HYPER VERSION
RECEIVED: *July 21, 2000*, ACCEPTED: *July 28, 2000*

On factorisation at small x^*

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ABSTRACT: We investigate factorisation at small x using a variety of analytical and numerical techniques. Previous results on factorisation in collinear models are generalised to the case of the full BFKL equation, and illustrated in the example of a collinear model which includes higher twist terms. Unlike the simplest collinear model, the BFKL equation leads to effective anomalous dimensions containing higher-twist pieces which grow as a (non-perturbative) power at small x. While these pieces dominate the effective splitting function at very small x they do not lead to a break-down of factorisation insofar as their effect on the predicted scaling violations remains strongly suppressed.

KEYWORDS: QCD, Deep Inelastic Scattering, Parton Model.

*Work supported by E.U. QCDNET contract FMRX-CT98-0194.

Contents

1. Introduction

The standard approach to calculating any quantity in Deep Inelastic Scattering (DIS) or Drell-Yan type processes is to assume factorisation, based on the operator product expansion. This allows us to consider parton distributions which evolve with scale as

$$\frac{\partial g_\omega(Q^2)}{\partial \ln Q^2} = \gamma_\omega(Q^2) g_\omega(Q^2) + \mathcal{O}\left(\frac{1}{Q^2}\right) , \tag{1.1}$$

where ω is the Mellin moment of the parton distribution. Cross sections are then expressed in terms of the parton distribution at scale Q^2, multiplied by a coefficient function, which is itself a function of only ω and $\alpha_\mathrm{s}(Q^2)$, plus corrections of order $1/Q^{2n}$.

A few years ago however, it was suggested in [1] that at small x (small ω) diffusion might place a limit on factorisation: namely that beyond a certain value of x factorisation would break down. The argument was the following: at moderate values of x parton evolution in Q^2 can be associated with a chain of emissions ordered in Q^2. Corrections to factorisation can arise from chains which evolve from some starting scale Q^2_{start} up to a scale Q^2, evolve down to a non-perturbative scale Λ^2 and then evolve back up to scale Q^2. At moderate x, such contributions are suppressed by a power of Λ^2/Q^2. At small x however, this process of going up and down in scale becomes an essential element of the evolution, and is no longer suppressed by powers of Q^2: so it would seemed that factorisation might break down at the values of x such that this diffusion enters into the non-perturbative region. More recently this has been used as an argument for parameterising and fitting the small-x behaviour of splitting functions, rather than trying to calculate them from first principles [2].

In [3] plausibility arguments were presented suggesting that there would be no such breakdown. Physically speaking, perhaps the simplest explanation is to say that evolution that, while going down in x drops below one's starting scale Q^2_{start}, contributes to the small-x part of the gluon distribution at the starting scale. Therefore it should not be included in the anomalous dimension in order to avoid counting it twice, so the anomalous dimension contains only evolution which remains above the starting scale, ensuring that it is perturbative.

However this explanation, and the arguments in [3], are not entirely satisfactory because they are not able to estimate the size or functional dependence on x of any violations of factorisation.

In [4], a model was studied which was qualitatively similar to BFKL in that it had diffusion, and which correctly reproduced collinear evolution at all orders. It was possible to show that this model leads to *exact factorisation*, namely that there were no $1/Q^2$ corrections at all. Thus diffusion on its own is not incompatible with factorisation.

This collinear model had two particularities: the branching kernel, when converted to γ space (the Mellin transform variable conjugate to transverse momentum) contained poles only at $\gamma = 0$ and $\gamma = 1$, i.e. the leading collinear and anti-collinear poles. The BFKL equation instead has poles at all integer values of γ. The second particularity was related to the scale of α_s: in the collinear model the scale was chosen to be $\max(k^2, k'^2)$, where k and k' are the exchanged transverse momenta before and after the branching. In the BFKL equation, the next-to-leading corrections seem to indicate that the correct scale is rather $q = |\vec{k} - \vec{k}'|$ [5] which has the property that it can go to zero even when both k and k' are large.

Thus there is a need for a study of factorisation within the BFKL equation. The complications mentioned above make this quite difficult to do analytically. We therefore adopt a two-pronged approach. In section 3 we consider an extension of the collinear model with additional poles at $\gamma = -1$ and $\gamma = 2$. Some of its gross features can be deduced analytically and can be expected to carry over to the full BFKL equation.

In section 4 we then present a simple numerical method which for the first time allows the extraction of the exact (effective — see below) anomalous dimensions in the full leading logarithmic BFKL equation, with all-order running coupling corrections, allowing a detailed study of factorisation not limited by the simplifying assumptions of any given particular model.

2. Understanding factorisation

We shall start off by discussing factorisation for the non-integrated gluon distribution, or equivalently for the gluon Green's function. Conceptually it it somewhat simpler — the generalisation to the integrated gluon distribution is then relatively straightforward.

We shall discuss factorisation by considering a number of models. We study a gluon Green's function $G_\omega(t,t_0)$, $t = \ln Q^2$, in analogy with the unintegrated gluon distribution in the proton. and consider as the non-perturbative aspects of the problem the value of t_0 (the lower hard scale of the process) and the regularisation of the running coupling in the infra-red. The latter will be represented by some scale $\bar{t}$ at which the coupling is cut off (we could equally have chosen to freeze the coupling at that scale). The small-x properties of the Green's function will depend on both t_0 and $\bar{t}$. More specifically the position of the leading pole in the ω-plane will depend on the value of $\bar{t}$ and its normalisation on both $\bar{t}$ and t_0.

If factorisation holds then the (non-integrated) effective anomalous dimension defined by

$$\tilde{\gamma}_\omega(t) = \frac{1}{G_\omega(t,t_0)}\frac{\partial G_\omega(t,t_0)}{\partial t}\,, \tag{2.1}$$

should be independent of both $\bar{t}$ and t_0 at least to within higher-twist terms suppressed at least as e^{-t}.

This is not quite a sufficient condition: indeed if the anomalous dimension contains higher-twist pieces which grow as a sufficiently large power of x then these could dominate the scaling violations making them impossible to predict. Since the Pomeron singularity will affect G_ω anyway, yielding a large power behaviour of this kind, the problem arises of understanding whether approximate factorisation is still preserved in the small-x region.

Since we are interested in pieces enhanced at small-x we need to understand the singularity structure of the anomalous dimension. There are two possible origins for singularities of $\tilde{\gamma}_\omega$. One possible origin is that at some ω, $\partial G_\omega/G_\omega$ contains a non-factorisable singularity, the other is for G_ω to be zero while ∂G_ω is finite.

Now, the simplest collinear model of [4] provided us with a mechanism by which the Pomeron singularity in G_ω can be consistent with exact factorisation: the leading singularity of $\tilde{\gamma}_\omega$ was found to come from a t-dependent *zero* of G_ω, leaving the Pomeron factorised away. We shall see in the following that this basic mechanism is still at work in collinear models with higher twist terms [6] and in the BFKL equation itself, thus suppressing the non-factorisable singularities.

2.1 Recalling the structure of the collinear model

The collinear model of ref. [4] gave a powerful handle for the study of anomalous dimensions. It was described by an equation of the form

$$\omega G_\omega(t,t_0) = \delta(t-t_0) + \int dt' K_2(t,t')\, G_\omega(t',t_0)\,, \tag{2.2}$$

whose kernel is

$$K_2(t,t') = \bar{\alpha}_{\mathrm{s}}(t)\Theta(t-t') + \bar{\alpha}_{\mathrm{s}}(t')\Theta(t'-t)e^{-(t'-t)}\,. \tag{2.3}$$

In γ-Mellin transform space with respect to $Q^2 = \exp(t)$, the leading order part of the kernel gives the following characteristic function:

$$\chi^{\text{2-pole}}(\gamma) = \frac{1}{\gamma} + \frac{1}{1-\gamma}\,, \tag{2.4}$$

for which reason it is also referred to as a 2-pole model. It has the convenient property that it can be expressed in terms of a second order differential equation, whose solutions have the following factorised form:

$$G_\omega(t,t_0) = F_\omega^L(t_0)F_\omega^R(t), \qquad t > t_0\,, \tag{2.5}$$

where F_ω^L and F_ω^R are the linearly independent solutions of the homogeneous equation (eq. (2.2) without the δ-function term), that are regular to the left (negative t_0) and to the right (positive t) respectively.

The fact that $F_\omega^R(t)$ is regular at large t means that it is independent of $\bar{t}$ for $t > \bar{t}$. It is free of singularities in ω-space, as illustrated in figure 1. All non-perturbative dependence, in particular the poles governing the high-energy behaviour of G, is contained in F_ω^L, also illustrated in figure 1.

One sees from these plots that zeroes of G_ω can arise from both F_ω^L and F_ω^R.[1] However since the t-dependence lies entirely in F_ω^R, the t_0-dependent, non-factorisable zeroes of F_ω^L lead to zeroes of both G_ω and of its t-derivative. Therefore they do not lead to divergences of the anomalous dimension. Only the t-dependent zeroes of F_ω^R lead to poles in $\gamma(\omega)$, $\partial_t F_\omega^R(t)$ not usually being zero when $F_\omega^R(t)$ is zero.

This fact has the consequence mentioned before: the Pomeron singularity, present in $F_\omega^L(t_0)$, does not occur in the anomalous dimension, which is singular at $\omega_c(t)$, the leading zero of F_ω^R (while we refer to the leading zero of $F_\omega^L(t_0)$ as $\omega_0(t_0)$).

[1]Note however that F_ω^L does not always have zeroes: their presence depends on a variety of factors such as the relative sign of successive divergences, which are determined by the non-perturbative parameters. Only in the situation of $t_0 \gg \bar{t}$ is at least one zero guaranteed, just to the left of the leading divergence.

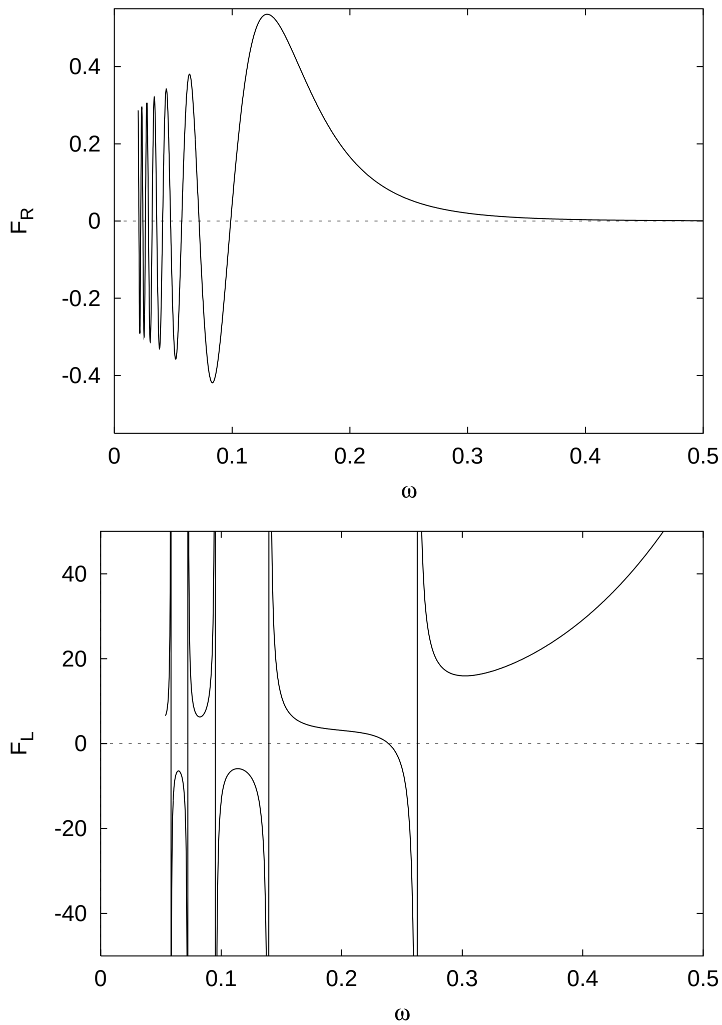

Figure 1: Illustration of the kind of the ω-dependence that arises in the right- and left-regular solutions of models based on second-order differential equations such as the two-pole collinear model. (The small-ω parts of the curves have been left out in order to improve the overall clarity of the plots).

3. The 4-pole collinear model

One of the main differences between the BFKL equation and the collinear model just-discussed lies in the presence in the BFKL equation of poles of $\chi(\gamma)$ at all integer values of γ. The poles beyond $\gamma = 0$ and $\gamma = 1$ can give rise to higher-twist effects, in which we are particularly interested in this article.

A useful stepping stone to the full BFKL equation is an equation containing the first subleading higher-twist parts, namely poles at $\gamma = -1$ and $\gamma = 2$. The equation for the Green's function is

$$\omega G_\omega(t, t_0) = \delta(t - t_0) + \int dt' K_4(t, t')\, G_\omega(t', t_0)\,, \tag{3.1}$$

where the kernel is

$$K_4(t,t') = \bar{\alpha}_s(t)\Theta(t-t')\left(1+e^{-(t-t')}\right) + \bar{\alpha}_s(t')\Theta(t'-t)\left(e^{-(t'-t)}+e^{-2(t'-t)}\right) - \\ - \frac{4}{3}\bar{\alpha}_s(t)\delta(t-t')\,. \tag{3.2}$$

The δ-function term is included so that the value of $\chi_0(1/2)$ be the same as for the 2-pole collinear model:

$$\chi^{4-\mathrm{pole}}(\gamma) = \frac{1}{1+\gamma} + \frac{1}{\gamma} + \frac{1}{1-\gamma} + \frac{1}{2-\gamma} - \frac{4}{3}\,. \tag{3.3}$$

The physics content of this formulation is quite similar to the leading and first-subleading collinear and anti-collinear parts of the BFKL equation. A slight difference exists in that the higher-order higher-twist pieces of the BFKL equation also contain a non-local dependence on α_s. Their inclusion would considerably increase the difficulty of the analytic treatment.

This model as it stands can be expressed as a fourth-order differential equation (or two coupled second order differential equations, etc.), using the same approach as in [4], as discussed in detail in the appendix. It is thus similar to the diffusion models with higher twist terms [6]. Without having to write down the full equations in detail, we describe here some important properties of its solutions.

There are four linearly-independent solutions of the homogeneous equation. We denote them by $F^{R,0}$, $F^{R,1}$, the leading and sub-leading twist right-regular solutions, which for large t go as a constant and e^{-t} respectively; and by $F^{L,a}$, $F^{L,b}$ for the two left-regular solutions. For $t > t_0$ the Green's function has the expression

$$G_\omega(t,t_0) = F_\omega^{L,a}(t_0)F_\omega^{R,0}(t) + F_\omega^{L,b}(t_0)F_\omega^{R,1}(t)\,. \tag{3.4}$$

as expected from [6], and proved in the appendix. There we also show how to determine the F^L's so as to satisfy the boundary conditions at $t = t_0$. The solution for $t < t_0$ can then also be obtained, by a simple exchange of variables. As in the 2-pole model, the right-regular solutions are independent of the non-perturbative parameters of the problem. The Pomeron singularity and the non-perturbative dependence all enter into the left-regular solutions.

It is convenient to divide the solution into its leading and higher-twist parts, G^0_ω and G^1_ω, corresponding to the first and second terms of (3.4) respectively. For large $t \gg t_0, \bar{t}$, we have that G^1_ω is strongly suppressed (by a relative amount $\sim e^{-t}$) compared to G^0_ω; G^0_ω should be qualitatively similar to the solution of the 2-pole model, having zeroes of perturbative and non-perturbative origin. We know therefore, from the start, that the violation of factorisation is uniformly of higher twist, just because of the additive nature of (3.4).

On the other hand, one can look at the same problem from the standpoint of the singularities of the anomalous dimension. Let us concentrate on the case with a discrete spectrum (running coupling with cutoff), in which we have to look for zeroes of G_ω. The fact that G^1_ω is strongly suppressed means that the distribution of zeroes of G_ω is determined essentially by the positions of the zeroes of G^0_ω. But in general where G^0_ω is zero, G^1_ω will be non-zero, causing the zeroes of G_ω to be slightly shifted compared to the zeroes of G^0_ω. The size of this shift will be related to the relative sizes of G^1_ω and G^0_ω, i.e. it will be of the order of of e^{-t}.

The effect of the shift will be different according to whether the zero is of perturbative or non-perturbative origin. In the case of the leading zero of perturbative origin, then the leading perturbative pole of the anomalous dimension is shifted, and its normalisation changes — both the effects are of order e^{-t} (the relation between relative changes to the normalisation and the position of the divergence depends on the nature of the t and ω-dependence of G^1, and so is difficult to predict).

The consequences of the rightmost (NP) zero $\omega_0(t_0)$ being shifted are somewhat more interesting, and possibly dangerous for factorisation. We recall that in the two-pole model, the non-perturbative zeroes did not lead to divergences of the anomalous dimension because the zero was in F^L and so the derivative $\partial_t G(t,t_0) = F^L(t_0)\partial_t F^R(t)$ was also zero when G was zero. This is no longer true when higher twist terms are present, and a singularity around ω_0 is expected. The exact value of ω_0, and indeed even the existence of this zero, depend rather subtly on the values of $\bar{t}$ and t_0. However, for t_0 significantly larger than $\bar{t}$ one expects that the zero exists, and is driven by the Pomeron term in $F^L(t_0)$, so that $\omega_{\mathbb{P}} - \omega_0 \sim e^{-t_0}$ is rather small and ω_0 may be leading.

To see in detail what happens in the 4-pole model if ω_0 is leading, we approximate $G^0_\omega(t,t_0)$ around its zero by $\mathcal{N}_0(\omega-\omega_0)F^{R,0}_{\omega_0}(t)$, where $\mathcal{N}_0 = \partial_\omega F^{L,a}_\omega(t_0)|_{\omega=\omega_0}$, and $G^1(t,t_0)$ by $\mathcal{N}_1 F^{R,1}_{\omega_0}(t)$. It is useful to define partial anomalous dimensions separately for $F^{R,0}_{\omega_0}(t)$ and $F^{R,1}_{\omega_0}(t)$:

$$\gamma^0_\omega(t) = \frac{\partial_t F^{R,0}_\omega(t)}{F^{R,0}_\omega(t)} = \frac{\bar{\alpha}_s}{\omega} + \cdots\,, \qquad \gamma^1_\omega(t) = \frac{\partial_t F^{R,1}_\omega(t)}{F^{R,1}_\omega(t)} = -1 + \frac{\bar{\alpha}_s}{\omega} + \cdots\,, \tag{3.5}$$

where we have neglected terms of order $(\bar{\alpha}_s/\omega)^2$ in the expansion of the anomalous dimensions. Thus $F^{R,1}_{\omega_0}(t)$ is suppressed compared to $F^{R,0}_{\omega_0}(t)$ by an amount

$$\frac{F^{R,1}_{\omega_0}(t)}{F^{R,0}_{\omega_0}(t)} \propto \exp\left[\int^t dt' \left(\gamma^1_{\omega_0}(t) - \gamma^0_{\omega_0}(t)\right)\right] \sim e^{-t+\mathcal{O}\left(\alpha_s(t)/\omega_0^2\right)}\,. \tag{3.6}$$

The rightmost zero of $G_\omega(t,t_0)$, ω_0', is therefore shifted compared to that of $F^{R,1}_\omega(t)$ by an amount

$$\omega_0' - \omega_0 \simeq -\frac{\mathcal{N}_1 F^{R,1}_{\omega_0}(t)}{\mathcal{N}_0 F^{R,0}_{\omega_0}(t)} \sim e^{-t+\mathcal{O}\left(\alpha_s/\omega_0^2\right)}\,. \tag{3.7}$$

At ω_0' the t derivative of G_ω is non-zero, leading to a pole of the anomalous dimension, whose residue $R(\omega_0')$ is

$$R(\omega_0') = (\gamma^1_{\omega_0}(t) - \gamma^0_{\omega_0}(t))\frac{\mathcal{N}_1 F^{R,1}_{\omega_0}(t)}{\mathcal{N}_0 F^{R,0}_{\omega_0}(t)} = \left(1+\mathcal{O}\left(\alpha_s^2/\omega^2\right)\right)(\omega_0'-\omega_0)\,. \tag{3.8}$$

So we have that the residue of the leading NP pole is the same as the shift of the zero and *both are higher twist*. In the case of the BFKL equation (section 4) one can test to see if this remains true, in order to verify that the same mechanisms are at work there as in the 4-pole model.[2]

Since ω_0, which may approach $\omega_{\mathbb{P}}$, is likely to be to the right of the perturbative pole ω_c, at small x this higher-twist non-perturbative contribution will dominate the anomalous dimension. At first sight this might seem to have worrying implications for the prediction of small-x scaling violations. But in the present model we know that this is not the case, because of the validity of (3.4). From the point of view of t-evolution we are saved by the fact that at small x, in the convolution of the splitting function with the non-perturbative input distribution (which grows as $\omega_{\mathbb{P}} > \omega_0$), the higher-twist part of the effective splitting function gives a contribution of order

$$\frac{e^{-t}}{\omega_{\mathbb{P}} - \omega_0},$$

while the perturbative contribution is of order

$$\frac{\alpha_s(t)}{\omega_{\mathbb{P}}},$$

both growing as $x^{-\omega_{\mathbb{P}}}$. We thus recover in the latter contribution the Pomeron part of the leading twist term in (3.4), that we know to be factorised. Thus there is always a value of Q^2 such that the higher-twist corrections can be ignored at all x.

A small point worth bearing in mind is that our analysis so far has always been for the anomalous dimensions related to unintegrated gluon distributions. In practice one is more interested in the anomalous dimensions of the integrated distributions. It turns out that their properties are very similar: this is because the n-pole models can be expressed in terms of n coupled linear differential equations, and the unintegrated and integrated gluon distributions are simply different linear combinations of the components of the equations.

4. The BFKL equation

We shall study the leading-order BFKL equation including a running coupling,

$$\mathcal{G}(x, k, Q_0) = \delta\left(\frac{k^2}{Q_0^2} - 1\right) + \int_x^1 \frac{dz}{z} \int \frac{d^2\vec{q}}{\pi q^2} \bar{\alpha}_s(q^2) \times \\ \times \left[\frac{k^2}{|\vec{k} - \vec{q}|^2} \mathcal{G}(x, |\vec{k} - \vec{q}|, Q_0) - \Theta(k - q) \mathcal{G}(x, k, Q_0)\right]. \quad (4.1)$$

[2] We note that for the BFKL equation including next-to-leading corrections, the relation between $\omega_0' - \omega_0$ and the residue of the pole can be modified by pieces of relative order α_s.

For notational convenience we have switched to using transverse momenta k, Q_0, rather than the logs of their squares t, t_0. The choice of the emitted transverse momentum as the scale for the running coupling, $\alpha_s(q^2)$, is suggested by the form of the NLO corrections to the kernel [5]. A normal DGLAP gluon distribution is expressed in terms of $\mathcal{G}(x, k, Q_0)$ via k-factorisation:

$$xg(x, Q^2) = \int^{Q^2} \frac{\mathrm{d}^2 k}{\pi k^2}\, \mathcal{G}(x, k, Q_0)\,. \tag{4.2}$$

In practice rather than solving the integral equation (4.1), it is easier to solve the related differential equation

$$\frac{\partial \mathcal{G}(x, k, Q_0)}{\partial \ln 1/x} = \int \frac{\mathrm{d}^2 \vec{q}}{\pi q^2}\, \bar{\alpha}_s(q^2) \left[\frac{k^2}{|\vec{k} - \vec{q}|^2} \mathcal{G}(x, |\vec{k} - \vec{q}|, Q_0) - \Theta(k - q)\mathcal{G}(x, k, Q_0) \right], \tag{4.3}$$

with initial condition $\mathcal{G}(1, k, Q_0) = \delta(k^2/Q_0^2 - 1)$.

4.1 The extraction of anomalous dimensions

Naively to obtain the effective splitting function, one would determine $\mathcal{G}$ and then solve for the function $P_{gg,\text{eff}}(z)$ such that

$$x\partial_{\ln Q^2} g(x, Q^2) = x \int_x^1 \frac{dz}{z} P_{gg,\text{eff}}(z) g(x/z, Q^2)\,. \tag{4.4}$$

However such an approach turns out to be subject to considerable numerical instabilities. The reason is that any method of solution for $\mathcal{G}$ introduces small errors (typically of the relative order of 10^{-2}–10^{-3}). When carrying out the deconvolution it generally turns out that $P(z)$ for small z contributes only a small amount to the scaling violations ($xg(x, Q^2)$ grows as $x^{-\omega_{\mathbb{P}}}$ whereas perturbatively, $P(z)$ grows as $x^{-\omega_c}$ and $\omega_c \ll \omega_{\mathbb{P}}$). When x is such that the small error on $\mathcal{G}(x)$ is of the same order as the contribution to the scaling violations from $P(x)$, then we no longer have a handle on the splitting function.

A solution is to choose an inhomogeneous term such that $g(x, Q^2)$ is independent of x. Then, for a given x, the convolution (4.4) is dominated by small z's and small errors on $g(x)$ are no longer amplified when translated to $P(x)$. We introduce $\mathcal{F}(x, k)$ as being the unintegrated gluon distribution which, integrated, gives $xg(x, Q^2) = 1$. It satisfies the equation

$$\begin{aligned}\frac{\partial \mathcal{F}(x, k)}{\partial \ln 1/x} = f(x, k)\delta\left(\frac{k^2}{Q_0^2 - 1}\right) + \int \frac{\mathrm{d}^2 \vec{q}}{\pi q^2}\, \bar{\alpha}_s(q^2) \times \\ \times \left[\frac{k^2}{|\vec{k} - \vec{q}|^2} \mathcal{F}(x, |\vec{k} - \vec{q}|) - \Theta(k - q)\mathcal{F}(x, k) \right],\end{aligned} \tag{4.5}$$

where $\mathcal{F}(1,k) = \delta(k^2/Q_0^2 - 1)$ and $f(x)$ is given implicitly by

$$f(x) = -\int^{Q^2} \frac{\mathrm{d}^2 k}{\pi k^2} \int \frac{\mathrm{d}^2 \vec{q}}{\pi q^2} \bar{\alpha}_{\mathrm{s}}(q^2) \left[\frac{k^2}{|\vec{k}-\vec{q}|^2} \mathcal{F}(x, |\vec{k}-\vec{q}|) - \Theta(k-q) \mathcal{F}(x,k) \right]. \quad (4.6)$$

It is trivial to verify that this leads to $xg(x, Q^2) = 1$. The form of $f(x)$ depends on the Q^2 value at which we intend to consider the splitting function and on the initial scale Q_0. Using (4.4) it is now simple to obtain the effective splitting function:

$$xP_{gg,\mathrm{eff}}(x) = \frac{\partial \mathcal{F}(x, Q)}{\partial \ln 1/x} . \quad (4.7)$$

This turns out to be numerically stable, at least until $xP_{gg,\mathrm{eff}}$ becomes comparable to the inverse of the machine precision.

Equation (4.5) is solved by discretising $\mathcal{F}(x, k)$ uniformally in $\ln k$ space, and then applying standard Runge-Kutta techniques for solving the resulting matrix differential equation. This method has the advantage over other potentially faster methods, such as a representation with a basis of Chebyshev polynomials [7, 8], that it quite easily accommodates the large variations that arise in the value of $\mathcal{F}(x, k)$ (and errors are at worst of the relative order of the discretisation interval).

For $\bar{\alpha}_{\mathrm{s}}$ we take the asymptotic formula, i.e.

$$\bar{\alpha}_{\mathrm{s}}(q^2) = \frac{\Theta(q^2 - \bar{Q}^2)}{b \ln q^2/\Lambda^2} , \quad (4.8)$$

where $b = 11/12$ (we work with zero flavours). The cutoff at small momenta corresponds to the regularisation prescription used in the previous section for the collinear model.

4.2 Results

This section has two aims. Firstly to demonstrate that for the BFKL equation the splitting function is a truly perturbative quantity, and that any non-perturbative dependence is higher-twist. And secondly, to show that our understanding of higher-twist effects as obtained from the 4-pole model, carries over to the BFKL equation.

Let us start by examining some concrete examples of effective splitting functions. Figure 2 illustrates the effective splitting function as a function of x in three situations, all with the same value of t, but different sets (a, b and c) of non-perturbative parameters, $\bar{t}$ and t_0. Going down in x from $x = 1$, one sees that initially the three splitting functions are almost identical (the inset with the larger scale reveals small differences between them). For moderately small x the splitting function actually decreases (this phenomenon was recently observed also by [9]), and then starts to grow as $x^{-\omega_c}$. The late onset of the power growth is related to the fact that in ω-space the PT pole of the anomalous dimension has a residue of order $\bar{\alpha}_{\mathrm{s}}^2$.

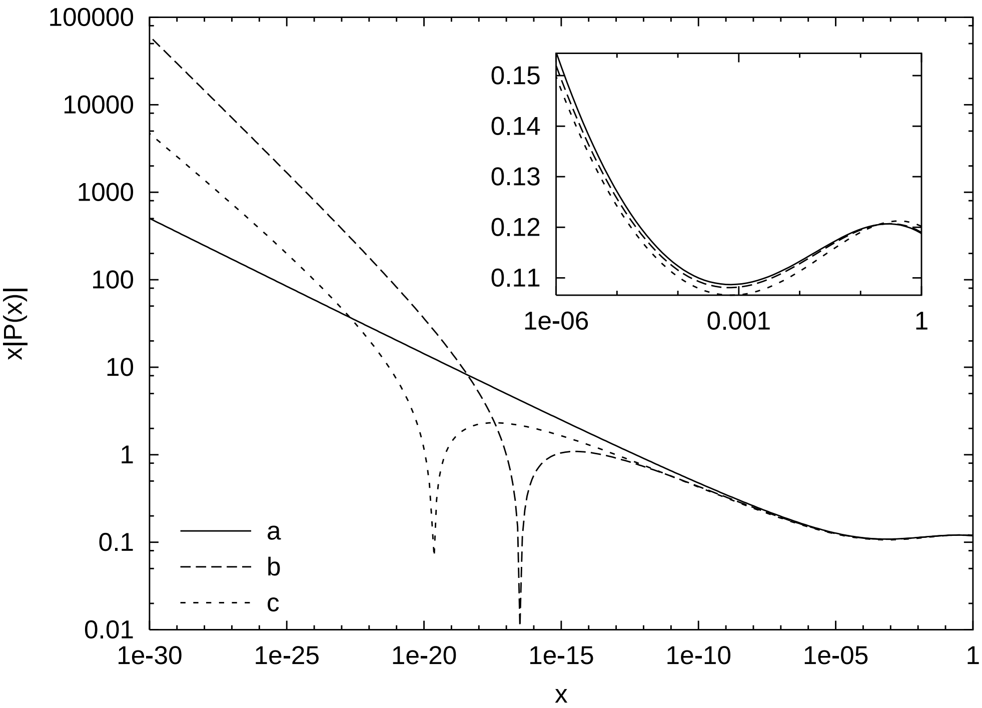

Figure 2: Examples of effective splitting functions for $t = 9.2$ (corresponding to $\bar{\alpha}_s \simeq 0.12$) and three different combinations of NP parameters: (a) has $\bar{t} = 1.0, t_0 = 2.0$; (b) has $\bar{t} = 1.0, t_0 = 3$ and (c) has $\bar{t} = 2.0, t_0 = 4.6$. The inset shows the same splitting functions on an enlarged scale.

At a certain point, two of the curves (b and c) change sign (since we use a logarithmic scale and plot the absolute value of the splitting function, the change of sign appears as a downward cusp) and start to grow with a much larger power (ω_0'). This is the non-perturbative higher-twist component of the splitting function discussed in the section 3. As can be seen the exact value of ω_0' depends on the non-perturbative parameters. Furthermore there are situations (curve a) in which there is no NP power growth at all, corresponding to the absence of NP zeroes in the Green's function.

Figure 2 is not sufficient to demonstrate that the NP corrections are truly higher twist. First we consider the 'PT part' of the splitting function. Figure 3 shows the ratio of two effective splitting functions, obtained with different non-perturbative parameter sets (a and b), chosen such that there is no component with the large NP power growth (which would complicate the interpretation of the ratio). We see that the NP parameters affect both the normalisation of the PT splitting function, and the exact value of the power growth, since the ratio grows as a power. This is as predicted in the 4-pole model, being due to the shift of the PT zeroes of the Green's function. We observe that the effect on the power is relatively small (note the x scale), and that from a practical point of view it will mostly be the effect on

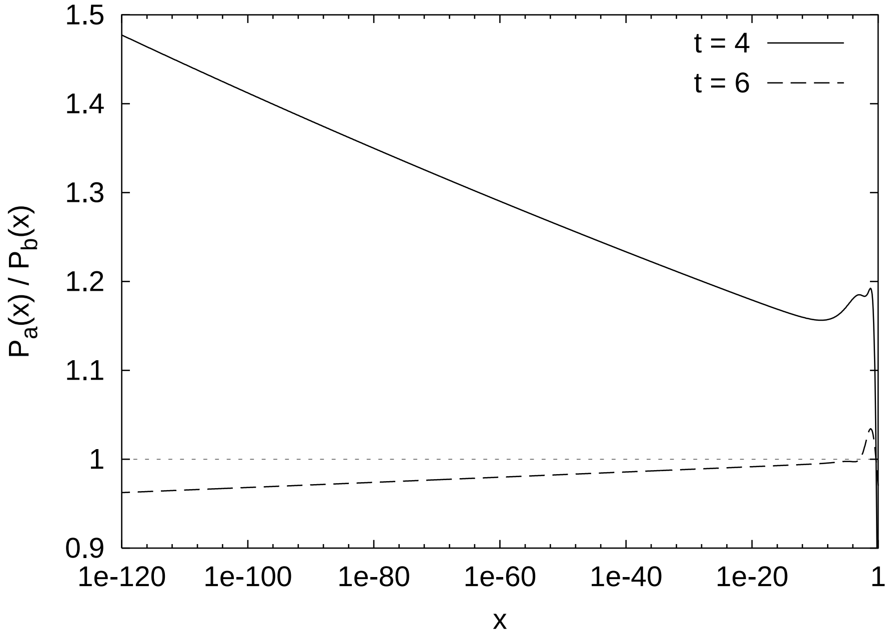

Figure 3: Ratio of splitting functions with the same t but different non-perturbative parameters: (a) $\bar{t} = 0.5, t_0 = 0.8$ and (b) $\bar{t} = 1.0, t_0 = 1.3$.

the normalisation that will be of interest. The curve at the higher value of t shows a significantly decreased dependence on the NP parameters, confirming that their effect is higher-twist.[3]

We also need to demonstrate that the NP power component is higher twist. Figure 4 shows the splitting function for three different t values, but with the same NP parameters. The initial parts of the splitting function now differ since the PT scales are different; the NP power growth has (approximately) the same power in the three cases, but its normalisation decreases rapidly with increasing t, roughly as e^{-t}, confirming that it too is higher twist.

One of the non-trivial features of the 4-pole model was the (approximate) equality between the normalisation of the NP power component, and the quantity $\omega_0' - \omega_0$, the difference between the exponent of the power growth and the position of the NP zero (obtained numerically from the exponent of the NP power growth in the limit $t \to \infty$). Both quantities are shown in figure 5 over a range of t values, illustrating very clearly their closeness (as well as their higher-twist nature). A detailed study reveals that the relative difference between them is consistent with a term of $\mathcal{O}\left(\alpha_s^2\right)$, as predicted in section 3. We note also that ω_0 is found to be below the NP exponent $\omega_{\mathbb{P}}$ characteristic of the Green's function, consistent with it being due to a zero to the left of the leading singularity.

[3]The attentive reader will have noticed that the modification of the PT exponent changes sign at the higher t value — indeed it turns out that the dependence on the NP parameters, while decreasing at least as fast as e^{-t}, has a non-trivial t dependence for moderate t values.

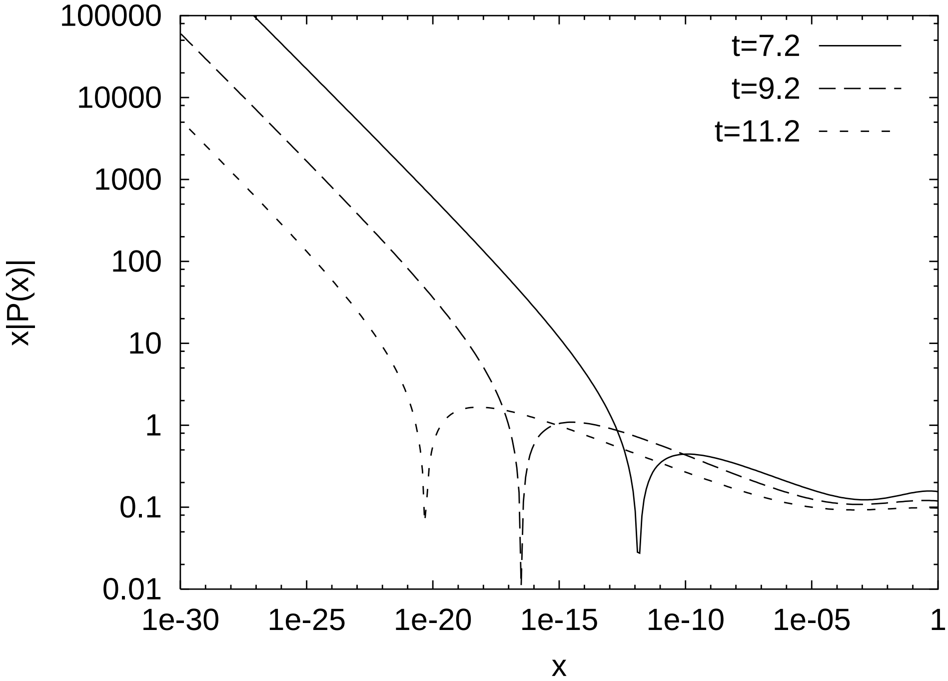

Figure 4: Examples of effective splitting functions for fixed non-perturbative parameters ($\bar{t} = 1$, $t_0 = 3$) and different t values, $t = 7.2, 9.2, 11.2$, corresponding to $\bar{\alpha}_{\rm s}$ values of 0.15, 0.12 and 0.097 respectively.

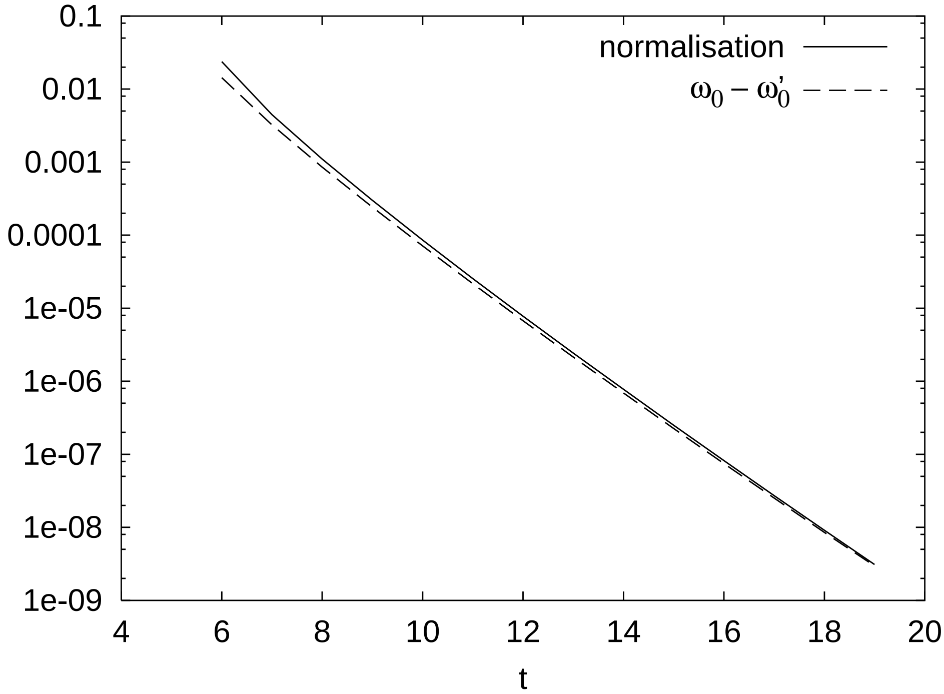

Figure 5: (Minus) the normalisation of the NP power component of the splitting function, and the difference $\omega_0 - \omega_0'$ between its exponent and the exponent in the limit $t \to \infty$. Shown for the NP parameters $\bar{t} = 2, t_0 = 5$.

5. Conclusions

For some time now there has been some debate as to whether diffusion might destroy small-x factorisation [1] or make it impossible to perturbatively predict the small-x splitting functions [2]. In [4] a model was presented which contained diffusion and the correct collinear limits, but also displayed the property of exact factorisation. It failed though to make any statement about (the functional dependence of) the magnitude of any higher-twist corrections, leaving open the possibility that they could come to dominate at small x, and thus still destroy factorisation.

This paper has presented a much more complete study of the problem, both by an extension of the model so that it includes leading higher-twist components, and by the development of numerical techniques for studying the effective splitting function in the full (LL with running coupling) BFKL equation.

The basic conclusion of these studies is that higher-twist effects, while present (and dominant at small x for the effective splitting function) are truly small *for scaling violations.* An explanation of this fact comes from the analysis of the 4-pole collinear model, whose Green's function is a sum of two terms, each of which with a NP Pomeron part factorised from the t-dependence, the second being uniformly of higher twist.

Although oversimplified, the collinear example is expected to exhibit the mechanism at work in a realistic BFKL equation also. In fact, it can be generalised to the case of $2n$-poles, for arbitrary n, the Green's function being a sum of n terms of higher and higher twist [6]. The basic point remains the fact that the Pomeron singularity, although non-perturbative, is stably factorised in front of the t-dependence in the leading twist term.

Acknowledgments

We wish to thank Martina Taiuti for several discussions in the early stages of this work. One of us (GPS) would also like to thank Guido Altarelli, Yuri Dokshitzer and Al Mueller for interesting conversations.

A. Green's function of the 4-pole model

Starting from eq. (3.3) there are various ways of deriving the differential equation of the model which differ by the treatment of the higher twist terms.

The simplest way is to insert the expression

$$b\omega t - \chi(\gamma) \tag{A.1}$$

into the γ-representation of the (regular) solution

$$\mathcal{F}_\omega(t) = e^{-\frac{1}{2}t} F_\omega(t) = \int \frac{d\gamma}{2\pi i} \exp\left[(\gamma - 1/2)\, t - \frac{1}{b\omega} \int^\gamma \chi(\gamma')\, d\gamma'\right] \tag{A.2}$$

and to notice that the result vanishes by partial integrations. The corresponding differential equation is thus obtained by the replacement $\gamma - 1/2 \to \partial_t$. By thus using the identity

$$\chi\left(\frac{1}{2}+\partial_t\right)+\frac{4}{3}=\frac{4\partial_t^2-3}{\partial_t^4-\frac{5}{2}\partial_t^2+\frac{9}{16}}=-\frac{\mathcal{N}_t}{\mathcal{D}_t} \tag{A.3}$$

and shifting the t variable to incorporate the constant term $4/3$, we obtain the Green's function equation

$$\begin{aligned}[\omega\mathcal{D}_t+\mathcal{N}_t\bar{\alpha}_\mathrm{s}(t)]\; g(t,t_0) &= [\omega\mathcal{D}_{t_0}+\mathcal{N}_{t_0}\bar{\alpha}_\mathrm{s}(t_0)]\; g(t,t_0)\\ &= -\frac{1}{\omega}\mathcal{N}_t\,\delta(t-t_0)\,,\end{aligned} \tag{A.4}$$

where $G_\omega(t,t_0)=\omega^{-1}\delta(t-t_0)+e^{\frac{1}{2}(t-t_0)}g_\omega(t,t_0)\bar{\alpha}_\mathrm{s}(t_0)$ is the gluon Green's function discussed in the main text.

Note first that $g(t,t_0)$ satisfies a fourth order differential equation in both the t and t_0 variables, for any low-t regularisation of $\bar{\alpha}_\mathrm{s}(t)$, starting from the $1/bt$ expression of asymptotic freedom (the regularisation depends on the $\bar{t}$ parameter of the main text). Some care is needed in order to treat the boundary conditions that g has to satisfy due to the peculiar distribution occurring in the right-hand side of eq. (A.4). This distribution can be taken into account by assuming $g(t,t_0)$ to be a linear combination of left (right) regular solutions of the homogeneous equation, as in (3.4), satisfying at $t=t_0$ the discontinuity requirements

$$\Delta g=\Delta g''=0\,,\quad \Delta g'=-\frac{4}{\omega^2}\,,\quad \Delta g'''=-\frac{7}{\omega^2}+\frac{16\bar{\alpha}_\mathrm{s}(t_0)}{\omega^3}\,,\qquad (t=t_0)\,. \tag{A.5}$$

Given an arbitrary basis $\mathcal{F}_L^0(t_0),\mathcal{F}_L^1(t_0)$ of left-regular solutions, the Green's function takes the form

$$\begin{aligned}g_\omega(t,t_0) = &\left(\mathcal{F}_R^0(t)\mathcal{F}_L^a(t_0)+\mathcal{F}_R^1(t)\mathcal{F}_L^b(t_0))\right)\Theta(t-t_0)+\\ &+\left(\mathcal{F}_L^0(t)\mathcal{F}_R^a(t_0)+\mathcal{F}_L^1(t)\mathcal{F}_R^b(t_0))\right)\Theta(t_0-t)\end{aligned} \tag{A.6}$$

and should also be symmetrical under t,t_0 interchange. Because of (A.6), the discontinuity conditions (A.5) at $t=t_0$ can be viewed as a system of four linear equations in the four unknowns $\mathcal{F}_L^a(t_0)$, $\mathcal{F}_L^b(t_0)$, $\mathcal{F}_R^a(t_0)$, $\mathcal{F}_R^b(t_0)$, which can be solved by standard methods.

As a consequence, the left-regular solutions $\mathcal{F}_L^a$, $\mathcal{F}_L^b$ occurring in (3.4) and satisfying (A.5) are given by the following expressions

$$\begin{aligned}\mathcal{F}_L^a(t) &= \frac{\det W(\Delta(t),\,\mathcal{F}_R^1(t),\,\mathcal{F}_L^0(t),\,\mathcal{F}_L^1(t))}{\det W(\mathcal{F}_R^0,\,\mathcal{F}_R^1,\,\mathcal{F}_L^0,\,\mathcal{F}_L^1)}\,,\\ \mathcal{F}_L^b(t) &= \frac{\det W(\mathcal{F}_R^0(t),\,\Delta(t),\,\mathcal{F}_L^0(t),\,\mathcal{F}_L^1(t))}{\det W(\mathcal{F}_R^0,\,\mathcal{F}_R^1,\,\mathcal{F}_L^0,\,\mathcal{F}_L^1)}\,.\end{aligned} \tag{A.7}$$

Here the W's are the Wronskian matrices of the corresponding functions, where in the numerators the column vector of derivatives is replaced, in the proper place, by the discontinuity vector ($i = 0, 1, 2, 3$)

$$\Delta(t) = (\Delta g^{(i)}) = -\frac{1}{\omega^2}\left(0, 4, 0, 7 - \frac{16\bar{\alpha}_{\rm s}(t)}{\omega}\right). \tag{A.8}$$

While the Wronskian in the denominator is constant, one can check that the numerators are indeed solutions of the basic homogeneous equation in (A.4). Furthermore the expressions (A.6) turn out to be independent of the choice of the left-regular basis, by the linearity properties of Wronskian matrices. Finally the Green's function is determined by symmetry for $t < t_0$.

The explicit determination of $\mathcal{F}_L^a(t_0)$ and $\mathcal{F}_L^b(t_0)$, and of the corresponding Pomeron singularity requires the solution of a matching problem for scattering in a fourth-order framework, depending on the regularisation of $\bar{\alpha}_{\rm s}(t)$ to the left. The outcome of this procedure is the separation of the left-regular solutions into irregular and regular ones on the right, as follows

$$\begin{aligned} \mathcal{F}_L^0(t_0) &= \mathcal{I}_R^0(t_0) + \sigma_{00}(\omega)\mathcal{F}_R^0(t_0) + \cdots, \\ \mathcal{F}_L^1(t_0) &= \mathcal{I}_R^1(t_0) + \sigma_{10}(\omega)\mathcal{F}_R^0(t_0) + \cdots, \end{aligned} \tag{A.9}$$

where $\sigma_{00}(\omega)$, $\sigma_{10}(\omega)$, ... are scattering coefficients which carry the (non-perturbative) Pomeron singularity and $\mathcal{I}_R^0 \sim (\mathcal{F}_R^0)^{-1}$, $\mathcal{I}_R^1 \sim (\mathcal{F}_R^1)^{-1}$ are the irregular solutions on the right. The relation of the non-perturbative to perturbative contributions in each of the terms of eq. (3.4) is thus very similar to that found in the two-pole model [4].

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