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BOUNDARIES, CROSSCAPS AND SIMPLE CURRENTS

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Abstract

Universal formulas for the boundary and crosscap coefficients are presented, which are valid for all symmetric simple current modifications of the charge conjugation invariant of any rational conformal field theory.

1. Boundaries and crosscaps

In this letter we report progress on the problem of finding boundaries and crosscaps for all conformal field theories that can be obtained as simple current invariants of a given rational conformal field theory (RCFT). This line of research was initiated by Cardy [1], who obtained the (bulk symmetry preserving) boundaries in the case where the torus partition function is the charge conjugation invariant. Later, in [2] the corresponding crosscap coefficients were obtained. In a series of subsequent papers [3–13] various more general situations were considered. In particular one may choose a different Klein bottle projection and a different modular invariant partition function (MIPF) $\chi_i Z_{ij} \bar{\chi}_j$ for the bulk theory.

The basic data one would like to determine are the set $\{m\}$ of Ishibashi labels, the set $\{a\}$ of boundary labels, a matrix $B_{m,a}$ of boundary coefficients and a vector Γ_m of crosscap coefficients. By "Ishibashi labels" we mean the labels of the Ishibashi states [14] that can propagate in the transverse (closed string) channel. There exists such a label for each primary field i that in the torus partition function is paired with its conjugate i^c. A difficulty arises when some of these terms in the torus partition function have a multiplicity larger than 1. In this case we must allow for multiplicities in the Ishibashi labels as well; the degeneracy is precisely given by Z_{ii^c}. The Ishibashi labels then consist of a pair (i, α), where i is a primary field label and α, which can take Z_{ii^c} different values, takes care of the degeneracy.

These data must satisfy a large collection of "sewing constraints" [15,16,4]. Unfortunately, most of them cannot be checked explicitly because they require detailed knowledge of fusing matrices, braiding matrices and OPE coefficients. However, there exists a set of simpler constraints, presumably a consequence of the sewing constraints, but certainly necessary, namely the requirement of positivity and integrality of the partition functions. These partition functions correspond to the torus, annulus, Möbius strip and Klein bottle surface. Each partition function is a linear combination of characters χ^i with certain arguments that depend on the surface under consideration, and with coefficients that depend on the choice of boundary. These coefficients are given by

$$\begin{aligned} A^i_{ab} &= \sum_{j,\alpha,\beta} S^i_j \, B_{(j,\alpha),a} \, B_{(j,\beta),b} \, g^{\alpha\beta}_j \,, \\ M^i_a &= \sum_{j,\alpha,\beta} P^i_j \, B_{(j,\alpha),a} \, \Gamma_{(j,\beta)} \, g^{\alpha\beta}_j \,, \\ K^i &= \sum_{j,\alpha,\beta} S^i_j \, \Gamma_{(j,\alpha)} \, \Gamma_{(j,\beta)} \, g^{\alpha\beta}_j \end{aligned} \tag{1}$$

for annulus, Möbius strip and Klein bottle, respectively. Here S is the usual modular transformation matrix of the RCFT, and $P = \sqrt{T} S T^2 S \sqrt{T}$ as introduced in [17]. Further, g_j is a "metric" in the space of degeneracy labels of the Ishibashi states belonging to j. This is part of the data to be determined. The torus partition function is described in terms of a non-negative integer matrix $Z = (Z_{ij})$. All these quantities must be integers, and the annulus coefficients as well as the combinations $\frac{1}{2}(Z_{ii} + K^i)$ and $\frac{1}{2}(A^i_{aa} + M^i_a)$ (the closed and open string partition function coefficients) must be non-negative integers. Furthermore A^0_{ab}, the boundary conjugation matrix (the label "0" refers to the vacuum), must be a permutation of order 2. In practice these conditions have turned out to be very restrictive. In addition to this one may wish to satisfy the "completeness conditions" [4], which in the present context is equivalent to requiring that the number of Ishibashi labels equals the number of boundary labels.

We pause to emphasize that what we wish to obtain is the complete set of boundaries and crosscaps that possess the symmetry $\bar{\mathfrak{A}}$ of the given unextended theory, even when the full bulk symmetry $\mathfrak{A}$ is larger due to the extension that is implied by the torus partition function. In other words, even though we express our results in terms of quantities of the underlying unextended theory, we are indeed studying boundaries and crosscaps of a CFT whose chiral algebra is $\mathfrak{A}$, not $\bar{\mathfrak{A}}$. ($\bar{\mathfrak{A}}$ and $\mathfrak{A}$ coincide if and only if the torus partition function is a pure automorphism invariant.) In particular our previous and present results include, in this sense, the case of "symmetry breaking boundaries" which preserve only part of the full bulk symmetry. Also note that in the of free boson case our results amount to finding D-branes (for boundary states) and orientifold planes (for crosscaps) that are not space-time filling, i.e. where some directions are Dirichlet, corresponding to the presence of a non-trivial automorphism for the boundary.

2. Simple current invariants

In principle one would like to determine the data listed above for arbitrary bulk modular invariants. A large subclass of the latter are the simple current invariants. What we will consider in this paper is in fact the complete class of (symmetric) [1] simple current modifications of the charge conjugation invariant. If the RCFT is real (in the sense that all fields are self-conjugate) this set nearly exhausts the possibilities, except for a few sporadic exceptional invariants. Complex RCFT's possess a second large set of invariants, namely the simple current modifications of the diagonal invariant. The diagonal invariant itself was discussed in [9] and was found to require additional data from a suitable orbifold theory. Its simple current modifications are obviously of interest as well, but they involve similar complications and are beyond the scope of this paper.

A complete classification of all simple current invariants of any RCFT has been achieved in [18, 19]. In various special cases, boundaries and crosscaps have already been studied. In particular, all cases where the MIPF is a pure extension of the chiral algebra were dealt with in [7, 10] as far as the boundaries are concerned. In [13] the crosscap coefficients were obtained for Z_2 and $Z_{\rm odd}$ extensions. Also pure automorphisms due to cyclic simple currents have been considered for boundaries [5] as well as crosscaps [12], building on pioneering work of [3, 4]. The general class of simple current invariants contains, however, some additional types of invariants, such as automorphims of pure extensions, and automorphisms generated by integer spin currents [20]. There are several motivations for trying to generalize the previous results. Simple current invariants appear abundantly in all practical applications of RCFT to string model building, and with the formulas we will present here a huge set of open string models becomes accessible to explicit computation. But in addition we expect that a general formula will provide additional insight in the conceptual issues involved in formulating RCFT on surfaces with boundaries and crosscaps.

Comparing the results obtained so far for pure extensions and pure automorphism invariants one notices a similarity between the formulas for crosscaps. The similarity between the boundary coefficients of the two cases is less obvious, but what they do have in common is that

[1] In this letter we will demand that the theory exists on unoriented surfaces, although it might be possible to relax this condition. This requires the torus partition function to be symmetric.

a crucial rôle is played by the so-called "fixed point resolution matrices". Our approach to the problem is as follows. We start with an *ansatz* for a general formula that includes all previous cases. This *ansatz* consists in particular of a prescription to determine the Ishibashi labels m, and the boundary labels a, plus a set of boundary coefficients B_{ma}. We then prove that B_{ma} has a left and a right inverse, so that it is a square matrix. This shows that the number of boundaries equals the number of Ishibashi labels, so that the set of boundaries is complete. We also compute the annulus coefficients and prove that they are integral.

Using integrality in the vacuum sector of the open string partition function we can then, following [21], determine the crosscap coefficients up to a collection of signs. Some of these signs are fixed by imposing integrality of K^i; some more signs are fixed by requiring integrality and positivity of the closed string partition function. On the other hand, some of the signs are not fixed by any constraint. They correspond to different Klein bottle choices, a possibility already encountered in previous cases. The final check is to compute the Möbius coefficients and verify open string integrality.

In this letter we will only present the results of this analysis. Proofs and further details will be postponed to a forthcoming publication [22]. We begin with the description of the torus partition function given in [19]. A general simple current invariant is characterized by a set of simple currents forming a finite abelian group $\mathcal{G}$, and a matrix X. The abelian group $\mathcal{G}$ is a product of k cyclic factors $\mathbb{Z}_{n_s}$, each generated by some current J_s. The monodromy matrix R of these generators is defined as $R_{st} := Q_s(J_t)$, where the monodromy charge Q_s is the combination $Q_s(i) = h_i + h_{J_s} - h_{J_s i} \bmod 1$ of conformal weights, plus a further constraint that fixes its diagonal elements modulo 2, depending on the conformal weight of the currents. The matrix X (defined modulo 1) must satisfy

$$X + X^T = R \tag{2}$$

and a certain quantization condition on the antisymmetric part of X, to be discussed below. The matrix X determines the matrix $Z = Z(\mathcal{G}, X)$ as follows: Z_{ij} is equal to the number of solutions J to the conditions[2]

$$\begin{aligned} &j = Ji\,, \quad J \in \mathcal{G} \quad \text{and} \\ &Q_K(i) + X(K,J) = 0 \bmod 1 \quad \text{for all } K \in \mathcal{G}\,. \end{aligned} \tag{3}$$

Here $X(K,J)$ is the number

$$X(K,J) \equiv \sum_{s,t} n_s\, m_t\, X_{st}\,, \tag{4}$$

with n_s and m_t obtained by expressing J and K through the generating currents J_s as $J = (J_1)^{n_1} \cdots (J_k)^{n_k}$, $K = (J_1)^{m_1} \cdots (J_k)^{m_k}$.

The restriction to symmetric invariants implies that X must be symmetric modulo integers. This leads to the much simpler equation $2X = R$, which determines X completely on the diagonal (since R is defined modulo 2), and modulo half-integers off-diagonally. The solutions can be described more precisely as follows. The matrix elements R_{st} and X_{st} are rationals satisfying the property that the products (no summation implied) $N_s R_{st}$, $N_s X_{st}$, $R_{st} N_t$ and $X_{st} N_t$ are integers, where N_s is the order of J_s. If N_s is odd, $R_{ss} N_s$ is always even, and hence X_{ss} is

[2] Clearly, it is sufficient to check the second condition for the cyclic group generators $J_s \in \mathcal{G}$.

determined. If N_s is even, $R_{ss}N_s$ may be odd. Then there is no solution for X_{ss}. In that case the current J_s does not belong to the "effective center", and cannot be used to build modular invariants. A second case in which $2X = R$ has no solutions is when N_s is even and N_sR_{st} is odd for some value of $t \neq s$. Then there are only non-symmetric invariants. In all other cases at least one solution exists. If both N_s and N_t are even one may shift the off-diagonal element X_{st} by a half-integer.

3. Ishibashi and boundary labels

The modular invariant $Z(\mathcal{G}, X)$ specified by X is to be multiplied with the charge conjugation matrix. Hence the Ishibashi states correspond to the *diagonal* elements of $Z(\mathcal{G}, X)$, counting multiplicities. The only currents that can contribute are those that satisfy $Ji = i$. They form a group, the stabilizer $\mathcal{S}_i$ of i. If this group is non-trivial, multiplicities larger than 1 may occur, possibly leading to Ishibashi label degeneracies. For pure extensions this was analyzed in [7,10], and the conclusion is that the Ishibashi label degeneracy is actually equal to the fixed point degeneracy.[3] It is natural to extend this result to the general case, and to label the degeneracy by the currents that cause it. Hence our *ansatz* for the Ishibashi labels is

$$m = (i, J); \quad J \in \mathcal{S}_i \;\; \text{with} \;\; Q_K(i) + X(K, J) = 0 \bmod 1 \;\; \text{for all } K \in \mathcal{G}\,. \tag{5}$$

This *ansatz* produces also the correct count for pure extension invariants, but the labelling chosen here is not the same as in [7, 10]. In those papers the dual basis – the characters ψ_α of $\mathcal{S}_i$ – was used for the degeneracy labels. This is not possible for pure automorphims because the currents satisfying (5) do not form a group in that case. For pure extensions the new basis differs by a Fourier transformation from the old one. This allows us to compute the degeneracy metric, given the fact that it was diagonal in the old basis. We find

$$g_j^{J,K} = \sum_{\alpha\beta} \psi_\alpha(J)\, \psi_\beta(K)\, \delta^{\alpha,\beta} = \delta^{J,K^c}\,. \tag{6}$$

Now we turn to the boundary labels. The results for pure extensions and automorphisms without fixed points is that the boundaries are in one-to-one correspondence with the complete set of $\mathcal{G}$ orbits (of arbitrary monodromy charge). As usual fixed points lead to degeneracies. For pure automorphism invariants due to a half-integer spin simple current the degeneracy was found to be given by the order of the stabilizer of the orbit, whereas for pure extensions it is the order of the untwisted stabilizer. The latter is defined as follows [23]. For every simple current J with fixed points there exists a "fixed point resolution matrix" S^J; these matrices can be used to express the unitary modular S-transformation matrix of the extended theory through quantities of the unextended theory. The matrices S^J are conjectured to be equal to the modular S-transformation matrices for the J-one-point conformal blocks on the torus, and are explicitly known for all WZW-models [24, 23], their simple current extensions [25] and also for coset conformal field theories. Elements of the matrix S^J whose labels are related by the action of a simple current K obey

$$S^J_{Ki,j} = F_i(K, J)\, \mathrm{e}^{2\pi \mathrm{i} Q_K(j)}\, S^J_{i,j}\,. \tag{7}$$

[3] This result is non-trivial because the degeneracy in the extended theory is in general *not* equal to the fixed point degeneracy, i.e. the order of the stabilizer, but rather to the size of a subgroup, the untwisted stabilizer.

The quantity F_i is called the simple current twist, and the untwisted stabilizer $\mathcal{U}_i$ is the subgroup of $\mathcal{S}_i$ of currents that have twist 1 with respect to all currents in $\mathcal{S}_i$. To combine the results for automorphisms and extensions we introduce a modified twist F_i^X by

$$F_i^X(K,J) := \mathrm{e}^{2\pi\mathrm{i}X(K,J)}\, F_i(K,J)^*\,, \tag{8}$$

and we define the *central stabilizer* $\mathcal{C}_i$ as

$$\mathcal{C}_i := \{J\in\mathcal{S}_i \,|\, F_i^X(K,J)=1 \text{ for all } K\in\mathcal{S}_i\}\,. \tag{9}$$

(The prescription (8) is motivated as follows. The modified twist is an alternating bihomomorphism i.e. obeys $F_i^X(J,J)=1$ for all $J\in\mathcal{G}$. Such bihomomorphisms F_i^X of an abelian group $\mathcal{G}$ are in one-to-one correspondence to cohomology classes $\mathcal{F}_i^X$ in $H^2(\mathcal{G},\mathrm{U}(1))$, thus leading to a cohomological interpretation [26]. In particular, the central stabilizer provides a basis of the center of the twisted group algebra $\mathbb{C}_{\mathcal{F}_i^X}\mathcal{S}_i$, which also motivates its name.)

The action (by the fusion product) of the simple currents in $\mathcal{G}$ organizes the labels i of the $\bar{\mathfrak{A}}$-theory into orbits. Moreover, in all known cases the boundary degeneracy is correctly decribed by the order of the central stabilizer, and hence this is our *ansatz* for the general case as well. In this case we choose the characters of $\mathcal{C}_i$ as the degeneracy labels. The boundaries are therefore given by

$$a=[i,\psi]\,, \tag{10}$$

where i is the label of a representative of a $\mathcal{G}$-orbit, and ψ a character of $\mathcal{C}_i$.

4. The boundary formula

Ishibashi states are nothing but conformal blocks for one-point correlation functions on the disk, i.e. specific two-point blocks on the sphere. But we can think of the Ishibashi state labelled by (i,J) also more like a *three*-point block on the sphere, with insertions i, i^c and J. (This is actually the natural interpretation when one wants to express such Ishibashi states in the three-dimensional topological picture that was established in [27].) Moreover, already from [1] it is known that the relation between Ishibashi and boundary states essentially expresses the effect of a modular S-transformation. Together with the previous observation, it is then natural to expect that the fixed point resolution matrices S^J appear in the boundary coefficients.

We are therefore ready to write down the following *ansatz* for the boundary coefficients:

$$B_{(i,J),[j,\psi]} = \sqrt{\frac{|\mathcal{G}|}{|\mathcal{S}_j|\,|\mathcal{C}_j|}}\,\frac{\alpha(J)\,S^J_{i,j}}{\sqrt{S_{0,i}}}\,\psi(J)^*\,, \tag{11}$$

where $\alpha(J)$ is a phase to be discussed later, but which must satisfy $\alpha(0)=1$. All previously studied cases are correctly reproduced by the remarkably simple formula (11). We have also verified that the matrix (11) has a left- and right-inverse, given by $(B^{-1})_{[j,\psi],(i,J)} = S_{0,i}\,B^*_{(i,J),[j,\psi]}$. This establishes in particular the result that the number of boundaries equals the number of Ishibashi labels, i.e. "completeness". This implies rather non-trivial relations involving the number of orbits of various kinds and the orders of stabilizers.

One can also check that the annuli obtained from (11) possess non-negative integral expansion coefficients A^i_{ab} with respect to the $\bar{\mathfrak{A}}$-characters χ_i. (We assume, as usual, that the

Verlinde formula produces non-negative integers both for the unextended and for the extended CFT.) When trying to express the annuli in terms of characters of (possibly twisted) representations of the *extended* chiral algebra $\mathfrak{A}$, one has to face the problem that their coefficients cannot be determined uniquely when the annuli are (as is usually done) considered only as functions of the variable τ associated to the Virasoro zero mode L_0. For reading off these annulus coefficients unambiguously, the introduction of additional variables – similar to the situation with full rather than Virasoro specialized characters – is required. This seems in fact to fit well with the above-mentioned interpretation of Ishibashi states (i,J) as three-point conformal blocks.

5. The crosscap formula

To compute the crosscap coefficients we use the special boundary corresponding to the vacuum orbit, which has degeneracy 1. The annulus coefficients for this boundary are easily found to be

$$A^i_{[0][0]} = \sum_{J\in\mathcal{G}} \delta_0^{Ji} \tag{12}$$

Positivity of annulus plus Möbius strip amplitudes then requires[4]

$$M^i_{[0]} = \sum_{J\in\mathcal{G}} \eta(J^c)\, \delta_0^{Ji}\,, \tag{13}$$

where $\eta(J^c)\in\{\pm1\}$. Using the formula for the Möbius amplitude and the fact that the matrix P is invertible we can now express most of the crosscap coefficients in terms of the signs η. The result is that for fields with $Q_K(i)=0$, for all $K\in\mathcal{G}$

$$\Gamma_{(i,J)} = \frac{1}{|\mathcal{G}|} \sum_{K\in\mathcal{G}} \eta(K)\, \frac{P_{K,i}}{\sqrt{S_{0,i}}}\, \delta^{J,0}\,. \tag{14}$$

Note that we only get information about the $J=0$ components of degenerate Ishibashi states,[5] because the boundary [0] is itself non-degenerate. In (14) we have postulated that $\Gamma_{(i,J)}=0$ for $J\neq0$. This postulate is based on known cases (where it can often be derived) and is justified by the consistency of the resulting Klein bottle. Comparison of the formula for the Möbius strip amplitude with (13) yields more information than just (14). We also find that the right-hand side of (14) must vanish if $Q_K(i)\neq0$ for some $K\in\mathcal{G}$. This implies relations between the signs $\eta(J)$. They can be derived using the relation

$$\begin{aligned} &P_{i,K^{2\ell}j} = \rho(\ell)\, \mathrm{e}^{\pi\mathrm{i}\Delta(2\ell,j)} \mathrm{e}^{2\pi\mathrm{i}\ell Q_K(i)} P_{ij} \\ &\text{with}\;\; \Delta(\ell,i) = h_{K^\ell i} - h_{K^\ell} - h_i + \ell Q_K(i)\,, \quad \rho(\ell) = \mathrm{e}^{\pi\mathrm{i}(r\ell+M_{2\ell})}, \quad M_\ell = h_{K^\ell} - \tfrac{r\ell(N-\ell)}{2N} \end{aligned} \tag{15}$$

for the matrix elements of P (N is the order of the current K). The number $\rho(\ell)\,\mathrm{e}^{\pi\mathrm{i}\Delta(2\ell,j)}$ is a sign, and the factors $\eta(J)$ must be chosen such that they cancel these signs. This is necessary

[4] The charge conjugation in the argument of η is for future convenience.

[5] Note that all Ishibashi states with $J=0$ satisfy $Q_K(i)=0$ for all $K\in\mathcal{G}$, so that (14) determines *all* such crosscap coefficients.

and sufficient to ensure the vanishing of the right hand side of (14) for some of the charges: namely all charges with respect to currents K that can be written as a square, $K = L^2$ for some $L \in \mathcal{G}$. These currents form a subgroup $\mathcal{G}_{\rm E}$ of $\mathcal{G}$, and will be called *even* currents henceforth. Note that any current of odd order is even. Vanishing of the expression for the remaining charges then turns out to yield no further conditions. This follows from the fact that $P_{ij} = 0$ if i and j have different charges with respect to a (half)-integer spin current of order 2. Since there is no further condition, the signs $\eta(J)$ remain unconstrained on the cosets $\mathcal{G}/\mathcal{G}_{\rm E}$.

The precise relation that the coefficients η have to satisfy can be written more conveniently by defining

$$\beta(J) := {\rm e}^{\pi {\rm i} h_J} \eta(J) \,. \tag{16}$$

We find then that for even currents $K = L^2$ (and any current J) [6]

$$\beta(KJ) = \beta(K)\beta(J)\, {\rm e}^{-2\pi {\rm i} Q_K(J)} = \beta(K)\beta(J)\, {\rm e}^{-2\pi {\rm i} X(K,J)} \,. \tag{17}$$

6. Integrality and positivity

We can now compute the Klein bottle and check integrality and positivity in the closed sector. It turns out that there are no further constraints as long as there are no fixed points. If, however, we assume that all allowed types of orbits actually do occur, in order to obtain a formula that is valid in all cases, then a further constraint is necessary, namely

$$\beta(KJ) = \beta(K)\beta(J)\, {\rm e}^{-2\pi {\rm i} X(K,J)} \,; \tag{18}$$

this is identical to (17), but this time also valid for odd currents. The number of free signs is therefore as follows. [7] The number of cosets $\mathcal{G}/\mathcal{G}_{\rm E}$ is 2^M with M the number of even cyclic factors of $\mathcal{G}$. Within each coset the signs $\eta(J)$ can be related using (17), but signs in different cosets are unrelated, so that there is a total of 2^M sign choices. If (18) is valid all signs can be expressed in terms of those of the generators of the even cyclic factors. This reduces the number of sign choices to M. Since this is the generic solution it is the one most likely to survive further consistency checks, but we cannot rule out the possibility that in theories were certain a priori allowed fixed points simply do not occur more general sign choices are permitted.

The last consistency condition follows from positivity and integrality of the open string sector, and concerns the phases $\alpha(J)$ introduced in (11). These phases do not appear in the Möbius strip partition function, and enter the annulus only as the combination $\alpha(J)\alpha(J^c)$. Such phases already occurred for the $\mathbb{Z}_2$ extensions and automorphism invariants discussed in [12] and [13], where they were found to be related to the sign choices in the crosscap. The same is true here, the precise relation being

$$\alpha(J)\,\alpha(J^c) = \beta(J) \,. \tag{19}$$

If J has fixed points it either has integer or half-integer spin. Since $\eta(J)$ is a sign, it follows from (16) that $\beta(J)$ is a sign for integer spin currents, and $\pm{\rm i}$ for half-integer spin currents. If

[6] The numbers ${\rm e}^{-2\pi {\rm i} Q_K(J)}$ furnish a two-cocycle on the quotient group $\mathcal{G}/\mathcal{G}_{\rm E}$. Formula (17) thus means that β forms a one-dimensional representation of the corresponding twisted group algebra, which is possible only when the cocycle is a coboundary; this is indeed the case.

[7] The overall sign of the crosscap coefficients is always free, and can be fixed by choosing $\eta(0) = 1$.

we fix the convention $\alpha(J)=\alpha(J^c)$, [8] we find that $\alpha(J)$ is a fourth root of unity for integral spin currents, and a primitive eight root of unity for half-integer spin currents. This resolves another apparent conflict between the earlier results for pure extensions and automorphisms. Namely, in the formulas of [7] for the former case the matrices S^J appear, whereas in the automorphism case in [5] a slightly different matrix appears, namely the modular transformation matrix of the relevant "orbit Lie-algebra" that was defined in [24]. Its definition involves folding a Dynkin diagram, a procedure that is only available for WZW models. In that case, the matrix differs from the fixed point resolution matrix S^J by a primitive eight root of unity, if J has half-integer spin, and by a fourth root of unity if J has integer spin. The present formalism allows us to use S^J in all cases; it has the additional advantage that S^J has a more general definition, and has been computed in more cases.

7. More general solutions

There is (at least) one further generalization possible whenever the RCFT under consideration has an additional simple current K that is not contained in $\mathcal{G}$. We can then generalize the results of [8] to obtain different Klein bottle projections and correspondingly different boundary coefficients. It turns out that K must satisfy the constraint

$$Q_J(K) = 0 \text{ for all } J\in\mathcal{G} \text{ with } J^2=0\,. \tag{20}$$

The modified formula for the boundaries is

$$B_{(i,J),[j,\psi]} = \sqrt{\frac{|\mathcal{G}|}{|\mathcal{S}_j|\,|\mathcal{C}_j|}}\,\frac{\alpha(J)\,S^J_{i,j}}{\sqrt{S_{K,i}}}\,\psi(J)^*\,, \tag{21}$$

and for the crosscap coefficients we find [9]

$$\Gamma_{i,J} = \frac{1}{\sqrt{|\mathcal{G}|}}\,\frac{1}{\sqrt{S_{K,i}}}\sum_{L\in\mathcal{G}}\eta(K,L)\,P_{KL,i}\,\delta^J_0\,. \tag{22}$$

The effect of the "Klein bottle current" K is to flip some signs of the Klein bottle projection. One finds nothing new (up to a permutation of the boundaries) if K is the square of another current, or if $K\in\mathcal{G}$. The signs $\eta(K,L)$ are given by

$$\eta(K,L) = \mathrm{e}^{\pi\mathrm{i}(h_K-h_{KL})}\beta(L)\,. \tag{23}$$

The coefficients $\beta(L)$ must satisfy the same condition (18) as in the case $K=0$, and the coefficients $\alpha(J)$ are related to phases $\beta(L)$ as in (19). Note that although the coefficients β satisfy the same product formula independent of the choice of the Klein bottle current K, the solutions *do* depend on K because of the additional requirement that $\eta(K,L)$ must be ±1. The phases $\alpha(L)$ are relevant only if L fixes some field. Then h_L is integer or half-integer and $Q_K(L) = h_K + k_L - h_{KL} \bmod 1 = 0 \bmod 1$. From (23) we find then that $\beta(L) = \mathrm{e}^{\pi\mathrm{i}h_L}\eta(K,L)$. Hence for any choice of K the coefficients $\alpha(J)$ are fourth (eight) roots of unity of integer (half-integer) spins, as before.

[8] For WZW models one has $S^J=S^{J^c}$, which makes it natural to impose the same condition on the phases.
[9] As explained in [8], any ambiguity in the choice of the square roots cancels out in the amplitudes.

8. Summary

The main results of this paper are the formulas (5) and (10), which specify the Ishibashi and boundary labels, as well as (11) and (14), which provide the boundary and crosscap coefficients, for a general simple current modular invariant that is based on the charge conjugation invariant. (In addition, the phases appearing in these expressions are subject to the constraints (18) and (19).) We do not have a proof that these results lead to consistent correlation functions on arbitrary Riemann surfaces. However, they do satisfy a set of quite non-trivial consistency conditions at the one-loop level, as well as the completeness conditions. Their simplicity and generality strongly suggest that this must indeed be the correct answer.

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