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LONGITUDINAL BEAM DYNAMICS

Application to synchrotron

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CONTENTS

Chapter 1	**Pages**
Introduction	4
Fields and forces	7
Acceleration by time-varying fields	10
Relativistic equations	14
Chapter 2	
A brief overview of some accelerators	18
Transit time factor	24
Main RF parameters	25
Momentum compaction factor	28
Transition energy	34
Chapter 3	
Equations related to a synchrotron	37
Synchronous particle	41
Synchrotron oscillations	42
Principle of phase stability	43
Chapter 4	
RF acceleration for synchronous particle	46
RF acceleration for non-synchronous particle	49
Small amplitude oscillations	53
Chapter 5	
Oscillations with Hamiltonian formalism	59
Limits of stable region	61
Adiabatic damping	67
References	69
Appendix I	71
Appendix II	72

PREFACE

A first course, concerning Accelerator Physics, was given at Grenoble in 1990 for the DEA (Diplôme d'Etudes Approfondies) “Instrumentation et Mesures” at Joseph Fourier University. It was repeated for 3 years at Grenoble and since 1994 it is given in Archamps to the students of JUAS (Joint Universities Accelerator School).

The longitudinal beam dynamics, presented here, is applied to the circular accelerators and mainly to the synchrotrons.

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CHAPTER 1

1.1 - INTRODUCTION

All particle accelerators are based on the interaction of the electric charge with static and dynamic electromagnetic fields. These fields are used over a large range of frequencies from static fields up to Radio Frequency (RF) fields in the GHz range.
In this note we discuss the basic principles which are used in circular accelerators for particle acceleration. An introduction for pedestrians [1] was written for the students of J. Fourier University (Grenoble).
The longitudinal beam dynamics for other types of accelerators (linacs and cyclotrons) are presented in [2] and in JUAS lectures. The JUAS bibliography is recalled at the end of this note. The CAS[1] yellow reports are a valuable source of information.

1.1.1 Betatron

Although this kind of accelerator is no longer used, it is worth mentioning as the first circular electron accelerator. This circular accelerator does not use a single RF cavity to accelerate the beam through many turns. It is based on the principle of acceleration by a time-varying magnetic field. It makes use of the transformer principle where the secondary is made by an electron beam circulating in a closed orbit.

1.1.2 Circular accelerators and linear accelerators

Three types of accelerator are distinguished according to the appearance of the particle trajectories.

i) In linear accelerators, particles travel only once through the RF structures aligned in a straight path. They are accelerated either by electrostatic fields or oscillating RF fields. All types of particles (leptons, hadrons, ions) are accelerated with linacs.

ii) In circular accelerators, particles travel on a closed orbit and periodically cross an RF cavity. They are accelerated through many revolutions. The word “circular” is generic and includes any shape of closed machine (circular, racetrack, square, diamond-shaped,). Circular accelerators can also accelerate all types of particles.

iii) In cyclotrons during acceleration, particles follow spiralling trajectories. The maximum radius corresponds to the final energy where particles are extracted from the accelerator. Although the particles do not follow a true closed orbit, this type of machine

[1] All acronyms used in this note are given in Appendix I

is often considered among the circular accelerators but it is not suited to the acceleration of lepton beams.

Linear and circular accelerators have very specific advantages and disadvantages and up to now, it is mainly the applications that impose the design of either one or the other.
Circular accelerators are based on the use of magnetic fields to guide the particles along a closed orbit. Only one or a few accelerating cavities are needed which are traversed by the charged particles many times during motion on the closed orbit. Therefore, in a circular accelerator the RF system is greatly simplified.
In this category of accelerators, one should consider accumulator rings. Such machines do not accelerate but accumulate (often antiparticles) at a given and fixed energy. Nevertheless the difference of mass between light particles (e.g. positron) and heavy particles (e.g. antiproton) has a significant consequence. While, in principle, for heavy particles, one does not need a RF system, for light particles it is indispensable to install an accelerating system of one or several RF cavities to compensate the losses due to the synchrotron radiations.
In both cases, an RF system is useful to capture or extract the beam and to do some RF manipulations in order to improve the stacking rate.

1.1.3 Synchrotron radiations

A slowly moving but accelerating charge radiates power P given by :

$$P = \frac{2}{3}\,\frac{r_e c}{E_0}\left\{\dot{p}_{//}^2 + \gamma^2 \dot{p}_{\perp}^2\right\} \tag{1}$$

where $\dot{p}_{//}$ and $\dot{p}_{\perp}$ are the components of the accelerating force parallel and perpendicular to the velocity respectively. If E is the total energy of the particle and E_0 the rest energy, one has $E = \gamma\, E_0$. The constants r_e and c are defined in Appendix II.
The lighter the particle, the stronger the radiation. An equal accelerating force, $\dot{\mathrm{p}} = \frac{\mathrm{dp}}{\mathrm{dt}}$, in each direction, provides a radiative emission γ^2 larger if the force is applied perpendicularly to the velocity of the particle.

The radiated energy, over a length L, is given by :

$$W = \int_o^{\pi} P dt = \int_o^{L} \frac{P}{\beta c} ds \quad . \tag{2}$$

With a constant magnetic field and an isomagnetic orbit (ρ = constant), one can establish the energy loss per turn W_t (in eV), for electrons :

$$W_t = 88 \times 10^3 \frac{E^4}{\rho} \tag{3}$$

where E is the beam energy (GeV) and ρ the bending radius (m).

Application to the LEP: As a circular accelerator, with a bending radius of 3026 m, LEP collides e^-/e^+ beams. In 1989, 128 classical RF cavities were installed into 2 regions of the machine to reach 45 GeV. According to (3), the energy loss was 126 MeV per turn. In 1999, an energy of 102 GeV per beam was reached. The RF system should now compensate an energy loss of 3.1 GeV per turn. All new installed RF cavities are superconducting cavities with an accelerating gradient of 6 MV/m.

1.1.4 Brief discussion for the future High Energy Physics

For High Energy Physics, a trade-off has to be found between the complexity of the detector and the difficulty of building an accelerator capable of reaching higher energies. For the detectors it is easier to record electron-positron collisions instead of proton-antiproton collisions because the background is much better.
For the accelerator it is more convenient to built a circular one to obtain high energy particle beams. But for a given energy the radiation losses become an issue and there is a limit where the compensation of the synchrotron radiation loss, with RF systems, become prohibitive.

The two largest machines existing in the world for lepton collisions are the SLC in California and the LEP at CERN.
The SLC is a linear collider with a length of 3.2 km. The existing accelerating cavities allow the beams to reach 50 GeV.
The LEP is a circular collider with a circumference of 27 km. In 1999, an energy of 102 GeV per beam was reached.
The future LHC will be installed in the LEP tunnel after having dismantled the LEP machine. The beam energy will be 7 TeV and will require a magnetic field of 8.3 T for the superconducting bending magnets.

The milestones with a possible future scenario are given below:
1989: SLC first beam (50 GeV)
1989: LEP first beam (45 GeV)
1998: End of SLC (50 GeV with polarised electrons)
2000: End of LEP (104 GeV)
2005: LHC first beam p^+/p^+ (7 TeV)) (approved in 1994)
2010: Linear Collider e^-/e^+ (up to 3 TeV) (?)
2030: Muons Collider μ^-/μ^+ (?)

With LEP, the size of circular machines has probably reached a limit for any type of particles.

1.2 - FIELDS AND FORCES

The equation of the motion for a particle with a charge *e* is :

$$\frac{d\vec{p}}{dt} = e\left(\vec{E} + \vec{v} \wedge \vec{B}\right) \tag{4}$$

where $\vec{p} = m\vec{v}$ is the momentum, $\vec{v}$ the velocity, $\vec{E}$ the electric field and $\vec{B}$ the magnetic induction which is usually called the magnetic field. The fields must satisfy Maxwell's equations.

The differential forms (in the vacuum) are recalled below :

$$\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho(\vec{r}, t) \tag{5}$$

$$\nabla \cdot \vec{B} = 0 \tag{6}$$

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{7}$$

$$\nabla \wedge \vec{B} = \mu_0 \vec{j}(\vec{r}, t) + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \quad . \tag{8}$$

∇ is the operator "nabla" $\quad \nabla = \vec{u}\frac{\partial}{\partial x} + \vec{v}\frac{\partial}{\partial y} + \vec{w}\frac{\partial}{\partial z} \quad .$ (9)

ρ and $\vec{j}$ are the charge and current density respectively.
ε_0 and μ_0 are the permittivity and the permeability of the free space. Their values are given in Appendix II.

The integral forms are given below and correspond to the different laws.

i) Gauss's law :

$$\int_S \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \int_V \rho \cdot dV \quad . \tag{10}$$

On the left-hand side, the integral is taken over a surface S while on the right-hand side it is taken through a volume V.

ii) No free magnetic poles :

$$\int_S \vec{B} \cdot d\vec{S} = 0 \quad . \tag{11}$$

iii) Faraday's law :

$$\oint_L \vec{E} \cdot d\vec{l} = -\int_S \frac{d\vec{B}}{dt} \cdot d\vec{S} \quad . \tag{12}$$

On the left-hand side, the integral is taken over a path length L .

iv) Ampere's law (modified by Maxwell)

$$\oint_L \vec{B} \cdot d\vec{l} = \mu_0 \int_S \vec{j} \cdot d\vec{S} + \frac{1}{c^2} \int_S \frac{d\vec{E}}{dt} \cdot d\vec{S} \quad . \tag{13}$$

1.2.1 Magnetic field

In a constant magnetic field $\vec{B}$, a charged particle e experiences a force $\vec{F}$ (Fig.1)

$$\vec{F} = e\vec{v} \wedge \vec{B} \quad . \tag{14}$$

Its direction is always perpendicular to the velocity $\vec{v}$ of the particle. This kind of force may modify the trajectory of the particle but cannot modify its velocity, i.e. its energy.

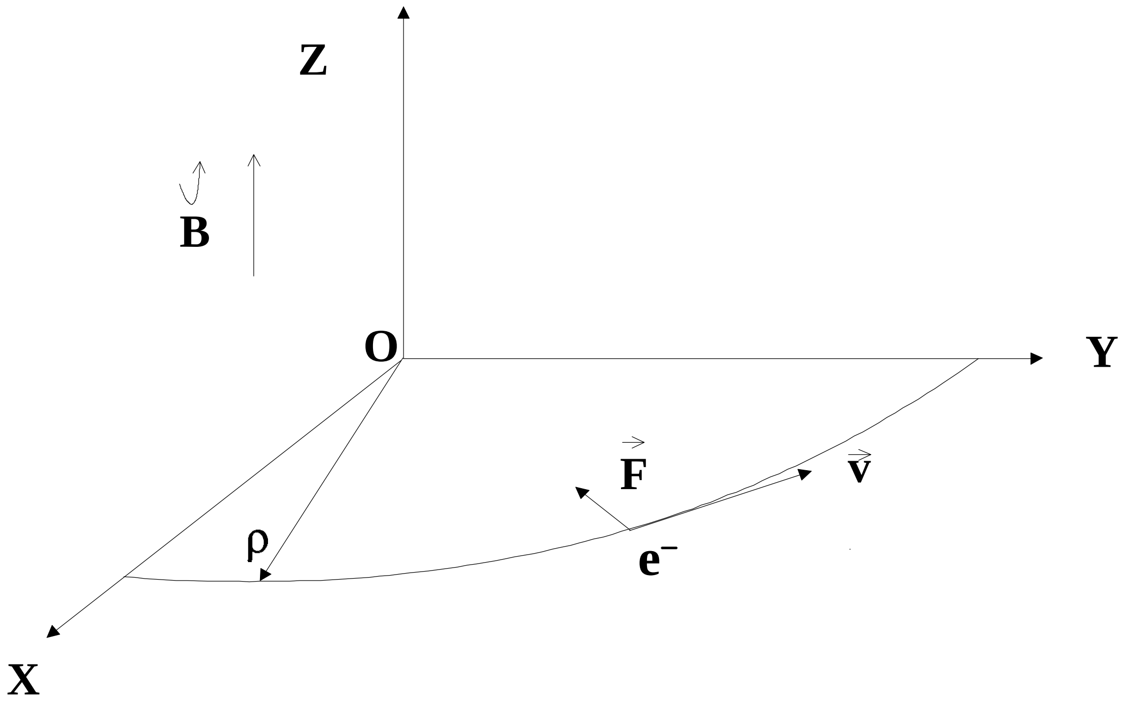


Figure 1: Lorentz force on a charged particle

In the XY plane, a particle, with a constant energy, describes a circle in equilibrium between the centripetal magnetic force and the centrifugal force :

$$e v B = \frac{m v^2}{\rho} \quad . \tag{15}$$

With the momentum, $p = mv$, one has :

$$B\,\rho = \frac{p}{e} \tag{16}$$

and the angular frequency is :

$$\omega = \frac{v}{\rho} = \frac{e}{m} B \quad . \tag{17}$$

$B\rho$ is called the "magnetic rigidity".

A quick analysis of equation (16) shows :

i) for a particle at constant momentum, if B increases, ρ decreases. Such a particle will be deflected more within a higher magnetic field.

ii) for a constant magnetic field, if p increases, the curvature radius ρ increases. Such a particle will be deflected less when its energy increases.

1.2.2 Electric field

In a constant electric field $\vec{E}$, a charged particle e experiences a force $\vec{F}$:

$$\vec{F} = e\vec{E} \quad . \tag{18}$$

Its direction is always parallel to the electric field. This kind of force can modify the trajectory of the particle but also its velocity, therefore its energy. It can be used to accelerate or decelerate particles.

The electric field, derived from a potential, is produced from a sinusoidal voltage applied to one or several cavities crossed by the particles. The sinusoidal wave presents many advantages. It is used in the RF cavities for circular machines but also in linear accelerators with high frequencies.

1.2.3 Comparison

A comparison can be made between magnetic and electric forces

$$\frac{F_{mag}}{F_{elec}} = \frac{evB}{eE} = \beta c \frac{B}{E} \quad .$$

In the LEP, the magnetic field is 0.1 T while the accelerating field is 6 MV/m. The ratio is a factor 5. In the CLIC, the accelerating electric field will be in the range of 100 MV/m. In principle, no dipoles are foreseen at high energies. However if a 1.5 TeV beam should be deflected towards a second detector, at the collision point, the integrated magnetic field inside such dipole would be 50 T.m for a deflection angle of 10 mrad. It will require a superconducting dipole. Possible values could be 5 m long and a magnetic field of 10 T. The ratio would be a factor 30.
For classical values such as B = 0.5 T and E = 10 MV/m and for relativistic particles, the magnetic force is 15 times higher than the electric force.

1.3 - ACCELERATION BY TIME-VARYING FIELDS

1.3.1 Acceleration by a time-varying magnetic field

We assume a cylindrical volume within which exists an uniform $\vec{B}$ field.
Here *uniform* means that $\vec{B}$ does not vary with position.
Let us suppose that a particle beam travels on a closed orbit perpendicular to the cylindrical axis inside this volume. Therefore the magnetic field, enclosed by the closed orbit of the beam, has a rotationally symmetry.
According to (7) or (12), a time varying $\vec{B}$ field gives rise to an electric field $\vec{E}$. (Faraday's law). A change of $\vec{B}$ leads to an electromotive force E_f

$$E_f = -\frac{d\Phi}{dt} \quad . \tag{19}$$

Φ is the magnetic flux through the closed orbit and the minus sign is given by the Lenz's law.

The induced force E_f corresponds to the electric field $\vec{E}$ taken all around the circuit (here the closed orbit)

$$E_f = \oint_L \vec{E} \cdot d\,\vec{l} \quad . \tag{20}$$

If L is a circular path with a radius R, the magnetic flux is given by :

$$\Phi = \pi R^2 B_z \quad .$$

where B_Z is the average magnetic field through the closed orbit.

Taking the time derivative :

$$\frac{d\Phi}{dt} = \pi R^2 \frac{dB_z}{dt} \quad .$$

$\vec{E}$ is tangential to the path of integration and from symmetry has the same value at all points on the closed orbit

$$\oint_L \vec{E} \cdot d\,\vec{l} = -\int_0^{2\pi} RE_\theta\, d\theta = -2\pi RE_\theta \quad .$$

In cylindrical coordinates $E_r = E_Z = 0$, and E_θ is the induced electric field (minus sign).

$$-2\,\pi\, R\, E_\theta \;=\; -\,\pi\, R^2 \frac{d\,B_z}{dt}$$
$$E_\theta \;=\; \frac{1}{2}\, R \frac{d\,B_z}{dt} \quad .$$

The accelerating force on the particle will be :

$$\frac{d\,p}{dt} \;=\; e\,E_\theta \;=\; \frac{1}{2}\, e\, R \frac{d\,B_z}{dt} \quad . \qquad (21)$$

Now if one wants that the particle remains on the closed orbit with a radius R, it must fulfil also (16). Taking the time derivative of it :

$$\frac{d\,p}{dt} \;=\; e\,R \frac{d\,B_f}{dt} \quad .$$

where B_f is the local field at the particle position.

Comparing the two last equations, the betatron condition is found after an integration :

$$B_f = \frac{1}{2}\, B_Z + cte \quad . \qquad (22)$$

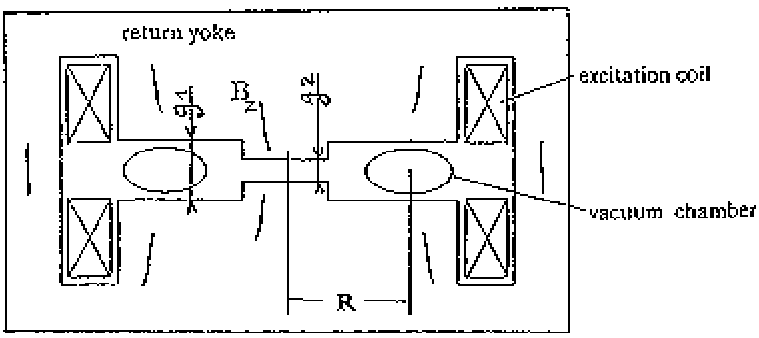


Figure 2: The betatron scheme [6]

Figure 2 shows two gaps of different aperture. One gap (g_1) where the beam is circulating has the magnetic field B_f along the closed orbit, the other gap (g_2), which is a part of the return yoke, is adjustable. It allows one to fulfil the betatron condition (22).

1.3.2 Acceleration by time-varying electric field

V_{RF} is the voltage, time dependent, applied across the gap *g* (Fig.3). An accelerating field E_Z is created as a function of the time and the position.

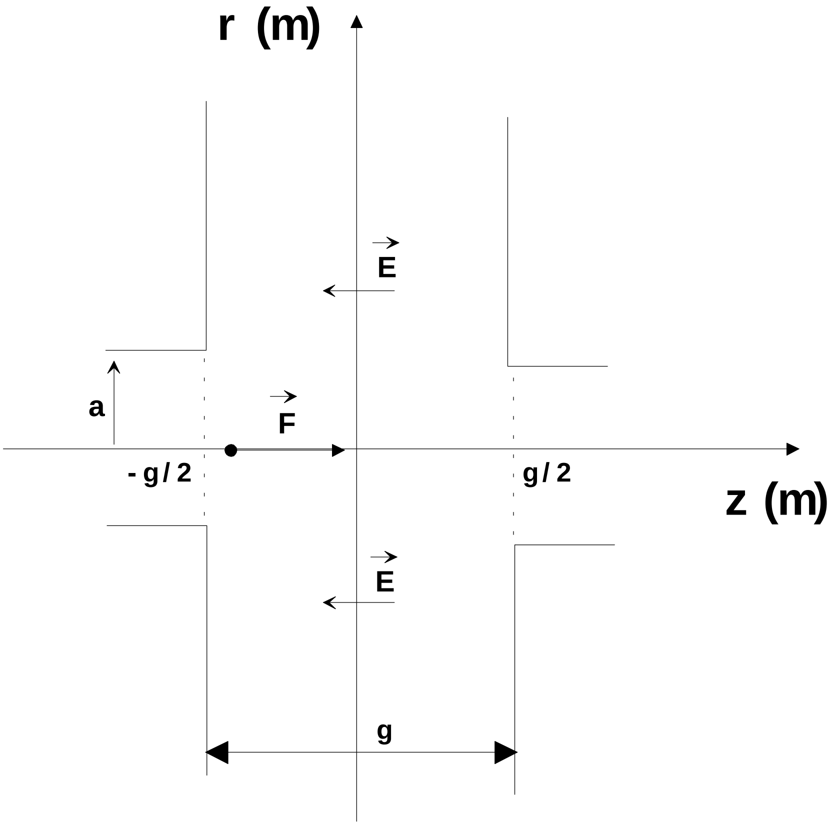


Figure 3: Schematic RF gap

Therefore a particle crossing the gap will receive an energy gain of :

$$\Delta E = e \int_{-g/2}^{g/2} E_z(z,t)dz \quad . \tag{23}$$

ΔE is expressed in [MeV] if E_Z is given in [MV/m] and *g* in [m].

In the cavity gap, the electric field can be expressed with separated variables as follows :

$$E_z(z,t) = E_1(z) \,.\, E_2(t) \quad .$$

E_1 (z) is the spatial component and E_2 (t) is the time component. The latter has a sinusoidal phase variation .

$$E_2\,(t) = E_0 \sin \phi$$

where

$$\phi = \int_{t_0}^{t} \omega_{RF} \cdot dt + \phi_0 \quad . \qquad (24)$$

ω_{RF} is the angular frequency.

An example is given by the Figure 4. It shows the electric field distribution inside a RF cavity which is a RF gun composed of one and a half cavity. The electrons are emitted from a photo-cathode in the vertical plane, on the left. Since there is cylindrical symmetry, only half of the cavity is plotted. The simulation code used here is called SUPERFISH.

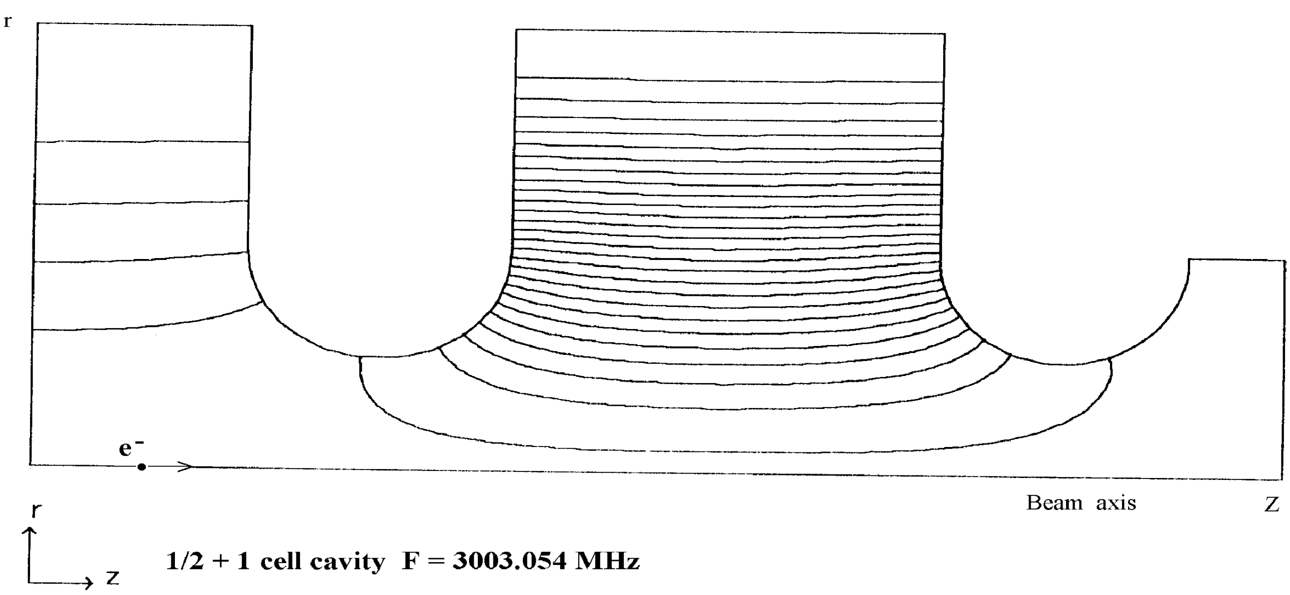


Figure 4: Electric field inside an RF cavity from SUPERFISH code

Note:

For historical reasons, the origin of the time is different for circular and linear accelerators.

i) For circular accelerators, this origin is taken at the zero crossing of the RF voltage with positive slope.

Phase of the particle = phase ϕ of the RF voltage when particle is at z = 0.

ii) For linear accelerators, this origin is taken at the crest of the RF voltage.

Phase of the particle = Phase $\phi - \pi/2$

1.4 - RELATIVISTIC EQUATIONS

1.4.1 Definitions and symbols

$$\beta = \frac{v}{c} \quad \textit{Normalized velocity} \tag{25}$$

$$\gamma = \frac{E}{E_0} \quad \frac{\textit{Total energy}}{\textit{Rest energy}} \tag{26}$$

$$E = mc^2 \qquad \textit{Total energy} \tag{27}$$

$$E_0 = m_0 c^2 \quad \textit{Rest energy} \tag{28}$$

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \quad \textit{Mass of the particle} \tag{29}$$

$$\gamma = \frac{m}{m_0} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}} \tag{30}$$

$$\gamma^2 = \frac{1}{1-\beta^2} \tag{31}$$

$$\beta^2 = \frac{\gamma^2 - 1}{\gamma^2} \tag{32}$$

$$(\beta\gamma)^2 = \gamma^2 - 1 \tag{33}$$

$$p = \frac{\beta\gamma\, E_0}{c} = \beta\gamma\, m_0\, c \quad . \tag{34}$$

1.4.2 Some useful relationships

i) Momentum
From (26) and (34), one has :

$$p\,c = \beta E = \beta\gamma\, E_0 \quad . \tag{35}$$

From (35), taken the square and using (32), one gets :

$$p^2 c^2 = E^2 - E_0^2 \quad . \tag{36}$$

ii) Energies
Let E_C be the kinetic energy. Then

$$E = E_0 + E_C \quad . \tag{37}$$

It can be written :

$$\gamma = 1 + \frac{E_c}{E_0} \quad . \tag{38}$$

$\gamma \approx 1$ Particle not relativistic
$\gamma > 1$ Particle relativistic
$\gamma >> 1$ Particle ultra-relativistic

iii) First derivatives

$$d\beta = \beta^{-1} \gamma^{-3} d\gamma$$

$$d(c\,p) = E_0 \gamma^3 d\beta$$

$$d\gamma = \beta \left(1-\beta^2\right)^{-3/2} d\beta \quad .$$

iv) Logarithmic derivatives

$$\frac{d\beta}{\beta} = (\beta\gamma)^{-2} \frac{d\gamma}{\gamma}$$

$$\frac{dp}{p} = \frac{\gamma^2}{\gamma^2 - 1} \frac{dE}{E} = \frac{\gamma}{\gamma + 1} \frac{dE_c}{E_c}$$

$$\frac{d\gamma}{\gamma} = \left(\gamma^2 - 1\right) \frac{d\beta}{\beta} \quad .$$

1.4.3 Units

All relationships are expressed in SI system. However the common units used in accelerator physics are based on the eV (electron-volt). Table 1 gives some current parameters :

Table 1: Units

Parameters	Units
Energy	*eV*
Mass	eV/c^2
Momentum	*eV/c*

1.4.4 An important relationship

The magnetic rigidity $B\rho$ is expressed from the momentum *p* by the following expression :

$$p \cong 0.3\, B\, \rho \qquad (39)$$

where *B* is in Tesla, ρ is in meters and *p* is in GeV/c.
The factor 0.3 comes from the change of units [$kg.m.s^{-1} \rightarrow$ GeV/c].

Note :

Let us compare three energies, assuming a black body behaviour. A factor 2 (due to reflection and other effects) will not change the order of magnitude :

- the energy stored in a lake,
- the energy in a flame of a candle,
- the energy in a collision e^+/e^- at the biggest collider in the world today.

i) Energy stored in the lake.
The power arriving at the earth's surface from the sun is roughly 1.4 W/cm^2. In a lake of 100 m^2, the power will be 1.4 MW.
During 3 hours of sunny period, the stored energy will be :

$$E_1 \cong 15 \text{ GJ} \ .$$

ii) Energy emitted by the flame of a candle.
Using the Stephan-Boltzmann law

$$M = \sigma T^4 \ \cdot$$

The Stephan constant is given in Appendix II.

Assuming that the temperature of the flame is around 700° C, then

$$T \cong 1000 \ °K \ .$$

Assuming that the area where is concentrated the heat is around 1 mm^2, then the radiated power is :

$$M = 5.6 \times 10^{-2} \ W \ .$$

During 3 hours, the produced energy will be :

$$E_2 \cong 605 \ J \ .$$

iii) Energy produced in a e^-/e^+ collision at 100 GeV in the LEP accelerator.

$$100 \times 10^{9} \times 1.6 \times 10^{-19} = 1.6 \times 10^{-8} \ J$$

$$E_3 \cong 16 \ nJ \ .$$

Although in the first case one has a tremendous amount of energy, we cannot use it easily, because it is not concentrated.
While in the second case, it is concentrated enough and we can make use of it in the every day life.
In the third case, it is a very small energy which could raise one gram of water by 3×10^{-6} degree. But since the particles are tiny, their kinetic energy can be transformed in a large variety of matter particles.

CHAPTER 2

2.1 - A BRIEF OVERVIEW OF SOME ACCELERATORS

2.1.1 Electrostatic accelerators

In such accelerators, the potential difference between two electrodes is used to accelerate particles. Current applications are the x-ray tubes used in medicine and in industry.
The thermionic guns are often used as electron sources for accelerators. The high voltage is in the range of a few kV to 100 kV and the current can vary between mA and A.
Figure 5 shows equipotentials and beam trajectories calculated from E-GUN code [3] for the LIL thermionic gun.
The electrons are emitted (from the left) by the thermionic cathode with a parabolic shape. The beam is focused inside the hole of the anode with a waist around z = 70 mm.

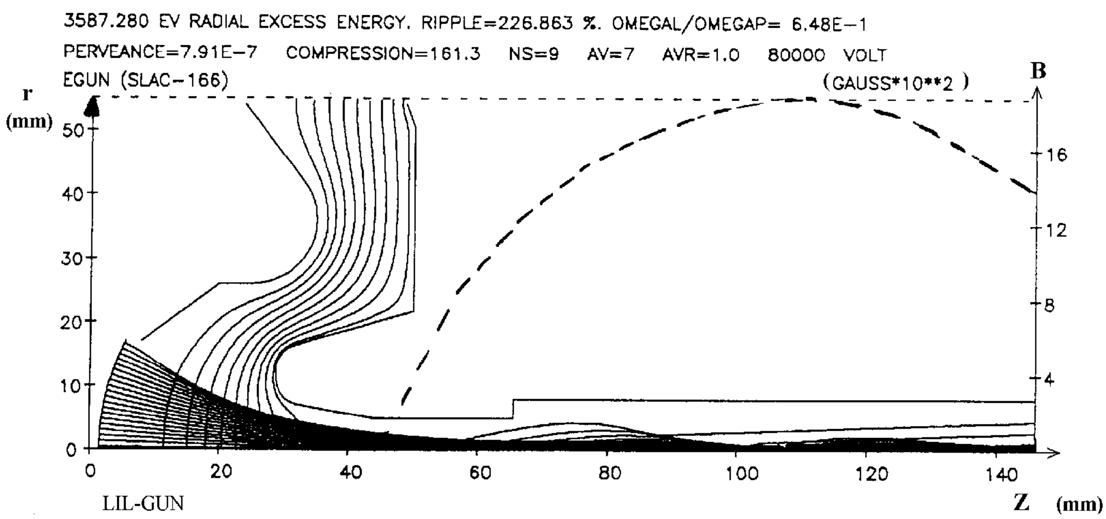


Figure 5: Electric field and beam trajectories inside an electrostatic accelerator computed from E-GUN code

2.1.2 Alvarez structure

This is a linear accelerator type which is still used to accelerate protons and heavy ions.
Figure 6 shows a schematic Alvarez structure. It is a section of circular waveguide loaded with drift tubes. Their length and position are chosen in order to have synchronism between the particles and the accelerating field while the resonant frequency of the cavity is kept constant. The axial accelerating field is concentrated in the gaps between drift tubes. The operating frequency is mainly 200 MHz. The energy range is around 50 MeV to 200 MeV.

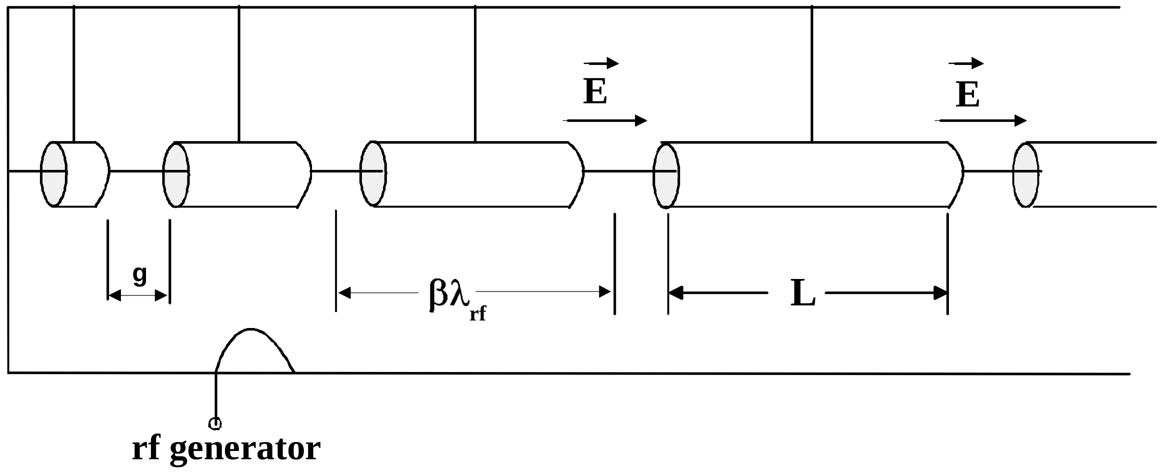


Figure 6: Alvarez linac structure [6]

Several modes of acceleration can be defined. The common value is the 2π-mode, which means that the electric field $\vec{E}$ has the same direction in two adjacent gaps at a given time.
In the 2π-mode, the synchronism condition is given by :

$$L = v_S T_{RF} = \beta_S \lambda_{RF} \quad . \tag{40}$$

L	is the length of the drift tube (not constant).
v_S	is the velocity of the particle
T_{RF}	is the RF period
λ_{RF}	is the free space wavelength of the operating frequency

The synchronism condition can be written :

$$\omega_{RF} = 2\pi \frac{v_s}{L} \quad .$$

2.1.3 Electron linac

At a few MeV, electrons are almost relativistic. Therefore above these energies, particles become ultra-relativistic and they approach a constant velocity v = c.
In a resonant cavity, the standing wave pattern can be expressed into two travelling waves : one is the forward wave which is in synchronism with the particle, the other one is the backward wave which has no effect on the energy gain of the particle.

The electric field can be expressed as :

$$E(z,t) = E_0 \; e^{i(\omega t - kz)}$$

where $k = \dfrac{2\pi}{\lambda_{RF}}$.

In order to accelerate particles continuously, it is necessary that the phase velocity equals the light velocity :

$$v_{ph} = \frac{\omega}{k} \cong c \quad .$$

An interesting aspect with travelling wave structures is that the RF power sources can be pulsed during a short period of time.
Therefore the peak power pulse can be higher and the accelerating field also.
Various sources (klystrons) have been developed in different ranges of frequencies. The common bands are :

L - band : ~ 1.5 GHz
S - band : ~ 3 GHz
C - band : ~ 6 GHz
X - band : ~ 12 GHz

Most of the linacs working today are in the S-band . The LIL accelerates e^- from 80 keV up to 500 MeV, while the SLC accelerates e^- from 120 keV up to 50 GeV.
Studies are made for future linear colliders using higher frequencies (NLC and JLC at 11.4 GHz, CLIC at 30 GHz, R&D at 90 GHz at Berlin and SLAC).

2.1.4 Cyclotron

The trajectory of the charged particles follows a spiral path (Fig.7) inside a magnetic field with a revolution frequency that depends on the particle properties and the strength of the magnetic field.
The particles are injected in the middle of the gap (Dee) where an oscillating RF voltage is applied. A magnetic field exists in the whole space where particles have to circulate. When the particle energy increases, the bending radius increases also.

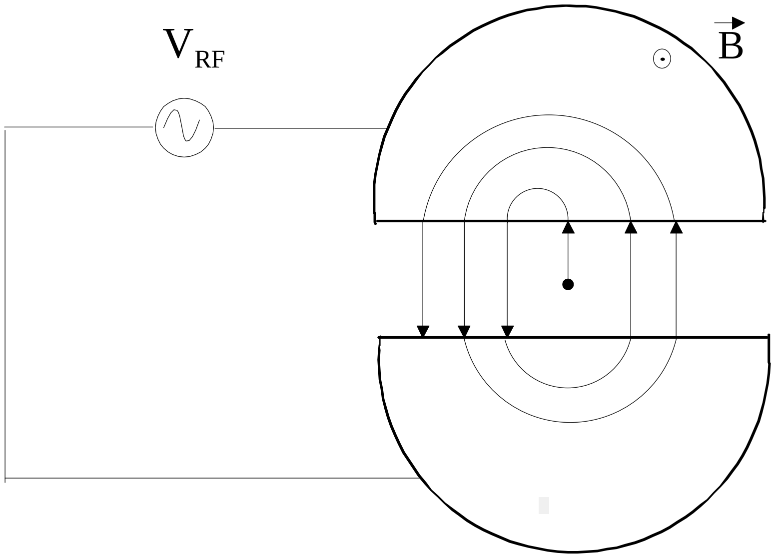


Figure 7: Cyclotron orbit

The synchronism condition is given by :

$$2\pi\rho_s = v_s T_{RF}$$

$$\omega_{RF} = \frac{2\pi}{T_{RF}} = \frac{v_s}{\rho_s} = \omega_s \quad .$$

By virtue of (17) :

$$\omega_s = \frac{e\,B}{\gamma\, m_0} \quad . \qquad (41)$$

In standard cyclotrons, the magnetic field B and the angular RF frequency ω_{RF} are constant.
Then the synchronism condition can be verified only for $\gamma \sim 1$. It implies non relativistic particles. It is the case for ions and protons at low energy.

Figure 8 shows a schematic cyclotron where the RF voltage is applied across the gap of the Dee. Particles travel in the median plane.

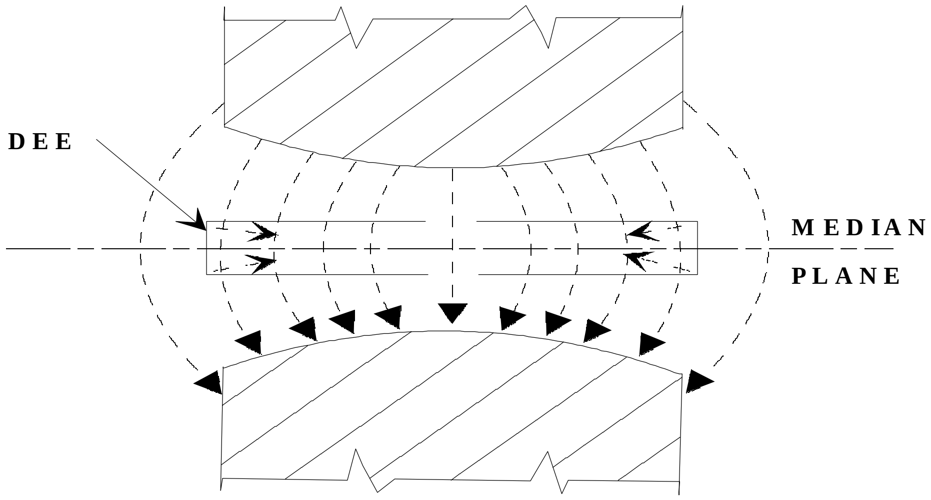


Figure 8: Classical cyclotron

If one wants to accelerate relativistic particles, it is necessary to fulfil the synchronism condition. From (41), one has :

$$\gamma\, \omega_{RF} = \frac{e\,B}{m_0} \quad .$$

The right-hand side term is constant. Then to verify this condition , one has to decrease the RF frequency when the particle energy increases.
It is done in the *Synchro-cyclotron.*

2.1.5 Synchrotron

This is a circular accelerator where particles follow a closed orbit and are accelerated each time they pass through a RF cavity installed somewhere on the closed orbit.
The synchronism condition is given by the equality between the revolution period of the particle T_S and the RF period T_{RF} (or one of its harmonic h) inside the cavity.

$$T_S = h\, T_{RF} \quad .$$

It can be written

$$\omega_s = \frac{v_s}{R} = \beta_s \frac{c}{R} = \frac{\omega_{RF}}{h} \quad . \qquad (42)$$

v_S is the particle velocity and R the constant orbit radius.

ω_S and ω_{RF} increase with energy. To keep particles on the closed orbit, the magnetic field should be varied with the time.

2.1.6 Parameters for circular accelerators

The basic principles, for the circular accelerators, are based on the two relations :

i) The Lorentz equation
Using (34) with (16), the curvature radius can be expressed as :

$$\rho = \frac{\gamma\ v\ m_0}{e B} \quad .$$

ii) The synchronicity equation
Using (41), the revolution frequency can be expressed as :

$$f = \frac{eB}{2\pi\gamma\, m_0} \quad .$$

According to the parameter we want to keep constant or let vary, one has different acceleration principles. They are summarised in Table 2.

Table 2: Parameters for some circular accelerators

Machine	Energy	Velocity	Field	Orbit	Frequency
	γ	v	B	ρ	f
Cyclotron	≈ 1	var.	const.	≈ v	const.
Synchro Cyclotron	var.	var.	$B(\rho)$	≈ p	$B(\rho) / \gamma(t)$
Proton/Ion Synchrotron	var.	var.	≈ p	R (*)	≈ v
Electron Synchrotron	var.	const.	≈ p	R (*)	const.

(*) See Chapter 3

2.2 - TRANSIT TIME FACTOR

The concept of this parameter will be illustrated with a simplified model.
Let $\hat{V}_{RF}$ be the maximum voltage across the gap g (see Figure 3) and assume that the corresponding electric field is independent of the longitudinal coordinate z. Then the accelerating field will be expressed by

$$E_z = \frac{\hat{V}_{RF}}{g} \sin(\omega t + \phi) \quad .$$

At t = 0, z = 0 and v ≠ 0. The energy gain is given by (23).

After integration, one finds :

$$\Delta E = e\hat{V}_{RF} \; T_a \sin \phi$$

where

$$T_a = \frac{\sin\left(\frac{\omega_{RF}\, g}{2v}\right)}{\left(\frac{\omega_{RF}\, g}{2v}\right)} \quad . \qquad (43)$$

T_a is called the *transit time factor*

T_a is always smaller than 1. If the gap g $\rightarrow$ 0 then $T_a \rightarrow$ 1, but an excessive local field gradient can produce RF breakdowns.

In the general case, the transit time factor is given by :

$$T = \frac{\int_{-\infty}^{+\infty} E(z,r)\cos\left(\frac{\omega_{RF}}{v}\, z\right) dz}{\int_{-\infty}^{+\infty} E(z,r)\, dz} \quad . \qquad (44)$$

This is the ratio between the peak energy gained by a particle with finite velocity v to the peak energy gained by a particle with infinite velocity $\propto$.
In others words, T shows the missing energy gain due to the finite velocity of the particle in a sinusoidal electric field. It is a reduction factor in energy gain.

2.3 - MAIN RF PARAMETERS

2.3.1 Amplitude, frequency, phase

To accelerate particles, longitudinal fields must be generated in the direction of the desired acceleration.
The efficiency of the acceleration depends mainly on the temporal variations of these fields.
The interaction of electromagnetic RF fields and charged particles assumes a wave of frequency ω propagating in the z-direction.
In a simple model, we saw that :

$$E\,(z\,,\,t) \;=\; E_1\,(z)\,.\,E_2\,(t) \quad .$$

E_2 (t) is expressed as :

$$E_2\,(t) \;=\; E_0\, sin\left(\int_{t_0}^{t}\omega_{RF}\,d\tau + \phi_0\right) \quad . \qquad (45)$$

The RF cavity which generates such field is characterised by the voltage amplitude, the frequency and the phase.

2.3.2 Harmonic number

It could be useful to have an integer multiple of the revolution frequency for the RF frequency.
This integer is called *harmonic number h*. Hence

$$\omega_{RF} \;=\; h\,\omega_s \quad . \qquad (46)$$

Table 3 gives some examples for CERN accelerators.

Table 3: Harmonic numbers

Machines	BS	PS	SPS
h	1	20	4620

Note :
All these CERN machines work also with different harmonic numbers.

2.3.3 Longitudinal acceptance

A concept very useful to describe longitudinal motion (as well as transversal motion) is the phase space. Don't confuse the trajectories in the phase space with the trajectories in the real space.

The classical pendulum can be described in the phase space where the coordinates are the angle θ (horizontal axis) and the momentum p (vertical axis). In such plane, the trajectory is the closed contour of the dashed area (Fig.9) assuming no friction and a small θ angle when the pendulum is spaced from its equilibrium position (vertical axis in the real space).

This analogy is useful to introduce the longitudinal phase space to describe the motion of particles in accelerators.

The first concept is the *emittance of the bunch.*

It is characterised by the area, in the longitudinal phase space, where the particles move (dashed area).

The second concept is the longitudinal *acceptance of the machine*.

It is characterised by another area, in the longitudinal phase space, where the parameters are given by the RF cavities and the accelerator optics.

Figure 9 shows an RF bucket and a bunch of particles inside of it, in the longitudinal phase space.

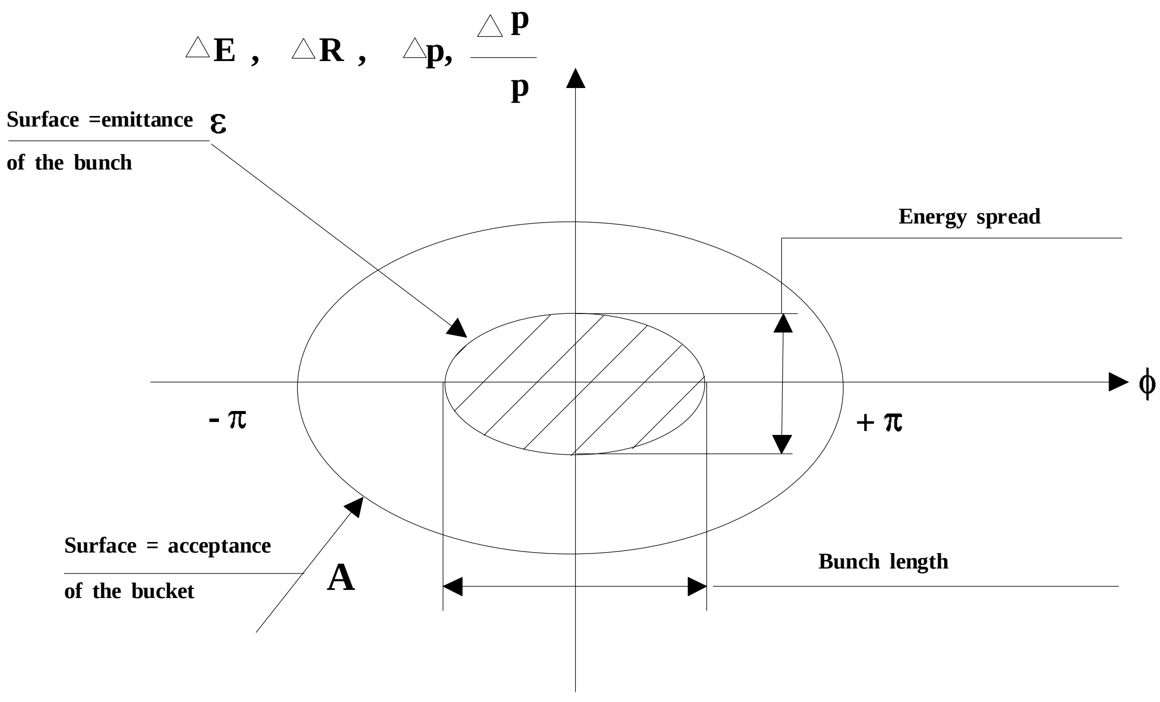


Figure 9: RF bucket and bunch area

The horizontal axis can have the phase, the time or the longitudinal position as variable, while the vertical axis has either a variation of energy, or momentum or closed orbit radius.
The area of the RF bucket determines the longitudinal acceptance A while the area of the bunch measures the longitudinal emittance ε of the beam.
In order to get a good capture efficiency of the bunch inside a bucket, the emittance should be a little bit smaller than the acceptance.

$$\varepsilon \leq A .$$

2.3.4 Adiabatic limit

The parameters of the motion have to be varied slowly compared to the synchrotron oscillations period. This parameter will be defined later on in the text.
Under this condition, a reversibility process is almost possible.
If Ω_S is the oscillations period, a coefficient α is defined as follows :

$$\alpha = \frac{1}{\Omega_s^2} \frac{d\,\Omega_s}{dt} \quad . \tag{47}$$

The process will be adiabatic if $\alpha << 1$.

An example of adiabatic debunching is given when the RF voltage is slowly decreased to zero from a stationary bucket. The beam (heavy particles) is debunched and forms a continuum path along the closed orbit. The reverse process is possible if the RF voltage is slowly increased up to the nominal value. The beam is bunched again (Fig.10).

An example of a non-adiabatic effect is given when the RF phase is shifted quickly by 180° from a stationary bucket. The particles are then at the limits of the buckets and start to be distributed along these limits.

A new quick shift of 180° brings back the particles to the centre of the bucket.
This effect is used to obtain either a bunch compression or a diminution of the momentum spread (Fig. 11).

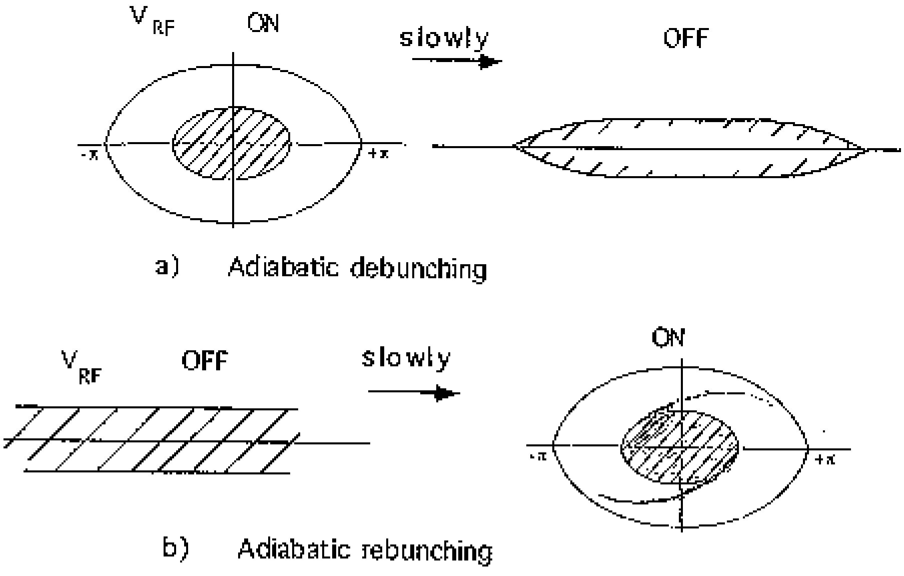


Figure 10: Adiabatic process

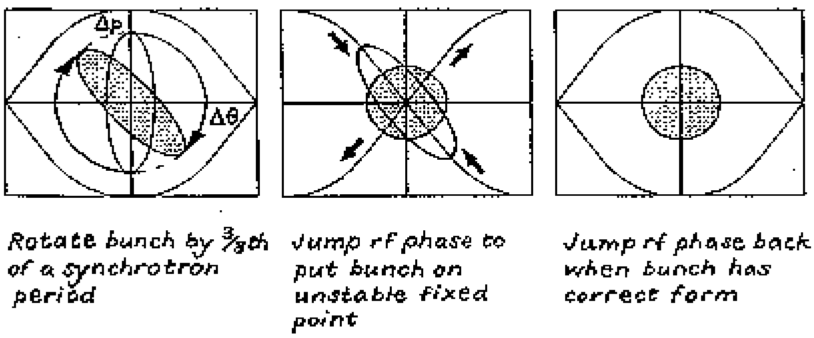


Figure 11: Non - adiabatic process [5]

2.4 - MOMENTUM COMPACTION FACTOR

2.4.1 Definition

The path length along a straight section is a function of the angle of the particle trajectory compared to the reference path. In linear dynamics, the second order corrections to the path length are neglected. Under these conditions, the contribution to the path length, in circular accelerator, comes from the dipole magnets. According to the beam energy, particles are curved differently and do not follow the same trajectory. A nominal closed orbit is defined for a nominal energy. For a given particle, a momentum deviation produces an orbit length variation. The momentum compaction is defined by the ratio :

$$\alpha_p = \frac{dC/C}{dp/p} = \frac{dR/R}{dp/p} \tag{48}$$

where $C = 2\pi R$ is the circumference of the closed orbit
α_p is a constant parameter. Its numerical value can be calculated from

$$\alpha_p = \frac{1}{L_0}\int_0^{L_0} \frac{D(s)}{\rho(s)}\, ds \tag{49}$$

where L_0 is the length of the reference closed orbit and D(s) the dispersion function in the bending magnets.

2.4.2 Expression of α_p versus energy E

From (34) $$E = \frac{pc}{\beta} \quad .$$

A logarithmic differentiation gives $\frac{dE}{E} = \beta^2 \frac{dp}{p}$ and according to the definition of α_p , one has :

$$\alpha_p = \beta^2 \frac{E}{R}\frac{dR}{dE} \quad . \tag{50}$$

2.4.3 Expression of α_p versus < B >

The average magnetic field is calculated as follow :

$$\langle B \rangle = \frac{1}{2\pi R} \oint_C B_f\, ds \tag{51}$$

where B_f is the magnetic field inside the dipoles.
By virtue of (16) :

$$\langle B \rangle = \frac{1}{2\pi R}\frac{p}{e} \oint \frac{ds}{\rho} \quad .$$

The integration along the closed orbit is taken only in the bending magnet, where ρ is not infinite. Therefore the value should be 2π.

Then

$$\langle B \rangle \;=\; \frac{p}{e}\left(\frac{1}{R}\right) \;=\; B_f\,\rho\left(\frac{1}{R}\right) \quad .$$

Finally

$$\langle B \rangle R \;=\; B_f\,\rho \;=\; \frac{p}{e} \quad . \tag{52}$$

A logarithmic differentiation of (52)

$$\frac{d\,\langle B \rangle}{\langle B \rangle} + \frac{d\,R}{R} \;=\; \frac{d\,p}{p} \quad .$$

Dividing by $\frac{\mathrm{d\,p}}{\mathrm{p}}$, one has :

$$\alpha_p \;=\; 1 - \frac{d\langle B \rangle / \langle B \rangle}{d\,p\,/\,p} \quad . \tag{53}$$

Another expression using the radius is :

$$\alpha_p \;=\; \frac{1}{1 + \dfrac{R}{\langle B \rangle}\dfrac{d\langle B \rangle}{d\,R}} \quad . \tag{54}$$

2.4.4 Expression of α_p for an isomagnetic guide field

$$d\;R \;=\; \frac{1}{2\,\pi\,\rho} \int_b x\,ds_0 \quad . \tag{55}$$

Integration is done in the bending magnets only. The variation of the path length can be expressed as

$$d\;R \;=\; \langle x \rangle_b \quad .$$

The position x is a function of $\frac{\mathrm{d\,p}}{\mathrm{p}}$ to the first order

$$x \;=\; D_x\,\frac{\Delta\,p}{p} \quad . \tag{56}$$

D_X is the dispersion function.

Then :

$$d\ R\ =\ \langle D_x \rangle \frac{\Delta p}{p} \quad . \tag{57}$$

$\langle D_x \rangle$ is numerically computed from lattice programs and the momentum compaction factor is given by :

$$\alpha_p \ = \ \frac{\langle D_x \rangle}{R} \quad . \tag{58}$$

Note :
The expression (56) shows that the beam dimensions will increase linearly with D_X.
However when $D_X = 0$, all particles will follow the same path whatever is their momentum spread.

2.4.5 Bunch compressor

Figure 12 shows the path length of particles inside 3 consecutive dipoles with momentum between p_1 and p_2.

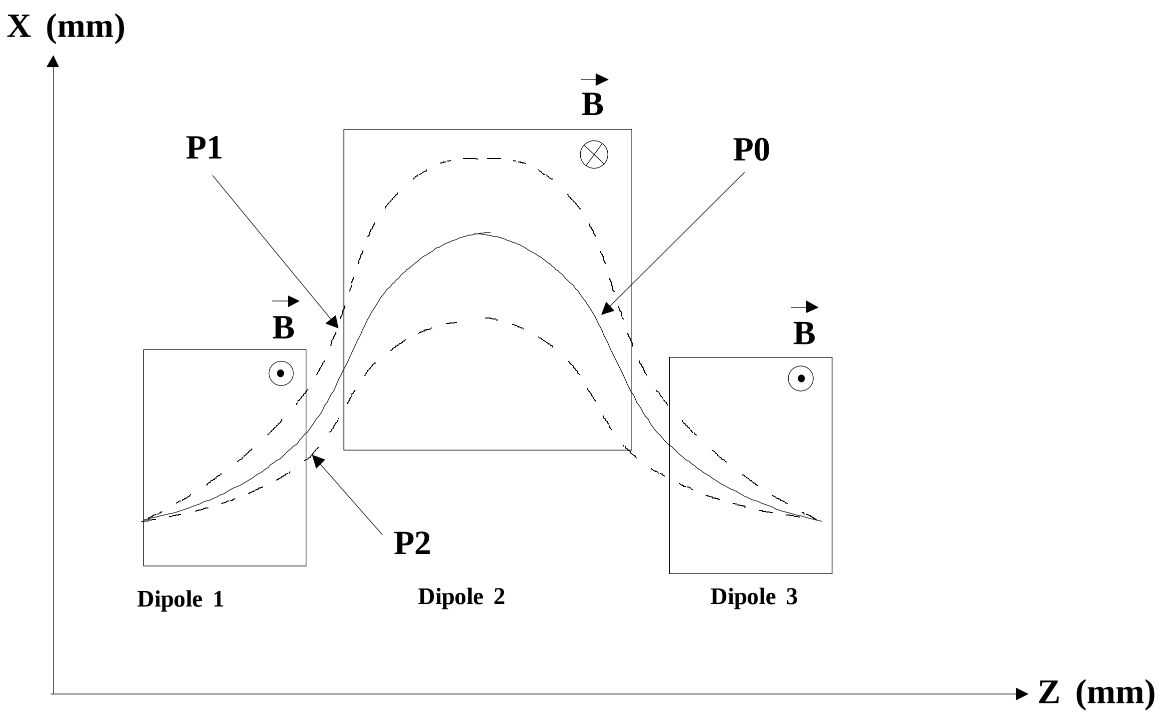


Figure 12: Classical bunch compressor

Particles of different energies inside a bunch travel on different path lengths.
Particles with lower momentum p_1 (in the head of the bunch) travel on a longer path.
Particles with higher momentum p_2 (in the tail of the bunch) travel on shorter path.

The path length difference is given by :

$$\Delta\ell = \left[4\rho \frac{\tan\alpha - \alpha}{\sin\alpha} + 2\lambda \tan^2\alpha \right] \frac{\Delta p}{p} \qquad (59)$$

α is the bending angle of each dipole ,
ρ the curvature radius of each dipole
λ the drift space between each dipole

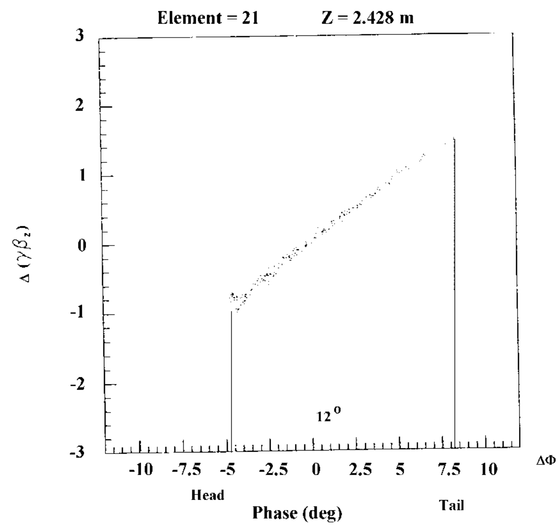
At the exit of the three dipoles, if B is correctly adjusted, head and tail will arrive at the same time (or the same phase).

Finally the dispersion function of the bending magnets allow compression of a bunch providing that the latter enters into the bunch compressor with the correct distribution of the energy spread $\frac{\Delta p}{p}$.

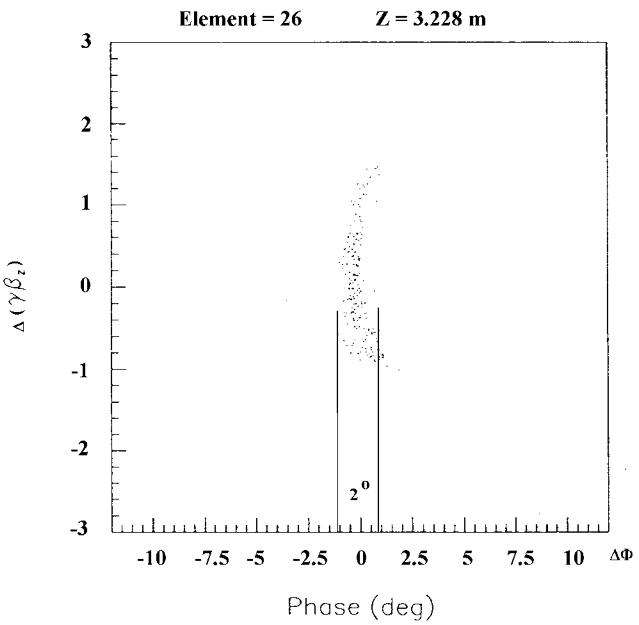
Figure 13a) shows the longitudinal phase space before the bunch compressor. The phase extension is 12° .

Figure 13b) shows the same thing after the bunch compressor. The phase extension is reduced to 2° .

These plots are the results of simulation using PARMELA code.



a) Before



b) After

Figure 13: Longitudinal phase space for a bunch compressor from PARMELA simulations

2.5 - TRANSITION ENERGY

2.5.1 Intuitive approach

The increase of energy has two contradictory effects :
i) an increase of the velocity
ii) an increase in the length of the trajectory

According to the variations of these two parameters, the revolution frequency evolves differently.
The angular frequency of a particle in a synchrotron can be written :

$$\omega = \frac{2\pi}{T} = \frac{2\pi v}{C} = 2\pi f \qquad (60)$$

where

T is the revolution period
C is the length of the circumference
v is the velocity of the particle
f is the revolution frequency

If the velocity increases faster than the length, the revolution frequency increases. In the opposite case, the revolution frequency decreases.
In general, for a given synchrotron, one makes the difference between low and high energy. At high energy, the velocity is close to the velocity of the light and practically does not change. At low energy, the increase of the velocity is more important than the variation of the trajectory.
Then, there is an intermediate energy for which the variation of the velocity is compensated by the variation of the trajectory. It is the transition energy. At this level, a variation of energy does not modify the frequency.

2.5.2 Quantitative approach

The dispersion of angular frequencies η is given by :

$$\eta = \frac{df/f}{dp/p} \quad . \qquad (61)$$

By differentiating (60), one obtains :

$$\frac{d\omega}{\omega} = -\frac{dT}{T} = \frac{d\beta}{\beta} - \frac{dC}{C} \quad .$$

The first term is calculated after differentiation of (34) which gives $\frac{dp}{p} = \gamma^2 \frac{d\beta}{\beta}$. The second term is given by the definition of the momentum compaction. Then $\frac{d\omega}{\omega} = \left(\frac{1}{\gamma^2} - \alpha_p\right)\frac{dp}{p}$. By virtue of (61), one has :

$$\eta = \frac{d\omega/\omega}{dp/p} = \frac{1}{\gamma^2} - \alpha_p \quad . \tag{62}$$

For a given machine, α_p is a fixed parameter. For a particle accelerator, γ is a variable parameter. Equation (62) shows that the dispersion of angular frequencies is equal to zero for

$$\alpha_p = \frac{1}{\gamma_t^2} \quad . \tag{63}$$

γ_t is the energy for which the variation of velocity is compensated by the variation of trajectory. It is the *transition energy*.

$$\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_t^2} \quad . \tag{64}$$

$\eta < 0$ when γ is large (high energies). It is called above the transition.
$\eta > 0$ when γ is small (low energies). It is called below the transition.

Note :

i) In many circular accelerators, the particles cross the transition energy when they are accelerated (Table 4).

Table 4: Transition energies for CERN accelerators

	PS	SPS	LHC
α_p	0.027	0.00186	0.000347
E_t (GeV)	5.7	21.7	50.4
Range (GeV) p^+	0.05 - 26	14 - 450	450 - 7000
Range (GeV) e^-	0.5 - 3.5	3.5 - 22	option

ii) In accumulator rings, the particles remain either above or below the transition energy.

iii) For linear accelerators, ρ —> $\propto$ then α_p —> 0 and η is always positive.

iv) According to the authors, one can find the following definition for η :

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} \quad .$$

Therefore the sign is opposite compared to our definition.

CHAPTER 3

3.1 - EQUATIONS RELATED TO A SYNCHROTRON

The following equations can be established for a synchrotron.

$$\frac{dp}{p} = \gamma_{tr}^{2} \frac{dR}{R} + \frac{dB}{B} \qquad (65)$$

$$\frac{dp}{p} = \gamma^{2} \frac{df}{f} + \gamma^{2} \frac{dR}{R} \qquad (66)$$

$$\frac{dB}{B} = \gamma_{tr}^{2} \frac{df}{f} + \left[1 - \left(\frac{\gamma_{tr}}{\gamma}\right)^{2}\right] \frac{dp}{p} \qquad (67)$$

$$\frac{dB}{B} = \gamma^{2} \frac{df}{f} + \left(\gamma^{2} - \gamma_{tr}^{2}\right) \frac{dR}{R} \qquad (68)$$

where

p is the momentum of the particle

R is the radius of the closed orbit

B is the magnetic field inside the dipoles

f is the revolution frequency

γ_t is the parameter for the transition energy.

3.1.1 Calculation of equation (65)

Let us integrate (48) :

$$\frac{d\,R}{R} = \alpha_{p} \frac{d\,p}{p}$$

$$ln\,R = \alpha_p \, ln\,p + cte \quad .$$

α_p is constant for a given synchrotron.

The solution is :

$$R = K\,p^{\alpha_p}$$

where K is a constant

The nominal closed orbit has a radius R_0 with the momentum p_0.

Then :

$$\frac{R}{R_0} = \left(\frac{p}{p_0}\right)^{\alpha_p} \quad . \tag{69}$$

For a given magnetic field, (16) should be verified for a particle (p_0, ρ_0) on the nominal closed orbit and for another particle (p, ρ).

Then $$\frac{p}{p_0} = \frac{\rho}{\rho_0} \quad .$$

Equation (16) can take the following expression for any particle :

$$p = e\,B\,\rho = e\,B\rho_0\left(\frac{p}{p_0}\right) \quad .$$

By virtue of (69) :

$$p = e\,B\,\rho_0\left(\frac{R}{R_0}\right)^{1/\alpha_p} \quad . \tag{70}$$

The derivation of (70) gives the result (65) taking into account (63) :

$$\frac{d\,p}{p} = \frac{d\,B}{B} + \gamma_{tr}^2\,\frac{d\,R}{R} \quad .$$

3.1.2 Calculation of equation (66)

A classical development between β and γ gives :

$$\frac{d\gamma}{\gamma} = \left(\gamma^2 - 1\right)\frac{d\beta}{\beta} \quad .$$

The logarithmic derivation of (35) provides :

$$\frac{d\,p}{p} = \frac{d\beta}{\beta} + \frac{d\gamma}{\gamma} \quad .$$

The expression of the momentum variation as a function of β becomes :

$$\frac{d\,p}{p} = \gamma^2\,\frac{d\beta}{\beta} \quad . \tag{71}$$

Equation (42) can be written :

$$2\pi f = \frac{\beta c}{R} \quad .$$

A logarithmic derivation gives :

$$\frac{d\beta}{\beta} = \frac{df}{f} + \frac{dR}{R} \quad . \tag{72}$$

Substituting (72) into (71), one has :

$$\frac{dp}{p} = \gamma^2 \frac{df}{f} + \gamma^2 \frac{dR}{R} \quad .$$

3.1.3 Calculation of equations (67) and (68)

The first one is obtained by removing $\frac{dR}{R}$ between (65) and (66).
The second one is obtained by removing $\frac{dp}{p}$ between (65) and (66).

3.1.4 Constant energy

In a synchrotron, if the RF voltage is adiabaticaly set to zero, one obtains a debunched beam after a certain time and dp = 0.
Using (65) and (66) one concludes : If *B* increases , *R* decreases and *f* increases**.**

3.1.5 Constant radius

With a radial loop control, the beam is maintained on the same orbit when the energy varies.

$$dR = 0 \quad .$$

Using (65) and (66), one has :

$$\frac{dp}{p} = \frac{dB}{B}$$

$$\frac{dp}{p} = \gamma^2 \frac{df}{f} \quad .$$

If *p* increases , *B* increases and *f* increases.

3.1.6 Magnetic flat-top

Let us consider a bunched beam and a constant magnetic field : $dB = 0$.

Putting this value in (65), (67) and (68) :

$$\frac{dp}{p} = \gamma_{tr}^2 \frac{dR}{R}$$

$$0 = \gamma^2 \frac{df}{f} + \left[1 - \left(\frac{\gamma_{tr}}{\gamma}\right)^2\right] \frac{dp}{p}$$

$$0 = \gamma^2 \frac{df}{f} + \left(\gamma^2 - \gamma_{tr}^2\right) \frac{dR}{R} \quad .$$

If p increases , R increases.

For the frequency f , it depends upon γ . If the beam energy is below the transition, then f increases. If the beam energy is above the transition , then f decreases. Equation (60) illustrates this effect according to the v value (close or not to the velocity of light).

3.1.7 Constant frequency

The beam is driven by an external oscillator which imposes the frequency.

$$df = 0 \; .$$

Putting this value in (66), (67) and (68) :

$$\frac{dp}{p} = \gamma^2 \frac{dR}{R}$$

$$\frac{dB}{B} = \left[1 - \left(\frac{\gamma_{tr}}{\gamma}\right)^2\right] \frac{dp}{p}$$

$$\frac{dB}{B} = \left(\gamma^2 - \gamma_{tr}^2\right) \frac{dR}{R} \quad .$$

p and R vary in the same direction. For the magnetic field B , it depends upon γ .
If the beam energy is below the transition, then B decreases.
If the beam energy is above the transition, then B increases.

Table 5 gives a summary of this discussion.

Table 5: Four conditions for a synchrotron

Beam	Parameter	Variations of p, B, R and f
Debunched	$\Delta p = 0$	B up R down f up
Fixed orbit	$\Delta R = 0$	B up p up f up
Magnetic flat-top	$\Delta B = 0$	p up R up f up ($\eta > 0$) f down ($\eta < 0$)
External oscillator	$\Delta f = 0$	B up p down R down ($\eta > 0$) p up R up ($\eta < 0$)

3.2 - SYNCHRONOUS PARTICLE

Figure 14 illustrates the possible effect of a sinusoidal voltage applied to two particles according to the respective phases.

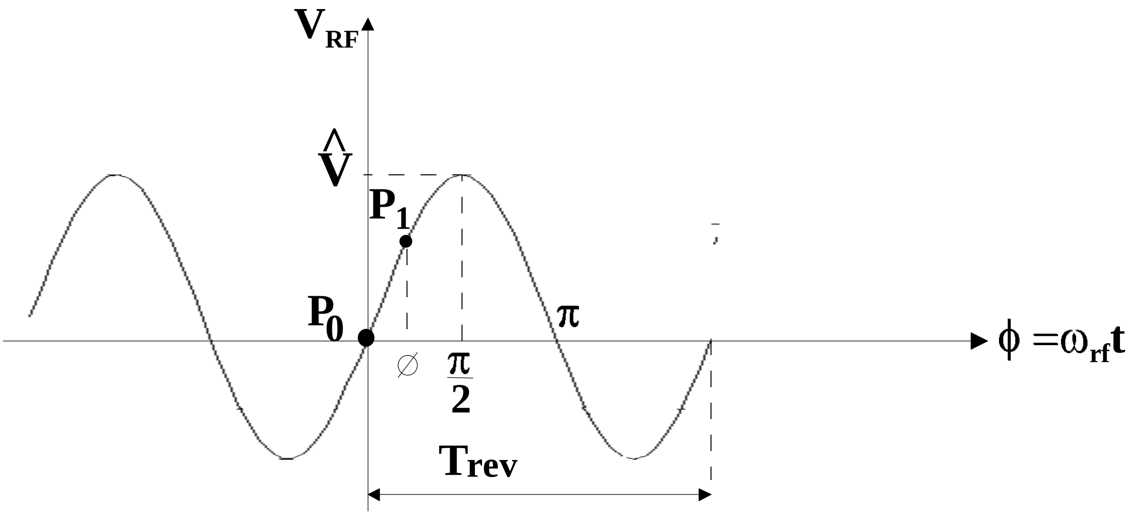


Figure 14: Sinusoidal voltage applied to two particles
(*$\phi = 0$* is at zero crossing for synchrotron)

Let us consider a particle P_0 turning in a synchrotron with an energy below the transition ($\eta > 0$). At each turn, it crosses a RF cavity with the voltage

$$V = \hat{V}_{RF} \sin\phi \quad .$$

If the revolution frequency of P_0 is equal to the RF frequency, and if P_0 arrives at the time t = 0 (phase zero), this particle is neither accelerated, nor decelerated.

We call P_0 *the synchronous particle.*

3.3 - SYNCHROTRON OSCILLATIONS

Now we consider another particle P_1 which enters the cavity with the phase ϕ_1, but with the same energy as P_0, i. e. the same velocity. (Fig. 14).
P_1 experiences a voltage

$$\hat{V}_{RF} \sin\phi_1$$

and therefore an accelerating field $\vec{E}$.
Keeping the assumption $\eta > 0$, let consider the two following cases :

a) $0 < \phi_1 < \pi$ P_1 is late compared to P_0 but undergoes an accelerating field and will then gain energy. The velocity increases and the revolution period decreases.
P_1 will be closer to P_0 the next turn.

b) $-\pi < \phi_1 < 0$ P_1 is in advance compared to P_0 but undergoes a decelerating field and will then lose energy. The velocity decreases and the revolution period increases.
P_1 will be closer to P_0 at the next turn.

When $0 < \phi_1 < \pi$, P_1 is coming closer to P_0 until they arrive simultaneously into the RF cavity. At this time the energy of P_1 is greater than the energy of P_0, i. e. a greater velocity. Then at the next turn, P_1 will have a negative phase ϕ_1. At each turn, the delay between P_1 and P_0 will increase. The negative phase ϕ_1 will continue to shift until the time where the velocity of P_1 equals the velocity of P_0.

Therefore P_1 is in the position described in the case b) above. P_1 will be closer to P_0 the next turn and so on.
An oscillatory phenomenon starts.
P_1 oscillates around the synchronous particle P_0 which has a constant energy and phase equals to zero.
The motions of P_1 are called *synchrotron oscillations*.
The frequency of such oscillations is well below the revolution frequency
$f_{rev} = 1/T_{rev}$

P_1 oscillates around P_0 both in energy and in position. The average energy of P_1 should be equal to the constant energy of P_0.

This oscillatory motion of P_1 is illustrated in Figure 15.

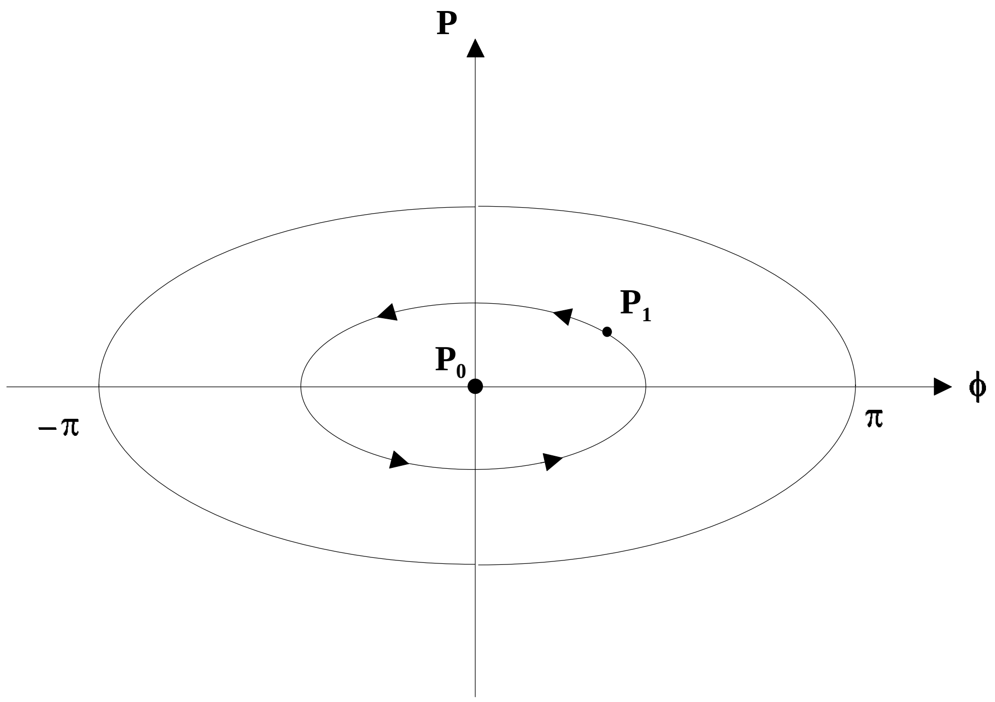


Figure 15: Synchrotron oscillations

The coordinates in Figure 15 are the momentum p and the phase $\phi = \omega t$.

However, as already explained in Figure 9, other units can be used in the longitudinal phase space.

3.4 - PRINCIPLE OF PHASE STABILITY

In the previous paragraph, we have considered a particle P_0 arriving in the RF cavity with a phase $\phi_0 = 0$.
Let us consider now a particle P_1 arriving with a phase different to zero (Fig. 16).
The energy gain is :

$$\Delta E = e\hat{V}_{RF} \sin\phi_1 \quad . \tag{73}$$

The velocity increases. Assuming $\eta > 0$ (below the transition), the path length increases, the revolution period decreases and the revolution frequency increases.

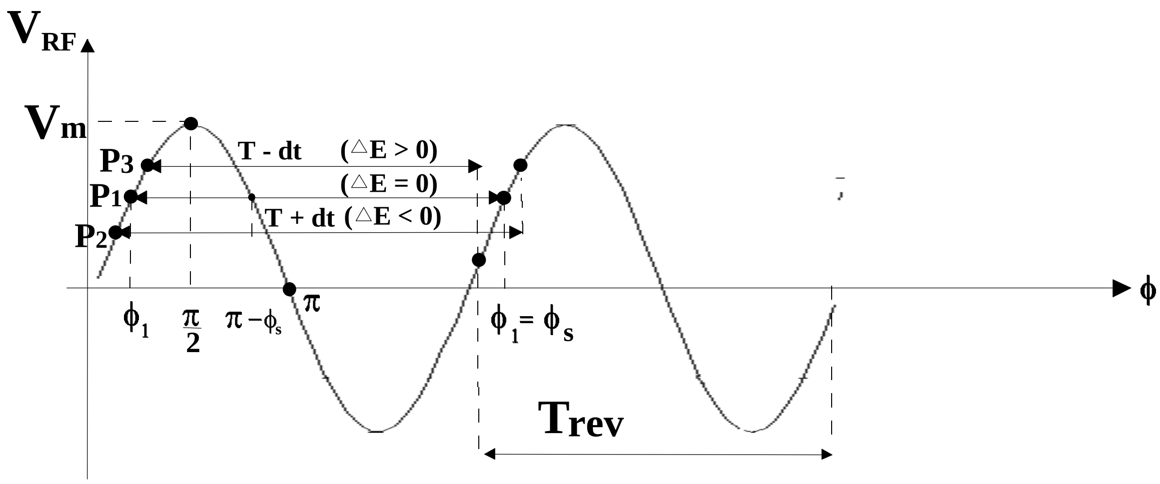


Figure 16: Synchronous particle with $\phi_S \neq 0$ and $\eta > 0$

Although P_1 gains energy, let us assume that P_1 arrives always with the same phase ϕ_1 in the RF cavity. P_1 will be the synchronous particle and its phase will be denoted as

$$\phi_1 = \phi_S \quad .$$

A particle P_2, with the same energy as P_1, but with $\phi_2 < \phi_1$, will receive a $\Delta E < 0$ (compared to P_1). The revolution period increases (T + dt). P_2 is coming closer to P_0 at each turn. When $\phi_2 = \phi_1$, then $E_2 < E_1$. The process continues until P_2 is at the position of P_3. As before, the same oscillating cycles are present, but around a synchronous particle which has a phase different to zero. Therefore, the symmetry around the vertical axis (Fig.15) disappears.
The phase variations of P_2 around ϕ_S do not produce the same energy gain if they take place at the crest or at the centre of the sinusoid (Fig.17).
In the Figure 16, a particle P_3, passing later than P_1, can gain energy only if its phase ϕ_3 respects the following condition :

$$\phi_S < \phi_3 < \pi - \phi_S \quad .$$

If $\phi_3 > \pi - \phi_S$, then the energy gain is not enough to move closer to P_1 and P_3 will move away from P_1.
$(\pi - \phi_S)$ is the beginning of an unstable phase (Fig. 17) .

For a particle P_2, in advance compared to P_1, the extremum of the stable phase will depend on the energy gain between $(\pi - \phi_S)$ and ϕ_S , which means it depends upon ϕ_S .
In consequence, there is a limit where the particles oscillate in phase and in energy around the synchronous particle. Above this limit, particles are lost.
In the phase space, this limit between the stable and unstable phase is called the *separatrix.*

This separatrix determines the RF bucket (Fig. 17)

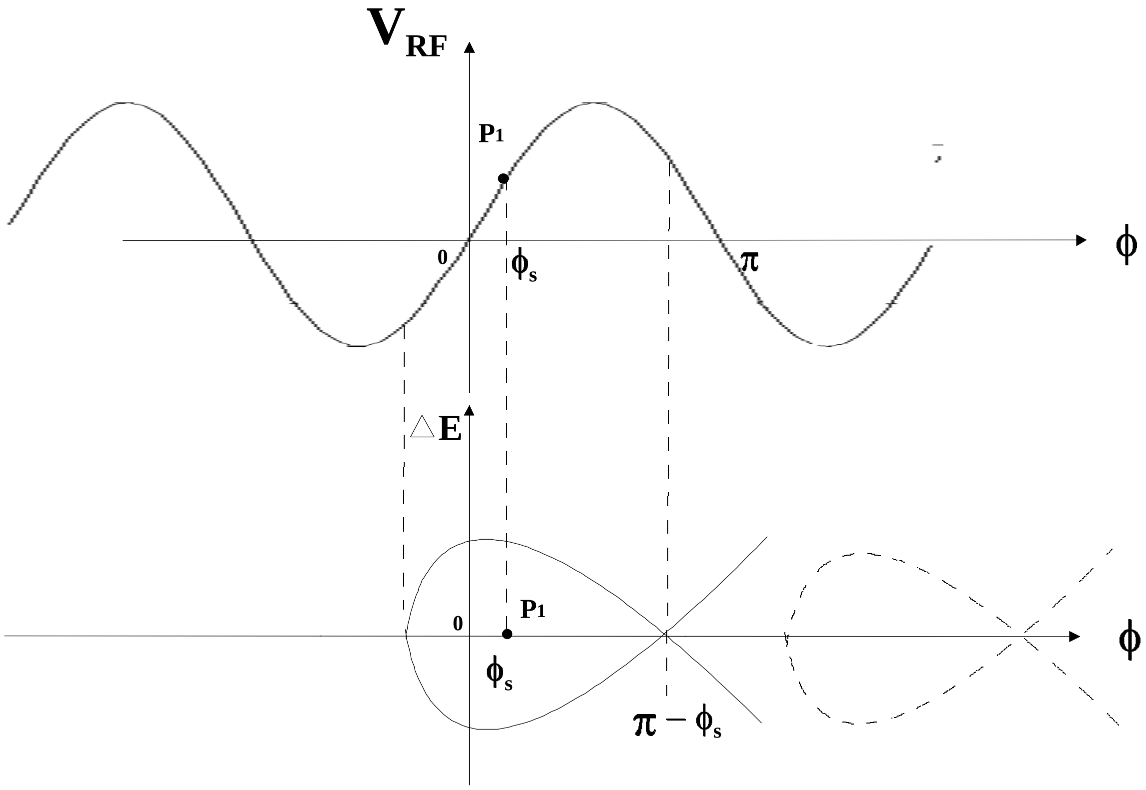


Figure 17: Amplitude variations and RF bucket

Note :
The same discussion is valid above the transition (η < 0). However ϕ_S will be on the negative slope of the RF voltage, instead of positive slope as in Figure 17.
Particles with higher momentum go slower, contrary to physical intuition (concept of negative mass effect).

CHAPTER 4

4.1 - RF ACCELERATION FOR SYNCHRONOUS PARTICLE

4.1.1 Energy gain per turn and phase of synchronous particle

The acceleration of charged particles comes from an oscillating electric field where the frequency is in synchronism with the revolution frequency.
In a RF cavity, we saw that the RF voltage is expressed as

$$V = \hat{V}_{RF} \ \sin\phi\,(t) \quad .$$

The synchronous particle arrives always with the same phase in the RF cavity and the equation of motion verifies the equation

$$\phi\,(t) = \phi_s = \text{constant} \quad . \tag{74}$$

In a synchrotron, the energy of the synchronous particle varies during the acceleration. Hence if one wants to keep this particle always on the same orbit (R = constant), it is necessary that the magnetic field varies with the time during the acceleration.
The differentiation of (52) provides

$$\frac{dp}{dt} = e\ R\,\frac{d\langle B\rangle}{dt} = eR\,\langle \dot{B}\rangle \quad . \tag{75}$$

The energy gain per turn could be obtained for the synchronous particle if the differentials are replaced by finite increments :

$$(\Delta p)_{turn} = eR\,\langle \dot{B}\rangle\,T_{rev} \quad . \tag{76}$$

T_{rev} is the revolution period

$$T_{rev} = \frac{1}{f_s} = \frac{2\pi R}{\beta c} \quad .$$

From the time derivative of (52), we can write

$$\langle \dot{B}\rangle R = \dot{B}_f\ \rho \quad .$$

From these last two equations (76) becomes :

$$(\Delta p)_{turn} = \frac{2\pi e R^2}{\beta c}\langle \dot{B}\rangle = \frac{2\pi e R}{\beta c}\rho \dot{B} \quad . \tag{77}$$

The index f of the magnetic field is removed for reasons of simplicity.
The derivative of (36) gives :

$$\Delta\left(E^2\right) = \Delta\left(p^2 c^2\right)$$

$$2\,E\,\Delta\,E = c^2\,2\,p\,\Delta\,p$$

$$\Delta\,E = \frac{p\,c^2}{E}\,\Delta\,p = \frac{m\,\beta\,c^3}{m\,c^2}\,\Delta p$$

$$\Delta\,E = \beta\,c\,\Delta\,p \quad . \tag{78}$$

Using the expression (77) of Δp, one gets :

$$(\Delta\,E)_{turn} = 2\,\pi\,e\,R\,\rho\,\dot{B} \quad . \tag{79}$$

If (73) is applied to the synchronous particle, then

$$(\Delta\,E)_{turn} = e\,\hat{V}_{RF}\,\sin\,\phi_s \quad . \tag{80}$$

The phase of the synchronous particle can be calculated from this expression.
If (80) is satisfied for ϕ_S, it will also be for $\pi - \phi_S$. However only one value corresponds to a stable dynamic equilibrium as we will see later on (equation 114).

4.1.2 RF frequency versus magnetic field

Now, the relationship between frequency and magnetic field is shown below.
For the synchronous particle, (17) gives :

$$\omega_s = \frac{e}{m}\langle B\rangle \quad .$$

By virtue of (52), one has

$$\omega_s = \frac{e}{m}\frac{\rho}{R}\,B \quad . \tag{81}$$

Using (46) into (81) :

$$\omega_s = \frac{\omega_{RF}}{h} = \frac{e}{m}\frac{\rho}{R} B \quad . \tag{82}$$

Using the classical relativistic equations (Chapter I), one can establish :

$$\omega_{RF} = \frac{hc}{R}\sqrt{\frac{B^2}{B^2 + (E_0 / ce\rho)^2}} \quad . \tag{83}$$

A magnetic field B_0 can be defined :

$$B_0 = \frac{E_0}{c\,e\,\rho} \quad . \tag{84}$$

For a given particle and a given accelerator B_0 is constant.
The expression (83) of the RF angular frequency becomes :

$$f_{RF} = \frac{h\,c}{2\,\pi\,R}\left(\frac{B}{B_0}\right)\frac{1}{\sqrt{1+\left(\frac{B}{B_0}\right)^2}} \quad . \tag{85}$$

4.1.3 Example of PS machine

In this synchrotron, there is one operation with protons where the magnetic field is increased by 2.4 T/s . The radius of the synchrotron R = 100 m . There are 100 dipoles along the circular accelerator with an effective length of 4.398 m each.

i) Radius of curvature calculation.
Since one should have a closed orbit, the total angle should be 2π . Then

$$\theta = \frac{100 \times 4.398}{\rho} = 2\,\pi$$

$$\rho = 70 \text{ m} \quad .$$

i) Energy gain per turn.
By virtue of (79) :

$$\begin{aligned}(\Delta E)_{turn} &= 2\pi \times 100 \times 70 \times 2.4 \\ &\cong 105\ keV \quad .\end{aligned}$$

In MKSA units, one gets :

$$(\Delta E)_{turn} \;=\; 105 \times 10^3 \times 1.6 \times 10^{-19} \;\cong\; 17 \;\; fJ \quad (femtoJoules) \quad .$$

iii) Minimum RF voltage.
It is calculated from (80).
The minimum is obtained if :

$$\sin \phi_S \;=\; 1, \;\; \text{i. e.} \;\; \phi_S \;=\; 90°$$

$$\left(\hat{V}_{RF}\right)_{\min} \;=\; 105\,kV \quad .$$

iv) RF frequency at ejection time.
When protons are ejected, the value of the magnetic field is B = 1.23 T.
From (84), B_0 = 0.0446 T.
Since B >> B_0 , the following approximation can be made :

$$f_{RF} \;\cong\; \frac{h\,c}{2\,\pi\,R} \quad .$$

With h = 20, the numerical calculation gives :

$$f_{RF} \;=\; 9.5 \text{ MHz} \quad .$$

4.2 - RF ACCELERATION FOR NON-SYNCHRONOUS PARTICLE

4.2.1 Definitions

The synchronous particle will be characterised with a *s* index.
The five variables, revolution frequency, RF phase, momentum, energy and azimuthal angle, of a generic particle, will be defined respectively as follows :

$$\begin{aligned} f &= f_S + \Delta f \\ \phi &= \phi_S + \Delta\phi \\ p &= p_S + \Delta p \\ E &= E_S + \Delta E \\ \theta &= \theta_S + \Delta\theta \quad . \end{aligned} \qquad (86)$$

The azimuthal position is given by :

$$ds \;=\; R\,d\theta \quad . \qquad (87)$$

A particle, passing later than the synchronous particle ($\Delta\theta < 0$), arrives in the RF cavity later ($\Delta\phi > 0$). If $f_S = f_{RF}$, then :

$$\Delta\phi \;=\; -\,\Delta\theta \quad . \tag{88}$$

If the harmonic number is greater than 1, by virtue of (46), one has :

$$\Delta\phi \;=\; -\,h\,\Delta\theta \quad . \tag{89}$$

Over one turn, θ varies by 2π, while ϕ varies by $2\pi h$.

4.2.2. Parameters versus $\dot{\phi}$

4.2.2.1 - Angular frequency

For a given particle, the azimuthal angle is given by

$$\theta(t) \;=\; \int_{t_0}^{t} \omega \;\; d\tau \quad . \tag{90}$$

If one calculates the variation $\Delta\theta$

$$\Delta\theta \;=\; \int_{t_0}^{t} (\omega - \omega_s)\,d\tau \;=\; \int_{t_0}^{t} \Delta\omega\, d\tau \quad .$$

Taking the time derivative

$$\frac{d}{dt}(\Delta\theta) \;=\; \Delta\omega \quad .$$

Using (89) :

$$\Delta\,\omega \;=\; -\frac{1}{h}\,\frac{d}{dt}(\Delta\,\phi) \;=\; -\frac{1}{h}\,\Delta\!\left(\frac{d\phi}{dt}\right)$$

$$\Delta\omega \;=\; -\frac{1}{h}\left[\frac{d\phi}{dt} - \frac{d\phi_s}{dt}\right] \quad .$$

According to (74) : $\dfrac{d\,\phi_s}{d\,t} \;=\; 0$

$$\Delta\,\omega \;=\; -\,\frac{1}{h}\;\frac{d\phi}{dt} \quad . \tag{91}$$

4.2.2.2 - Momentum

According to (61), one can write :

$$\eta \quad = \quad \frac{d\,\omega\,/\,\omega}{d\;p\,/\,p} \quad .$$

Replacing $\Delta\omega$ by its expression (91), one has :

$$\Delta\,p \quad = \quad -\frac{p_s}{\eta\,h\,\omega_s}\,\frac{d\,\phi}{dt} \quad . \tag{92}$$

4.2.2.3 - Energy

Taking the momentum derivative in (36), one can write

$$\frac{dE}{dp} \quad = \quad v$$

which is true for finite variations

$$\frac{\Delta E}{\Delta p} \quad = \quad v \quad = \quad \omega\;R \quad . \tag{93}$$

By virtue of (92) :

$$\Delta\,E \quad = \quad -\frac{R\,p_s}{\eta\,h}\,\frac{d\,\phi}{dt} \quad . \tag{94}$$

4.2.3 Equation of motion

The equation (80) gives the energy gain per turn for the synchronous particle.
From (93), the momentum gain per turn is given by

$$(\Delta\,p)_{turn} \quad = \quad \frac{e}{\omega_s\;R_s}\,\hat{V}_{RF}\;sin\,\phi_s \quad . \tag{95}$$

The average increase per time unit is given by dividing by the revolution period T .

$$\left(\frac{\Delta p}{T}\right)_{turn} \quad = \quad \dot{p}_s \quad = \quad \frac{e}{2\,\pi\;R_s}\,\hat{V}_{RF}\,sin\;\phi_s \quad . \tag{96}$$

One can write :

$$2\,\pi\;R_s\;\dot{p}_s \quad = \quad e\,\hat{V}_{RF}\;sin\,\phi_s \quad . \tag{97}$$

This equation is also verified by any particle.

If we consider (97) applied to a non-synchronous particle and subtract the same equation applied to the synchronous particle, one has :

$$2\pi\left(R\dot{p}-R_s\dot{p}_s\right) = e\hat{V}_{RF}\left(\sin\phi-\sin\phi_s\right) \quad . \qquad (98)$$

Developing the left-hand side :

$$R\dot{p}-R_s\dot{p}_s = R\frac{dp}{dt}-R_s\frac{dp_s}{dt} \quad .$$

R and p can be expressed with reference to the synchronous particle.

$$R\dot{p}-R_s\dot{p}_s = \left(R_s+\Delta R\right)\frac{d}{dt}\left(p_s+\Delta p\right)-R_s\frac{d}{dt}\left(p_s\right)$$

$$\cong R_s\frac{d}{dt}\left(\Delta p\right)+\Delta R\frac{d}{dt}\left(p_s\right)$$

The second order term is neglected.

For many reasons, the lattice is designed with dispersion-free regions around the RF cavities ($D_X = 0$)

It implies that particles pass on the same orbit, whatever their energies and they receive the same energy gain. Therefore $\Delta R = 0$.

One can write :

$$2\pi\left(R\dot{p}-R_s\dot{p}_s\right) = 2\pi R_s\frac{d}{dt}\left(\Delta p\right) \quad .$$

Hence (98) has the following expression :

$$\frac{d\left(\Delta p\right)}{dt} = \frac{e\hat{V}_{RF}}{2\pi R_S}\left(\sin\phi-\sin\phi_s\right) \quad . \qquad (99)$$

From (92) and (99), the motion of the non-synchronous particle is given by the system of differential equations :

$$\left\{\begin{array}{lll}\dfrac{d\left(\Delta p\right)}{dt} & = & \dfrac{e\hat{V}_{RF}}{2\pi R_S}\left(\sin\phi-\sin\phi_s\right) \qquad (100)\\[2ex] \dfrac{d\phi}{dt} & = & -\dfrac{\eta h\omega_s}{p_s}\Delta p \qquad (101)\end{array}\right.$$

the variables are the momentum Δp and the phase ϕ.

Note :
For ultra-relativistic particles, the energy gain is quasi-constant and it is not necessary to impose a dispersion-free region for RF cavities.
In synchrotrons, the betatron acceleration force due to the time variation of the magnetic field is, in general, self compensated.
A development can be found in [5].
A simplified expression is given by :

$$\begin{cases} \dfrac{d(\Delta p)}{dt} = A\left(\sin\phi - \sin\phi_s\right) & (102) \\ \dfrac{d\phi}{dt} = B\ \Delta p & (103) \end{cases}$$

where

$$A = \frac{e\,\hat{V}_{RF}}{2\,\pi\,R_s} \qquad (104)$$

$$B = -\frac{\eta\,h\,\omega_s}{p_s} = -\frac{\eta\,h\,\beta_s\,c}{p_s\,R_s} \quad . \qquad (105)$$

4.3 - SMALL AMPLITUDE OSCILLATIONS

4.3.1 First approximation

Equation (102) can be written :

$$\frac{d}{dt}\left[\frac{1}{B}\frac{d\phi}{dt}\right] - [A]\left(\sin\phi - \sin\phi_s\right) = 0 \quad . \qquad (106)$$

We assume that variations in time of quantities between square brackets are slow compared to the variations $\Delta\phi = \phi - \phi_S$.
Hence equation (106) becomes in the first approximation ($B \neq 0$) :

$$\frac{d^2\phi}{dt^2} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0 \qquad (107)$$

where

$$\Omega_s^2 = -A\,B\cos\phi_s = \frac{e\,\hat{V}_{RF}\ \eta\,h\,\beta_s\,c}{2\,\pi\,p_s\,R_s^2}\cos\phi_s \qquad (108)$$

or

$$\Omega_s^2 = \Omega_0^2\cos\phi_s \quad . \qquad (109)$$

We have $$\Omega_0^2 \quad = \quad - A\, B \quad = \quad \frac{e\, \hat{V}_{RF}\ \eta\, h\, \beta_s\ c}{2\, \pi\, p_s\ R_s^2} \quad . \tag{110}$$

This quantity can be expressed in energy instead of momentum

$$\frac{\beta_s\ c}{p_s\ R_s^2} \quad = \quad \frac{\beta_s\ c}{m_s\ v_s\ R_s^2} \quad = \quad \frac{v_s}{m_s\ v_s\ R_s^2} \quad = \quad \frac{1}{m_s\ R_s^2} \quad = \quad \frac{c^2}{E_s\ R_s^2} \quad .$$

Then

$$\Omega_0^2 \quad = \quad \frac{e\, \hat{V}_{RF}\ \eta\, h\, c^2}{2\, \pi\, E_s\ R_s^2} \quad . \tag{111}$$

In this first approximation, the equation of motion is given by the second order differential equation in phase ϕ.

$$\frac{d^2\phi}{dt^2} + \Omega_0^2\ \left(\sin\phi - \sin\phi_s\right) \quad = \quad 0 \quad . \tag{112}$$

4.3.2 Second approximation

We consider particles which remain close to the synchronous particle. Their variations in phase and in energy are small. We develop $\sin\phi$:

$$\sin\phi \ = \ \sin(\phi_S + \Delta\phi) \ = \ \sin\phi_S \cos\Delta\phi \ + \ \cos\phi_S \sin\Delta\phi$$

To the first order :

$$\sin\phi \quad \cong \quad \sin\phi_s + \cos\phi_s\ \Delta\phi$$

and

$$\left(\sin\phi - \sin\phi_s\right) \quad \cong \quad \Delta\phi\ \cos\phi_s \quad .$$

To the second order :

$$\frac{d^2\phi}{dt^2} \quad = \quad \frac{d^2\left(\Delta\phi\right)}{dt^2} \quad .$$

Under these assumptions, (107) becomes :

$$\frac{d^2\left(\Delta\phi\right)}{dt^2} + \Omega_s^2\ \Delta\phi \quad = \quad 0 \quad . \tag{113}$$

For small amplitude oscillations, we obtain the equation of a harmonic oscillator where Ω_S is the angular frequency.

4.3.3 Stability condition

According to the harmonic oscillator equation (113), the stability is obtained under the condition that Ω_s^2 is real positive, which implies that the right-hand side of (108) should be positive . The only parameters which could be negative are η and $\cos\phi_S$.
Discussion about the condition :

$$\eta \cos \phi_S > 0 \quad . \tag{114}$$

We saw in (80) that the synchronous acceleration admits two distinct phases ϕ_S and $\pi - \phi_S$. The condition (114) imposes the following conditions on the synchronous particle :

$$\begin{aligned}
&\gamma < \gamma_{tr}, \quad \eta > 0, \quad \cos\phi_s > 0, \quad 0 < \phi_s < \frac{\pi}{2} \\
&\gamma > \gamma_{tr}, \quad \eta < 0, \quad \cos\phi_s < 0, \quad \frac{\pi}{2} < \phi_s < \pi \quad .
\end{aligned}$$

Before the transition , the stable synchronous particle crosses the cavity during the rise time of the accelerator field . After the transition , this particle crosses the cavity during the falling time of the accelerator field .
We remark that for the two cases mentioned above , the variations range of ϕ_S always gives $\sin\phi_S > 0$.
This corresponds to an acceleration of the synchronous particle .

Now when $\sin\phi_S < 0$ it will correspond to a deceleration of the synchronous particle.

In order to fulfil the stability condition (114), one should have :

$$\begin{aligned}
&\gamma > \gamma_{tr}\ , \quad \eta < 0\ , \quad \cos\phi_s < 0\ , \quad \pi < \phi_s < \frac{3\pi}{2} \\
&\gamma < \gamma_{tr}\ , \quad \eta > 0\ , \quad \cos\phi_s > 0\ , \quad \frac{3\pi}{2} < \phi_s < 2\pi \quad .
\end{aligned}$$

All these cases are summarised on the Figure 18.

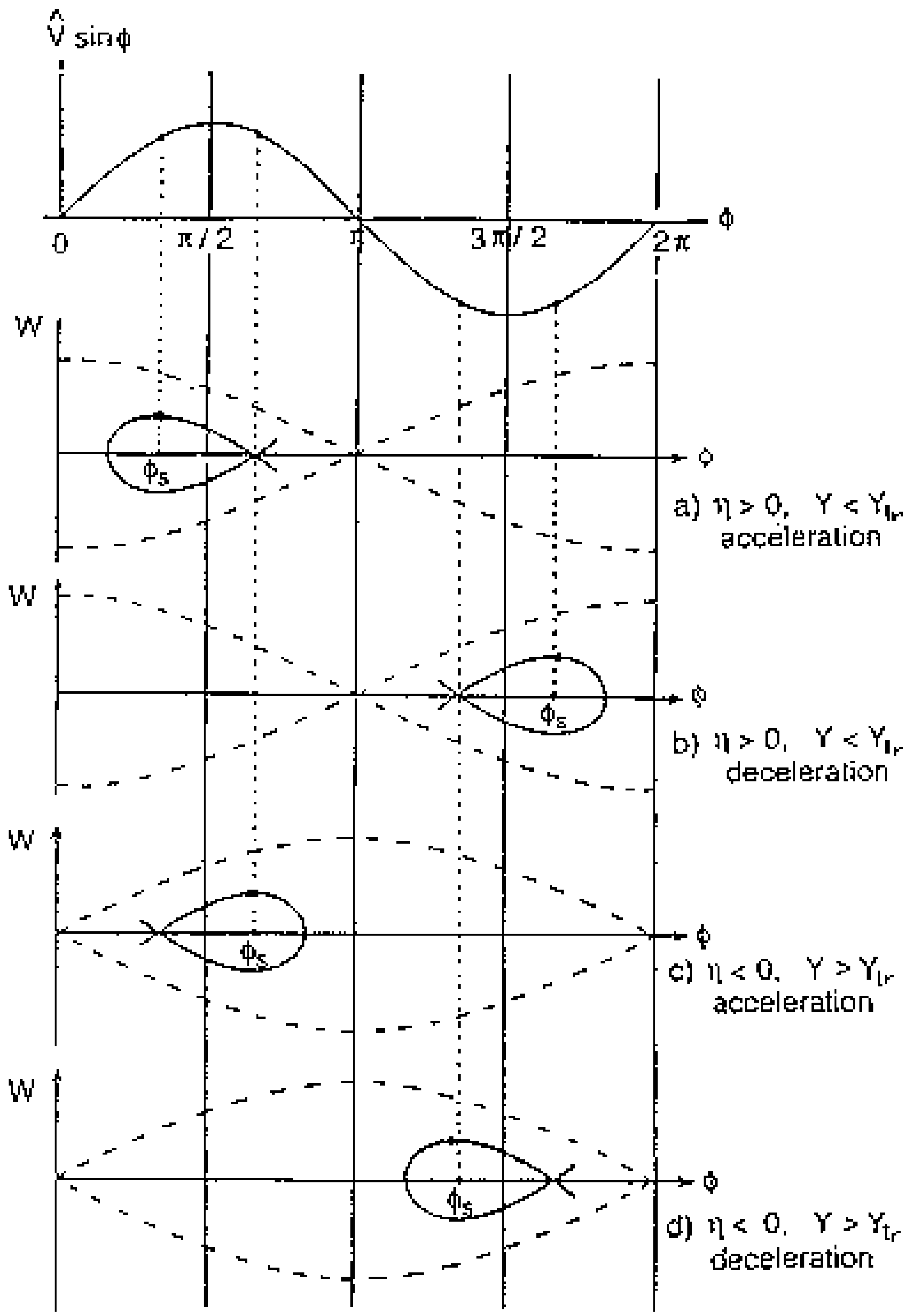


Figure 18: Phasing of moving and stationary buckets relative to RF voltage [4]

Notes :

i) At the transition energy $\eta \to 0$ *and* $\Omega_s^2 \to 0$, there is no longer phase stability , in the first approximation.

ii) In a synchrotron, when the particles have to cross the transition energy , the RF system must quickly shift its phase from ϕ_S to $\pi - \phi_S$, in order to maintain the dynamic stability through the transition.

iii) In the case of the lepton machines, (synchrotrons or accumulators), where the velocity of particles is almost equal to c , we have :

$$\eta \cong -\alpha_p \quad .$$

4.3.4 Longitudinal phase space

The solution of the harmonic oscillator equation expressed by the second order differential equation (113) is :

$$\Delta\phi = \Delta\hat{\phi}\, sin\left(\Omega_s t + \phi_0\right) \quad . \qquad (115)$$

By virtue of (103), one has

$$\frac{d\phi}{dt} = B\,\Delta p = \frac{d(\Delta\phi)}{dt} \quad .$$

The time derivative of (115) is

$$\frac{d\,(\Delta\phi)}{dt} = \Delta\hat{\phi}\,\Omega_s\, cos\left(\Omega_s t + \theta_0\right) = B\,\Delta p \quad .$$

The longitudinal phase space variables are given by :

$$\begin{cases} \Delta\phi = \Delta\hat{\phi}\, sin\left(\Omega_s t + \phi_0\right) \\ \Delta p = \Delta\hat{p}\, cos\left(\Omega_s t + \phi_0\right) \end{cases} \qquad (116)$$

with

$$\Delta\hat{p} = \frac{\Omega_s}{B}\Delta\hat{\phi} \quad .$$

The motion is a stable oscillation around the phase ϕ_S for small $\Delta\phi$ and assuming that condition (114) is fulfilled.

4.3.5 Synchrotron oscillations frequency for lepton machines

In these machines, either synchrotrons or accumulators, $v_S \cong c$ and the following simplifications can be applied :

$$\beta \cong 1, \quad \eta \cong -\alpha_p , \quad \omega_s \cong \frac{c}{R_s} , \quad p_s \cong \frac{E_s}{c} \quad .$$

Therefore (108) becomes :

$$\Omega_s^2 \quad = \quad \frac{-e \hat{V}_{RF} \, \alpha_p \, h \, c^2}{2 \, \pi \, E_s \, R_s^2} \; \cos \phi_s \quad . \qquad (117)$$

The synchrotron tune is defined as :

$$Q_s = \frac{\Omega_s}{\omega_s} = \frac{\Omega_s}{c} R_s \quad .$$

From (117) :

$$Q_s \quad = \quad \sqrt{\frac{-e \, \hat{V}_{RF} \; \alpha_p \; h}{2 \, \pi \, E_s} \; \cos \phi_s} \quad .$$

In lepton machines, R_S and ω_S do not change and the RF frequency is also constant.

The maximum phase extension $\Delta \hat{\phi}$ from (116), is :

$$\Delta \hat{\phi} = \frac{B}{\Omega_s} \Delta \hat{p} \quad .$$

It can be expressed as a function of the synchrotron tune using (105) :

$$\Delta \hat{\phi} = \frac{\alpha_p \; h}{Q_s} \frac{\Delta \hat{p}}{p_s} \quad .$$

CHAPTER 5

5.1 - OSCILLATIONS WITH HAMILTONIAN FORMALISM

The aim of this chapter is to illustrate the limits of the stable regions for the large amplitude oscillations and the adiabatic damping for the small amplitude oscillations using the Hamiltonian formalism.

5.1.1 Notion of invariant

In Physics, the notion of invariant is important and useful. One example is given by Liouville's theorem. It says that an elementary volume of the phase space is conserved as the isolated system undergoes transformations. Nevertheless its shape changes in general.
Another way to express this theorem is : an arbitrary area (p,q) in the phase space is conserved in a canonical transformation.
In the longitudinal phase space, with changing parameters, the stable trajectories do not exactly close over one cycle of synchrotron oscillation.
Therefore the area conservation is not obvious.
We introduce the concept of the Action integral.
The Boltzmann-Ehrenfest theorem specifies that a non dissipative oscillatory system with slowly changing parameters has its canonical variables which are evolving so that the action integral remains constant.
One can write :

$$I = \int_T p \, dq = constant$$

p and q are the canonical variables.
T is an oscillation period.

5.1.2 Hamiltonian equation

Since the invariance exists only for canonical variables, it is necessary to define them.
Let $H(\phi,W,t)$ be the Hamiltonian of the motion. Then the canonical variables should fulfil the following conditions :

$$\begin{cases} \dfrac{d\phi}{dt} = \dfrac{\partial H}{\partial W} \\ \\ \dfrac{dW}{dt} = -\dfrac{\partial H}{\partial \phi} \quad . \end{cases} \qquad (118)$$

We define the "energy" variable W which is the canonically conjugate variable of the "position" ϕ as :

$$W = \frac{\Delta E}{f} \tag{119}$$

where ΔE = energy gain and f = frequency.

The new variable has the dimension of an "action" : it is a product of energy with time. Several expressions can be given to W :

$$W = \frac{\Delta E}{f} = 2\pi \frac{\Delta E}{\omega} = 2\pi R \Delta p = C \Delta p \quad . \tag{120}$$

The differential equations system (102) (103) can be written :

$$\begin{cases} \dfrac{dW}{dt} = AC\left(\sin\phi - \sin\phi_S\right) \\ \\ \dfrac{d\phi}{dt} = \dfrac{B}{C} W \quad . \end{cases} \tag{121}$$

These equations can be derived from the Hamiltonian :

$$H(\phi, W, t) = A\,C\left[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right] + \frac{1}{2}\frac{B}{C} W^2 \quad .$$

One can verify that the conditions (118) are fulfiled for this Hamiltonian.

Taking into account (104) and (105), the Hamiltonian can be written :

$$H(\phi, W, t) = e\hat{V}_{RF}\left[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right] - \frac{h\eta\,\omega_s}{4\pi p_s R_s} W^2 \quad . \tag{122}$$

η, β_S, p_S, R_S, ϕ_S and $\hat{V}_{RF}$ can be a function of the time. However their variations are slow compared to the synchrotron oscillations.
In this case, the Hamiltonian will be time independent

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0 \quad .$$

5.1.3 Adiabatic condition

Using (47), one can write

$$\alpha \quad = \quad \frac{1}{\Omega_s^2} \, \frac{d\Omega_s}{dp} \, \frac{dp}{dt} \quad .$$

By virtue of (96) and (108), one has :

$$\frac{1}{2} \left[\frac{e\hat{V}_{RF} \quad \cos\phi_s}{2\pi \;\; h \;\; \eta \;\; p_s \;\; \beta_s \;\; c} \right]^{\frac{1}{2}} \quad tg\ \phi_s <<1 \quad .$$

In the synchroton, if $\hat{V}_{RF}$ and ϕ_S are constant during the acceleration, then this adiabatic condition is respected.

Note :
If the beam energy goes through the transition energy, then this condition is no longer respected ($\eta = 0$).

5.2 - LIMITS OF STABLE REGION

5.2.1 Qualitative approach

In Figure 15, the plots given for the P_1 trajectory and for the external contour are for a constant Hamiltonian. It is the same for the plot given in Figure 17 which crosses the phase $\pi - \phi_S$.
Figure 19 shows families of curves above the transition ($\eta < 0$). Particles moves along the trajectories (in time) in the direction of the arrows.

i) Let us consider Figure 19 a).
We have a stationary bucket :

$$\phi_S = \pi \quad .$$

The separatrix joints the adjacent unstable points. There is no communication between the 2 half planes. Inside the stationary buckets, one has a stable region. For a given Hamiltonian, we have 2 points where W = 0. Outside of these buckets, one has an unstable region with one point where W = 0, at $\phi = \pm k\pi$ (k integer).

ii) Let us consider Figure 19 b).
We have a moving bucket

$$\phi_S \neq \pi \qquad (\phi_S = 150^\circ) \ .$$

Inside the stable region, we have 2 points where W = 0.
Outside, we have the unstable region with one point where W = 0.
Particles can pass from one half plane (above W_0) to the other (below W_0).

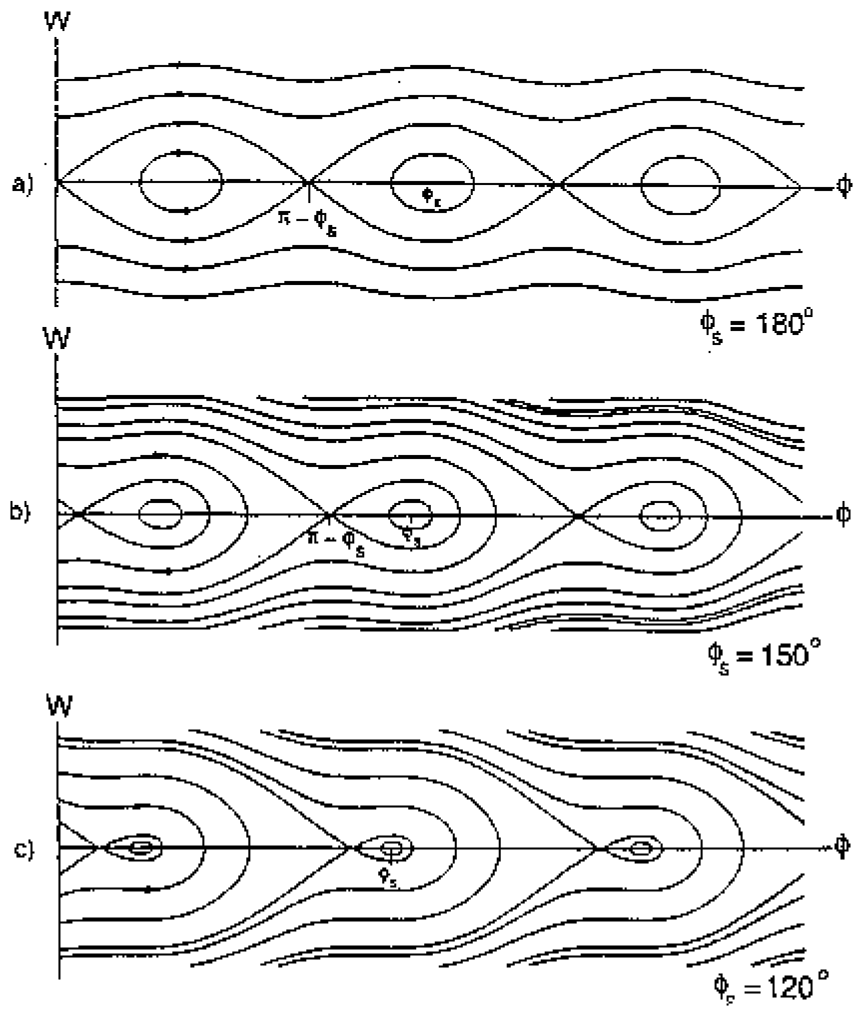


Figure 19: Trajectories for a constant Hamiltonian ($\gamma > \gamma_{tr}$) from [4]

5.2.2 Quantitative approach

Particles moving along the trajectories of the longitudinal phase plane fulfil the following conditions :

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{\partial H}{\partial W} = -\frac{h\,\eta\,\omega_s}{2\pi\, p_s\, R_s} W \tag{123}$$

$$\dot{W} = \frac{dW}{dt} = -\frac{\partial H}{\partial \phi} = e\,\hat{V}_{RF}\left[\sin\phi - \sin\phi_s\right] . \tag{124}$$

We discuss the different points over an interval 2π.

i) $\phi = \phi_S$ $W = 0$

It is the synchronous particle.

By virtue of (123) and (124), one has :

$$\dot{\phi} = \dot{W} = 0 \ .$$

We will have an extremum for W.

ii) In the stable region, we have 2 points where $W = 0$.

According to (123), it implies an extremum for ϕ .

iii) $\phi = \pi - \phi_S$ $W = 0$.

It is the unstable point belonging to the separatrix and corresponding also to an extremum of ϕ.

iv) $\phi_S = \pi$ or $\phi_S = 0$.

By virtue of (96), $\dot{p}_s = 0$. Therefore there is no acceleration.

It corresponds to the stationary bucket.

The limits of oscillations, in W, inside the stable regions are provided by $\dot{W} = 0$.

By virtue of (124) one has 2 solutions :

ϕ_S and $\pi - \phi_S$ (unstable point).

For $\phi = \phi_S$, (122) gives:

$$H\left(\hat{W},\phi\right) = -\frac{h\ \eta\omega_s}{4\pi\ p_s\ R_s}\hat{W}^2 \quad . \qquad (125)$$

Since the Hamiltonians are constant, one can equate (122) and (125) and the limits $\hat{W}$, in energy, are given by :

$$\hat{W}^2 = W^2 - \frac{e\hat{V}_{RF}\ \left(4\pi\ p_s R_s\right)}{h\eta\omega_s}\left[\cos\phi - \cos\phi_s + \left(\phi - \phi_s\right)\sin\phi_s\right] \quad . \qquad (126)$$

The limits of oscillations, in ϕ , inside the stable regions are provided by $\dot{\phi} = 0$.

Equation (123) implies W = 0 and from (122), one has :

$$H\left(W,\hat{\phi}\right) = e\hat{V}_{RF}\left[\cos\hat{\phi} - \cos\phi_s + \left(\hat{\phi} - \phi_s\right)\sin\phi_s\right] \qquad (127)$$

Equating the two Hamiltonians (122) and (127), one obtains a transcendental equation in $\hat{\phi}$:

$$\cos\hat{\phi} - \cos\phi_s + \left(\hat{\phi} - \phi_s\right)\sin\phi_s = \left(\cos\phi - \cos\phi_s\right) + \left(\phi - \phi_s\right)\sin\phi_s - \frac{h\eta\omega_s}{4\pi R_s\ p_s e\hat{V}_{RF}}W^2 \quad . \qquad (128)$$

5.2.3 Special cases

5.2.3.1 - Stationary bucket

Let us consider $\eta > 0$. In this case $\phi_S = 0$.

The maximum $\hat{\phi}$ is obtained when W = 0, and the maximum $\hat{W}$ is obtained when $\phi = \phi_S = 0$.

Using (126), one has

$$\hat{W}^2 = -\frac{e\hat{V}_{RF}\left(4\pi\, p_s R_s\right)}{h\eta\omega_s}\left[\cos\hat{\phi}-1\right]$$

$$= -\frac{e\hat{V}_{RF}\, 4\pi\, p_s R_s}{h\eta\omega_s}\left[2\sin^2\frac{\hat{\phi}}{2}\right]$$

$$\hat{W}^2 = K^2 \sin^2\frac{\hat{\phi}}{2} \quad . \qquad (129)$$

Taking the square root :

$$\hat{W} = \pm K \sin\frac{\hat{\phi}}{2} \quad .$$

Note :

1) Equation (129) has the form of the Hamiltonian for the classical pendulum.

2) The ratio $\frac{\hat{W}}{\hat{\phi}}$ is very important for the longitudinal matching of bunches into buckets.

5.2.3.2 - Small oscillations in the moving bucket.

Assume that $\Delta\phi = \phi - \phi_S << 1$.
Then the quantity between brackets of (122) is simplified :

$$[\ \ ] \cong -\frac{1}{2}\left(\Delta\phi\right)^2 \cos\phi_s \quad .$$

Putting this expression into (126), with W = 0 , one has :

$$\Delta\phi = \pm\left[\frac{h\,\eta\,\omega_s}{2\pi R_s\, p_s\, e\hat{V}_{RF}\cos\phi_s}\right]^{\frac{1}{2}} \hat{W} \quad .$$

5.2.4 Equation of separatrix

One of the limits of the stability region is given by :

W = 0 and $\phi_1 = \pi - \phi_s$.

Putting these values into (126), one can obtain an extremum of W which belongs to the separatrix :

$$\hat{W}_{sep}^2 = \frac{e\hat{V}_{RF}\left(4\pi\, p_s R_s\right)}{h\eta\omega_s}\left[2\cos\phi_s - \left(\pi - 2\phi_s\right)\sin\phi_s\right] \quad . \qquad (130)$$

This other limit in phase ϕ_{sep} is given by putting (130) into (126) with $\phi = \phi_{sep}$ and W = 0

$$\cos\phi_{sep} + \phi_{sep}\ \sin\phi_{sep} = \left(\pi - \phi_s\right)\sin\phi_s - \cos\phi_s \quad .$$

The equation of the separatrix is established by multiplying (107) by $\dot{\phi}$ (t).

$$\dot{\phi}\ \ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\dot{\phi}\ \sin\phi - \dot{\phi}\ \sin\phi_s\right) = 0 \quad .$$

A first integral is :

$$\frac{1}{2}\ \phi^2 - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\ \sin\phi_s\right) = \text{constant} \, .$$

By virtue of (123) and (105) :

$$\frac{B^2}{8\,\pi^2\ R_s^2}\, W^2 - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\ \sin\phi_s\right) = constant \, . \qquad (131)$$

The constant is determined by a point which belongs to the separatrix.
For example : W = 0 and $\phi_1 = \pi - \phi_s$.
The equation (131) becomes :

$$\frac{B^2}{8\,\pi^2\ R_s^2}\, W^2 - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \cos\phi_s + \left(\phi - \pi + \phi_s\right)\sin\phi_s\right) = 0 \quad . \qquad (132)$$

This is the equation of the separatrix in the longitudinal phase space.

5.3 - ADIABATIC DAMPING

Applying the action integral, one has

$$I = \oint W \, d\phi = \oint W \frac{d\phi}{dt} dt = constant . \qquad (133)$$

We consider small amplitude oscillations and we average the product $W \frac{d\phi}{dt}$ over one synchrotron oscillation period.

$$I \cong \left\langle W \frac{d\phi}{dt} \right\rangle T_S$$

$$= \left\langle W\dot{\phi} \right\rangle \frac{2\pi}{\Omega_S} = \text{constant.}$$

Using (108) and (123), one has

$$K_1 < W^2 > = constant$$

where

$$K_1 = \left[\frac{2\pi \, h\eta\omega_s}{R_s \, p_s \, e\hat{V}_{RF} \, cos\phi_s} \right]^{\frac{1}{2}} .$$

For sinusoidal variations, one can write :

$$\left\langle W^2 \right\rangle = \frac{1}{T_s} \int_0^{T_s} \hat{W}^2 \, sin^2 \, \Omega_s \, t \, dt \quad .$$

After development, one has :

$$\left\langle W^2 \right\rangle = \frac{1}{2} \hat{W}^2 \quad . \qquad (134)$$

Including all constant parameters of K_1 into the right-hand side constant, one has:

$$\hat{W} = \left[\frac{\hat{V}_{RF} p_s \, R_s \, cos\,\phi_s}{\eta\omega_s} \right]^{\frac{1}{4}} \times \, constant . \qquad (135)$$

The adiabatic variations of the phase are obtained in the similar way :

$$I = \oint \Delta\phi \cdot dW = \oint \Delta\phi \frac{dW}{dt}\, dt = \text{constant} \ .$$

$$I = \left\langle \Delta\phi \cdot \dot{W} \right\rangle \frac{2\pi}{\Omega_s} \ .$$

We saw that development for small amplitudes of the right-hand side of (124) provides :

$$\dot{W} \cong e \hat{V}_{RF}\, \Delta\phi \cos\phi_s \ .$$

Then : $\quad K_2 \left\langle \Delta\phi^2 \right\rangle = constant$

where $\quad K_2 = 2\pi \left[\dfrac{2\pi R_s\, p_s\, e\, \hat{V}_{RF} \cos\phi_s}{h\,\eta\,\omega} \right]^{\frac{1}{2}} \ .$

Including all constant parameters of K_2 into the right-hand side constant, and with the same equation as (134) applied to $\Delta\phi$, one has

$$\hat{\Delta\phi} = \left[\frac{\eta\,\omega_s}{R_s\, p_s\, \hat{V}_{RF} \cos\phi_s} \right]^{\frac{1}{4}} \times constant \, .$$

Conclusion : when p_s increases, $\hat{\Delta\phi}$ decreases as $p_s^{-1/4}$. It is called the *adiabatic damping* of the synchrotron oscillations.

However one can see that the product $\hat{W} \, . \, \hat{\Delta\phi}$ remains constant while p_s changes.

Therefore the Liouville's theorem holds.

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APPENDIX I

List of acronyms

AA	=	Antiproton Accumulator
AD	=	Antiproton Decelerator
BS	=	Booster Synchrotron
CAS	=	CERN Accelerator School
CERN	=	Conseil Européen pour la Recherche Nucléaire (1954) European Organization for Nuclear Research (2000)
CLIC	=	Compact Linear Collider (CERN study)
JUAS	=	Joint Universities Accelerator School
JLC	=	Joint Linear Collider (KEK study - Japan)
LEP	=	Large Electron Positron collider
LHC	=	Large Hadron Collider
LIL	=	LEP Injector Linac
NLC	=	Next Linear Collider (SLAC study - California)
PS	=	Proton Synchrotron
RF	=	Radio Frequency
SI	=	Système International
SLAC	=	Stanford Linear Accelerator Center
SLC	=	Stanford Linear Collider
SPS	=	Super Proton Synchrotron

APPENDIX II

Some physical constants

Symbols	Meaning	Value	Units
c	velocity of light	2.99792458×10^{8}	$m.s^{-1}$
e	electric charge	1.6021×10^{-19}	C
h	Planck's constant	6.6256×10^{-34}	J.s
m_e	rest mass of electron	9.1091×10^{-31}	kg
		0.511006×10^{-3}	GeV/c^2
m_p	rest mass of proton	1.6725×10^{-27}	kg
		0.93826	GeV/c^2
r_e	classical electron radius	2.8178×10^{-15}	m
μ_0	permeability of free space	$4 \pi \times 10^{-7}$	$H.m^{-1}$
ε_0	permittivity of free space	8.8542×10^{-12}	$F.m^{-1}$
k	Boltzmann's constant	1.38066×10^{-23}	$J.°K^{-1}$
σ	Stephan's constant	5.670×10^{-8}	$W.m^{-2}.°K^{-4}$

$$\varepsilon_0 \mu_0 c^2 = 1$$