AEI-2000-043
hep-th/0008063
CERN-TH/2000-230

Conformal and Quasiconformal Realizations of Exceptional Lie Groups*

M. Günaydin ‡

CERN, Theory Division
1211 Geneva 23, Switzerland

E-mail: murat.gunaydin@cern.ch

K. Koepsell, H. Nicolai

Max-Planck-Institut für Gravitationsphysik,
Albert-Einstein-Institut,
Mühlenberg 1, D-14476 Potsdam, Germany

E-mail: koepsell@aei.mpg.de, nicolai@aei.mpg.de

Abstract

We present a nonlinear realization of $E_{8(8)}$ on a space of 57 dimensions, which is quasiconformal in the sense that it leaves invariant a suitably defined "light cone" in $\mathbb{R}^{57}$. This realization, which is related to the Freudenthal triple system associated with the unique exceptional Jordan algebra over the split octonions, contains previous conformal realizations of the lower rank exceptional Lie groups on generalized space times, and in particular a conformal realization of $E_{7(7)}$ on $\mathbb{R}^{27}$ which we exhibit explicitly. Possible applications of our results to supergravity and M-Theory are briefly mentioned.

AUGUST 2000

*This work was supported in part by the NATO collaborative research grant CRG. 960188.
‡Work supported in part by the National Science Foundation under grant number PHY-9802510.
Permanent address: Physics Department, Penn State University, University Park, PA 16802, USA.

1 Introduction

It is an old idea to define generalized space-times by association with Jordan algebras J, in such a way that the space-time is coordinatized by the elements of J, and that its rotation, Lorentz, and conformal group can be identified with the automorphism, reduced structure, and the linear fractional group of J, respectively [6, 7, 8]. The aesthetic appeal of this idea rests to a large extent on the fact that key ingredients for formulating relativistic quantum field theories over four dimensional Minkowski space extend naturally to these generalized space times; in particular , the well-known connection between the positive energy unitary representations of the four dimensional conformal group $SU(2,2)$ and the covariant fields transforming in finite dimensional representations of the Lorentz group $SL(2,\mathbb{C})$ [21, 20] extends to all generalized space-times defined by Jordan algebras [10]. The appearance of exceptional Lie groups and algebras in extended supergravities and their relevance to understanding the non-perturbative regime of string theory have provided new impetus; indeed, possible applications to string and M-Theory constitute the main motivation for the present investigation.

In this paper, we will present a novel construction involving the maximally extended Lie group $E_{8(8)}$, which contains all previous examples of generalized space-times based on exceptional Lie groups, and at the same time goes beyond the framework of Jordan algebras. More precisely, we show that there exists a quasi-conformal nonlinear realization of $E_{8(8)}$ on a space of 57 dimensions[1]. This space may be viewed as the quotient of $E_{8(8)}$ by its maximal parabolic subgroup [11]; there is no Jordan algebra directly associated with it, but it can be related to a certain Freudenthal triple system which itself is associated with the "split" exceptional Jordan algebra $J_3^{\mathbb{O}_S}$ where $\mathbb{O}_S$ denote the split real form of the octonions $\mathbb{O}$. It furthermore admits an $E_{7(7)}$ invariant norm form $\mathcal{N}_4$, which gets multiplied by a (coordinate dependent) factor under the nonlinearly realized "special conformal" transformations. Therefore the light cone, defined by the condition $\mathcal{N}_4 = 0$, is actually invariant under the full $E_{8(8)}$, which thus plays the role of a generalized conformal group. By truncation we obtain quasiconformal realizations of other exceptional Lie groups. Furthermore, we recover previous conformal realizations of the lower rank exceptional groups (some of which correspond to Jordan algebras). In particular, we give a completely explicit conformal Möbius-like nonlinear realization of $E_{7(7)}$ on the 27-dimensional space associated with the exceptional Jordan algebra $J_3^{\mathbb{O}_S}$, with linearly realized subgroups $F_{4(4)}$ (the "rotation group") and $E_{6(6)}$ (the "Lorentz group"). Although in part this result is implicitly contained in the existing literature on Jordan algebras, the relevant transformations have never been exhibited explicitly so far, and are here presented in the basis that is also used in maximal supergravity theories.

[1] A nonlinear realization will be referred to as "quasiconformal" if it is based on a five graded decomposition of the underlying Lie algebra (as for $E_{8(8)}$); it will be called "conformal" if it is based on a three graded decomposition (as e.g. for $E_{7(7)}$).

The basic concepts are best illustrated in terms of a simple and familiar example, namely the conformal group in four dimensions [21], and its realization via the Jordan algebra $J_2^{\mathbb{C}}$ of hermitean 2×2 matrices with the hermiticity preserving commutative (but non-associative) product

$$a \circ b := \tfrac{1}{2}(ab + ba) \tag{1}$$

(basic properties of Jordan algebras are summarized in appendix A). As is well known, these matrices are in one-to-one correspondence with four-vectors x^μ in Minkowski space via the formula $x \equiv x_\mu \sigma^\mu$ where $\sigma^\mu := (1, \vec{\sigma})$. The "norm form" on this algebra is just the ordinary determinant, i.e.

$$\mathcal{N}_2(x) := \det x = x_\mu x^\mu \tag{2}$$

(it will be a higher order polynomial in the general case). Defining $\bar{x} := x_\mu \bar{\sigma}^\mu$ (where $\bar{\sigma}^\mu := (1, -\vec{\sigma})$) we introduce the Jordan triple product on $J_2^{\mathbb{C}}$:

$$\begin{aligned} \{a\, b\, c\} &:= (a \circ \bar{b}) \circ c + (c \circ \bar{b}) \circ a - (a \circ c) \circ \bar{b} \\ &= \tfrac{1}{2}(a\bar{b}c + c\bar{b}a) = \langle a, b\rangle c + \langle c, b\rangle a - \langle a, c\rangle b \end{aligned} \tag{3}$$

with the standard Lorentz invariant bilinear form $\langle a, b\rangle := a_\mu b^\mu$. However, it is not generally true that the Jordan triple product can be thus expressed in terms of a bilinear form.

The automorphism group of $J_2^{\mathbb{C}}$, which is by definition compatible with the Jordan product, is just the rotation group $SU(2)$; the structure group, defined as the invariance of the norm form up to a constant factor, is the product $SL(2, \mathbb{C}) \times \mathcal{D}$, i.e. the Lorentz group and dilatations. The conformal group associated with $J_2^{\mathbb{C}}$ is the group leaving invariant the light-cone $\mathcal{N}_2(x) = 0$. As is well known, the associated Lie algebra is $su(2, 2)$, and possesses a three-graded structure

$$\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{+1}\,, \tag{4}$$

where the grade $+1$ and grade -1 spaces correspond to generators of translations P^μ and special conformal transformations K^μ, respectively, while the grade 0 subspace is spanned by the Lorentz generators $M^{\mu\nu}$ and the dilatation generator D. The subspaces $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$ can each be associated with the Jordan algebra $J_2^{\mathbb{C}}$, such that their elements are labelled by elements $a = a_\mu \sigma^\mu$ of $J_2^{\mathbb{C}}$. The precise correspondence is

$$U_a := a_\mu P^\mu \in \mathfrak{g}^{+1} \qquad \text{and} \qquad \tilde{U}_a := a_\mu K^\mu \in \mathfrak{g}^{-1}\,. \tag{5}$$

By contrast, the generators in $\mathfrak{g}^0$ are labeled by *two* elements $a, b \in J_2^{\mathbb{C}}$, viz.

$$S_{ab} := a_\mu b_\nu (M^{\mu\nu} + \eta^{\mu\nu} D)\,. \tag{6}$$

The conformal group is realized non-linearly on the space of four-vectors $x \in J_2^{\mathbb{C}}$, with a Möbius-like infinitesimal action of the special conformal transformations

$$\delta x^\mu = 2\langle c, x\rangle x^\mu - \langle x, x\rangle c^\mu \tag{7}$$

with parameter c^μ. All variations acquire a very simple form when expressed in terms of above generators: we have

$$
\begin{aligned}
U_a(x) &= a\,, \\
S_{ab}(x) &= \{a\,b\,x\}\,, \\
\tilde{U}_c(x) &= -\tfrac{1}{2}\{x\,c\,x\}\,,
\end{aligned}
\tag{8}
$$

where $\{\ldots\}$ is the Jordan triple product introduced above. From these transformations it is elementary to deduce the commutation relations

$$
\begin{aligned}
[U_a,\tilde{U}_b] &= S_{ab}\,, \\
[S_{ab},U_c] &= U_{\{abc\}}\,, \\
[S_{ab},\tilde{U}_c] &= \tilde{U}_{\{bac\}}\,, \\
[S_{ab},S_{cd}] &= S_{\{abc\}\,d} - S_{\{bad\}\,c}\,.
\end{aligned}
\tag{9}
$$

(of course, these could have been derived directly from those of the conformal group). As one can also see, the Lie algebra $\mathfrak{g}$ admits an involutive automorphism ι exchanging $\mathfrak{g}^{+1}$ and $\mathfrak{g}^{-1}$ (hence, $\iota(K^\mu) = P^\mu$).

The above transformation rules and commutation relations exemplify the structure that we will encounter again in section 3 of this paper: the conformal realization of $E_{7(7)}$ on $\mathbb{R}^{27}$ presented there has the same form, except that $J_2^{\mathbb{C}}$ is replaced by the exceptional Jordan algebra $J_3^{\mathbb{O}_S}$ over the split octonions $\mathbb{O}_S$. While our form of the nonlinear variations appears to be new, the concomitant construction of the Lie algebra itself by means of the Jordan triple product has been known in the literature as the Tits-Kantor-Koecher construction [24, 13, 17], and as such generalizes to other Jordan algebras. The generalized linear fractional (Möbius) groups of Jordan algebras can be abstractly defined in an analogous manner [18], and shown to leave invariant certain generalized p-angles defined by the norm form of degree p of the underlying Jordan algebra [14, 9]. However, explicit formulas of the type derived here have never before appeared in the literature.

While this construction works for the exceptional Lie algebras $E_{6(6)}$, and $E_{7(7)}$, as well as other Lie algebras admitting a three graded structure, it fails for $E_{8(8)}$, $F_{4(4)}$, and $G_{2(2)}$, for which a three grading does not exist. These algebras possess only a five graded structure

$$
\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2}\,.
\tag{10}
$$

Our main result, to be described in section 2, states that a "quasiconformal" realization is still possible on a space of dimension $\dim(\mathfrak{g}^1)+1$ if the top grade spaces $\mathfrak{g}^{\pm 2}$ are one-dimensional. Five graded Lie algebras with this property are closely related to the so-called Freudenthal Triple Systems [4, 22], which were originally invented to provide alternative constructions of the exceptional Lie groups[2]. This

[2]The more general Kantor-Triple-Systems for which $\mathfrak{g}^{\pm 2}$ have more than one dimension, will not be discussed in this paper.

relation will be made very explicit in the present paper. The novel realization of $E_{8(8)}$ which we will arrive at contains various other constructions of exceptional Lie algebras by truncation, including the conformal realizations based on a three graded structure. For this reason, we describe it first in section 2, and then show how the other cases can be obtained from it.

Whereas previous attempts to construct generalized space-times mainly focussed on generalizing Minkowski space-time and its symmetries, the physical applications that we have in mind here are of a somewhat different nature, and inspired by recent developments in superstring and M-Theory. More specifically, the generalized "space-times" presented here could conceivably be identified with certain internal spaces arising in supergravity and superstring theory. As an example, recall that the solitonic degrees of freedom of $d = 4, N = 8$ supergravity carry 28 electric and 28 magnetic charges, which appear as central charges in the $N = 8$ superalgebra, and combine into the $\mathbf{56}$ representation of $E_{7(7)}$ (this is a non-trivial fact, because the superalgebra initially "knows" only about the R symmetry $SU(8)$). Central charges and their solitonic carriers have been much discussed in the recent literature because it is hoped that they may provide a window on M-Theory and its non-perturbative degrees of freedom. They also play an important role in the microscopic description of black hole entropy: for maximally extended $N = 8$ supergravity, the latter is conjectured to be given by an $E_{7(7)}$ invariant formula [12], which reproduces the known results in all cases studied so far. This formula is formally identical to our eq. (25) defining a light-cone in $\mathbb{R}^{57}$, which suggests that the 57th component of our $E_{8(8)}$ realization should be interpreted as the entropy. While the latter is only $E_{7(7)}$ invariant, the formula defining it actually possesses a bigger nonlinearly realized quasiconformal invariance under $E_{8(8)}$!

For applications to M-Theory it would be important to obtain the exponentiated version of our realization. One might reasonably expect that modular forms with respect to a fractional linear realization of the arithmetic group $E_{8(8)}(\mathbb{Z})$ will have a role to play; in this case, such forms would consequently depend on 28 complex variables and one real one. The 57 dimensions in which $E_{8(8)}$ acts might alternatively be interpreted as a generalized Heisenberg group, in which case the 57th component would play the role of a variable parameter $\hbar$. The action of $E_{8(8)}(\mathbb{Z})$ on the 57 dimensional Heisenberg group would then constitute the invariance group of a generalized Dirac quantization condition. This observation is also in accord with the fact that the term modifying the vector space addition in $\mathbb{R}^{57}$ (cf. eq.(23)), which is required by $E_{8(8)}$ invariance, is just the cocycle induced by the standard canonical commutation relations on an (28+28)-dimensional phase space.

2 Quasiconformal Realization of $E_{8(8)}$

2.1 $E_{7(7)}$ decomposition of $E_{8(8)}$

We will start with the maximal case, the exceptional Lie group $E_{8(8)}$, and its quasiconformal realization on $\mathbb{R}^{57}$, because this realization contains all others by truncation. Our results are based on the following five graded decomposition of $E_{8(8)}$ with respect to its $E_{7(7)} \times \mathcal{D}$ subgroup

$$\begin{array}{ccccccccc} \mathfrak{g}^{-2} & \oplus & \mathfrak{g}^{-1} & \oplus & \mathfrak{g}^{0} & \oplus & \mathfrak{g}^{+1} & \oplus & \mathfrak{g}^{+2} \\ \mathbf{1} & \oplus & \mathbf{56} & \oplus & (\mathbf{133} \oplus \mathbf{1}) & \oplus & \mathbf{56} & \oplus & \mathbf{1} \end{array} \tag{11}$$

with the one-dimensional group $\mathcal{D}$ consisting of dilatations. $\mathcal{D}$ itself is part of an $SL(2,\mathbb{R})$ group, and the above decomposition thus corresponds to the decomposition $\mathbf{248} \rightarrow (\mathbf{133},\mathbf{1}) \oplus (\mathbf{56},\mathbf{2}) \oplus (\mathbf{1},\mathbf{3})$ of $E_{8(8)}$ under its subgroup $E_{7(7)} \times SL(2,\mathbb{R})$.

In order to write out the $E_{7(7)}$ generators, it is convenient to further decompose them w.r.t. the maximal compact subgroup of $E_{7(7)}$, which is SU(8). In this basis, the Lie algebra of $E_{7(7)}$ is spanned by the SU(8) generators $G^i{}_j$, which are antihermitean and traceless, together with the antisymmetric and complex self-dual generators G^{ijkl}, transforming in the $\mathbf{70}$ and $\mathbf{63}$ representation of SU(8), respectively:

$$\begin{aligned} (G^{ijkl})^\dagger &= \tfrac{1}{24}\epsilon_{ijklmnpq} G^{mnpq} \ := \ G_{ijkl}\,, \\ G_i{}^j &\equiv (G^i{}_j)^\dagger = -G^j{}_i\,, \end{aligned}$$

with SU(8) indices $1 \leq i, j, \ldots \leq 8$. The commutation relations are

$$\begin{aligned} \left[G^i{}_j\,,\,G^k{}_l\right] &= \delta^k_j\, G^i{}_l - \delta^i_l\, G^k{}_j\,, \\ \left[G^i{}_j\,,\,G^{klmn}\right] &= -4\,\delta^{[k}_j\, G^{lmn]i} - \tfrac{1}{2}\,\delta^i_j G^{klmn}\,, \\ \left[G^{ijkl}\,,\,G^{mnpq}\right] &= -\tfrac{1}{36}\,\epsilon^{ijkls[mnp}\, G^{q]}{}_s\,. \end{aligned}$$

The fundamental $\mathbf{56}$ representation of E_7 is spanned by the anti-symmetric complex tensors Z^{ij} and their complex conjugates

$$Z_{ij} \ := \ (Z^{ij})^*\,.$$

The action of $E_{7(7)}$ is given by

$$\begin{aligned} \delta Z^{ij} &= \Lambda^i{}_k Z^{kj} - \Lambda^j{}_k Z^{ki} + \Sigma^{ijkl} Z_{kl}\,, \\ \delta Z_{ij} &= \Lambda^k{}_i Z_{jk} - \Lambda^k{}_j Z_{ik} + \Sigma_{ijkl} Z^{kl}\,. \end{aligned} \tag{12}$$

In order to extend $E_{7(7)} \times \mathcal{D}$ to the full $E_{8(8)}$, we must enlarge $\mathcal{D}$ to an $SL(2,\mathbb{R})$ with generators (E, F, H) in the standard Chevalley basis, together with 2×56 further generators (F_{ij}, F^{ij}) and (E_{ij}, E^{ij}), where, of course,

$$F^{ij} \ = \ (F_{ij})^* \quad \text{and} \quad E^{ij} \ = \ (E_{ij})^*\,.$$

However, under hermitean conjugation, we have

$$F^{ij} \;=\; (E_{ij})^\dagger \quad \text{and} \quad E^{ij} \;=\; (F_{ij})^\dagger \,.$$

Similarly, $E^* = E$ and $F^* = F$, but $E^\dagger = F$.

The grade $-2, -1, 1$ and 2 subspaces in the above decomposition correspond to the subspaces $\mathfrak{g}^{-2}$, $\mathfrak{g}^{-1}$, $\mathfrak{g}^1$, and $\mathfrak{g}^2$ in (11), respectively:

$$F \oplus \{F^{ij}, F_{ij}\} \oplus \{G^{ijkl},\, G^i{}_j\,;\, H\} \oplus \{E^{ij}, E_{ij}\} \oplus E \tag{13}$$

The grading may be read off from the commutators with H

$$\begin{aligned} [H\,,\,E] \;&=\; 2\,E\,, & [H\,,\,F] \;&=\; -2\,F\,,\\ \big[H\,,\,E^{ij}\big] \;&=\; E^{ij}\,, & \big[H\,,\,F^{ij}\big] \;&=\; -F^{ij}\,,\\ [H\,,\,E_{ij}] \;&=\; E_{ij}\,, & [H\,,\,F_{ij}] \;&=\; -F_{ij}\,. \end{aligned}$$

Under $\mathrm{SU}(8)$ the new generators transform as

$$\begin{aligned} \Big[G^i{}_j\,,\,E^{kl}\Big] \;&=\; -\delta^k_j\,E^{il} + \delta^l_j\,E^{ik} + \tfrac{1}{4}\delta^i_j\,E^{kl}\,,\\ \big[G^i{}_j\,,\,E_{kl}\big] \;&=\; -\delta^i_l\,E_{jk} + \delta^i_k\,E_{jl} + \tfrac{1}{4}\delta^i_j\,E_{kl}\,,\\ \Big[G^i{}_j\,,\,F^{kl}\Big] \;&=\; -\delta^k_j\,F^{il} + \delta^l_j\,F^{ik} + \tfrac{1}{4}\delta^i_j\,F^{kl}\,,\\ \big[G^i{}_j\,,\,F_{kl}\big] \;&=\; -\delta^i_l\,F_{jk} + \delta^i_k\,F_{jl} + \tfrac{1}{4}\delta^i_j\,F_{kl}\,. \end{aligned}$$

The remaining non-vanishing commutation relations are given by

$$[E, F] = H$$

and

$$\begin{aligned} \big[G^{ijkl}\,,\,E_{mn}\big] \;&=\; \delta^{[ij}_{mn}\,E^{kl]}\,, & \big[G^{ijkl}\,,\,E^{mn}\big] \;&=\; -\tfrac{1}{24}\,\varepsilon^{ijklmnpq}\,E_{pq}\,,\\ \big[G^{ijkl}\,,\,F_{mn}\big] \;&=\; -\delta^{[ij}_{mn}\,F^{kl]}\,, & \big[G^{ijkl}\,,\,F^{mn}\big] \;&=\; \tfrac{1}{24}\,\varepsilon^{ijklmnpq}\,F_{pq}\,, \end{aligned}$$

$$\begin{aligned} \big[E^{ij}\,,\,F^{kl}\big] \;&=\; -12\,G^{ijkl}\,, & \big[E^{ij}\,,\,E_{kl}\big] \;&=\; 2\,\delta^{ij}_{kl}\,E\,,\\ \big[E^{ij}\,,\,F_{kl}\big] \;&=\; -4\,\delta^{[i}_{[k}G^{j]}{}_{l]} - \delta^{ij}_{kl}\,H\,, & \big[F^{ij}\,,\,F_{kl}\big] \;&=\; -2\,\delta^{ij}_{kl}\,F\,,\ \cdot\\ \big[E\,,\,F^{ij}\big] \;&=\; -E^{ij}\,, & \big[F\,,\,E^{ij}\big] \;&=\; -F^{ij}\,. \end{aligned}$$

To see that we are really dealing with the maximally split form of E_8, let us count the number of compact generators: in addition to the 63 generators of $SU(8)$, there are $56+1$ anti-hermitean generators $(E^{ij} - F_{ij})$, $(E_{ij} - F^{ij})$ and $(E - F)$, giving a total of 120 generators corresponding to the maximal compact subgroup $SO(16)$.

An important role is played by the symplectic invariant of two $\mathbf{56}$ representations. It is given by

$$\langle X, Y\rangle \;:=\; \mathrm{i}\,(X^{ij}Y_{ij} - X_{ij}Y^{ij})\,. \tag{14}$$

The second important structure which we need to introduce is the triple product. This is a trilinear map $\mathbf{56} \times \mathbf{56} \times \mathbf{56} \longrightarrow \mathbf{56}$, which associates to three elements X, Y and Z another element transforming in the $\mathbf{56}$ representation, denoted by (X,Y,Z), and defined by

$$\begin{aligned}
(X,Y,Z)^{ij} \;:=\; & -8\mathrm{i}\, X^{i\overline{k} Y_{kl} Z^{l}j} - 8\mathrm{i}\, Y^{i\overline{k} X_{kl} Z^{l}j} - 8\mathrm{i}\, Y^{i\overline{k} Z_{kl} X^{l}j} \\
& -2\mathrm{i}\, Y^{ij} X^{kl} Z_{kl} - 2\mathrm{i}\, X^{ij} Y^{kl} Z_{kl} - 2\mathrm{i}\, Z^{ij} Y^{kl} X_{kl} \\
& + \tfrac{\mathrm{i}}{2}\, \epsilon^{ijklmnpq} X_{kl} Y_{mn} Z_{pq} \,. \qquad (15)
\end{aligned}$$

A somewhat tedious calculation[3] shows that this triple product obeys the relations

$$\begin{aligned}
(X,Y,Z) \;&=\; (Y,X,Z) + 2\,\langle X,Y\rangle\, Z\,, \\
(X,Y,Z) \;&=\; (Z,Y,X) - 2\,\langle X,Z\rangle\, Y\,, \\
\langle (X,Y,Z)\,, W\rangle \;&=\; \langle (X,W,Z)\,, Y\rangle - 2\,\langle X,Z\rangle\,\langle Y,W\rangle\;, \\
(X,Y,(V,W,Z)) \;&=\; (V,W,(X,Y,Z)) + ((X,Y,V)\,, W, Z) \\
&\qquad + (V,(Y,X,W)\,,Z)\;. \qquad (16)
\end{aligned}$$

We note that the triple product (15) could be modified by terms involving the symplectic invariant, such as $\langle X,Y\rangle Z$; the above choice has been made in order to obtain agreement with the formulas of [3].

While there is no (symmetric) quadratic invariant of $E_{7(7)}$ in the $\mathbf{56}$ representation, a real quartic invariant $\mathcal{I}_4$ can be constructed by means of the above triple product and the bilinear form; it reads

$$\begin{aligned}
\mathcal{I}_4(Z^{ij}, Z_{ij}) \;:=\; & \tfrac{1}{12}\,\langle (Z,Z,Z)\,, Z\rangle \\
\equiv\; & 4\, Z^{ij} Z_{jk} Z^{kl} Z_{li} - Z^{ij} Z_{ij} Z^{kl} Z_{kl} \\
& + \tfrac{1}{24}\, \epsilon^{ijklmnpq} Z_{ij} Z_{kl} Z_{mn} Z_{pq} \\
& + \tfrac{1}{24}\, \epsilon_{ijklmnpq} Z^{ij} Z^{kl} Z^{mn} Z^{pq} \\
\equiv\; & \mathcal{I}_4(Z^{ij}, Z_{ij})^* \,. \qquad (17)
\end{aligned}$$

2.2 Quasiconformal nonlinear realization of $E_{8(8)}$

We will now exhibit a nonlinear realization of $E_{8(8)}$ on the 57-dimensional vector space with basis $\mathcal{Z} := (Z^{ij}, Z_{ij}, z)$, where z is real, and again $Z^{ij} = (Z_{ij})^*$. While z is an $E_{7(7)}$ singlet, the remaining 56 variables transform linearly under $E_{7(7)}$. Thus $\mathcal{Z}$ forms the $\mathbf{56} \oplus \mathbf{1}$ representation of E_7. In writing the transformation rules we will always omit the transformation parameters in order not to make the formulas (and notation) too cumbersome. To recover the infinitesimal variations, one must simply contract the formulas with the appropriate transformation parameters.

[3] Which relies heavily on the Schouten identity $\varepsilon_{[ijklmnpq} X_{r]s} = 0$.

The generator H acts by scale transformations:

$$
\begin{aligned}
G^i{}_j(Z^{kl}) &= 2\,\overline{\delta_j^k Z^{il}}\,, & G^i{}_j(z) &= 0\,, \\
G^{ijkl}(Z^{mn}) &= \tfrac{1}{24}\epsilon^{ijklmnpq} Z_{pq}\,, & G^{ijkl}(z) &= 0\,, \\
H(Z^{ij}) &= Z^{ij}\,, & H(z) &= 2\,z\,.
\end{aligned}
\tag{18}
$$

The E generators act as translations on $\mathcal{Z}$; we have

$$
E(Z^{ij}) \;=\; 0\,, \qquad E(z) \;=\; 1\,. \tag{19}
$$

and

$$
\begin{aligned}
E^{ij}(Z^{kl}) &= 0\,, & E^{ij}(z) &= -\mathrm{i}\,Z^{ij}\,, \\
E_{ij}(Z^{kl}) &= \delta_{ij}^{kl}\,, & E_{ij}(z) &= \mathrm{i}\,Z_{ij}\,.
\end{aligned}
\tag{20}
$$

By contrast, the F generators are realized nonlinearly:

$$
\begin{aligned}
F(Z^{ij}) &= \tfrac{1}{6}\,(Z,Z,Z)^{ij} - Z^{ij}\,z \\
&\equiv -4\mathrm{i}\,\overline{Z^{ik} Z_{kl} Z^{lj}} - \mathrm{i} Z^{ij} Z^{kl} Z_{kl} \\
&\quad + \tfrac{\mathrm{i}}{12}\epsilon^{ijklmnpq} Z_{kl} Z_{mn} Z_{pq} - Z^{ij}\,z\,, \\
F(z) &= \mathcal{I}_4(Z^{ij}, Z_{ij}) - z^2 \\
&\equiv 4\,Z^{ij} Z_{jk} Z^{kl} Z_{li} - Z^{ij} Z_{ij} Z^{kl} Z_{kl} \\
&\quad + \tfrac{1}{24}\,\epsilon^{ijklmnpq} Z_{ij} Z_{kl} Z_{mn} Z_{pq} \\
&\quad + \tfrac{1}{24}\,\epsilon_{ijklmnpq} Z^{ij} Z^{kl} Z^{mn} Z^{pq} - z^2\,.
\end{aligned}
\tag{21}
$$

Observe that the form of the r.h.s. is dictated by the requirement of $E_{7(7)}$ covariance: $(F(Z^{ij}), F(Z_{ij}))$ and $F(z)$ must still transform as the $\mathbf{56}$ and $\mathbf{1}$ of $E_{7(7)}$, respectively. The action of the remaining generators is likewise $E_{7(7)}$ covariant:

$$
\begin{aligned}
F^{ij}(Z^{kl}) &= 4\mathrm{i}\,\overline{Z^{ki} Z^{jl}} - \tfrac{\mathrm{i}}{4}\,\epsilon^{ijklmnpq} Z_{mn} Z_{pq}\,, \\
F_{ij}(Z^{kl}) &= 8\mathrm{i}\,\overline{\delta_i^k Z_{jm} Z^{ml}} + \mathrm{i}\,\delta_{ij}^{kl} Z^{mn} Z_{mn} + 2\mathrm{i}\,Z_{ij} Z^{kl} + \delta_{ij}^{kl}\,z\,, \\
F^{ij}(z) &= 4\,Z^{ik} Z_{kl} Z^{lj} + Z^{ij} Z^{kl} Z_{kl} - \tfrac{1}{12}\,\epsilon^{ijklmnpq} Z_{kl} Z_{mn} Z_{pq} - \mathrm{i}\,Z^{ij}\,z\,, \\
F_{ij}(z) &= 4\,Z_{ik} Z^{kl} Z_{lj} + Z_{ij} Z^{kl} Z_{kl} - \tfrac{1}{12}\,\epsilon_{ijklmnpq} Z^{kl} Z^{mn} Z^{pq} + \mathrm{i}\,Z_{ij}\,z\,.
\end{aligned}
\tag{22}
$$

Clearly, $E_{7(7)}$ covariance considerably constrains the expressions that can appear on the r.h.s., but it does not fix them uniquely: as for the triple product (15) one could add further terms involving the symplectic invariant. However, all ambiguities are removed by imposing closure of the algebra, and we have checked by explicit computation that the above variations do close into the full $E_{8(8)}$ algebra in the basis given in the previous section. This is a crucial consistency check.

The term “quasiconformal realization” is motivated by the existence of a norm form that is left invariant up to a (possibly coordinate dependent) factor under all transformations. To write it down we must first define a nonlinear “difference” between two points $\mathcal{X} \equiv (X^{ij}, X_{ij}\,;\,x)$ and $\mathcal{Y} \equiv (Y^{ij}, Y_{ij}\,;\,y)$; curiously, the standard difference is *not* invariant under the translations (E^{ij}, E_{ij})! Rather, we must choose

$$\delta(\mathcal{X},\,\mathcal{Y}) \;:=\; (X^{ij}-Y^{ij}, X_{ij}-Y_{ij}\,;\,x-y+\langle X, Y\rangle)\,. \tag{23}$$

This difference still obeys $\delta(\mathcal{X},\mathcal{Y})=-\delta(\mathcal{Y},\mathcal{X})$ and thus $\delta(\mathcal{X},\mathcal{X})=0$, and is now invariant under (E^{ij}, E_{ij}) as well as E; however, it is no longer additive. In fact, with the sum of two vectors being defined as $\delta(\mathcal{X},-\mathcal{Y})$, the extra term involving $\langle X, Y\rangle$ can be interpreted as the cocycle induced by the standard canonical commutation relations. In this way, the requirement of $E_{8(8)}$ invariance becomes linked to quantization!

The relevant invariant is a linear combination of z^2 and the quartic $E_{7(7)}$ invariant $\mathcal{I}_4$, viz.

$$\mathcal{N}_4(\mathcal{Z}) \;\equiv\; \mathcal{N}_4(Z^{ij}, Z_{ij}; z) \;:=\; \mathcal{I}_4(Z) + z^2\,, \tag{24}$$

In order to ensure invariance under the translation generators, we consider the expression $\mathcal{N}_4(\delta(\mathcal{X},\mathcal{Y}))$, which is manifestly invariant under the linearly realized subgroup $E_{7(7)}$. Remarkably, it also transforms into itself up to an overall factor under the action of the nonlinearly realized generators. More specifically, we find

$$\begin{aligned}
F\Big(\mathcal{N}_4(\delta(\mathcal{X},\mathcal{Y}))\Big) \;&=\; -2\,(x+y)\,\mathcal{N}_4(\delta(\mathcal{X},\mathcal{Y}))\\
F^{ij}\Big(\mathcal{N}_4(\delta(\mathcal{X},\mathcal{Y}))\Big) \;&=\; -2\mathrm{i}\,(X^{ij}+Y^{ij})\,\mathcal{N}_4(\delta(\mathcal{X},\mathcal{Y}))\\
H\Big(\mathcal{N}_4(\delta(\mathcal{X},\mathcal{Y}))\Big) \;&=\; 4\,\mathcal{N}_4(\delta(\mathcal{X},\mathcal{Y}))
\end{aligned}$$

Therefore, for every $\mathcal{Y} \in \mathbb{R}^{57}$ the “light cone” with base point $\mathcal{Y}$, defined by the set of $\mathcal{X} \in \mathbb{R}^{57}$ obeying

$$\mathcal{N}_4(\delta(\mathcal{X},\mathcal{Y})) = 0\,, \tag{25}$$

is preserved by the full $E_{8(8)}$ group, and in this sense, $\mathcal{N}_4$ is a “conformal invariant” of $E_{8(8)}$. We note that the light cones defined by the above equation are not only curved hypersufaces in $\mathbb{R}^{57}$, but get deformed as one varies the base point $\mathcal{Y}$. The existence of a fourth order conformal invariant of $E_{8(8)}$ is noteworthy in view of the fact that no irreducible fourth order invariant exists for the linearly realized $E_{8(8)}$ group (the next invariant after the quadratic Casimir being of order eight).

2.3 Relation with Freudenthal Triple Systems

We will now rewrite the nonlinear transformation rules in another form in order to establish contact with mathematical literature. Both the bilinear form (14) and the triple product (15) already appear in [3], albeit in a very different guise. That work starts from 2×2 "matrices" of the form

$$A = \begin{pmatrix} \alpha_1 & x_1 \\ x_2 & \alpha_2 \end{pmatrix} , \tag{26}$$

where α_1, α_2 are real numbers and x_1, x_2 are elements of a simple Jordan algebra J of degree three. There are only four simple Jordan algebras J of this type, namely the 3×3 hermitian matrices over the four division algebras, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. The associated matrices are then related to non-compact forms of the exceptional Lie algebras F_4, E_6, E_7, and E_8, respectively. For simplicity, let us concentrate on the maximal case $J_3^{\mathbb{O}_S}$, when the matrix A carries 1+1+27+27 = 56 degrees of freedom. This counting suggests an obvious relation with the $\mathbf{56}$ of $E_{7(7)}$ and its decomposition under $E_{6(6)}$, but more work is required to make the connection precise. To this aim, [3] defines a symplectic invariant $\langle A, B \rangle$, and a trilinear product mapping three such matrices A, B and C to another one, denoted by (A, B, C). This triple system differs from a Jordan triple system in that it is not derivable from a binary product. The formulas for the triple product in terms of the matrices A, B and C given in [3] are somewhat cumbersome, lacking manifest $E_{7(7)}$ covariance. For this reason, instead of directly verifying that our prescription (15) and the one of [3] coincide, we have checked that they satisfy identical relations: a quick glance shows that the relations (T1)–(T4) [3] are indeed the same as our relations (16), which are manifestly $E_{7(7)}$ covariant.

To rewrite the transformation formulas we introduce Lie algebra generators U_A and $\tilde{U}_A$ labeled by the above matrices, as well as generators S_{AB} labeled by a pair of such matrices. For the grade ± 2 subspaces we would in general need another set of generators K_{AB} and $\tilde{K}_{AB}$ labeled by two matrices, but since these subspaces are one-dimensional in the present case, we have only two more generators K_a and $\tilde{K}_a$ labelled by one real number a. In the same vein, we reinterpret the 57 coordinates $\mathcal{Z}$ as a pair (Z, z), where Z is a 2×2 matrix of the type defined above. The variations then take the simple form

$$\begin{aligned}
K_a(Z) &= 0 , \\
K_a(z) &= 2\, a , \\
U_A(Z) &= A , \\
U_A(z) &= \langle A, Z \rangle , \\
S_{AB}(Z) &= (A, B, Z) , \\
S_{AB}(z) &= 2 \langle A, B \rangle \, z , \\
\tilde{U}_A(Z) &= -\tfrac{1}{2} (Z, A, Z) + A z , \\
\tilde{U}_A(z) &= \tfrac{1}{6} \langle (Z, Z, Z) , A \rangle + \langle A, Z \rangle \, z , \\
\tilde{K}_a(Z) &= \tfrac{2}{3}\, a \, (Z, Z, Z) + 2\, a Z z ,
\end{aligned}$$

$$\tilde{K}_a(z) \;=\; \tfrac{1}{3}\, a\, \langle (Z,Z,Z)\,, Z\rangle + 2\, a z^2\,, \tag{27}$$

From these formulas it is straightforward to determine the commutation relations of the transformations. To expose the connection with the more general Kantor triple systems we write

$$K_{AB} \equiv K_{\langle A,B\rangle} \tag{28}$$

in the formulas below. The consistency of this specialization is ensured by the relations (16). By explicit computation one finds

$$\begin{aligned}
[U_A, \tilde{U}_B] &= S_{AB}\,, \\
[U_A, U_B] &= K_{AB}\,, \\
[\tilde{U}_A, \tilde{U}_B] &= \tilde{K}_{AB}\,, \\
[S_{AB}, U_C] &= U_{(A,B,C)}\,, \\
[S_{AB}, \tilde{U}_C] &= -\tilde{U}_{(B,A,C)}\,, \\
[K_{AB}, \tilde{U}_C] &= -U_{(A,C,B)} + U_{(B,C,A)}\,, \\
[\tilde{K}_{AB}, U_C] &= -\tilde{U}_{(B,C,A)} + \tilde{U}_{(A,C,B)}\,, \\
[S_{AB}, S_{CD}] &= S_{(A,B,C)D} + S_{C(B,A,D)}\,, \\
[S_{AB}, K_{CD}] &= -K_{(A,B,C)D} - K_{C(A,B,D)}\,, \\
[S_{AB}, \tilde{K}_{CD}] &= -\tilde{K}_{(B,A,C)D} - \tilde{K}_{C(B,A,D)}\,, \\
[K_{AB}, \tilde{K}_{CD}] &= S_{(A,C,B)D} - S_{(B,C,A)D} - S_{(A,D,B)C} + S_{(B,D,A)C}\,.
\end{aligned} \tag{29}$$

For general K_{AB}, these are the defining commutation relations of a Kantor triple system, and, with the further specification (28), those of a Freudenthal triple system (FTS). Freudenthal introduced these triple systems in his study of the metasymplectic geometries associated with exceptional groups [5]; these geometries were further studied in [1, 3, 22, 16]. A classification of FTS's may be found in [16], where it is also shown that there is a one-to-one correspondence between simple Lie algebras and simple FTS's with a non-degenerate bilinear form. Hence there is a quasiconformal realization of every Lie group acting on a generalized lightcone.

3 Truncations of $E_{8(8)}$

For the lower rank exceptional groups contained in $E_{8(8)}$, we can derive similar conformal or quasiconformal realizations by truncation. In this section, we will first give the list of quasiconformal realizations contained in $E_{8(8)}$. In the second part of this section, we consider truncations to a three graded structure, which will yield conformal realizations. In particular, we will work out the conformal realization of $E_{7(7)}$ on a space of 27 dimensions as an example, which is again the maximal example of its kind.

3.1 More quasiconformal realizations

All simple Lie algebras (except for $SU(2)$) can be given a five graded structure (10) with respect to some subalgebra of maximal rank and associate a triple system with the grade $+1$ subspace [15, 2]. Conversely, one can construct every simple Lie algebra over the corresponding triple system.

The realization of E_8 over the FTS defined by the exceptional Jordan algebra can be truncated to the realizations of E_7, E_6, and F_4 by restricting oneself to subalgebras defined by quaternionic, complex, and real Hermitian 3×3 matrices. Analogously the non-linear realization of $E_{8(8)}$ given in the previous section can be truncated to non-linear realizations of $E_{7(7)}$, $E_{6(6)}$, and $F_{4(4)}$. These truncations preserve the five grading. More specifically we find that the Lie algebra of $E_{7(7)}$ has a five grading of the form:

$$E_{7(7)} = \overline{\mathbf{1}} \oplus \overline{\mathbf{32}} \oplus (SO(6,6) \oplus \mathcal{D}) \oplus \mathbf{32} \oplus \mathbf{1} \tag{30}$$

Hence this truncation leads to a nonlinear realization of $E_{7(7)}$ on a $\mathbf{33}$ dimensional space. Note that this is not a minimal realization of $E_{7(7)}$. Further truncation to the $E_{6(6)}$ subgroup preserving the five grading leads to:

$$E_{6(6)} = \overline{\mathbf{1}} \oplus \overline{\mathbf{20}} \oplus (SL(6,\mathbb{R}) \oplus \mathcal{D}) \oplus \mathbf{20} \oplus \mathbf{1} \tag{31}$$

This yields a nonlinear realization of $E_{6(6)}$ on a $\mathbf{21}$ dimensional space, which again is not the minimal realization. Further reduction to $F_{4(4)}$ preserving the five grading

$$F_{4(4)} = \overline{\mathbf{1}} \oplus \overline{\mathbf{14}} \oplus (Sp(6,\mathbb{R}) \oplus \mathcal{D}) \oplus \mathbf{14} \oplus \mathbf{1} \tag{32}$$

leads to a minimal realization of $F_{4(4)}$ on a fifteen dimensional space. One can further truncate F_4 to a subalgebra $G_{2(2)}$ while preserving the five grading

$$G_{2(2)} = \overline{\mathbf{1}} \oplus \overline{\mathbf{4}} \oplus (SL(2,\mathbb{R}) \oplus \mathcal{D}) \oplus \mathbf{4} \oplus \mathbf{1}\,, \tag{33}$$

which then yields a nonlinear realization over a five dimensional space. One can go even futher and truncate G_2 to its subalgebra $SL(3,\mathbb{R})$

$$SL(3,\mathbb{R}) = \overline{\mathbf{1}} \oplus \overline{\mathbf{2}} \oplus (SO(1,1) \oplus \mathcal{D}) \oplus \mathbf{2} \oplus \mathbf{1}\,, \tag{34}$$

which is the smallest simple Lie algebra admitting a five grading. We should perhaps stress that the nonlinear realizations given above are minimal for $G_{2(2)}$, $F_{4(4)}$, and $E_{8(8)}$ which are the only simple Lie algebras that do not admit a three grading and hence do not have unitary representations of the lowest weight type.

The above nonlinear realizations of the exceptional Lie algebras can also be truncated to subalgebras with a three graded structure, in which case our nonlinear realization reduces to the standard nonlinear realization over a JTS. This truncation we will describe in section 3.2 in more detail.

With respect to $E_{6(6)}$ the quasiconformal realization of $E_{8(8)}$ (11) decomposes as follows:

$$
\begin{array}{ccccccccc}
\mathbf{1} & \oplus & \mathbf{56} & \oplus & (\mathbf{133} \oplus \mathbf{1}) & \oplus & \mathbf{56} & \oplus & \mathbf{1} \\
 & & & & \mathbf{1} & & & & \\
 & & \mathbf{1} & & \oplus & & \mathbf{1} & & \\
 & & \oplus & & \mathbf{27} & & \oplus & & \\
 & & \mathbf{27} & & \oplus & & \mathbf{27} & & \\
\mathbf{1} & & \oplus & & \mathbf{78} & & \oplus & & \mathbf{1} \\
 & & \overline{\mathbf{27}} & & \oplus & & \overline{\mathbf{27}} & & \\
 & & \oplus & & \overline{\mathbf{27}} & & \oplus & & \\
 & & \mathbf{1} & & \oplus & & \mathbf{1} & & \\
 & & & & \mathbf{1} & & & &
\end{array}
$$

e7e8emb.pst

The $\mathbf{27}$ of grade $+1$ subspace and the $\overline{\mathbf{27}}$ of grade -1 subspace close into the $E_{6(6)} \oplus \mathcal{D}$ subalgebra of grade zero subspace and generate the Lie algebra of $E_{7(7)}$. Similarly $\overline{\mathbf{27}}$ of grade +1 subspace together with the $\mathbf{27}$ of grade -1 subspace form another $E_{7(7)}$ subalgebra of $E_{8(8)}$. Hence we have four different $E_{7(7)}$ subalgebras of $E_{8(8)}$:

i) $E_{7(7)}$ subalgebra of grade zero subspace which is realized linearly.

ii) $E_{7(7)}$ subalgebra preserving the 5-grading, which is realized nonlinearly over a 33 dimensional space

iii) $E_{7(7)}$ subalgebra that acts on the $\mathbf{27}$ dimensional subspace as the generalized conformal generators.

iv) $E_{7(7)}$ subalgebra that acts on the $\overline{\mathbf{27}}$ dimensional subspace as the generalized conformal generators.

Similarly for $E_{7(7)}$ under the $SL(6,\mathbb{R})$ subalgebra of the grade zero subspace the $\mathbf{32}$ dimensional grade $+1$ subspace decomposes as

$$\mathbf{32} = \mathbf{1} + \overline{\mathbf{15}} + \mathbf{15} + \mathbf{1}\,.$$

The $\mathbf{15}$ from grade $+1\,(-1)$ subspace together with $\overline{\mathbf{15}}\ (\mathbf{15})$ of grade $-1\,(+1)$ subspace generate a nonlinearly realized $SO(6,6)$ subalgebra that acts as the generalized conformal algebra on the $\mathbf{15}\ (\overline{\mathbf{15}})$ dimensional subspace.

For $E_{6(6)}$, $F_{4(4)}$, $G_{2(2)}$, and $SL(3,\mathbb{R})$ the analogous truncations lead to nonlinear conformal subalgebras $SL(6,\mathbb{R})$, $Sp(6,\mathbb{R})$, $SO(2,2)$, and $SL(2,\mathbb{R})$, respectively.

3.2 Conformal Realization of $E_{7(7)}$

As a special truncation the quasiconformal realization of $E_{8(8)}$ contains a conformal realization of $E_{7(7)}$ on a space of 27 dimensions, on which the $E_{6(6)}$ subgroup of $E_{7(7)}$ acts linearly. The main difference is that the construction is now based on a three-graded decomposition (4) of $E_{7(7)}$ rather than (10) – hence the realization is "conformal" rather than "quasiconformal". The relevant decomposition can be directly read off from the figure: we simply truncate to an $E_{7(7)}$ subalgebra in such a way that the grade ± 2 subspace can no longer be reached by commutation. This requirement is met only by the two truncations corresponding to the diagonal lines in the figure; adding a singlet we arrive at the desired three graded decomposition of $E_{7(7)}$

$$\mathbf{133} = \mathbf{27} \oplus (\mathbf{78} \oplus \mathbf{1}) \oplus \overline{\mathbf{27}} \tag{35}$$

under its $E_{6(6)} \times \mathcal{D}$ subgroup.

The Lie algebra $E_{6(6)}$ has USp(8) as its maximal compact subalgebra. It is spanned by a symmetric tensor $\tilde{G}^{ij}$ in the adjoint representation $\mathbf{36}$ of USp(8) and a fully antisymmetric symplectic traceless tensor $\tilde{G}^{ijkl}$ transforming under the $\mathbf{42}$ of USp(8); indices $1 \leq i, j, \ldots \leq 8$ are now USp(8) indices and all tensors with a tilde transform under USp(8) rather then SU(8). $\tilde{G}^{ijkl}$ is traceless with respect to the real symplectic metric $\Omega_{ij} = -\Omega_{ji} = -\Omega^{ij}$ (thus $\Omega_{ik}\Omega^{kj} = \delta_i^j$). The symplectic metric also serves to pull up and down indices, with the convention that this is always to be done from the left.

The $E_{6(6)}$ generators are most simply recovered from those of $E_{7(7)}$: we have

$$\begin{array}{ccccccc} G^{ijkl} & =: & \tilde{G}^{ijkl} & + & 3\mathrm{i}\,\Omega^{[ij}V^{kl]} & + & \Omega^{[ij}\Omega^{kl]}\tilde{H} \\ \mathbf{70} & \rightarrow & \mathbf{42} & + & \mathbf{27} & + & \mathbf{1} \end{array} \tag{36}$$

and (with $G^{ij} := \Omega^{ik}G_k{}^j$)

$$\begin{array}{ccccc} G^{ij} & =: & \tilde{G}^{ij} & + & \mathrm{i}\,U^{ij} \\ \mathbf{63} & \rightarrow & \mathbf{36} & + & \mathbf{27} \end{array} , \tag{37}$$

where $\tilde{G}^{ij}$ is symmetric and U^{ij} antisymmetric; by definition all antisymmetric tensors on the r.h.s. are thus symplectic traceless. The generators $\tilde{G}^{ijkl}, \tilde{G}^{ij}$ form a $E_{6(6)}$ subalgebra; $\tilde{H}$ is the extra dilatation generator. The translation generators $\tilde{E}^{ij}$ and the nonlinearly realized generators $\tilde{F}^{ij}$, transforming as $\mathbf{27}$ and $\overline{\mathbf{27}}$, respectively, are defined by taking the following linear combinations of the remaining generators U^{ij} and V^{ij}:

$$\begin{aligned} \tilde{E}^{ij} &:= U^{ij} + V^{ij} , \\ \tilde{F}^{ij} &:= U^{ij} - V^{ij} . \end{aligned}$$

Unlike for $E_{8(8)}$, there is no need here to distinguish the generators by the position of their indices, since the corresponding generators are linearly related by means of the symplectic metric.

The fundamental $\mathbf{27}$ of $E_{6(6)}$ (on which we are going to realize a nonlinear action of $E_{7(7)}$) is given by the traceless anti-symmetric tensor $\tilde{Z}^{ij}$ transforming as

$$
\begin{aligned}
\tilde{G}^{i}{}_{j}(\tilde{Z}^{kl}) \;&=\; 2\,\delta^{k}_{j}\tilde{Z}^{il}\,, \\
\tilde{G}^{ijkl}(\tilde{Z}^{mn}) \;&=\; \tfrac{1}{24}\varepsilon^{ijklmnpq}\tilde{Z}_{pq}\,,
\end{aligned}
\tag{38}
$$

where

$$
\tilde{Z}_{ij} \;:=\; \Omega_{ik}\Omega_{jl}\tilde{Z}^{kl} = (\tilde{Z}^{ij})^{*}\,.
$$

Likewise, the $\overline{\mathbf{27}}$ representation transforms as

$$
\begin{aligned}
\tilde{G}^{i}{}_{j}(\bar{Z}^{kl}) \;&=\; 2\,\delta^{k}_{j}\bar{Z}^{il}\,, \\
\tilde{G}^{ijkl}(\bar{Z}^{mn}) \;&=\; -\tfrac{1}{24}\varepsilon^{ijklmnpq}\bar{Z}_{pq}\,.
\end{aligned}
\tag{39}
$$

Because the product of two $\mathbf{27}$'s contains no singlet, there exists no quadratic invariant of $E_{6(6)}$; however, there is a cubic invariant given by

$$
\mathcal{N}_3(\tilde{Z}) := \tilde{Z}^{ij}\tilde{Z}_{jk}\tilde{Z}^{kl}\Omega_{il}\,. \tag{40}
$$

As we already mentioned, both the $\mathbf{27}$ and the $\overline{\mathbf{27}}$ are contained in the $\mathbf{56}$ of E_7; we have

$$
\begin{array}{ccccccccc}
Z^{ij} & =: & \tilde{Z}^{ij} & + & \mathrm{i}\,\bar{Z}^{ij} & + & \Omega^{ij}\tilde{Z} & + & \mathrm{i}\,\Omega^{ij}\bar{Z} \\
\mathbf{56} & \to & \mathbf{27} & + & \overline{\mathbf{27}} & + & \mathbf{1} & + & \mathbf{1}
\end{array}\,,
$$

where, of course

$$
\Omega_{ij}\tilde{Z}^{ij} \;=\; \Omega_{ij}\bar{Z}^{ij} \;=\; 0\,.
$$

We are now ready to give the conformal realization of $E_{7(7)}$ on the 27 dimensional space spanned by the $\tilde{Z}^{ij}$. As the action of the linearly realized $E_{6(6)}$ subgroup has already been given, we list only the remaining variations. As before $\tilde{E}^{ij}$ acts by translations:

$$
\tilde{E}^{ij}(\tilde{Z}^{kl}) = -\Omega^{i[k}\Omega^{l]j} - \tfrac{1}{8}\Omega^{ij}\Omega^{kl} \tag{41}
$$

and $\tilde{H}$ by dilatations

$$
\tilde{H}(\tilde{Z}^{ij}) = \tilde{Z}^{ij}\,. \tag{42}
$$

The $\overline{\mathbf{27}}$ generators $\tilde{F}^{ij}$ are realized nonlinearly:

$$
\begin{aligned}
\tilde{F}^{ij}(\tilde{Z}^{kl}) \;:=\; & -2\,\tilde{Z}^{ij}(\tilde{Z}^{kl}) + \Omega^{i[k}\Omega^{l]j}(\tilde{Z}^{mn}\tilde{Z}_{mn}) + \tfrac{1}{8}\,\Omega^{ij}\Omega^{kl}(\tilde{Z}^{mn}\tilde{Z}_{mn}) \\
& + 8\,\tilde{Z}^{km}\tilde{Z}_{mn}\Omega^{n[i}\Omega^{j]l} - \Omega^{kl}(\tilde{Z}^{im}\Omega_{mn}\tilde{Z}^{nj})
\end{aligned}
\tag{43}
$$

The norm form needed to define the $E_{7(7)}$ invariant "light cones" is now constructed from the cubic invariant of $E_{6(6)}$. Then $\mathcal{N}_3(\tilde{X}-\tilde{Y})$ is manifestly invariant under $E_{6(6)}$ and under the translations $\tilde{E}^{ij}$ (observe that there is no need to introduce a nonlinear difference unlike for $E_{8(8)}$). Under $\tilde{H}$ it transforms by a constant factor, whereas under the action of $\tilde{F}^{ij}$ we have

$$\tilde{F}^{ij}\Big(\mathcal{N}_3(\tilde{X}-\tilde{Y})\Big) = (\tilde{X}^{ij}+\tilde{Y}^{ij})\mathcal{N}(\tilde{X}-\tilde{Y})\,. \tag{44}$$

Thus the light cones in $\mathbb{R}^{27}$ with base point $\tilde{Y}$

$$\mathcal{N}_3(\tilde{X}-\tilde{Y}) = 0 \tag{45}$$

are indeed invariant under $E_{7(7)}$. They are still curved hypersurfaces, but in contrast to the $E_{8(8)}$ light-cones constructed before, they are no longer deformed as one varies the base point $\tilde{Y}$.

The connection to the Jordan Triple Systems of appendix A can now be made quite explicit, and the formulas that we arrive at in this way are completely analogous to the ones given in the introduction. We first of all notice that we can again define a triple product in terms of the $E_{6(6)}$ representations; it reads

$$\begin{aligned}\{\tilde{X}\,\tilde{Y}\,\tilde{Z}\}^{ij} \;=\; & 16\,\tilde{X}^{i\overline{k\,}}\overline{\tilde{Z}_{kl}\tilde{Y}^{lj}} + 16\,\tilde{Z}^{i\overline{k\,}}\overline{\tilde{X}_{kl}\tilde{Y}^{lj}} + 4\,\Omega^{ij}(\tilde{X}^{kl}\tilde{Y}_{lm}\tilde{Z}^{mn}\Omega_{kn}) \\ & + 4\,\tilde{X}^{ij}\tilde{Y}^{kl}\tilde{Z}_{kl} + 4\,\tilde{Y}^{ij}\tilde{X}^{kl}\tilde{Z}_{kl} + 2\,\tilde{Z}^{ij}\tilde{X}^{kl}\tilde{Y}_{kl}\,.\end{aligned} \tag{46}$$

This triple product can be used to rewrite the conformal realization. Recalling that a triple product with identical properties exists for the 27-dimensional Jordan algebra $J_3^{\mathbb{O}_S}$, we now now consider $\tilde{Z}$ as an element of $J_3^{\mathbb{O}_S}$. Next we introduce generators labeled by elements of $J_3^{\mathbb{O}_S}$, and define the variations

$$\begin{aligned} U_a(\tilde{Z}) \;&=\; a\,, \\ S_{ab}(\tilde{Z}) \;&=\; \{a\,b\,\tilde{Z}\}\,, \\ \tilde{U}_c(\tilde{Z}) \;&=\; = -\tfrac{1}{2}\{\tilde{Z}\,c\,\tilde{Z}\}\,, \end{aligned} \tag{47}$$

for $a, b, c \in J_3^{\mathbb{O}_S}$. It is straightforward to check that these reproduce the commutation relations listed in the introduction with the only difference that $J_2^{\mathbb{C}}$ has been replaced by $J_3^{\mathbb{O}_S}$.

Appendix A Jordan Triple Systems

Let us first recall the defining properties of a Jordan algebra. By definition these are algebras equipped with a commutative (but non-associative) binary product $a \circ b = b \circ a$ satisfying the Jordan identity

$$(a \circ b) \circ a^2 = a \circ (b \circ a^2)\,. \tag{A.1}$$

A Jordan algebra with such a product defines a so-called Jordan triple system (JTS) under the Jordan triple product

$$\{a\,b\,c\} = a \circ (\tilde{b} \circ c) + (a \circ \tilde{b}) \circ c - \tilde{b} \circ (a \circ c)\,,$$

where $\tilde{}$ denotes a conjugation in J corresponding to the operation † in $\mathfrak{g}$. The triple product satisfies the identities (which can alternatively be taken as the defining identities of the triple system)

$$\begin{aligned} &\{a\,b\,c\} = \{c\,b\,a\}\,, \\ &\{a\,b\,\{c\,d\,x\}\} - \{c\,d\,\{a\,b\,x\}\} - \{a\,\{d\,c\,b\}\,x\} + \{\{c\,d\,a\}\,b\,x\} = 0\,. \end{aligned} \tag{A.2}$$

The Tits-Kantor-Koecher (TKK) construction [24, 13, 17] associates every JTS with a 3-graded Lie algebra

$$\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{+1}\,, \tag{A.3}$$

satsifying the formal commutation relations:

$$\begin{aligned} \left[\mathfrak{g}^{+1}\,,\,\mathfrak{g}^{-1}\right] &= \mathfrak{g}^{0}\,, \\ \left[\mathfrak{g}^{+1}\,,\,\mathfrak{g}^{+1}\right] &= 0\,, \\ \left[\mathfrak{g}^{-1}\,,\,\mathfrak{g}^{-1}\right] &= 0\,. \end{aligned}$$

With the exception of the Lie algebras G_2, F_4, and E_8 every simple Lie algebra $\mathfrak{g}$ can be given a three graded decomposition with respect to a subalgebra $\mathfrak{g}^0$ of maximal rank.

By the TKK construction the elements U_a of the $\mathfrak{g}^{+1}$ subspace of the Lie algebra are labelled by the elements $a \in J$. Furthermore every such Lie algebra $\mathfrak{g}$ admits an involutive automorphismι, which maps the elements of the grade $+1$ space onto the elements of the subspace of grade -1:

$$\iota(U_a) =: \tilde{U}_a \in \mathfrak{g}^{-1} \tag{A.4}$$

To get a complete set of generators of $\mathfrak{g}$ we define

$$\begin{aligned} [U_a, \tilde{U}_b] &= S_{ab}\,, \\ [S_{ab}, U_c] &= U_{\{abc\}}\,, \end{aligned} \tag{A.5}$$

where $S_{ab} \in \mathfrak{g}^0$ and $\{abc\}$ is the Jordan triple product under which the space J is closed.

The remaining commutation relations are

$$
\begin{aligned}
[S_{ab}, \tilde{U}_c] &= \tilde{U}_{\{bac\}} , \\
[S_{ab}, S_{cd}] &= S_{\{abc\}d} - S_{c\{bad\}},
\end{aligned}
\tag{A.6}
$$

and the closure of the algebra under commutation follows from the defining identities of a JTS given above.

The Lie algebra generated by S_{ab} is called the structure algebra of the $JTS\ J$, under which the elements of J transform linearly. The traceless elements of this action of S_{ab} generate the reduced structure algebra of J. There exist four infinite families of hermitian JTS's and two exceptional ones [23, 19]. The latter are listed in the table below (where $M_{1,2}(\mathbb{O})$ denotes 1×2 matrices over the octonions, i.e. the octonionic plane)

J	G	H
$M_{1,2}(\mathbb{O}_S)$	$E_{6(6)}$	$SO(5,5)$
$M_{1,2}(\mathbb{O})$	$E_{6(-14)}$	$SO(8,2)$
$J_3^{\mathbb{O}_S}$	$E_{7(7)}$	$E_{6(6)}$
$J_3^{\mathbb{O}}$	$E_{7(-25)}$	$E_{6(-26)}$

Here we are mainly interested in the real form $J_3^{\mathbb{O}_S}$, which corresponds to the split octonions $\mathbb{O}_S$ and has $E_{7(7)}$ and $E_{6(6)}$ as its conformal and reduced structure group, respectively.

References

[1] B. Allison, J. Faulkner. A Cayley-Dickson process for a class of structurable algebras. *Trans. Am. Math. Soc.*, **283**, 185 (1984)

[2] I. Bars, M. Günaydin. Construction of Lie algebras and Lie superalgebras from ternary algebras. *J. Math. Phys.*, **20**, 1977 (1979)

[3] J. Faulkner. A construction of Lie algebras from a class of ternary algebras. *Trans. Am. Math. Soc.*, **155**, 397 (1971)

[4] H. Freudenthal. Beziehungen der E_7 und E_8 zur Oktavenebene. I. *Nederl. Akad. Wet., Proc., Ser. A*, **57**, 218 (1954)

[5] H. Freudenthal. Oktaven, Ausnahmegruppen und Oktavengeometrie. *Geom. Dedicata*, **19**, 7 (1985)

[6] M. Günaydin. Exceptional realizations of lorentz group: Supersymmetries and leptons. *Nuovo Cim.*, **A29**, 467 (1975)

[7] M. Günaydin. Quadratic jordan formulation of quantum mechanics and construction of lie (super)algebras from jordan (super)algebras. *Ann. Israel Phys. Soc*, **3** (1980)

[8] M. Günaydin. The exceptional superspace and the quadratic jordan formulation of quantum mechanics. In J. Schwarz (editor), *Elementary particles and the universe*, pages 99–119. Cambridge University Press (1989)

[9] M. Günaydin. Generalized conformal and superconformal group actions and jordan algebras. *Mod. Phys. Lett. A*, **8**, 1407 (1993)

[10] M. Günaydin. AdS/CFT dualities and the unitary representations of non-compact groups: Wigner versus Dirac (1999). To appear in Turkish Journal of Physics, hep-th/0005168

[11] A. Joseph. Minimal realizations and spectrum generating algebras. *Commun. Math. Phys.*, **36**, 325 (1974)

[12] R. Kallosh, B. Kol. E(7) symmetric area of the black hole horizon. *Phys. Rev.*, **D53**, 5344 (1996)

[13] I. Kantor. Classification of irreducible transitively differential groups. *Sov. Math., Dokl.*, **5**, 1404 (1965)

[14] I. Kantor. Nonlinear transformation groups defined by general norms of Jordan algebras. *Sov. Math., Dokl.*, **8**, 176 (1967)

[15] I. Kantor. Models of exceptional Lie algebras. *Sov. Math., Dokl.*, **14**, 254 (1973)

[16] I. Kantor, I. Skopets. Some results on Freudenthal triple systems. *Sel. Math. Sov.*, **2**, 293 (1982)

[17] M. Koecher. Imbedding of Jordan algebras into Lie algebras. I. *Am. J. Math.*, **89**, 787 (1967)

[18] M. Koecher. Ueber eine Gruppe von rationalen Abbildungen. *Invent. Math.*, **3**, 136 (1967)

[19] O. Loos. *Jordan pairs*. Lecture Notes in Math. Springer, Berlin-Heidelberg (1975)

[20] G. Mack. All unitary ray representations of the conformal group SU(2,2) with positive energy. *Commun. Math. Phys.*, **55**, 1 (1977)

[21] G. Mack, A. Salam. Finite component field representations of the conformal group. *Ann. Phys.*, **53**, 174 (1969)

[22] K. Meyberg. Eine Theorie der Freudenthalschen Tripelsysteme. I, II. *Nederl. Akad. Wet., Proc., Ser. A*, **71**, 162 (1968)

[23] E. Neher. On the classification of Lie and Jordan triple systems. *Commun. Algebra*, **13**, 2615 (1985)

[24] J. Tits. Une classe d'algebres de Lie en relation avec les algebres de Jordan. *Nederl. Akad. Wet., Proc., Ser. A*, **65**, 530 (1962)