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Living Inside a Hedgehog: Higher-dimensional Solutions that Localize Gravity

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Abstract

We consider spherically symmetric higher-dimensional solutions of Einstein's equations with a bulk cosmological constant and n transverse dimensions. In contrast to the case of one or two extra dimensions we find no solutions that localize gravity when $n \geq 3$, for strictly local topological defects. We discuss global topological defects that lead to compactification and estimate the corrections to Newton's law. We show that the introduction of a bulk "hedgehog" magnetic field leads to a regular geometry and localizes gravity on the 3-brane with either a positive, zero or negative bulk cosmological constant. The corrections to Newton's law on the 3-brane are parametrically the same as for the case of one transverse dimension.

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1 Introduction

A lot of attention has been devoted recently to alternatives [1]–[6] of Kaluza-Klein compactification [7]. In particular, our spacetime can be associated with some topological defect - 3 brane, embedded in a higher-dimensional spacetime with non-compact extra dimensions. It is usually assumed that the matter fields are localized on the brane because of the specific dynamics of solitons in string theory - D-branes [8]. Moreover, in Ref. [6] it was shown that the gravity of a domain wall in 5-dimensional anti-deSitter (AdS) spacetime has a 4-dimensional character for the particles living on the brane, provided that the domain wall tension is fine tuned to a bulk cosmological constant. The corrections to Newton's gravity law are generically small for macroscopic scales (see, however, [9] for a more complicated construction involving several branes). A similar statement is true for a local string living in 6-dimensional AdS space [10] (or more general constructions [11]).

The aim of the present paper is to generalize the results of Ref. [10] to the case when the number n of transverse dimensions is larger than two. This happens to be not as trivial as one expects. The reason is that the transverse spaces with $n \leq 2$, and $n \geq 3$ extra dimensions are qualitatively different, at least in the spherically symmetric setup we are interested in. In contrast to the case with $n \geq 3$, for $n = 1$ the extra space is flat while for $n = 2$ the extra space can be curved, but is still conformally flat.

We will consider three different possibilities. The first possibility is called a strictly local defect. By strictly local we mean the situation when the stress-energy tensor of the defect is zero outside the core (or, for the more realistic situation of a "fat" brane, exponentially falling outside the core). Here, we were not able to find any geometry leading to the compactification of gravity, contrary to the $n = 1$ and $n = 2$ cases.

The second possibility is related to the so called global defects. In this case one assumes that there exists a scalar field with, say $O(N)$ ($N \geq n$) global symmetry which is spontaneously broken. Outside the string core this field may have a hedgehog type configuration, which gives a specific contribution to the energy-momentum tensor outside the defect. This case was studied in [12, 13] for $n = 2$ and in [14] for higher dimensions, where the solutions with an exponential warp factor were found. We compute the corrections to Newton's law in this case and study the boundary conditions at the core of the global string. Furthermore, a generalization of these metric solutions is also defined.

The third possibility is related to configurations of the monopole type, where outside the defect there exists a magnetic field (or, its generalization to higher dimensions - p-form Abelian gauge field). We consider different spherically symmetric ansatz and define those that lead to gravity localization on the 3-brane and a regular geometry in the bulk. We also discuss the corrections to Newton's law for these solutions.

2 Einstein equations with a 3-brane source

In D-dimensions the Einstein equations with a bulk cosmological constant Λ_D and stress-energy tensor T_{AB} are

$$R_{AB} - \frac{1}{2} g_{AB} R = \frac{1}{M_D^{n+2}} \left(\Lambda_D g_{AB} + T_{AB} \right) \;, \tag{1}$$

where M_D is the reduced D-dimensional Planck scale. We will assume that there exists a solution that respects 4d Poincare invariance. A D-dimensional metric satisfying this ansatz for n transverse spherical coordinates with $0 \leq \rho < \infty$, $0 \leq \{\theta_{n-1}, \ldots, \theta_2\} < \pi$ and $0 \leq \theta_1 < 2\pi$, is

$$ds^2 = \sigma(\rho) g_{\mu\nu} dx^\mu dx^\nu - d\rho^2 - \gamma(\rho) d\Omega_{n-1}^2 \,, \tag{2}$$

where the metric signature of $g_{\mu\nu}$ is $(+,-,-,-)$ and $d\Omega_{n-1}^2$ is defined recursively as

$$d\Omega_{n-1}^2 = d\theta_{n-1}^2 + \sin^2\theta_{n-1} d\Omega_{n-2}^2 \;, \tag{3}$$

with $d\Omega_0^2 = 0$. At the origin $\rho = 0$ we will assume that there is a 3-brane, whose source is described by a stress-energy tensor T_B^A with nonzero components

$$T_\nu^\mu = \delta_\nu^\mu f_0(\rho), \quad T_\rho^\rho = f_\rho(\rho), \quad \text{and} \quad T_\theta^\theta = f_\theta(\rho) \;. \tag{4}$$

Here we have introduced three source functions f_0, f_ρ, and f_θ which depend only on the radial coordinate ρ and by spherical symmetry all the angular source functions are identical, where we have defined $\theta \equiv \theta_{n-1}$. Using the metric ansatz (2) and the stress-energy tensor (4), the Einstein equations become

$$\frac{3}{2}\frac{\sigma''}{\sigma} + \frac{3}{4}(n-1)\frac{\sigma'}{\sigma}\frac{\gamma'}{\gamma} + \frac{1}{8}(n-1)(n-4)\frac{\gamma'^2}{\gamma^2} + \frac{1}{2}(n-1)\frac{\gamma''}{\gamma} - \frac{1}{2\gamma}(n-1)(n-2)$$
$$= -\frac{1}{M_D^{n+2}}(\Lambda_D + f_0(\rho)) + \frac{\Lambda_{phys}}{M_P^2}\frac{1}{\sigma} \;, \tag{5}$$

$$\frac{3}{2}\frac{\sigma'^2}{\sigma^2} + (n-1)\frac{\sigma'}{\sigma}\frac{\gamma'}{\gamma} + \frac{1}{8}(n-1)(n-2)\frac{\gamma'^2}{\gamma^2} - \frac{1}{2\gamma}(n-1)(n-2)$$
$$= -\frac{1}{M_D^{n+2}}(\Lambda_D + f_\rho(\rho)) + \frac{2\Lambda_{phys}}{M_P^2}\frac{1}{\sigma} \;, \tag{6}$$

$$2\frac{\sigma''}{\sigma} + \frac{1}{2}\frac{\sigma'^2}{\sigma^2} + (n-2)\frac{\sigma'}{\sigma}\frac{\gamma'}{\gamma} + \frac{1}{8}(n-2)(n-5)\frac{\gamma'^2}{\gamma^2} + \frac{1}{2}(n-2)\frac{\gamma''}{\gamma} - \frac{1}{2\gamma}(n-2)(n-3)$$
$$= -\frac{1}{M_D^{n+2}}(\Lambda_D + f_\theta(\rho)) + \frac{2\Lambda_{phys}}{M_P^2}\frac{1}{\sigma} \;, \tag{7}$$

where the $'$ denotes differentiation $d/d\rho$ and the Einstein equations arising from all the angular components simply reduce to the one angular equation (7). The constant Λ_{phys} represents the physical 4-dimensional cosmological constant, where

$$R_{\mu\nu}^{(4)} - \frac{1}{2} g_{\mu\nu} R^{(4)} = \frac{\Lambda_{phys}}{M_P^2} g_{\mu\nu} \;. \tag{8}$$

The system of equations (5)–(7) describes the generalization of the setup considered in [2, 6, 10], to the case where there are n transverse dimensions, together with a nonzero cosmological constant in 4-dimensions. If we eliminate two of the equations in (5)–(7) then the source functions satisfy

$$f_\rho' = 2\frac{\sigma'}{\sigma}(f_0 - f_\rho) + \frac{n-1}{2}\frac{\gamma'}{\gamma}(f_\theta - f_\rho) \ , \tag{9}$$

which is simply a consequence of the conservation of the stress-energy tensor $D_M T^M_N = 0$. In general the Ricci scalar corresponding to the metric ansatz (2) is

$$R = 4\frac{\sigma''}{\sigma} + \frac{\sigma'^2}{\sigma^2} + 2(n-1)\frac{\sigma'}{\sigma}\frac{\gamma'}{\gamma} + (n-1)\frac{\gamma''}{\gamma} + \frac{1}{4}(n-1)(n-4)\frac{\gamma'^2}{\gamma^2} \\ -(n-1)(n-2)\frac{1}{\gamma} - \frac{4}{\sigma}\frac{\Lambda_{\rm phys}}{M_P^2} \ . \tag{10}$$

The boundary conditions at the origin of the transverse space are assumed to be

$$\sigma'\big|_{\rho=0} = 0 \ , \quad (\sqrt{\gamma})'\big|_{\rho=0} = 1 \quad \text{and} \quad \gamma\big|_{\rho=0} = 0 \ , \tag{11}$$

which is consistent with the usual regular solution in flat space. We have set $\sigma(0) = 1$, since the arbitrary integration constant corresponds to an overall rescaling of the coordinates x^μ. Following [15], we can integrate over the disk of small radius ϵ containing the 3-brane, and define various components of the brane tension per unit length as

$$\mu_i = \int_0^\epsilon d\rho\, \sigma^2 \gamma^{(n-1)/2} f_i(\rho) \ . \tag{12}$$

where $i = 0, \rho, \theta$. Using the system of equations (5)–(7) we obtain the following boundary conditions

$$\sigma\sigma'\sqrt{\gamma^{n-1}}\big|_0^\epsilon = \frac{2}{(n+2)}\frac{1}{M_D^{n+2}}\Big((n-2)\mu_0 - \mu_\rho - (n-1)\mu_\theta\Big) \ , \tag{13}$$

and

$$\sigma^2\sqrt{\gamma^{n-2}}(\sqrt{\gamma})'\big|_0^\epsilon = -\frac{1}{(n+2)}\frac{1}{M_D^{n+2}}\Big(4\mu_0 + \mu_\rho - 3\mu_\theta\Big) \ , \tag{14}$$

where it is understood that the limit $\epsilon \to 0$ is taken. The equations (13) and (14) are the general conditions relating the brane tension components to the metric solution of the Einstein equations (5)–(7), and lead to nontrivial relationships between the components of the brane tension per unit length. In particular, these conditions on the brane tension components reduce to the relations obtained for $n = 2$ [10]. Furthermore, by analogy with the solution for local strings we can identify (13) as the gravitational mass per unit length and (14) as the angular deficit per unit length. Thus the source for the 3-brane, in general curves the transverse space.

From the Einstein term in the D-dimensional Lagrangian we can obtain the effective four-dimensional Planck mass. Using the spherically symmetric metric ansatz (2), the four-dimensional reduced Planck mass is given by

$$M_P^2 = \mathcal{A}_n M_D^{n+2} \int_0^\infty d\rho \, \sigma \, \gamma^{(n-1)/2} \ . \tag{15}$$

where $\mathcal{A}_n$ is the surface area of an n-dimensional unit sphere. We are interested in obtaining solutions to the Einstein equations (5)–(7) such that a finite four-dimensional Planck mass is obtained. This leads to various possible asymptotic behaviours for the metric warp factors σ and γ in the limit $\rho \to \infty$. Below, we will concentrate only on the case when the 4-dimensional cosmological constant is zero, $\Lambda_{phys} = 0$.

3 Strictly local defect solutions

First, let us assume that the functions $f_i(\rho)$ are zero outside the core of the topological defect. In order to obtain a finite 4-dimensional Planck scale, one requires a solution of the system of equations (5)–(7) for which the function $\sigma\gamma^{(n-1)/2}$ goes to zero when $\rho \to \infty$. For $n = 1$ and $n = 2$ the solutions are known to exist, see [6] and [10] correspondingly. However, when $n \geq 3$ the structure of the equations is qualitatively different, because now there is a $1/\gamma$ term. Thus, there is no simple generalization of the solutions found for $n = 1, 2$.

To neutralize the effect of the $1/\gamma$ term, one can look for asymptotic solutions for which γ is a positive constant. However, one can easily check that the system of equations (5)–(7) does not allow a solution for which γ tends to a constant when $\rho \to \infty$, and σ is a negative exponential.

Alternatively, we can assume that there is an asymptotic solution for which $\gamma \to \infty$ but σ tends to zero faster than $\gamma^{(n-1)/2}$ and omit the troublesome $1/\gamma$ term from the equations of motion. In this case the set of equations (5)–(7) can be simply reduced to a single equation, as in 6D case [2, 10]:

$$z'' = -\frac{dU(z)}{dz} \ , \tag{16}$$

where the potential $U(z)$ is given by

$$U(z) = \frac{(n+3)}{4(n+2)} \frac{\Lambda_D}{M_D^{n+2}} z^2 \ . \tag{17}$$

With this parametrisation the metric functions $\sigma(\rho)$ and $\gamma(\rho)$ can be written in terms of $z(\rho)$ as

$$\sigma = |z'|^{(2-\sqrt{(n+2)(n-1)})/(n+3)} \, |z|^{(2+\sqrt{(n+2)(n-1)})/(n+3)} \tag{18}$$

$$\gamma = |z'|^{6/(1-n+2\sqrt{(n+2)(n-1)})} \, |z|^{6/(1-n-2\sqrt{(n+2)(n-1)})} \ . \tag{19}$$

Solving equation (16) with the potential (17) gives the general solution

$$z(\rho) = d_1\, e^{-\frac{1}{4}(n+3)c\rho} + d_2\, e^{\frac{1}{4}(n+3)c\rho} \ , \tag{20}$$

where d_1, d_2 are constants and we take $\Lambda_D < 0$. This solution is a generalization of the pure exponential solution considered earlier and in Ref. [14], see below. In this picture we can think of particle motion under the influence of the potential (17) with position $z(\rho)$ and "time" ρ.

Since $1 - n + 2\sqrt{(n+2)(n-1)} > 0$ and $1 - n - 2\sqrt{(n+2)(n-1)} < 0$, the metric factor γ can be large in two cases. In the first case $z(\rho)$ is zero for some ρ_0. However, this point is only a coordinate singularity (the Ricci scalar is regular at this point) and the metric can be extended beyond ρ_0, leading then to an exponentially rising solution for both σ and γ. This, unfortunately, is not interesting for compactification. In the second case both z' and z are non-zero and increase exponentially for large ρ. Thus, there is no possibility of a finite Planck scale in this case either.

Similarly, we were unable to find solutions in the reverse case when γ vanishes at infinity. Moreover, even if such solutions were to exist, they would likely lead to a singular geometry (naked singularity), because the Ricci scalar contains a $1/\gamma$ term, see eq. (10).

4 Bulk scalar field

4.1 Global topological defects

The other possibility is to consider defects with different types of "hair" i.e. with non-zero stress-energy tensor outside the core of the defect. We start with global topological defects. In fact, we have little to add to this question as it has been extensively studied in [13, 14, 16], so we just list a number of explicit solutions (see also [17]). Again, for simplicity we will restrict to the case where the four-dimensional cosmological constant $\Lambda_{phys} = 0$, and assume that

$$\sigma(\rho) = e^{-c\rho} \ . \tag{21}$$

Consider n scalar fields ϕ^a with a potential

$$V(\phi) = \lambda(\phi^a\phi^a - v^2)^2 \ , \tag{22}$$

where v has mass dimension $(n+2)/2$. Then the potential minimum is at $\phi^a\phi^a = v^2$. The defect solution has a "hedgehog" configuration outside the core

$$\phi^a(\rho) = v d^a \ , \tag{23}$$

where d^a is a unit vector in the extra dimensions, $d^n = \cos\theta_{n-1},\ d^{n-1} = \sin\theta_{n-1}\cos\theta_{n-2}, \ldots.$

The scalar field gives an additional contribution to the stress-energy tensor in the bulk with components

$$T_\mu^\nu = (n-1)\frac{v^2}{2\gamma}\delta_\mu^\nu \,, \tag{24}$$

$$T_\rho^\rho = (n-1)\frac{v^2}{2\gamma} \,, \tag{25}$$

$$T_\theta^\theta = (n-3)\frac{v^2}{2\gamma} \,. \tag{26}$$

Now, for $\Lambda_D < 0$ and $v^2 > (n-2)M_D^{n+2}$ the following solution leads to the localization of gravity and a regular geometry in the bulk [13, 14]

$$c = \sqrt{\frac{2(-\Lambda_D)}{(n+2)M_D^{n+2}}} \,, \tag{27}$$

$$\gamma = \frac{1}{c^2}\left(\frac{v^2}{M_D^{n+2}} - n + 2\right) \,, \tag{28}$$

provided that the brane tension components satisfy the conditions

$$-c\sqrt{\gamma^{n-1}} = \frac{2}{(n+2)}\frac{1}{M_D^{n+2}}\Big((n-2)\mu_0 - \mu_\rho - (n-1)\mu_\theta\Big) \,, \tag{29}$$

and

$$\delta_{n2} = \frac{1}{(n+2)}\frac{1}{M_D^{n+2}}\Big(4\mu_0 + \mu_\rho - 3\mu_\theta\Big) \,. \tag{30}$$

When $v^2 = (n-2)M_D^{n+2}$ the $1/\gamma$ terms are eliminated from the system of equations (5)–(7) and the exponential solution to the coupled set of equations (5)–(7) can then be found with

$$\gamma(\rho) = R_0^2\sigma(\rho), \quad c = \sqrt{\frac{8(-\Lambda_D)}{(n+2)(n+3)M_D^{n+2}}} \,, \tag{31}$$

where R_0 is an arbitrary length scale. As expected, the negative exponential solution (21) requires that $\Lambda_D < 0$. Notice that the exponential solution (31) only requires the "hedgehog" scalar field configuration in the bulk for transverse spaces with dimension $n \geq 3$. No such configuration is needed for the 5d [6] and 6d cases [10], which only require gravity in the bulk.

The Ricci scalar corresponding to the negative exponential solution with vanishing four-dimensional cosmological constant is

$$R = (n+3)(n+4)\frac{c^2}{4} - (n-1)(n-2)\frac{e^{c\rho}}{R_0^2} \,, \tag{32}$$

which diverges when $\rho = \infty$. Thus we see that for $n \geq 3$ the space is no longer a constant curvature space and in fact has a singularity at $\rho = \infty$. This is also confirmed by looking at the other curvature invariants, $R_{AB}R^{AB}$ and $R_{ABCD}R^{ABCD}$. Only for the 5d and 6d cases do we obtain a constant curvature anti-deSitter space. The appearance of a singularity is similiar to case of the global-string defect [12].

The metric solution (31) can also be written in the form

$$ds^2 = z^2 g_{\mu\nu} dx^\mu dx^\nu - z^2 R_0^2 d\Omega_{n-1}^2 - \frac{4}{c^2 z^2} dz^2 \ . \tag{33}$$

where $z = \exp(-\frac{c}{2}\rho)$. In these coordinates the origin $\rho = 0$ is now mapped to $z = 1$ and the 3-brane source is spread around the surface area of a n-dimensional sphere of radius R_0. This confirms our previous suggestion [10] that for $n \geq 2$ transverse dimensions the 3-brane can be identified with $\mathcal{M}_{n+2}/S^{n-1}$, where $\mathcal{M}_{n+2}$ is a $(n+2)$-brane.

Requiring that our exponential solution satisfy the boundary conditions (13) and (14), leads to the relation

$$\mu_\theta + \mu_\rho = \frac{1}{2}(n+2) R_0^{n-1} M_D^{n+2} c \ , \tag{34}$$

where μ_0 satisfies

$$\mu_0 = \mu_\theta + M_D^4 \, \delta_{n2} \ , \tag{35}$$

and μ_ρ remains undetermined. Thus for $n > 2$ we simply have $\mu_0 = \mu_\theta$.

4.2 Corrections to Newton's Law

For the solution (27),(28) the corrections to Newton's law are parametrically the same as for 5d case, since γ is a constant. On the other hand, the singular solution (31) will ultimately require that the singularity is smoothed by string theory corrections (perhaps similiar to the nonsingular deformations considered in [18]). Assuming that this is the case, then the corrections to Newton's law on the 3-brane can be calculated by generalizing the calculation presented in [6, 10] (see also [19]).

In order to see that gravity is only localized on the 3-brane, let us now consider the equations of motion for the linearized metric fluctuations. We will only concentrate on the spin-2 modes and neglect the scalar modes, which needs to be taken into account together with the bending of the brane [20]. The vector modes are massive as follows from a simple modification of the results in Ref. [21]. For a fluctuation of the form $h_{\mu\nu}(x,z) = \Phi(z) h_{\mu\nu}(x)$ where $z = (\rho, \theta)$ and $\partial^2 h_{\mu\nu}(x) = m_0^2 h_{\mu\nu}(x)$ we can separate the variables by defining $\Phi(z) = \sum_{l_i m} \phi_m(\rho) Y_{l_i}(\phi, \theta_i)$. The radial modes satisfy the equation [21]

$$-\frac{1}{\sigma \gamma^{(n-1)/2}} \partial_\rho \left[\sigma^2 \gamma^{(n-1)/2} \, \partial_\rho \phi_m \right] = m^2 \phi_m \ , \tag{36}$$

where $m_0^2 = m^2 + \Delta^2/R_0^2$ contains the contributions from the angular momentum modes l_i. The differential operator (36) is self-adjoint provided that we impose the boundary conditions

$$\phi'_m(0) = \phi'_m(\infty) = 0 \ , \tag{37}$$

where the modes ϕ_m satisfy the orthonormal condition

$$\mathcal{A}_n \int_0^\infty d\rho\, \sigma\, \gamma^{(n-1)/2}\, \phi_m^* \phi_n = \delta_{mn} \ . \tag{38}$$

Using the specific solution (21),(31), the differential operator (36) becomes

$$\phi''_m - \frac{(n+3)}{2}\, c\, \phi'_m + m^2 e^{c\rho} \phi_m = 0 \ . \tag{39}$$

This equation is the same as that obtained for the 5d domain wall solution [6], when $n = 1$ and the local stringlike solution [10] when $n = 2$. We see that each extra transverse coordinate augments this coefficient by 1/2. When $m = 0$ we clearly see that $\phi_0(\rho) =$ constant is a solution. Thus we have a zero-mode tensor fluctuation which is localized near the origin $\rho = 0$ and is normalizable.

The contribution from the nonzero modes will modify Newton's law on the 3-brane. In order to calculate this contribution we need to obtain the wavefunction for the nonzero modes at the origin. The nonzero mass eigenvalues can be obtained by imposing the boundary conditions (37) on the solutions of the differential equation (39). The solutions of (39) are

$$\phi_m(\rho) = e^{\frac{c}{4}(n+3)\rho} \left[C_1 J_{\frac{1}{2}(n+3)}(\frac{2m}{c} e^{\frac{c}{2}\rho}) + C_2 Y_{\frac{1}{2}(n+3)}(\frac{2m}{c} e^{\frac{c}{2}\rho}) \right] \ , \tag{40}$$

where C_1, C_2 are constants and $J_{\frac{1}{2}(n+3)}, Y_{\frac{1}{2}(n+3)}$ are the usual Bessel functions. Imposing the boundary conditions (37) at a finite radial distance cutoff $\rho = \rho_{\rm max}$ (instead of $\rho = \infty$) will lead to a discrete mass spectrum, where for $k = 1, 2, 3, \ldots$ we obtain

$$m_k \simeq c(k + \frac{n}{4})\frac{\pi}{2} e^{-\frac{c}{2}\rho_{\rm max}} \ . \tag{41}$$

With this discrete mass spectrum we find that in the limit of vanishing mass m_k,

$$\phi^2_{m_k}(0) = \frac{1}{\mathcal{A}_n R_0} \frac{\pi c}{8} \frac{(n+1)^2}{\Gamma^2[(n+3)/2]} \left(\frac{m_k}{c}\right)^n e^{-\frac{c}{2}\rho_{\rm max}} \ , \tag{42}$$

where $\Gamma[x]$ is the gamma-function. On the 3-brane the gravitational potential between two point masses m_1 and m_2, will receive a contribution from the discrete nonzero modes given by

$$\Delta V(r) \simeq G_N \frac{m_1 m_2}{r} \frac{\pi(n+1)}{4\Gamma^2[(n+3)/2]} \sum_k e^{-m_k r} \left(\frac{m_k}{c}\right)^n e^{-\frac{c}{2}\rho_{\rm max}} \ , \tag{43}$$

where G_N is Newton's constant. In the limit that $\rho_{\rm max} \to \infty$, the spectrum becomes continuous and the discrete sum is converted into an integral. The contribution to the gravitational potential then becomes

$$V(r) \simeq G_N \frac{m_1 m_2}{r}\left[1+\frac{1}{2c^{n+1}}\frac{n+1}{\Gamma^2[(n+3)/2]}\int_0^\infty dm\, m^n e^{-mr}\right] \tag{44}$$

$$= G_N \frac{m_1 m_2}{r}\left[1+\frac{\Gamma[n+2]}{2\Gamma^2[(n+3)/2]}\frac{1}{(cr)^{n+1}}\right] \tag{45}$$

Thus we see that for n transverse dimensions the correction to Newton's law from the bulk continuum states grows like $1/(cr)^{n+1}$. This correction becomes more suppressed as the number of transverse dimensions grows, because now the gravitational field of the bulk continuum modes spreads out in more dimensions and so their effect on the 3-brane is weaker.

5 Bulk p-form field

The global topological defects considered in the previous section inevitably contain massless scalar fields – Nambu-Goldstone bosons associated with the spontaneous breakdown of the global symmetry. Thus, the stability of these configurations is far from being obvious.

We will now consider the possibility of introducing other types of bulk fields (p-form field $A_{\mu_1\ldots\mu_p}$), which directly lead to a regular geometry. The stability of the corresponding configurations may be insured simply by the magnetic flux conservation. The D-dimensional action is

$$S=\int d^D x\sqrt{|g|}\left(\frac{1}{2}M_D^{n+2}R-\frac{\Lambda_D}{M_D^{n+2}}+(-1)^p\frac{1}{4}F_{\mu_1\ldots\mu_{p+1}}F^{\mu_1\ldots\mu_{p+1}}\right) . \tag{46}$$

The energy-momentum tensor associated with the p-form field configuration is given by

$$T^A_B=(-1)^{p+1}\left(\frac{1}{4}\delta^A_B F_{\mu_1\ldots\mu_{p+1}}F^{\mu_1\ldots\mu_{p+1}}-\frac{p+1}{2}F^A{}_{\mu_1\ldots\mu_p}F_B{}^{\mu_1\ldots\mu_p}\right) . \tag{47}$$

A solution to the equation of motion for the p-form field when $p=n-2$ is

$$F_{\theta_1\ldots\theta_{n-1}}=Q(\sin\theta_{n-1})^{(n-2)}\ldots\sin\theta_2 , \tag{48}$$

where Q is the charge of the field configuration and all other components of F are equal to zero. In fact, this "hedgehog" field configuration is the generalization of the magnetic field of a monopole. The stress-energy tensor associated with this p-form field in the bulk is

$$T^\mu_\nu = \frac{(n-1)!}{4}\frac{Q^2}{\gamma^{n-1}}\delta^\mu_\nu , \tag{49}$$

$$T^\rho_\rho = -T^\theta_\theta=\frac{(n-1)!}{4}\frac{Q^2}{\gamma^{n-1}} . \tag{50}$$

Let us assume a solution of the form

$$\sigma(\rho) = e^{-c\rho} \quad \text{and} \quad \gamma = \text{constant} . \tag{51}$$

With this ansatz we see from the Ricci scalar that the transverse space will have a constant curvature and the effective four-dimensional Planck constant will be finite. If we substitute this ansatz and also include the contribution of the p-form bulk field to the stress-energy tensor, the Einstein equations (5)–(7), with $\Lambda_{phys} = 0$ are reduced to the following two equations for the metric factors outside the 3-brane source

$$(n-1)!\frac{Q^2}{\gamma^{n-1}} - \frac{1}{2\gamma}(n-2)(n+2) + \frac{\Lambda_D}{M_D^{n+2}} = 0 , \tag{52}$$

$$c^2 = -\frac{1}{2}\frac{\Lambda_D}{M_D^{n+2}} + \frac{1}{4\gamma}(n-2)^2 . \tag{53}$$

We are interested in the solutions of these two equations which are exponential, $c^2 > 0$ and do not change the metric signature, $\gamma > 0$. Remarkably, solutions to these equations exist for which these conditions can be simultaneously satisfied. Let us first consider the case $n = 2$. Then the solutions reduce simply to

$$\frac{Q^2}{\gamma} = -\frac{\Lambda_6}{M_6^4} , \tag{54}$$

$$c^2 = -\frac{1}{2}\frac{\Lambda_6}{M_6^4} . \tag{55}$$

Thus, for $\Lambda_6 < 0$ we see that there is solution satisfying (51). This solution is different from the local string defect [10]. In this case the brane tension components satisfy

$$\mu_0 = \mu_\theta + (1 - \frac{Q}{2\sqrt{2}})M_6^4 , \tag{56}$$

where μ_ρ remains undetermined. This reduces to the condition satisfied by the local string solution when $Q = 0$.

Next we consider the case $n = 3$. Only the '+' solution to the quadratic equation (52) gives rise to a solution with both $c^2 > 0$ and $\gamma > 0$. This solution can be written in the form

$$\frac{Q^2}{\gamma} = \frac{1}{4}\left[\frac{5}{2} + \sqrt{(\frac{5}{2})^2 - 8Q^2\frac{\Lambda_7}{M_7^5}}\right] , \tag{57}$$

$$Q^2 c^2 = \frac{1}{16}\left[\frac{5}{2} - 8Q^2\frac{\Lambda_7}{M_7^5} + \sqrt{(\frac{5}{2})^2 - 8Q^2\frac{\Lambda_7}{M_7^5}}\right] . \tag{58}$$

When $Q^2\Lambda_7/M_7^5 < 25/32$ we obtain real solutions which are plotted in Figure 1. In fact requiring $c^2 > 0$ we see that there are solutions not only for $\Lambda_7 < 0$, but also for $\Lambda_7 \geq 0$, provided that $Q^2\Lambda_7/M_7^5 < 1/2$. Thus, the bulk cosmological constant does not need to be negative in order to localize gravity.

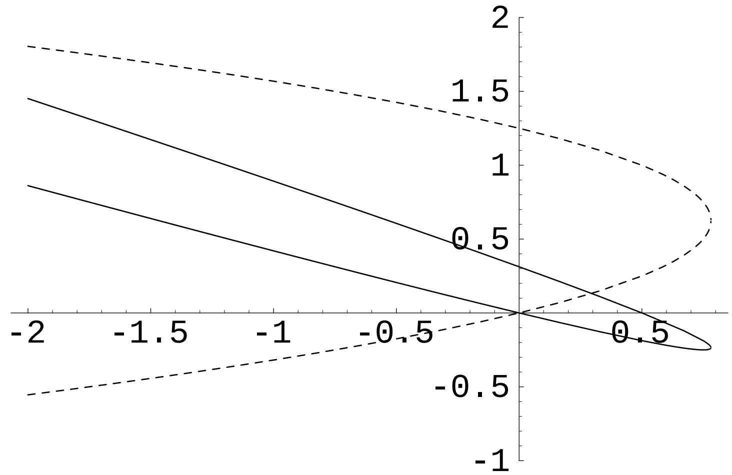

Figure 1: *The $n = 3$ solution for Q^2/γ (dashed line) and Q^2c^2 (solid line), as a function of $Q^2\Lambda_7/M_7^5$. Only the branches with $c^2 > 0$ and $\gamma > 0$ lead to solutions that localize gravity.*

Similarly, solutions for which $c^2 > 0$ and $\gamma > 0$ exist for values of $n > 3$. Again, we find that solutions with both positive, zero and negative bulk cosmological constants exist. In general for these type of solutions, the 3-brane tension components satisfy

$$\mu_0 = \mu_\theta - \frac{c}{2}\gamma^{(n-1)/2} M_D^{n+2} , \tag{59}$$

where μ_ρ remains undetermined.

The nice property of these solutions (51) is that since γ is a constant the Ricci scalar does not blow up at any point in the transverse space. In particular for the $n = 2$ solution the Ricci scalar is $R = -5/2\Lambda_6/M_6^4$, while for $n = 3$ it is

$$R = -\frac{1}{2Q^2}\left[\frac{35}{16} + Q^2\frac{\Lambda_7}{M_7^5} + \frac{7}{8}\sqrt{(\frac{5}{2})^2 - 8Q^2\frac{\Lambda_7}{M_7^5}}\right] . \tag{60}$$

The space is again a constant curvature space, but it is not necessarily anti-deSitter. This is also confirmed by checking the higher-order curvature invariants $R_{AB}R^{AB}$ and $R_{ABCD}R^{ABCD}$.

Finally, we see from (36) that the equation of motion for the spin-2 radial modes using the solution (51) is qualitatively similar to the 5d case. The constant γ factor effectively plays no role in the localization of gravity. Thus, the corrections to Newton's law will be suppressed by $1/r^2$ for all solutions $n \geq 3$.

Another possible solution for the p-form in the bulk includes

$$F_{\mu_1\ldots\mu_n} = \epsilon_{\mu_1\ldots\mu_n}\kappa(\rho) , \tag{61}$$

where

$$\kappa(\rho) = Q\frac{\gamma^{(n-1)/2}}{\sigma^2}(\sin\theta_{n-1})^{(n-2)}\ldots\sin\theta_2 . \tag{62}$$

In this case the contribution to the stress-energy tensor is

$$T^\mu_\nu = \frac{n!}{4}\frac{Q^2}{\sigma^4}\delta^\mu_\nu \ , \tag{63}$$

$$T^\rho_\rho = T^\theta_\theta = -\frac{n!}{4}\frac{Q^2}{\sigma^4} \ . \tag{64}$$

This contribution does not appear to make the solution of the Einstein equations any easier.

The above two p-form solutions have only included components in the transverse space. If we also require the p-form field to transform nontrivially under the 3-brane coordinates, then we can have

$$F_{0123\theta_1\ldots\theta_{n-1}} = Q(\sin\theta_{n-1})^{(n-2)}\ldots\sin\theta_2 \ , \tag{65}$$

where all other components of F are zero. The components of the stress-energy tensor for this field configuration are

$$T^\mu_\nu = \frac{(n+3)!}{4}\frac{Q^2}{\sigma^4\gamma^{n-1}}\delta^\mu_\nu \tag{66}$$

$$T^\rho_\rho = -T^\theta_\theta = -\frac{(n+3)!}{4}\frac{Q^2}{\sigma^4\gamma^{n-1}} \ . \tag{67}$$

Again, there is no simple solution of the Einstein equations with the inclusion of this contribution.

Finally, one can also generalize the solution (61)

$$F_{\mu\nu\alpha\beta a_1\ldots a_n} = \epsilon_{\mu\nu\alpha\beta a_1\ldots a_n}\kappa(\rho) \ , \tag{68}$$

where

$$\kappa(\rho) = Q\sigma^2\gamma^{(n-1)/2}(\sin\theta_{n-1})^{(n-2)}\ldots\sin\theta_2 \ , \tag{69}$$

and the stress energy tensor

$$T^M_N = \frac{(n+4)!}{4}Q^2\delta^M_N \ , \tag{70}$$

is a constant. Thus, we see that the addition of a $n+4$-form field is equivalent to adding a bulk cosmological constant.

6 Conclusion

We have seen that higher-dimensional solutions exist which can localize gravity to the 3-brane. The generalization of the exponential solution found in Ref. [6, 10], only exists when a scalar field with "hedgehog" configuration is added to the bulk. In this case the transverse space no longer has constant curvature, and furthermore it develops a singularity at $\rho = \infty$. The corrections to Newton's law on the 3-brane are suppressed by $1/(cr)^{n+1}$. If, however γ is a constant then regular solutions with an exponential warp factor do exist.

However, if instead a $n-1$-form field configuration is added in the bulk, which generalizes the magnetic field of a monopole, then solutions which localize gravity can be found for positive, zero and negative bulk cosmological constant. In this case, the transverse space has constant curvature but is not an anti-deSitter space. The corrections to Newton's law have the same form as in the original model [6]. Furthermore, the addition of an $n+4$-form field in the bulk is equivalent to adding a bulk cosmological constant.

Given that p-form fields are generic in string theories, it would be interesting to study whether the exponential solutions that we have found here can be realized in an effective supergravity theory. It is encouraging that the embedding of dilatonic "global cosmic strings" in string theory has recently been considered in [22, 18].

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