z_k are the global coordinates of the k-th local node.

Since the nodes of neighboring elements coincide, the number of the global nodes is less than the product of the number of elements and $n_n^{(e)}$. It will be denoted by n_n. A *global node shape function* N_i (i = 1, 2, ..., n_n) is associated with each of the global nodes and is defined as

$$N_i(x,y,z) = \begin{cases} N_k^{(e_j)}(x,y,z) \;\; \text{in the } j\text{-th element, if the global node } i \text{ coincides with its local node } k, \\ 0 \;\; \text{in an element, if the global node } i \text{ does not coincide with any of its local nodes.} \end{cases} \tag{45}$$

The global node shape functions are continuous in Ω since they have the same variation in both elements along any interface between two elements. They have a property similar to Eq. (43) with respect to the global nodes:

$$N_i(x,y,z) = \begin{cases} 1 \text{ at the global node } i, \\ 0 \text{ at all other global nodes,} \end{cases} \qquad i = 1, 2, ..., n_n. \tag{46}$$

The n_n global node shape functions are linearly independent, but there is a linear interdependence among their gradients. In fact, the sum of all nodal basis functions is 1:

$$\sum_{i=1}^{n_n} N_i = 1, \tag{47}$$

as it is obvious from the fact that the function identically equal to 1 can be exactly represented with the aid of node shape functions (the constant is the lowest order polynomial) and all its nodal values are 1. The following is obtained after taking the gradient of (47):

$$\sum_{i=1}^{n_n} gradN_i = 0\,. \tag{48}$$

This means that the maximal number of linearly independent gradients of the global node shape functions is n_n-1, i.e. the number of tree edges in the graph defined by the finite element mesh. It follows from the linear independence of the global node shape functions that n_n-1 of their gradients are in fact linearly independent.

Let us now consider the numerical solution of the weak form (20) of the reduced scalar potential formulation of magnetostatic fields. Let the global nodes in the finite element mesh which are not on the surface Γ_H (where a Dirichlet boundary condition is prescribed) obtain the order numbers 1, 2, ..., n and those on Γ_H the order numbers n+1, n+2, ..., n_n. Let further Φ_i denote the value of the reduced magnetic scalar potential in the global node i. Obviously, the values Φ_i, i = n+1, n+2, ..., n_n are known from the Dirichlet boundary condition (15) and, due to the interpolatory property (46) of the global node shape functions, the expression

$$\Phi_D(x,y,z) = \sum_{i=n+1}^{n_n} \Phi_i N_i(x,y,z) \tag{49}$$

is a known function approximately satisfying Eq. (15). Furthermore, the functions N_i, i = 1, 2, ..., n satisfy the homogeneous Dirichlet boundary condition (17) required of the weighting functions. Therefore, an approximation of the scalar potential in the form

$$\Phi(x,y,z) \approx \Phi^{(n)}(x,y,z) = \sum_{i=1}^{n_n} \Phi_i N_i(x,y,z) = \Phi_D(x,y,z) + \sum_{i=1}^{n} \Phi_i N_i(x,y,z) \tag{50}$$

is suitable for a numerical solution based on the weak form (20). Indeed, it satisfies the Dirichlet boundary condition (15) independent of the choice of the n unknown nodal potential values Φ_i, i = 1, 2, ..., n.

The relevant numerical method is called *Galerkin's procedure* and is constituted by writing the weak form (20) with the scalar potential function replaced by the approximation (50) and using the functions N_i, i = 1, 2, ..., n as weighting functions. This leads to a system of algebraic equations for the n unknowns:

$$\int_\Omega gradN_i \cdot \mu grad\Phi^{(n)} d\Omega = \int_\Omega gradN_i \cdot \mu \mathbf{T} d\Omega + \int_{\Gamma_B} N_i b d\Gamma\,, \quad i = 1, 2, ..., n, \tag{51}$$

or, with the known quantities brought to the right hand side:

$$\sum_{j=1}^{n} \Phi_j \int_{\Omega} gradN_i \cdot \mu gradN_j d\Omega$$

$$= \int_{\Omega} gradN_i \cdot \mu \mathbf{T} d\Omega + \int_{\Gamma_B} N_i b d\Gamma - \int_{\Omega} gradN_i \cdot \mu grad\Phi_D d\Omega\,, \quad i = 1, 2, ..., n. \tag{52}$$

The matrix of this equations system is obviously symmetric and also sparse, since the support of the global node shape functions extends over a few finite elements only. The matrix is also positive definite. The system can be solved advantageously with the aid of iterative techniques. The most widely spread method is that of preconditioned conjugate gradients [10].

4.2 Edge based elements

An *element edge shape function* $\mathbf{N}_k^{(e_j)}$ (k = 1, 2, ..., $n_e^{(e)}$) is associated with each of the $n_e^{(e)}$ edges of the j-th finite element. They are usually defined in a local coordinate system and are low order vector polynomials of the local coordinates ξ, η, ζ and of their gradients. This results in the important property that the gradients of the node shape functions are in the space spanned by the edge shape functions or, in other words, the gradients of the node shape functions can be written as linear combinations of the edge shape functions. The latter are constructed to obey the equations

$$\int_{edge_l} \mathbf{N}_k^{(e_j)} \cdot d\mathbf{l} = \begin{cases} 1 \text{ if } l = k, \\ 0 \text{ otherwise,} \end{cases} \quad k = 1, 2, ..., n_e^{(e)}. \tag{53}$$

where $edge_l$ is the l-th edge of the element.

The transformation between local and global coordinates is established with the aid of the element node shape functions as in Eq. (44).

Since the edges of neighboring elements coincide, the number of the global edges is less than the product of the number of elements and $n_e^{(e)}$. It will be denoted by n_e. A *global edge shape function* $\mathbf{N}_i$ (i = 1, 2, ..., n_e) is associated with each of the global edges and is defined as

$$\mathbf{N}_i(x, y, z) = \begin{cases} \mathbf{N}_k^{(e_j)}(x, y, z) \text{ in the } j\text{-th element, if the global edge } i \text{ coincides with its local edge } k, \\ 0 \text{ in an element, if the global edge } i \text{ does not coincide with any of its local edges.} \end{cases}$$

(54)

The tangential components of the global edge shape functions are continuous in Ω since they have the same variation in both elements along any interface between two elements. They have a property similar to Eq. (53) with respect to the global edges:

$$\int_{edge_j} \mathbf{N}_i \cdot d\mathbf{l} = \begin{cases} 1 \text{ if } j = i, \\ 0 \text{ otherwise,} \end{cases} \quad i = 1, 2, ..., n_e. \tag{55}$$

The n_e global edge shape functions are linearly independent, but there are linear interdependencies among their curls. Indeed, since the gradients of the nodal basis functions are in the function space spanned by the edge basis functions, we have the following n_n-1 linearly independent relations:

$$grad N_i = \sum_{k=1}^{n_e} c_{ik} \mathbf{N}_k \,,\ i=1,2,...,n_n\text{-}1 \tag{56}$$

where

$$\sum_{k=1}^{n_e} c_{ik}^2 > 0\,,\ i=1,2,...,n_n\text{-}1. \tag{57}$$

Taking the curl of each of the equations in (56) results in

$$\sum_{k=1}^{n_e} c_{ik}\, curl \mathbf{N}_k = \mathbf{0}\,,\ i=1,2,...,n_n\text{-}1. \tag{58}$$

Together with (57) and with the linear independence of the equations in (56), this implies that the maximal number of linearly independent curls of the edge basis functions is n_e -(n_n-1), i.e. the number of cotree edges in the graph of the finite element mesh. Since there are no more linearly independent gradients in the space spanned by the edge basis functions than n_n-1, not less than n_e -(n_n-1) of the curls of the edge basis functions are linearly independent.

Let us now consider the numerical solution of the weak form (33) of the ungauged vector potential formulation of magnetostatic fields. Let the global edges in the finite element mesh which are not on the surface Γ_B (where a Dirichlet boundary condition is prescribed) obtain the order numbers 1, 2, ..., n and those on Γ_B the order numbers n+1, n+2, ..., n_e. Let further A_i denote the integral of the magnetic vector potential over the global edge i. Obviously, the values A_i, i = n+1, n+2, ..., n_e are known from the Dirichlet boundary condition (26) and, due to the interpolatory property (53) of the global edge shape functions, the expression

$$\mathbf{A}_D(x,y,z) = \sum_{i=n+1}^{n_e} A_i \mathbf{N}_i(x,y,z) \tag{59}$$

is a known function approximately satisfying Eq. (26). Furthermore, the functions $\mathbf{N}_i$, i = 1, 2, ..., n satisfy the homogeneous Dirichlet boundary condition (31) required of the weighting functions. Therefore, an approximation of the vector potential in the form

$$\mathbf{A}(x,y,z) \approx \mathbf{A}^{(n)}(x,y,z) = \sum_{i=1}^{n_e} A_i \mathbf{N}_i(x,y,z) = \mathbf{A}_D(x,y,z) + \sum_{i=1}^{n} A_i \mathbf{N}_i(x,y,z) \tag{60}$$

is suitable for a numerical solution based on the weak form (33). Indeed, it satisfies the Dirichlet boundary condition (26) independent of the choice of the n unknown integral values A_i, i = 1, 2, ..., n.

The application of Galerkin's method is constituted by writing the weak form (33) with the vector potential function replaced by the approximation (60) and using the functions $\mathbf{N}_i$, i = 1, 2, ..., n as weighting functions. This leads to a system of algebraic equations for the n unknowns:

$$\int_\Omega curl \mathbf{N}_i \cdot \nu curl \mathbf{A}^{(n)} d\Omega = \int_\Omega \mathbf{N}_i \cdot \mathbf{J} d\Omega + \int_{\Gamma_H} \mathbf{N}_i \cdot \mathbf{K} d\Gamma\,,\ i = 1, 2, ..., n, \tag{61}$$

or, with the known quantities brought to the right hand side:

$$\sum_{j=1}^{n} A_j \int_{\Omega} curl\mathbf{N}_i \cdot \nu curl\mathbf{N}_j d\Omega = \int_{\Omega} \mathbf{N}_i \cdot \mathbf{J} d\Omega + \int_{\Gamma_H} \mathbf{N}_i \cdot \mathbf{K} d\Gamma - \int_{\Omega} curl\mathbf{N}_i \cdot \nu curl\mathbf{A}_D d\Omega ,$$

$$i = 1, 2, ..., n. \tag{62}$$

The matrix of this equations system is obviously symmetric and also sparse, since, similarly to the node shape functions, the support of the global edge shape functions extends over a few finite elements only. The matrix is also positive semidefinite, i.e. all its eigenvalues are nonnegative, but some of them are zero. The singularity of the matrix follows immediately from the linear interdependencies between the curls of the global edge shape functions written in Eq. (58). Since the method of conjugate gradients can cope efficiently with positive semidefinite (singular) matrices [11] provided the right hand side of the equations system is consistent, the robustness of the numerical scheme is ensured. Note that this would not be the case if node shape functions were used to approximate each component of the vector potential: the matrix in the equations system (62) were then not singular but, because of the small eigenvalues approximating zero, extremely ill conditioned.

The form (62) of the Galerkin equations does not ensure the consistence of the right hand side. Using the weak form (36) instead of (33), however, results in the equations system

$$\sum_{j=1}^{n} A_j \int_{\Omega} curl\mathbf{N}_i \cdot \nu curl\mathbf{N}_j d\Omega = \int_{\Omega} curl\mathbf{N}_i \cdot \mathbf{T} d\Omega - \int_{\Omega} curl\mathbf{N}_i \cdot \nu curl\mathbf{A}_D d\Omega , \; i = 1, 2, ..., n. \tag{63}$$

In this form, the right hand side is obviously consistent, since the same linear interdependence among its elements is present as among the rows of the left hand side. Consequently, the form (63) of the Galerkin equations must be used in the numerical realization.

5. REDUCED VECTOR POTENTIAL

The total vector potential formulation described in section 3.2 with its numerical solution by means of edge finite elements presented in section 4.2 has the disadvantage that the shape of the coils has to be exactly modeled by finite elements. If this is not the case, then the precision of the right hand side of the Galerkin equations becomes very low as it can be seen in Eq. (62). Indeed, the numerical integration of the product of the edge shape functions and of the current density cannot be carried out precisely if **J** is discontinuous within the finite elements. In addition, it is desirable that the part of the field due to the conductors be computed analytically using the Biot-Savart Law and only the part due to the iron be obtained numerically with the aid of the method of finite elements.

The necessity of representing the shape of coils by the finite element mesh can be avoided by introducing a reduced vector potential $\mathbf{A}_r$ [3] as

$$\mathbf{B} = \mu_0 \mathbf{H}_S + curl\mathbf{A}_r \quad \text{in } \Omega \tag{64}$$

where $\mathbf{H}_S$ is the Biot-Savart field defined in (21). Also, $\mu_0\mathbf{H}_S$ is the magnetic field due to the coils in free space and, hence, $curl\mathbf{A}_r$ is the field resulting from the presence of iron.

The choice (64) automatically satisfies Maxwell's equation (2) since the divergence of the Biot-Savart field is zero. Ampere's Law (1) remains to be solved. Taking account of the material relationship (3) and of the fact that the curl of $\mathbf{H}_S$ is **J**, it has the form

$$curl(\nu curl\mathbf{A_r}) = curl\mathbf{H_S} - curl(\nu\mu_0\mathbf{H_S}) \quad \text{in } \Omega. \tag{65}$$

This second order partial differential equation is similar to Eq. (23). In air regions, the right hand side is obviously zero.

The boundary condition (5) becomes

$$\nu curl\mathbf{A_r}\times\mathbf{n} = \mathbf{K} - \nu\mathbf{H_S}\times\mathbf{n} \quad \text{on } \Gamma_H, \tag{66}$$

a nonhomogeneous Neumann boundary condition. Expressing the condition (4) with the aid of the reduced vector potential,

$$\mathbf{n}\cdot curl\mathbf{A_r} = b - \mu_0\mathbf{n}\cdot\mathbf{H_S} \quad \text{on } \Gamma_B \tag{67}$$

is obtained. This is equivalent to the Dirichlet boundary condition

$$\mathbf{n}\times\mathbf{A_r} = \mathbf{a} \text{ on } \Gamma_B \tag{68}$$

where **a** satisfies two conditions similar to (27) and (28).

To obtain the weak form of the boundary value problem (65), (66), (68), the satisfaction of the Dirichlet boundary condition (68) is assumed and (65), (66) are written by requiring that

$$\int_\Omega \mathbf{w}\cdot[curl(\nu curl\mathbf{A_r})]d\Omega + \int_{\Gamma_H}\mathbf{w}\cdot(\nu curl\mathbf{A_r}\times\mathbf{n})d\Gamma$$

$$= \int_\Omega \mathbf{w}\cdot[curl\mathbf{H_S} - curl(\nu\mu_0\mathbf{H_S})]d\Omega + \int_{\Gamma_H}\mathbf{w}\cdot(\mathbf{K} - \nu\mu_0\mathbf{H_S}\times\mathbf{n})d\Gamma \tag{69}$$

is satisfied with **w** being an *arbitrary* vector weighting function that obeys the homogeneous Dirichlet boundary condition (31)

Using the identities

$$\int_\Omega \mathbf{w}\cdot[curl(\nu curl\mathbf{A_r})]d\Omega = \int_\Omega curl\mathbf{w}\cdot\nu curl\mathbf{A_r}d\Omega - \oint_{\Gamma_B+\Gamma_H}\mathbf{w}\cdot(\nu curl\mathbf{A_r}\times\mathbf{n})d\Gamma, \tag{70}$$

$$\int_\Omega \mathbf{w}\cdot[curl(\nu\mu_0\mathbf{H_S})]d\Omega = \int_\Omega curl\mathbf{w}\cdot\nu\mu_0\mathbf{H_S}d\Omega - \oint_{\Gamma_B+\Gamma_H}\mathbf{w}\cdot(\nu\mu_0\mathbf{H_S}\times\mathbf{n})d\Gamma, \tag{71}$$

$$\int_\Omega \mathbf{w}\cdot curl\mathbf{H_S}d\Omega = \int_\Omega curl\mathbf{w}\cdot\mathbf{H_S}d\Omega - \oint_{\Gamma_B+\Gamma_H}\mathbf{w}\cdot(\mathbf{H_S}\times\mathbf{n})d\Gamma, \tag{72}$$

assuming similarly to (34) that

$$\mathbf{H_S}\times\mathbf{n} = \mathbf{K} \quad \text{on } \Gamma_H \tag{73}$$

and using the boundary condition (31), the following can be stated:

A solution of the boundary value problem (65), (66), (68) is any function $\mathbf{A}_\mathrm{r}$ satisfying the Dirichlet boundary condition (68) if the weak form

$$\int_\Omega curl\mathbf{w} \cdot \nu curl\mathbf{A}_\mathrm{r} d\Omega = \int_\Omega curl\mathbf{w} \cdot (\mathbf{H}_\mathrm{S} - \nu\mu_0 \mathbf{H}_\mathrm{S}) d\Omega \tag{74}$$

holds for any function **w** satisfying Eq. (31).

In order to solve the weak form (74) by Galerkin's method using edge finite elements, let the global edges in the finite element mesh which are not on the surface Γ_B (where a Dirichlet boundary condition is prescribed) obtain the order numbers 1, 2, ..., n and those on Γ_B the order numbers n+1, n+2, ..., n_e. Let further A_i denote the integral of the reduced magnetic vector potential over the global edge i. Obviously, the values A_i, $i = n+1, n+2, ..., n_e$ are known from the Dirichlet boundary condition (68) and, due to the interpolatory property (53) of the global edge shape functions, the expression

$$\mathbf{A}_D(x,y,z) = \sum_{i=n+1}^{n_e} A_i \mathbf{N}_i(x,y,z) \tag{75}$$

is a known function approximately satisfying Eq. (68). Furthermore, the functions $\mathbf{N}_i$, $i = 1, 2, ..., n$ satisfy the homogeneous Dirichlet boundary condition (31) required of the weighting functions. Therefore, an approximation of the reduced vector potential in the form

$$\mathbf{A}_\mathrm{r}(x,y,z) \approx \mathbf{A}_\mathrm{r}^{(n)}(x,y,z) = \sum_{i=1}^{n_e} A_i \mathbf{N}_i(x,y,z) = \mathbf{A}_D(x,y,z) + \sum_{i=1}^{n} A_i \mathbf{N}_i(x,y,z) \tag{76}$$

is suitable for a numerical solution based on the weak form (74). Indeed, it satisfies the Dirichlet boundary condition (68) independent of the choice of the n unknown integral values A_i, $i = 1, 2, ..., n$.

The application of Galerkin's method is again constituted by writing the weak form (74) with the vector potential function replaced by the approximation (76) and using the functions $\mathbf{N}_i$, $i = 1, 2, ..., n$ as weighting functions. This leads to a system of algebraic equations for the n unknowns:

$$\int_\Omega curl\mathbf{N}_i \cdot \nu curl\mathbf{A}_\mathrm{r}^{(n)} d\Omega = \int_\Omega curl\mathbf{N}_i \cdot (\mathbf{H}_\mathrm{S} - \nu\mu_0 \mathbf{H}_\mathrm{S}) d\Omega\,, \quad i = 1, 2, ..., n, \tag{77}$$

or, with the known quantities brought to the right hand side:

$$\sum_{j=1}^{n} A_j \int_\Omega curl\mathbf{N}_i \cdot \nu curl\mathbf{N}_j d\Omega = \int_\Omega curl\mathbf{N}_i \cdot (\mathbf{H}_\mathrm{S} - \nu\mu_0 \mathbf{H}_\mathrm{S}) d\Omega - \int_\Omega curl\mathbf{N}_i \cdot \nu curl\mathbf{A}_D d\Omega\,,$$

$$i = 1, 2, ..., n. \tag{78}$$

The matrix of this equations system is the same as in Eq. (62) obtained in the case of the total vector potential. The consistence of the right hand side is obvious, so the singularity of the matrix does not impair the robustness of the numerical scheme.

6. CONCLUSIONS

The numerical solution of magnetostatic fields by means of the finite element method can be based either on a reduced magnetic scalar potential or on a magnetic vector potential. The first option can be realized by node based finite elements and the second one by edge based ones. The precision of the vector potential formulation is superior to that of the method based on a scalar potential in highly permeable iron parts. The use of edge based finite elements for the numerical solution of the boundary value problem in terms of an ungauged vector potential leads to an equations system with a singular matrix. Choosing a suitable form of the right hand side, it can be made to be consistent and, hence, the iterative solution of the equations system is robust. Taking account of the field of the coils in free space with the aid of the Biot-Savart Law, a reduced magnetic vector potential can be introduced. This eliminates the necessity of modeling the shape of the coils by the finite element mesh and results in high precision since the finite element solution represents the iron induced fields only. Consequently, the reduced vector potential formulation is especially suitable for the analysis of superconducting magnets including iron parts such as the LHC magnets.

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