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Boundary Fixed Points, Enhanced Gauge Symmetry and Singular Bundles on K3

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Abstract

We investigate certain fixed points in the boundary conformal field theory representation of type IIA D-branes on Gepner points of $K3$. They correspond geometrically to degenerate brane configurations, and physically lead to enhanced gauge symmetries on the world-volume. Non-abelian gauge groups arise if the stabilizer group of the fixed points is realized projectively, which is similar to D-branes on orbifolds with discrete torsion. Moreover, the fixed point boundary states can be resolved into several irreducible components. These correspond to bound states at threshold and can be viewed as (non-locally free) sub-sheaves of semi-stable sheaves. Thus, the BCFT fixed points appear to carry two-fold geometrical information: on the one hand they probe the boundary of the instanton moduli space on $K3$, on the other hand they probe discrete torsion in D-geometry.

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1. Introduction

Conformal field theory on world-sheets with boundary (BCFT) has proven to be a powerful tool for studying the quantum geometry of D-branes. In particular, investigating exactly solvable tensor products of $\mathcal{N}=2$ minimal models [1,2] has provided important insight in the D-geometry of Calabi-Yau threefolds, in the domain of strong quantum corrections [3-12]. The purpose of the present letter is to investigate extra gauge symmetries stemming from certain fixed points in the BCFT. As these arise already in type II compactifications on $K3$, we will focus here on this particularly simple situation where all properly constructed BCFT states should have a well-defined, classical geometric interpretation; however, most of our CFT considerations directly apply also to general n-folds.

The problem can be stated as follows. Using the methods developed in [3], given some Fermat ("Gepner") point on a $K3$ surface one can easily find a list of boundary states that correspond to D-brane configurations $\mathcal{E}$ with RR-charges†

$$v(\mathcal{E}) \;=\; (q_4, q_2, q_0) \;\equiv\; (\mathrm{rk}, c_1, r + \tfrac{1}{2}{c_1}^2 - c_2) \in H^{\mathrm{even}}(K3, \mathbb{Z}) \; . \tag{1}$$

It is in fact possible to extract more bundle data than just the charges from the BCFT, like for instance the number of moduli of a configuration. For some configurations it was found in [5,6] that the number ν of vacuum states in the open string sector is larger than one. Physically, this corresponds to extra gauge fields on the world-volume and mathematically to a degenerate bundle or sheaf. If there are $\widetilde{\nu} > 1$ $U(1)$ factors in the gauge group (where $1 \leq \widetilde{\nu} \leq \nu$), such configurations should be considered as reducible because each $U(1)$ corresponds to an independent center-of-mass degree of freedom of a multi-brane system. There is no fundamental distinction between a multi-brane system and a bound state at threshold with the same overall charges, as these configurations live on the same continuous branch of moduli space.

Our aim is to investigate such degenerate configurations with $\nu > 1$, which correspond to singular boundary points of the instanton moduli space where extra gauge fields appear. In particular, we would like to find what the charge vectors $v^{(i)}$ ($i=1, ..., \widetilde{\nu}$) of the individual components of a reducible configuration are. This first of all requires understanding the CFT origin of the non-trivial multiplicities. As we will see, they are rooted in certain "simple current" fixed points that generically appear on the boundary but not in the bulk CFT. This reflects the fact that the relevant

† As usual, c_i denote Chern classes and rk the rank of the corresponding bundle or sheaf.

geometric singularities do not appear on the manifold, but rather in bundles over it. Upon resolving the fixed points, we will obtain the brane charges $v^{(i)}$ of the irreducible components; they turn out to have a canonical description in terms of sub-sheaves of semi-stable sheaves.

We will also find that while ν is given by the order of the stabilizer $\mathcal{S}$ of the fixed point, the number $\widetilde{\nu}$ of irreducible components is given by the order of the so-called [13] untwisted stabilizer $\mathcal{U}$. If the stabilizer is realized only projectively on the fixed point, then $\nu/\widetilde{\nu} \equiv N^2 > 1$, which leads to enhanced $U(N)$ gauge symmetry on the world-volume. This is analogous to D-branes on orbifolds with discrete torsion [14,15], and provides an interesting mechanism for obtaining non-abelian gauge symmetries in type II string compactifications, within the conformal field theory of $\mathcal{N}=2$ minimal models.

2. Fixed points in B-type $\mathcal{N}=2$ boundary CFT

We consider the CFT describing the internal part of a Gepner model for an n-fold (where n is the complex dimension of the Calabi-Yau space) at the Fermat point. It is constructed out of the tensor product of r $\mathcal{N}=2$ minimal models with levels k_i, suitably projected to ensure worldsheet supersymmetry and to allow (after the GSO projection) for supersymmetry in the external spacetime [16]. While A- and B-type boundary conditions [17] for such models have been studied over the past few years, the resolution of fixed points under these projections has not yet been fully worked out.

More concretely, for A-type states (associated to real submanifolds) the algebraic problems associated with fixed points were pointed out in [18], analyzed in an example in [8], and solved in [19]. For B-type states, associated to holomorphic geometry, fixed point phenomena were first noticed in a geometrical context in [5]. Specifically, recall that the B-type states in [1] were (partially) labelled by integers $\vec{L}=(L_1,...,L_r)$, where $0 \leq L_i \leq [k_i/2]$. It was observed in [5,6,8] that whenever a label L_i reaches $k_i/2$, then by virtue of field identifications there is an extra copy of the vacuum state contributing to open string amplitudes. Each such vacuum state corresponds to a gauge field on the brane world-volume [3]. The details depend on whether $n+r$ is even or odd [6], and altogether it is found that the total number of such vacuum states is:

$$\nu \;=\; 2^{\widetilde{\ell}}\,, \qquad \widetilde{\ell} \;=\; \begin{cases} \ell & n+r \text{ odd} \\ \ell-1 & n+r \text{ even}, \ \ell>0 \\ 0 & n+r \text{ even}, \ \ell=0\ , \end{cases} \tag{2}$$

where ℓ is the number of L_i equal to $k_i/2$. When all k_j are odd, as for example in the case of the quintic threefold, then ν is always equal to one and the phenomena that we are going to discuss do not appear.

The peculiarity of labels $L_i = k_i/2$ has been recognized before in a different context, namely boundary operators in su(2) WZW models [20] with $D_{\rm odd}$-type modular invariant. It has been traced to the fixed point of the simple current that generates the modular invariant [20,21,22]. This simple current has non-integer conformal dimension and therefore does not lead to a fixed point in the bulk theory. However it leads to a fixed point in the boundary CFT, and as a consequence the boundary state splits into a pair of states. Our aim is to resolve the analogous fixed points in $\mathcal{N}=2$ Gepner models, and find what the charges of the resolved boundary states are.

2.1. Simple currents in the Gepner-Greene-Plesser construction

At the level of chiral CFT, the problem of constructing B-type boundary conditions is characterized by both an increase in chiral symmetry (guaranteeing integrality of $U(1)$ charge) and a partial breaking of the chiral symmetry (because of "twisted gluing conditions" [1], or non-trivial "automorphism type" [2]). To deal with these complications in a direct manner would require the development of new CFT techniques. However, mirror symmetry exchanges the A- and B-type boundary conditions, so that we can more easily construct B-type states as A-type states in the mirror model. As is well-known, the mirror model can be obtained, according to the Greene-Plesser [23] construction, by modding out all phase symmetries, and BCFT constructions allowing to deal with such a situation are available right now.

For the closed string spectrum it is sufficient to study the action of the orbifold group on the primary fields of the theory. However, for obtaining modular data, and a fortiori boundary conditions and the open string spectrum, it is essential to understand the chiral realization of the symmetries. This can be done with simple current techniques. We will not describe those techniques in any detail here, but rather refer to ref. [24] for simple currents in general, and to [19] for the application of simple currents to the construction of A-type boundary conditions in Gepner models, as well as for further references. The general theory of boundary conditions in simple current invariants will appear elsewhere [25]. (See also the appendix of the present paper.)

For a single $\mathcal{N}=2$ minimal model at level k (whose primary fields are labelled, up to field identification, by (l,m,s)), the most important simple currents are listed in Table 1.

simple current	(l,m,s)	order	conf. weight
v	(0,0,2)	2	3/2
s	(0,1,1)	$4h$ or $2h$	$k/8h$
p	(0,2,0)	h	$k+1/h$
f	(k,0,0)	2	$k/4$

Table 1: *The most important simple currents of $\mathcal{N}=2$ minimal models, for odd or even level k, and with $h=k+2$.*

In more familiar terms, we note that v is just the primary field that contains the world-sheet supercurrent(s), s is the primary field that contains the spectral flow operator, while the monodromies of p give the phase symmetries. The simple current f is distinguished because it is the only one with potential fixed points. It acts on primary fields as $f(l,m,s)=(k-l,m,s)$, and thus, if k is even, then f fixes all fields of the form $(l,m,s)=(k/2,m,s)$. By field identification we can write:

$$f=(k,0,0)\equiv(0,h,2)=p^{h/2}v\ . \tag{3}$$

When forming the tensor product of minimal models, we add a subscript i to indicate the factor. The Calabi-Yau projection, which turns the tensor product into the exact solution of the Calabi-Yau sigma model, can be thought of as a simple current extension by the currents $w_i=v_1v_i,\ i=2,...,r$ and by $u=v_1^{n+r}\prod_i p_i$. To obtain the mirror model, one must in addition include into the simple current group all invariant phase symmetries, i.e., all combinations $v_1^\epsilon\prod {p_i}^{\pi_i}$ that satisfy

$$\sum_i \pi_i/h_i+\epsilon/2\in\mathbb{Z}\ , \tag{4}$$

where $\pi_i=0,...,h_i-1$ and $\epsilon=0$ if $n+r$ is even, and $\epsilon=0,1$ if $n+r$ is odd.†

† The possibility of having $\epsilon\neq0$ is usually neglected in the literature, because it is irrelevant for the computation of the closed string NS spectrum. This is no longer true in the open string sector.

As a first step in analyzing the fixed points, we need to know the stabilizer of a given B-type boundary condition. This is rather simple. The only candidate fixed point boundary conditions are those with $L_i = k_i/2$, for some i. So all we need to do is to determine which combinations of f_i are allowed phase symmetries. When $n+r$ is odd, it follows from (3) and (4) that every f_i is allowed. Therefore the total number of phase symmetries leaving $\vec{L}$ fixed is 2^ℓ, where ℓ is the number of j with $L_j = k_j/2$. When $n+r$ is even, only pairs $f_i f_j$ with $i \neq j$ are allowed phase symmetries, so the order of the stabilizer is $2^{\ell-1}$. Thus, we see that ν as defined in (2) is indeed precisely given by the order of the stabilizer $\mathcal{S}$ of $\vec{L}$.

2.2. Projective representation and non-abelian gauge symmetry

An important result from the general theory [25] is that the number$^\diamond$ of independent boundary states associated to a given $\vec{L}$ is not given by the order ν of the stabilizer $\mathcal{S}$, but rather by the order $\widetilde{\nu}$ of the *untwisted* (or central) stabilizer $\mathcal{U}$, which differs from ν multiplicatively by a square number:

$$\nu \;=\; \widetilde{\nu}\, N^2 \,. \tag{5}$$

This equation means that a fixed point boundary state can be resolved into $\widetilde{\nu}$ independent components that are not further decomposable. It is the analogue of the relation $|\Gamma| = \sum_{i=1}^{N_R} (d_{R_i})^2$ that was derived for orbifolds [14,26,15], where Γ is the discrete group that is modded out and d_{R_i} is the dimension of the irreducible (projective) representation R_i of Γ. In our context the rôle of Γ is played by the stabilizer $\mathcal{S}$, and $\widetilde{\nu}$ is the number of irreducible representations. In the present BCFT construction, these all have the same dimension N.‡

More concretely, the untwisted stabilizer is associated with an alternating bihomomorphism (a commutator two-cocycle describing an element of $H^2(\mathcal{S}, U(1))$), i.e., a pairing

$$E:\;\; \mathcal{S} \times \mathcal{S} \to \mathbb{C}^\times$$

$\diamond$ More precisely, the number of $\mathbb{Z}_K$ orbits, where $K = \mathrm{l.c.m.}(k_i+2)$.

$\ddagger$ More precisely, when forming tensor products of ordinary superconformal minimal models, $N = 2^{[\widetilde{\ell/2}]}$ is always a power of two (see the appendix). Other values of N, given by powers of K, should be possible by using $\mathcal{N}=2$ coset models based on $SU(K)$.

compatible with the group law and equal to one on the diagonal. The untwisted stabilizer is

$$\mathcal{U} := \{\Pi \in \mathcal{S};\ E(\Xi,\Pi) = 1\ \forall\ \Xi \in \mathcal{S}\}\ . \tag{6}$$

In general, E is the product of a (not necessarily alternating) bihomomorphism F computed from the fixed point modular matrices, and the relative monodromies of the currents:[♮]

$$E\ =\ F\,\mathrm{e}^{2\pi\mathrm{i}X}\ . \tag{7}$$

If E is non-trivial (which means that $\nu/\widetilde{\nu} = N^2 > 1$), we can have only a projective realization of the stabilizer group (this may be called "discrete torsion" [27,28]). The Hilbert space can then be written as $\mathcal{H} = \mathcal{V} \otimes \mathcal{H}'$, where $\mathcal{V} = \mathbb{C}^N$ is the representation space for an N^2-dimensional projective representation of $\mathcal{S}$ (the natural action of $\mathcal{S}$ on $\mathcal{V}$ is by multiplication with $N \times N$ matrices). Accordingly, each of the $\widetilde{\nu}$ vertex operators that describe the emission of $U(1)$ gauge bosons on the boundary, gains $N \times N$ additional "internal" indices, thereby forming vertex operators associated with $U(N)$ gauge symmetry.

The physical picture underlying (5) thus is that the collection of ν gauge fields splits into $\widetilde{\nu}$ separate families, each containing N^2 gauge fields carrying the adjoint representation of $U(N)$.

Accordingly, the fixed point D-brane boundary states split into $\widetilde{\nu}$ independent "N-fold bound states", each of which realizes a $U(N)$ gauge symmetry on its world volume. This is analogous to the argumentation in [14] where D-branes on orbifolds with discrete torsion were considered. Specifically it it was argued [14,15] that discrete torsion in the open string sector can be attributed to a flat but topologically non-trivial background B-field on a torsion 2-cycle. The net effect of this is that the minimal wrapping number of a D-brane is N, because configurations with charge less than N are not allowed [15] due to global world-sheet anomalies [29,30]. We find that this consistency condition is naturally encoded in the BCFT, in that the $\widetilde{\nu}$ independent boundary states cannot be decomposed into further boundary states with smaller charges.

It would be interesting to more explicitly see how the fixed point boundary states with $N > 1$ probe the torsion part of the 2-homology [31]. More broadly, it may also be possible to give them a meaning in terms of twisted K-theory groups, which seems to be the appropriate framework for D-branes in a B-field background [29,32]. These issues are however beyond the scope of the present paper.

♮ See Appendix A for the details of the computation of F, X, E and $\mathcal{U}$ in Gepner models.

2.3. RR charges of resolved boundary states

We now explain the computation of the RR charges of the B-type boundary states that arise from resolving fixed points, and in particular how one finds the corresponding geometric brane charge vectors $v^{(i)}$, $i=1,...,\widetilde{\nu}$. In general the RR charge is given, up to a normalization, by the one-point amplitude on the disk with boundary condition a, with the insertion of the bulk vertex operator $\Phi_{\rm RR}$ of a massless RR state. Since the boundary state $||a\rangle\rangle$ simply encodes the information about all such one-point functions, we can write the RR-charge suggestively as an inner product

$$q_{\rm RR}(a) \propto \langle \Phi_{\rm RR}\rangle_a = \langle \Phi_{\rm RR}||a\rangle\rangle$$

Furthermore, expanding the boundary state in a basis of Ishibashi states,

$$||a\rangle\rangle = \sum_i \mathcal{B}_{ia}|i\rangle\rangle \ , \tag{8}$$

we see that the RR charges are (up to a normalization) nothing else than the reflection ("Cardy") coefficients $\mathcal{B}_{ia}$ of the *massless* RR Ishibashi fields in the expansion of a. Of course, the full expansion contains many more terms, but those correspond to massive, non-topological components that we are not interested in.

The problem with this computation is that the resulting charges are not properly normalized and that there is no immediate connection to a geometric basis of the charge lattice. Both normalization and basis can be fixed by computing the intersection form, given in CFT by the index $tr(-1)^F$, and comparing it with the geometric intersection form at the Gepner point [3]. It was found in [3] that there is a very simple relation between the intersection numbers of the $L_i = 0$ states and the intersection form of the periods in the $\mathbb{Z}_K$-symmetric basis. Thus, once the ambiguities are fixed for the $\vec{L} = 0$ states, the charges of all remaining boundary states, and in particular of the resolved ones, can be determined from the reflection coefficients.

For the reflection coefficients of the resolved B-type boundary states (constructed as A-type states in the mirror modular invariant associated to the simple current group, as explained above) we find explicitly:

$$\begin{aligned}\mathcal{B}_{(\lambda,\mu,f)(\vec{L},M,S,\Psi)} = \sqrt{\frac{|\mathcal{G}^{\rm mirr}|}{\nu_{\vec{L}}\widetilde{\nu}_{\vec{L}}}}\,\Psi(f)\prod_{i\notin I_f} 2\sqrt{\frac{2}{h_i}}\,\sin\!\left[\pi\frac{(l_i+1)(L_i+1)}{h_i}\right] \\ \times\prod_{i\in I_f} {\rm e}^{-2\pi{\rm i}\,3k_i/16}\prod_{i=1}^{r}\frac{1}{\sqrt{2h_i}}\,{\rm e}^{2\pi{\rm i}Mm_it_i/2h_i}\;\;\frac{1}{2^r}\,{\rm e}^{-2\pi{\rm i}(Ss_1+\sum_{i=2}^{r}S^2s_i)/4}\end{aligned} \tag{9}$$

In this expression, (λ,μ,f) labels the Ishibashi states, which according to the general theory requires $f=\prod_{i\in I_f} f_i$ to be a simple current which fixes the bulk field label $(\lambda,\mu)=(l_1,...,l_r,m_1,...,m_r,s_1,...,s_r)$. (In most cases, f will simply be the identity field, but non-trivial f's are possible when $l_i=k_i/2$ for some i). The combination (λ,μ,f) must in addition possess the right relative monodromy to cancel the discrete torsion with respect to all symmetries of the Gepner-Greene-Plesser construction outlined above. Furthermore, Ψ is a character of the untwisted stabilizer of $\vec{L}$, $M=0,...,2K-1$ measures the unbroken spacetime supersymmetry†, and $t=(t_1,...,t_r)$ is the combination with minimal (non-integer) $U(1)$ charge, i.e. $\sum_i t_i/h_i=1/K$. The label S distinguishes branes ($S=0,1$) from anti-branes ($S=2,3$). Moreover, a crucial ingredient is the factor $\mathrm{e}^{-2\pi \mathrm{i} 3k_i/16}$ which comes from the resolution of the fixed points. Its form is inferred from known fixed point matrices [24,13] for the simple currents in su(2) WZW models. Finally, $|\mathcal{G}^{\mathrm{mirr}}|$ is the order of the Gepner-Greene-Plesser simple current group.

Note that formula (9) gives the minimal D-brane charges, namely the charges when there is no discrete torsion. As mentioned above, when there is discrete torsion, there are "N-fold bound states" which cannot be decomposed further, and consistency [30] requires that the allowed charges are an integral multiple of (9), i.e.,

$$Q \;=\; N\,Q_{\min} \;=\; N\,\mathcal{B}\,. \tag{10}$$

This is the BCFT analog of the formula $Q=d_R/|\Gamma|$ for D-branes on orbifolds with discrete torsion [26].

3. An example

We consider the Gepner model with $(k_1,k_2,k_3)=(4,4,4)$, which geometrically corresponds to a $K3$ defined by the equation $\sum_{i=1}^3 {x_i}^6+{x_4}^2=0$ in $\mathbb{P}(1,1,1,3)[6]$. It figures as fiber in a CY threefold that was investigated in [6,7]. In these references, the appearance of brane configurations with $\nu=1,2,4,8$ was noticed, and this was one of the motivations for the present investigation.

† M runs over even or odd values depending on the parity of $\vec{L}$.

In order to determine the RR charge vectors from the reflection coefficients (9), we first need to have the complete set of massless Ishibashi RR states. For the model at hand, the following massless RR fields can couple to B-type boundary states:

$$\begin{aligned} \Phi^{(1)}_{RR} &= \left[\phi^L_{(0,1,1)}\phi^R_{(0,-1,-1)}\right]^3 \\ \Phi^{(2)}_{RR} &= \left[\phi^L_{(0,-1,-1)}\phi^R_{(0,1,1)}\right]^3 \\ \Phi^{(3)}_{RR} &= \left[\phi^L_{(2,3,1)}\phi^R_{(2,-3,-1)}\right]^3 \end{aligned} \tag{11}$$

where $\phi^{L,R}_{(l,m,s)}$ are left- and right-moving primary fields. Using (9), we find that the charges of the orbit of $\vec{L}=(0,0,0)$, $S=0$ states in this basis are, up to normalization ($\rho := \mathrm{e}^{2\pi\mathrm{i}/6}$):

$$Q^{\mathrm{BCFT}}_{(0,0,0)} = \begin{pmatrix} 1 & 1 & 1 \\ \rho & -\rho^2 & -1 \\ \rho^2 & -\rho & 1 \end{pmatrix} . \tag{12}$$

The columns correspond to the three Ramond ground states (11), while the rows correspond to the different M labels, i.e. $M=0,2,4$ for $\vec{L}=(0,0,0)$. We have suppressed the lower half of this matrix ($M=6,8,10$), because it is simply the negative of the upper half and so represents the anti-branes. From [6], we know the analytic continuation from the Gepner to the geometric charge basis at large radius, and by inverting (12), we can easily find the matrix H that furnishes the change of basis:

$$H = \frac{1}{3}\begin{pmatrix} \rho^2 & 2 & -\rho \\ -\rho & 2 & \rho^2 \\ 4 & -4 & 4 \end{pmatrix} . \tag{13}$$

Indeed, multiplying (12) from the right with H, we obtain,

$$v^i_{(0,0,0)} = Q^{\mathrm{BCFT}}_{(0,0,0)} H = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 2 & -1 \\ 1 & -2 & 2 \end{pmatrix} ,$$

which gives back the charges of the $L_i=0$ states as given in [6,7] (up to a slight change of basis).

To compute the charges of the remaining states, we notice that the ambiguity in the normalization of (12) rests inside each column. Also, compared to equation (9), we have omitted the factors $\left(\sin^3\frac{\pi}{6}, \sin^3\frac{\pi}{6}, \sin^3\frac{3\pi}{6}\right)$. Such ambiguities just change the normalization of each charge, and can be adjusted by a redefinition of the basis change (13) that connects the BCFT charges with the geometric brane charges. With this normalization in mind, one can easily compute the charges of the boundary states with $\vec{L} \neq (0,0,0)$. With the help of trigonometric identities one can thereby recover the charges found in [3,5], and listed in the Table below.

For example, the charges of states with $\vec{L} = (2,2,2)$ are given by

$$
\begin{aligned}
Q^{\mathrm{BCFT}}_{(2,2,2)} &= \begin{pmatrix} \left(\frac{\sin\frac{3\pi}{6}}{\sin\frac{\pi}{6}}\right)^3 & \left(\frac{\sin\frac{3\pi}{6}}{\sin\frac{\pi}{6}}\right)^3 & \left(\frac{\sin\frac{9\pi}{6}}{\sin\frac{3\pi}{6}}\right)^3 \\ \left(\frac{\sin\frac{3\pi}{6}}{\sin\frac{\pi}{6}}\right)^3\rho & \left(\frac{\sin\frac{3\pi}{6}}{\sin\frac{\pi}{6}}\right)^3\rho^5 & \left(\frac{\sin\frac{9\pi}{6}}{\sin\frac{3\pi}{6}}\right)^3\rho^3 \\ \left(\frac{\sin\frac{3\pi}{6}}{\sin\frac{\pi}{6}}\right)^3\rho^2 & \left(\frac{\sin\frac{3\pi}{6}}{\sin\frac{\pi}{6}}\right)^3\rho^4 & \left(\frac{\sin\frac{9\pi}{6}}{\sin\frac{3\pi}{6}}\right)^3 \end{pmatrix} \\
&= \begin{pmatrix} 8 & 8 & -1 \\ 8\rho & -8\rho^2 & 1 \\ 8\rho^2 & -8\rho & -1 \end{pmatrix} = (g^{-1}+1+g)^3\, Q^{\mathrm{BCFT}}_{(0,0,0)} \ ,
\end{aligned} \tag{14}
$$

where $g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$ is the appropriate shift matrix. This yields the following RR charges in the geometrical basis:

$$
v^i_{(2,2,2)} \ = \ Q^{\mathrm{BCFT}}_{(2,2,2)} H \ = \ \begin{pmatrix} -4 & 12 & -4 \\ -4 & 4 & 4 \\ -4 & -4 & 4 \end{pmatrix} \ . \tag{15}
$$

More precisely, because of the fixed points, these charges are the unresolved *overall* charges of reducible brane configurations. We now would like to know into how many irreducible components these (and also the other $\nu > 1$) states split, and what the charge vectors v of these components are. In the following, we will explicitly work out the relevant combinatorics of the twisted and untwisted stabilizers in our example. More details regarding the general case can be found in the appendix.

The phase symmetries (4) we divide by in the Greene-Plesser construction are generated by the following simple currents:

$$
\begin{aligned}
\Pi_1 &= p_1 p_2^5 \\
\Pi_2 &= v_1 p_1^3 \ .
\end{aligned} \tag{16}
$$

As is easy to see, and follows from the general discussion after eq. (4), the simple current group generated by the currents (16) together with the w_i and $u = v_1 p_1 p_2 p_3$, contains all three currents f_1, f_2, and f_3. Because of the permutation symmetry, there are then three different possible stabilizers in our example:

$$\mathcal{S} = \begin{cases} \mathbb{Z}_2 = \{1, f_1\} & \vec{L} = (2,*,*)\ (\nu = 2) \\ \mathbb{Z}_2^2 = \{1, f_1, f_2, f_1 f_2\} & \vec{L} = (2,2,*)\ (\nu = 4) \\ \mathbb{Z}_2^3 = \{1, f_1, f_2, f_3, f_1 f_2, f_1 f_3, f_2 f_3, f_1 f_2 f_3\} & \vec{L} = (2,2,2)\ (\nu = 8) \end{cases} \tag{17}$$

(where $* = 0, 1$).

To determine the untwisted stabilizer, we need to determine the bihomomorphism (7). In the case of our interest, all levels are equal to zero modulo 4. It is then easy to see that the fixed point matrices in (9) satisfy the same simple current relations as the modular S matrix [24]. Therefore, the bihomomorphism F which would measure the deviation from the usual simple current relations, is identically equal to one. Furthermore, the matrix of relative monodromies on the stabilizer is given by

$$X(f_i, f_j) = X(p_i^{h_i/2}, p_j^{h_j/2}) + X(v_i, v_j) = -\frac{\delta_{ij}}{h_i}\frac{h_i}{2}\frac{h_j}{2} + \frac{1}{2} = \begin{cases} 0 \bmod \mathbb{Z} & i = j \\ 1/2 \bmod \mathbb{Z} & i \neq j \end{cases}$$

on the generators of the stabilizers. Thus, E is given by

$$E(f_i, f_j) = (-1)^{1+\delta_{ij}} .$$

Consider the untwisted stabilizer (6) for the case $\nu = 4$ in (17). Since $E(f_1, f_2) = E(f_2, f_1) = E(f_1 f_2, f_1) = -1$, we see that in fact no non-trivial element of the stabilizer $\mathbb{Z}_2^2$ is in the kernel of E, and the untwisted stabilizer is trivial. More generally we find for all the other cases:

$$\mathcal{U} = \begin{cases} \mathbb{Z}_2 = \{1, f_1\}\ , & \widetilde{\nu} = 2 \quad (\text{for } \nu = 2) \\ \{\text{id}\}\ , & \widetilde{\nu} = 1 \quad (\text{for } \nu = 4) \\ \mathbb{Z}_2 = \{1, f_1 f_2 f_3\}\ , & \widetilde{\nu} = 2 \quad (\text{for } \nu = 8)\ , \end{cases}$$

which corresponds to $N = 1, 2, 2$, respectively, in (5). According to our general reasoning, we thus find non-abelian gauge groups for the $\nu = 4, 8$ boundary states, namely $U(2)$ and $U(2) \times U(2)$, respectively.

The general result, computed in the appendix, is that for Gepner models, $\widetilde{\nu} = 1$ when $\widetilde{\ell}$ is even, $\widetilde{\nu} = 2$ when $\widetilde{\ell}$ is odd, and hence $N = 2^{[\widetilde{l}/2]}$.

To determine the resolved charge vectors from equation (9), we need to know the complete labels of the RR ground states (11), i.e., not only the bulk labels, but also the currents associated with them. Now $\Phi_{RR}^{(1)}$ and $\Phi_{RR}^{(2)}$ obviously have trivial stabilizer, so they must be combined with the identity simple current. However, for $\Phi^{(3)}_{RR}$ the stabilizer is $\mathbb{Z}_2 = \{1, f_1 f_2 f_3\}$, and there is a slight ambiguity as to what we mean actually by $\Phi^{(3)}_{RR}$. By considering the fibration of the K3 to a CY threefold, it appears that the only consistent choice is to associate $\Phi^{(3)}_{RR}$ with the identity as well.

Having fixed the labels, and taking into account (9) and (10), we then find that the charge vectors of the resolved boundary states are simply given by $1/\widetilde{\nu}$ times the charge vectors of the unresolved states. That is, the coefficients of their expansion into massless RR Ishibashi states (8) turn out to be the same (up to an overall factor of $1/\widetilde{\nu}$) as for the unresolved boundary states.[†] This is fortunate, since, as we will discuss in the next section, it is precisely what we expect from the geometry of semi-stable sheaves.

Note that the untwisted stabilizer (i.e., the discrete torsion) plays an important rôle in determining what the charges of the resolved D-brane states are. For example, for the fixed point boundary states with $\nu = 8$ the unresolved charges $v^i_{(2,2,2)}$ in (15) are multiples of four, and one might have been tempted to believe that the resolved states have 1/4 of these charges. However, the resulting charges cannot describe physical states, as the complex dimension of the moduli spaces, given by the Mukai formula $\mu(v) = \langle v, v\rangle + 2$ [33], turns out to be fractional; in other words, the charges are not properly quantized. This is a reflection of the fact, as mentioned above, that in the presence of discrete torsion, $\nu/\widetilde{\nu} \equiv N^2 > 1$, we have "$N$-fold bound states" whose charges are N times larger (cf., (10)); for the $\nu = 8$ states we have $\widetilde{\nu} = 2$ so that the resolved states have charges which are one-half of $v^i_{(2,2,2)}$ in (15).

We have summarized all the relevant data in Table 2, which is the refinement of a similar table in ref. [6], where the subtleties of the untwisted stabilizer were not taken care of.

† Due to the independence of the $\widetilde{\nu}$ resolved states, these must then differ in the massive, non-topological Ishibashi expansion components.

L_i	$v(\mathcal{E}) = \big(r, c_1(\mathcal{E}), r+\frac{1}{2}{c_1}^2(\mathcal{E})-c_2(\mathcal{E})\big)$			ν	$\widetilde{\nu}$	G
[0,0,0]	(1,0,1)	(1,-2,2)	(2,-2,1)	1	1	$U(1)$
[1,0,0]	(1,0,-1)	(1,-2,0)	(0,2,-1)	1	1	$U(1)$
[1,1,0]	(1,-4,1)	(1,2,-2)	(2,-2,-1)	1	1	$U(1)$
[1,1,1]	(3,0,-3)	(3,-6,0)	(0,6,-3)	1	1	$U(1)$
[2,0,0]	(2,0,0)	(2,-4,2)	(0,0,-2)	2	2	$U(1)\times U(1)$
[2,1,0]	(2,0,-2)	(2,-4,0)	(0,4,-2)	2	2	$U(1)\times U(1)$
[2,1,1]	(2,-8,2)	(2,4,-4)	(4,-4,-2)	2	2	$U(1)\times U(1)$
[2,2,0]	(4,-4,0)	(0,4,-4)	(0,4,0)	4	1	$U(2)$
[2,2,1]	(4,0,-4)	(4,-8,0)	(0,8,-4)	4	1	$U(2)$
[2,2,2]	(4,-12,4)	(4,-4,-4)	(4,4,-4)	8	2	$U(2)\times U(2)$

Table 2: *Labels and unresolved RR brane charges of boundary states on the $K3$ surface in $\mathbb{P}(1,1,1,3)[6]$. Furthermore, ν denotes the order of the stabilizer of the fixed points, which gives the total number of gauge fields, while $\widetilde{\nu}$ is the order of the untwisted stabilizer, which gives the number of $U(N)$ factors and irreducible components (with charges given by $1/\widetilde{\nu}$ of the overall charges). On the right we list the unbroken gauge groups G, as implied by the discrete torsion. Geometrically, configurations with $\widetilde{\nu}>1$ correspond to strictly semi-stable sheaves.*

4. Boundary fixed points and semi-stable sheaves

We now discuss the resolution of the boundary CFT fixed points from the viewpoint of space-time geometry. Note first of all that the simple currents f_i generically have non-integer dimensions, and when this is the case, the fixed points cannot appear in the bulk, but only on the boundary. Geometrically this should mean that these fixed points do not correspond to singularities of the manifold, but rather to the degeneration of bundles over it.

This touches upon an interesting mathematical issue, namely the compactification of the moduli space of instantons (mathematically: holomorphic vector bundles, or more generally torsion-free coherent sheaves) on $K3$. Recall that for a given Mukai charge vector $v(\mathcal{E})$ (1), the complex dimension of the moduli space is

$\mu(v(\mathcal{E})) = \langle v(\mathcal{E}), v(\mathcal{E})\rangle + 2$ [33]. Over this space the structure of bundles or sheaves $\mathcal{E}$ changes, and can in particular degenerate.

To be specific, consider for example rank two configurations with charges† $v=(2,0,2-2k)$, $k\geq 2$ (which indeed appear in the BCFT construction of the $K3$ we discussed in the present paper, see Table 2 and [6,7]). These correspond to $SU(2)$ bundles with instanton number $c_2=2k$, whose moduli spaces have dimension $\mu=2rc_2-2\dim G=8k-6$. As is well-known, these moduli spaces are (almost) the same as the moduli spaces of $SU(2)$ $\mathcal{N}=2$ gauge theories with $N_m=2rc_2$ hypermultiplets. At generic points the Higgs VEVs break the gauge symmetry completely ($\nu=\widetilde{\nu}=1$); however at the origin, where the VEV's of all hypermultiplets vanish, there is an extra unbroken $SU(2)$ gauge group ($\nu=4$, $\widetilde{\nu}=1$). In terms of the brane picture, this degeneration corresponds to small instantons, ie., point-like $D0$-branes. One can also envision a degeneration where just an extra $U(1)$ factor appears, which then would correspond to a reducible configuration with $\nu=\widetilde{\nu}=2$. In particular, the degeneration into a reducible line bundle, $\mathcal{E}\sim L\oplus L^{-1}$, was discussed in [35]. Physically, this means that the brane configuration splits into two irreducible components with non-zero first Chern class: $v^{(\pm)}=(1,\pm c_1,1-k)$.

In the present paper, we have found that resolving BCFT fixed points amounts to decomposing a reducible brane system into irreducible components in the simplest manner, namely into building blocks with identical charges. Specifically, what we find is, for example, that a rank two bundle with charges $(2,0,2-2k)$ and $\nu=2$ splits into two configurations, each with charges $(1,0,1-k)$. Unlike the above-mentioned degeneration $v\to v^{(+)}\oplus v^{(-)}$ [35], such a degeneration cannot be described in terms of ordinary $U(1)$ line bundles (since these would imply $c_1\neq 0$), but it does not have to.

Rather, it is known [36] that the moduli space of stable bundles on $K3$ is naturally compactified by adding strictly semi-stable sheaves, which is more general than bundles.◇ Physically, this amounts to including point-like degrees of freedom, in the present example $(k-1)$ $D0$-branes sitting on each of two $D4$-branes.‡ Such strictly (Gieseker) semi-stable sheaves $\mathcal{E}$ have the property [33] that they have proper subsheaves $\mathcal{E}'$ with $\mathrm{rk}\,\mathcal{E}'<\mathrm{rk}\,\mathcal{E}$ and charges $v(\mathcal{E}')$, such that the normalized Mukai vectors

† The compactification of the moduli space of such bundles has been investigated in ref. [34].

◇ For related considerations in the physics literature, see e.g., [37,38,39,40,41].

‡ Since this configuration is expected to exist also when the $K3$ is large, this should have a conventional interpretation, for example in terms of a background B-field.

$v(\mathcal{E})/\mathrm{rk}\,\mathcal{E}$ and $v(\mathcal{E}')/\mathrm{rk}\,\mathcal{E}'$ are equal. This can only occur when $\gcd(q_4,q_2,q_0)>1$ (for primitive charge vectors, $\gcd(q_4,q_2,q_0)=1$, the moduli space is already compact). Physically this condition corresponds to collinear central charges and thus to bound states at threshold.

Precisely this structure, namely the degeneration into components with charges $v^{(i)}=v(\mathcal{E})/\widetilde{\nu}$, is what we find from CFT. While expected on general grounds, this is nevertheless reassuring, since a priori the resolution of boundary fixed points might have given something else and turned out to be incompatible with a geometric picture, in particular with the picture of decomposing into sub-sheaves. We thus meet another instance where abstract properties of 2d superconformal field theory possess an identifiable, concrete geometrical meaning when translated into the space-time picture.

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Appendix A. Untwisted stabilizers

We discuss the general fixed point combinatorics for B-type boundary states in Gepner models and in particular derive the general rule $\widetilde{\nu}=2^{\frac{1-(-1)^{\widetilde{\ell}}}{2}}$ for the order of the untwisted stabilizer. Together with $\nu=2^{\widetilde{\ell}}$, this implies $N=2^{[\widetilde{\ell}/2]}$.

We first recall from [13] the definition of the bihomomorphism F on the stabilizer. Consider some CFT with modular S matrix S_{ab} and some simple current group $\mathcal{G}$. For all $J\in\mathcal{G}$, one can define a fixed point matrix S^J_{ab} between fields a,b with $J\in\mathcal{S}_a\cap\mathcal{S}_b$, where $\mathcal{S}_a\subseteq\mathcal{G}$ is the stabilizer of the primary field a. While the S matrix satisfies the usual simple current relation

$$S_{a,Kb}=\mathrm{e}^{2\pi\mathrm{i}Q_K(a)}S_{ab}\ ,$$

where $Q_K(a)$ is the monodromy charge of a with respect to K, this is generically violated for the fixed point matrices, and the violation is measured by F, i.e.:

$$S^J_{a,Kb} = \mathrm{e}^{2\pi\mathrm{i}Q_K(a)} S_{ab}\, F^*_b(K,J) \ .$$

The second ingredient in the bihomomorphism (7) is the pairing X (defined modulo $\mathbb{Z}$). The symmetric part of X is determined by the conformal weights of the currents. According to the general results of [42], the antisymmetric part of X can be freely chosen (with some restrictions) and together with the choice of simple current group $\mathcal{G}$ determines the modular invariant♮.

We now compute F in the situation of the main text, considering the case $n+r$ even first. For a given $\vec{L}$, we denote $I := \{i; L_i{=}k_i/2\}$, distinguish some $a_0 \in I$ (assuming $I \neq \emptyset$), denote the corresponding simple current by f_0, and let the stabilizer $\mathcal{S}_{\vec{L}}$ be generated by $f_{0a} = f_0 f_a$, with $a \in I$. We have $\mathcal{S}_{\vec{L}} \cong (\mathbb{Z}_2)^{\ell-1}$, where $\ell = |I|$. To compute the twisting of simple current relations, consider $(\vec{L},M,S) = (..,k_0/2,..,k_a/2,..,k_b/2,..,...)$ and $(\lambda,\mu) = (...,k_0/2,..,k_a/2,..,l_b,...,...)$.* Explicitly, the relevant $SU(2)$ part of the fixed point matrix is, up to irrelevant factors,

$$S^{f_{0a}}_{\lambda,\vec{L}} \sim \prod_{i\neq a_0,a} \sin\left[\pi\frac{(l_i+1)(L_i+1)}{h_i}\right] \prod_{i=a_0,a} \mathrm{e}^{-2\pi\mathrm{i}\,3k_i/16} \ ,$$

which is a part of the expression (9). Then one easily finds:

$$S^{f_{0a}}_{\lambda,f_{0b}\vec{L}} = \begin{cases} (-1)^{l_b} S^{f_{0a}}_{\lambda,\vec{L}} & a \neq b \\ S^{f_{0a}}_{\lambda,\vec{L}} & a = b \ . \end{cases}$$

We conclude that

$$F_{\vec{L}}(f_{0b},f_{0a}) = \begin{cases} (-1)^{k_0/2} & a \neq b \\ (-1)^{k_0/2+k_a/2} & a = b \ , \end{cases}$$

which gives the first part of (7).

To find the correct choice of X requires a careful analysis of the Greene-Plesser construction. In turns out that to obtain the mirror modular invariant as a simple

♮ Notice that in [42], the antisymmetric part of X is called “discrete torsion” on the simple current group. In the main part of the present paper, we have chosen to call E the “discrete torsion” on the stabilizer.

∗ We only divide k's by two when they are even.

current invariant, one has to define X by $X(p_a,p_b)=\delta_{ab}h_a/2$ and $X(v_a,v_b)=1/2$. Recalling $f_a=p_a^{h/2}v_a$, one finds:

$$X(f_{0a},f_{0b}) = \frac{h_0}{4} + \frac{h_a}{4}\,\delta_{ab}\ .$$

Putting things together, we thus have:

$$E_{\vec{L}}(f_{0a},f_{0b}) = F_{\vec{L}}(f_{0a},f_{0b})\,\mathrm{e}^{2\pi\mathrm{i}X(f_{0a},f_{0b})} = (-1)^{1+\delta_{ab}}\ .$$

Computing the untwisted stabilizer (6) is now an easy exercise. Consider some $\phi=\prod_{b\in I'} f_{0b}\in\mathcal{S}_{\vec{L}}$, with $I'\subseteq I\setminus\{a_0\}$.

$$E_{\vec{L}}(f_{0a},\phi) = \begin{cases} (-1)^{|I'|} & a\notin I' \\ (-1)^{|I'|-1} & a\in I'\ . \end{cases}$$

For ϕ to be in $\mathcal{U}_{\vec{L}}$, we must require $E_{\vec{L}}(f_{0a},\phi)=1$ for all a. This is only possible if $I'=\emptyset$, or if $I'=I\setminus\{a_0\}$ and $|I|=\ell$ is even. We conclude:

$$\mathcal{U}_{\vec{L}} = \begin{cases} \mathbb{Z}_2 & \ell \text{ even}, \ \ell\neq 0 \\ \{\mathrm{id}\} & \ell \text{ odd or } \ell=0\ . \end{cases}$$

The combinatorics for the boundary states in the case $n+r=$ odd can be mapped to $n+r=$ even by appending a trivial factor with $k_0=0$. Put differently, the above derivation still holds by letting $\mathcal{S}_{\vec{L}}$ be generated by $f_{0a}=f_a$, without distinguishing any particular $a\in I$. Simple current twists and monodromy carry over mutatis mutandis, but the final result is somewhat different: $\phi\in\mathcal{U}_{\vec{L}}$ if either $I'=\emptyset$ or $I'=I$, and $|I|$ odd. We conclude:

$$\mathcal{U}_{\vec{L}} = \begin{cases} \mathbb{Z}_2 & \ell \text{ odd} \\ \{\mathrm{id}\} & \ell \text{ even}\ . \end{cases}$$

The claims then follow.

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