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Mikheyev-Smirnov-Wolfenstein Effect for Linear Electron Density

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Abstract

When the electron density is a linear function of distance, it is known that the MSW equations for two neutrino species can be solved in terms of known functions. It is shown here that more generally, for any number of neutrino species, these MSW equations can be solved exactly in terms of single integrals. While these integrals cannot be expressed in terms of known functions, some of their simple properties are obtained. Application to the solar neutrino problem is briefly discussed.

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1 Introduction

In studying the Mikheyev-Smirnov-Wolfenstein (MSW) effect [1] due to the coherent forward scattering of neutrinos by electrons in matter, it is often instructive to consider first special cases where the electron density is taken to be a simple function of distance. It is the purpose of the present paper to investigate perhaps the simplest case: the case where the electron density is a linear function of distance.

The problem of the linear electron density is formulated in Sec. 2. The case of two neutrino species has a long history [2], and the solution, as reviewed in Sec. 3, can be expressed in terms of parabolic cylinder functions—see, for example, Chapter VIII of [3], or equivalently confluent hypergeometric functions. However, the solution in this form is specific to the case of two neutrino species, and is not convenient for generalizations to more neutrino species. Physically, this generalization is essential because there are at least three types of neutrinos. Therefore, Sec. 4 is devoted to treating in a different way the MSW differential equations for linear electron density and two neutrino species. On the one hand, this alternative method must lead to the same solutions as those in Sec. 3; on the other hand, this new treatment can be readily generalized to any number of neutrino species. This case of linear electron density but any number of neutrino species forms the main content of the present paper, and various aspects of this case are treated in Sec. 5 and Sec. 6.

2 Formulation of the Problem

Let there be N types of neutrinos, denoted by ν_1, ν_2, ... ν_N, where ν_1 is the neutrino of the first generation, i.e., the one that forms the SU(2) doublet with the electron. It is assumed that ν_1 is the only neutrino which interacts differently with the electron because of the exchange of the intermediate boson W, while the others neutrinos ν_2, ν_3, ... ν_N all have the same interaction with the electron. Thus, the neutrino mass matrix M [4] is an $N \times N$ matrix. The eigenvalues of M give the N neutrino masses μ.

In analyzing the MSW effect, the neutrino masses are usually taken to be much smaller than the momentum p of the neutrino. Under this assumption, because

$$(p^2 + \mu^2)^{1/2} \sim p + \frac{1}{2p}\,\mu^2, \tag{2.1}$$

it is M^2 that enters in the differential equation for the MSW effect. Let $\Psi(x)$ be the N-component neutrino wave function, then this differential equation is

$$i\frac{d}{dx}\,\Psi(x) = \left[W(x) + \frac{1}{2p}\,M^2\right]\Psi(x) \tag{2.2}$$

where $W(x)$ is an $N \times N$ matrix whose only non-zero element is

$$[W(x)]_{11} = \sqrt{2}\,G_{\rm F} N_e(x), \tag{2.3}$$

with $G_{\rm F}$ the Fermi weak-interaction constant and $N_e(x)$ the electron density at the point x.

The terminology "linear electron density" is used to mean that $N_e(x)$ is a linear function of x.

Since $N_e(x)$ is the density of electrons, it cannot be negative. Therefore, the MSW differential equation (2.2) is physically meaningful only for the half-line of x where $N_e(x) \geq 0$. On the other hand, when the neutrino or the electron, but not both, is replaced by its antiparticle, the quantity $[W(x)]_{11}$ of Eq. (2.3) changes sign. Therefore, the complementary half-line of x describes this slightly different physical situation. For this reason, Eq. (2.2) is to be studied for the entire range of x, from $-\infty$ to $+\infty$.

For the present case of the linear electron density, Eq. (2.2) can be reduced, for a given value of p, to the dimensionless standard form

$$i\frac{d}{dt}\psi(t) = A(t)\psi(x), \tag{2.4}$$

where

$$\psi(t) = \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \\ \psi_3(t) \\ \vdots \\ \psi_N(t) \end{bmatrix} \tag{2.5}$$

and

$$A(t) = \begin{bmatrix} -t & a_2 & a_3 & \dots & a_N \\ a_2 & b_2 & 0 & \dots & 0 \\ a_3 & 0 & b_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_N & 0 & 0 & \dots & b_N \end{bmatrix}. \tag{2.6}$$

This is accomplished as follows.

(i) To change the independent variable from x to t, there is a shift in origin and a rescaling with possibly a reversal of sign.

(ii) To change the dependent variable from Ψ to ψ, there is a rotation in the second to Nth component and an introduction of exponential factors with possibly some minus signs.

Furthermore, from (i) and (ii), the elements of the matrix $A(t)$ can be chosen to satisfy the conditions

$$\sum_{j=2}^{N} b_j = 0, \tag{2.7}$$

$$b_2 \leq b_3 \leq b_4 \leq \ldots \leq b_{N-1} \leq b_N, \tag{2.8}$$

and

$$a_j \geq 0 \qquad \text{for} \quad j = 2, 3, 4, \ldots N. \tag{2.9}$$

Consider the following special cases.

(a) If, for some j, say j_0, $a_{j_0} = 0$, then it is seen from Eqs. (2.5) and (2.6) that ψ_{j_0} is decoupled from the other ψ_j's. Thus, this special case of N types of neutrinos is reduced to a problem with $N - 1$ types of neutrinos.

(b) If, again for some j, say j_0, $b_{j_0} = b_{j_0+1}$, then a rotation can be carried out between ψ_{j_0} and ψ_{j_0+1} such that, after this additional rotation, the new a_{j_0} is zero. Thus, this special case of $b_{j_0} = b_{j_0+1}$ can be reduced to the above case of $a_{j_0} = 0$, and hence again this second special case of N types of neutrinos is reduced to a problem with $N - 1$ types of neutrinos.

It is therefore sufficient to study the ordinary differential equation (2.4) with Eqs. (2.5) and (2.6) under the condition (2.7) together with

$$b_2 < b_3 < b_4 < \ldots < b_{N-1} < b_N \tag{2.10}$$

and

$$a_j > 0 \qquad \text{for} \quad j = 2, 3, 4, \ldots N. \tag{2.11}$$

In view of the inequality (2.10), it turns out to be convenient to define symbolically

$$b_1 = -\infty \qquad \text{and} \quad b_{N+1} = +\infty. \tag{2.12}$$

3 Case $N = 2$

Let us review first the well-known case of the MSW effect for two types of neutrinos [1, 2]. By Eqs. (2.2)–(2.7), the MSW equations are

$$i\frac{d}{dt}\begin{bmatrix}\psi_1(t)\\ \psi_2(t)\end{bmatrix} = \begin{bmatrix}-t & a_2\\ a_2 & 0\end{bmatrix}\begin{bmatrix}\psi_1(t)\\ \psi_2(t)\end{bmatrix}, \tag{3.1}$$

or more explicitly

$$i\frac{d}{dt}\psi_1(t) = -t\psi_1(t) + a_2\psi_2(t), \tag{3.2}$$

$$i\frac{d}{dt}\psi_2(t) = a_2\psi_1(t), \tag{3.3}$$

with $a_2 > 0$. A second-order ordinary differential equation for $\psi_1(t)$ is obtained by applying d/dt to Eq. (3.2) and using Eq. (3.3):

$$\frac{d^2\psi_1(t)}{dt^2} - it\,\frac{d\psi_1(t)}{dt} + (a_2^2 - i)\psi_1(t) = 0. \tag{3.4}$$

In order to remove the first-derivative term, let

$$\psi_1(t) = e^{it^2/4}\,\phi_1(t) \tag{3.5}$$

Then the equation for $\phi_1(t)$ is

$$\frac{d^2\phi_1(t)}{dt^2} + (\tfrac{1}{4}t^2 + a_2^2 - \tfrac{1}{2}i)\phi_1(t) = 0 \tag{3.6}$$

Two linearly independent solutions of this Eq. (3.6) are the parabolic cylinder functions [3]

$$D_\rho(\pm e^{i\pi/4}\, t), \tag{3.7}$$

where

$$\rho = -ia_2^2 - 1. \tag{3.8}$$

Parabolic cylinder functions are special cases of the confluent hypergeometric function [5], the relation being

$$D_\rho(z) = 2^{(\rho-1)/2}\, e^{-z^2/4}\, z\, \Psi(\tfrac{1}{2} - \tfrac{1}{2}\rho, \tfrac{3}{2}; \tfrac{1}{2}z^2). \tag{3.9}$$

Since the confluent hypergeometric functions Ψ and Φ satisfy the same second-order differential equation, the general solution of Eq. (3.6) is

$$\psi_1(t) = t\big[C\, \Phi(1 + \tfrac{1}{2}ia_2^2, \tfrac{3}{2}; \tfrac{1}{2}it^2) + C'\, \Psi(1 + \tfrac{1}{2}ia_2^2, \tfrac{3}{2}; \tfrac{1}{2}it^2)\big]. \tag{3.10}$$

This is one convenient form for the solution for $N = 2$.

4 Case $N = 2$—an Alternative Approach

In the existing treatment in the literature for linear electron density and two types of neutrinos [1, 2] as reviewed in Sec. 3, the crucial step is to recognize that the second-order differential equation (3.4) can be solved exactly in terms of known higher transcendental functions, either parabolic cylinder functions or confluent hypergeometric functions. More generally, for N types of neutrinos, the corresponding differential equation is of Nth order. Even for $N = 3$, the third-order differential equation is not one for any well-known transcendental function. Therefore, in order to be able to generalize the treatment of $N = 2$ to larger values of N, we must recast the solution of Sec. 3 so that parabolic cylinder functions and confluent hypergeometric functions do not play an essential role.

A useful question to ask is the following: In what way is the linear electron density especially simple? The answer must be sought in Eq. (2.6), from which it is seen that the independent variable t appears only in one matrix element, and furthermore, it appears only linearly in that element. This implies that, if Fourier transform is applied to the differential equation (2.4), the differentiation with respect to the Fourier-transform variable appears only once. Hence it is expected that an explicit expression can be obtained for the Fourier transform of ψ.

Let

$$F(\zeta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, e^{i\zeta t}\, \psi_1(t), \tag{4.1}$$

then it follows from Eq. (3.4) that $F(\zeta)$ satisfies the first-order differential equation

$$-\zeta^2 F(\zeta) - \frac{d}{d\zeta}[-i\zeta F(\zeta)] + (a_2^2 - i)F(\zeta) = 0,$$

where we have omitted all terms from $t = \pm\infty$. This differential equation simplifies immediately to

$$i\zeta \frac{dF(\zeta)}{d\zeta} - (\zeta^2 - a_2^2)F(\zeta) = 0,$$

or

$$\frac{1}{F(\zeta)} \frac{dF(\zeta)}{d\zeta} = \frac{i}{\zeta}(a_2^2 - \zeta^2). \tag{4.2}$$

Integration over ζ gives

$$F(\zeta) = \text{const.}\, e^{-i\zeta^2/2}\, \zeta^{ia_2^2}. \tag{4.3}$$

From the inequality (2.11), it is seen that the function on the right-hand side of Eq. (4.3) has a singularity at

$$\zeta = 0 = b_2. \tag{4.4}$$

Therefore the constant in (4.3) can take on different values for ζ positive and for ζ negative. In other words, the differential equation (4.2) is really two differential equations, one for $\zeta > 0$ and the other for $\zeta < 0$, consistent with the fact that the right-hand side of Eq. (4.2) has a singularity at $\zeta = 0$. With this observation, it is natural to define

$$F_1(\zeta) = \begin{cases} e^{-i\zeta^2/2}\, (-\zeta)^{ia_2^2} & \text{for } \zeta < 0, \\ 0 & \text{for } \zeta > 0, \end{cases} \tag{4.5a}$$

and

$$F_2(\zeta) = \begin{cases} 0 & \text{for } \zeta < 0, \\ e^{-i\zeta^2/2}\, \zeta^{ia_2^2} & \text{for } \zeta > 0. \end{cases} \tag{4.5b}$$

Inverting the Fourier transform (4.1), this choice leads to

$$\begin{aligned} \psi_1^{(1)}(t) &= \int_{-\infty}^{0} d\zeta\, e^{-i\zeta t}\, e^{-i\zeta^2/2}\, (-\zeta)^{ia_2^2}\,, \\ \psi_1^{(2)}(t) &= \int_{0}^{\infty} d\zeta\, e^{-i\zeta t}\, e^{-i\zeta^2/2}\, \zeta^{ia_2^2}\,. \end{aligned} \tag{4.6}$$

With the notation (2.12), these two formulas (4.6) can be written as

$$\psi_1^{(n)}(t) = \int_{b_n}^{b_{n+1}} d\zeta\, e^{-i\zeta t}\, e^{-i\zeta^2/2}\, |\zeta|^{ia_2^2}, \tag{4.7}$$

for $n = 1, 2$.

It remains to show that both $\psi_1^{(1)}(t)$ and $\psi_1^{(2)}(t)$ are confluent hypergeometric functions of the correct parameters and argument. For this purpose, it is convenient to define

$$\begin{aligned} \psi_c(t) &= \int_0^\infty d\zeta\, \cos(\zeta t) e^{-i\zeta^2/2}\, \zeta^{ia_2^2}, \\ \psi_s(t) &= \int_0^\infty d\zeta\, \sin(\zeta t) e^{-i\zeta^2/2}\, \zeta^{ia_2^2}, \end{aligned} \tag{4.8}$$

so that it follows from Eqs. (4.6) that

$$\begin{aligned}\psi_1^{(1)}(t) &= \psi_c(t) + i\psi_s(t), \\ \psi_1^{(2)}(t) &= \psi_c(t) - i\psi_s(t).\end{aligned} \tag{4.9}$$

It is found that

$$\psi_c(t) = e^{-i\pi/4}\, e^{\pi a_2^2/4}\, 2^{(-1+ia_2^2)/2}\, \Gamma(\tfrac{1}{2} + \tfrac{1}{2}ia_2^2)\, \Phi(\tfrac{1}{2} + \tfrac{1}{2}ia_2^2, \tfrac{1}{2}; \tfrac{1}{2}it^2) \tag{4.10}$$

and

$$\psi_s(t) = -i\, e^{\pi a_2^2/4}\, 2^{ia_2^2/2}\, \Gamma(1 + \tfrac{1}{2}ia_2^2)\, t\, \Phi(1 + \tfrac{1}{2}ia_2^2, \tfrac{3}{2}; \tfrac{1}{2}it^2). \tag{4.11}$$

There are various ways to verify Eqs. (4.10) and (4.11), including carrying out power series expansions in t for the left-hand and right-hand sides.

Finally, we note from Eq. (7) on p. 257 of reference [5] that

$$\Psi(a, c; x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\, \Phi(a, c; x) + \frac{\Gamma(c-1)}{\Gamma(a)}\, x^{1-c}\, \Phi(a-c+1, 2-c; x). \tag{4.12}$$

Therefore, the results of Sec. 3 and this section are the same.

5 General values of N

The procedure of Sec. 4 for $N = 2$ can be generalized in a straightforward way to larger values of N. Indeed, this is the major advantage over the previously known ones as reviewed in Sec. 3. This generalization to arbitrary values of N is to be carried out in this section. Thus, the differential equations (2.4) need to be solved under the constraints (2.7), (2.10), and (2.11).

By Eqs. (2.5) and (2.6), the Eqs. (2.4) are more explicitly

$$i\,\frac{d\psi_1(t)}{dt} = -t\,\psi_1(t) + \sum_{j=2}^{N} a_j\psi_j(t) \tag{5.1}$$

and, for $k = 2, 3, 4 \ldots N$,

$$\left(i\,\frac{d}{dt} - b_k\right)\psi_k(t) = a_k\psi_1(t). \tag{5.2}$$

In order to get a differential equation for $\psi_1(t)$, apply the operator

$$\prod_{k=2}^{N}\left(i\,\frac{d}{dt} - b_k\right)$$

to Eq. (5.1). By Eq. (5.2), this gives

$$\left[\prod_{k=2}^{N}\left(i\,\frac{d}{dt} - b_k\right)\right]\left(i\,\frac{d}{dt} + t\right)\psi_1(t) = \sum_{j=2}^{N} a_j^2 \prod_{\substack{k=2\\k\neq j}}^{N}\left(i\,\frac{d}{dt} - b_k\right)\psi_1(t). \tag{5.3}$$

Eq. (5.3) is an Nth-order ordinary differential equation for $\psi_1(t)$; it reduces to Eq. (3.4) when $N = 2$.

Following Sec. 4, define the Fourier transform $F(\zeta)$ of $\psi_1(t)$ by Eq. (4.1), then the first-order differential equation for $F(\zeta)$ is

$$\left[\prod_{k=2}^{N}(\zeta - b_k)\right]\left(\zeta - i\,\frac{d}{d\zeta}\right)F(\zeta) = \sum_{j=2}^{N} a_j^2 \prod_{\substack{k=2\\k\neq j}}^{N}(\zeta - b_k)F(\zeta), \tag{5.4}$$

or

$$\left(\zeta - i\,\frac{d}{d\zeta}\right)F(\zeta) = \sum_{j=2}^{N}\frac{a_j^2}{\zeta - b_j}\,F(\zeta), \tag{5.5}$$

or

$$\frac{1}{F(\zeta)}\,\frac{dF(\zeta)}{d\zeta} = i\left(-\zeta + \sum_{j=2}^{N}\frac{a_j^2}{\zeta - b_j}\right). \tag{5.6}$$

This Eq. (5.6) is the generalization of the previous Eq. (4.2) for $N = 2$. Integration over ζ gives the generalization of Eq. (4.3):

$$F(\zeta) = \text{const.}\, e^{-i\zeta^2/2}\prod_{j=2}^{N}(\zeta - b_j)^{ia_j^2}. \tag{5.7}$$

From (2.11), the function on the right-hand side of this Eq. (5.7) has singularities at

$$\zeta = b_j \tag{5.8}$$

for $j = 2, 3, \ldots N$. Therefore, define for $n = 1, 2, 3 \ldots N$

$$F_n(\zeta) = \begin{cases} e^{-i\zeta^2/2}\prod_{j=2}^{N}|\zeta - b_j|^{ia_j^2} & \text{for } b_n < \zeta < b_{n+1}, \\ 0 & \text{otherwise.}\end{cases} \tag{5.9}$$

Inverting the Fourier transform (4.1) then gives the desired N linearly independent solutions of the differential equation (5.3) as

$$\psi_1^{(n)}(t) = \int_{b_n}^{b_{n+1}} d\zeta\, e^{-i\zeta t}\, e^{-i\zeta^2/2}\prod_{j=2}^{N}|\zeta - b_j|^{ia_j^2} \tag{5.10}$$

for $n = 1, 2, 3, \ldots N$. In both Eq. (5.9) and Eq. (5.10), the notation of (2.12) has been used. The general solution of (5.3) is of course

$$\psi_1(t) = \sum_{n=1}^{N} C_n\,\psi_1^{(n)}(t) \tag{5.11}$$

with arbitrary constants C_n.

The other components of the $\psi(t)$ of (2.5) can be easily obtained also, and the result is

$$\psi(t) = \sum_{n=1}^{N} C_n \, \psi^{(n)}(t), \tag{5.12}$$

where

$$\psi^{(n)}(t) = \int_{b_n}^{b_{n+1}} d\zeta \, e^{-i\zeta t} \, e^{-i\zeta^2/2} \left(\prod_{j=2}^{N} |\zeta - b_j|^{ia_j^2} \right) \begin{bmatrix} 1 \\ \frac{a_2}{\zeta - b_2} \\ \frac{a_3}{\zeta - b_3} \\ \vdots \\ \frac{a_{N-1}}{\zeta - b_{N-1}} \\ \frac{a_N}{\zeta - b_N} \end{bmatrix}. \tag{5.13}$$

6 Limiting Behaviors for Large Distances

The next task is to obtain the limiting behaviors of the various components of the wave function when the distance x is large, either positive or negative. In other words, the problem is to find the limiting behaviors of the $\psi_j^{(n)}(t)$, as given explicitly by Eq. (5.13), both for $t \to -\infty$ and $t \to \infty$, with all the a's and b's fixed. It is important to remember that these two limits correspond to different physical problems, as discussed after Eq. (2.3).

The consideration here will be limited to the part of the asymptotic behavior that does not vanish as $t \to \pm\infty$. This is the physically interesting part. There are two possible types of contributions, from points of stationary phase and from end points of integration.

6.1 Points of Stationary Phase

From Eq. (5.13), the points of stationary phase are determined by

$$\frac{\partial}{\partial \zeta}(-\zeta t - \tfrac{1}{2}\zeta^2) = 0 \tag{6.1}$$

or

$$\zeta = -t. \tag{6.2}$$

In Eq. (6.1), the additional phase due to the factor

$$\prod_{j=2}^{N} |\zeta - b_j|^{ia_j^2}$$

is not included because the a_j and b_j are all fixed while $t \to \pm\infty$. Eq. (6.2) implies that this point of stationary point is relevant only to:

- $\psi_1^{(1)}(t)$ as $t \to \infty$, and
- $\psi_1^{(N)}(t)$ as $t \to -\infty$

in view of Eq. (5.13). For example, when $j > 1$, $\psi_j^{(1)}(t)$ as $t \to \infty$ and $\psi_j^{(N)}(t)$ as $t \to -\infty$ both behave as $1/t$ in absolute value so far as the contribution from this point of stationary phase (6.2) is concerned.

6.2 End Points of Integration

It is seen from Eq. (5.13) that, when $k \geq 2$, there is an extra factor of

$$\frac{a_k}{\zeta - b_k} \tag{6.3}$$

associated with $\psi_k^{(n)}(t)$. But the range of integration for this $\psi_k^{(n)}(t)$ as given by Eq. (5.13) is from b_n to b_{n+1}. Therefore, the contribution from these end points of integration can lead to a non-zero answer only when the index k appearing in the expression (6.3) agrees with either n or $n+1$. In other words, there are non-zero contributions as $t \to \pm\infty$ only to $\psi_k^{(k-1)}(t)$ [i.e., $n = k-1$] and $\psi_k^{(k)}(t)$ [i.e., $n = k$]. These particular components are given by

$$\begin{aligned}
\psi_k^{(k-1)}(t) &= \int_{b_{k-1}}^{b_k} d\zeta\, e^{-i\zeta t}\, e^{-i\zeta^2/2} \left[\prod_{j=2}^{k-1} (\zeta - b_j)^{ia_j^2}\right] \left[\prod_{j=k}^{N} (b_j - \zeta)^{ia_j^2}\right] \frac{-a_k}{b_k - \zeta}\,, \\
\psi_k^{(k)}(t) &= \int_{b_k}^{b_{k+1}} d\zeta\, e^{-i\zeta t}\, e^{-i\zeta^2/2} \left[\prod_{j=2}^{k} (\zeta - b_j)^{ia_j^2}\right] \left[\prod_{j=k+1}^{N} (b_j - \zeta)^{ia_j^2}\right] \frac{a_k}{\zeta - b_k}.
\end{aligned} \tag{6.4}$$

These Eq. (6.4) are exact.

Since the important contributions come from the vicinity of $\zeta = b_k$, all the ζ's in Eq. (6.4) can be replaced approximately by b_k except in the factors $e^{-i\zeta t}$, $b_k - \zeta$, and $\zeta - b_k$. Therefore

$$\begin{aligned}
\psi_k^{(k-1)}(t) &\sim e^{-ib_k^2/2} \left[\prod_{\substack{j=2 \\ j\neq k}}^{N} |b_j - b_k|^{ia_j^2}\right] (-a_k) \int^{b_k} d\zeta\, e^{-i\zeta t}\, (b_k - \zeta)^{-1+ia_k^2}\,, \\
\psi_k^{(k)}(t) &\sim e^{-ib_k^2/2} \left[\prod_{\substack{j=2 \\ j\neq k}}^{N} |b_j - b_k|^{ia_j^2}\right] a_k \int_{b_k} d\zeta\, e^{-i\zeta t}\, (\zeta - b_k)^{-1+ia_k^2}\,,
\end{aligned} \tag{6.5}$$

or

$$\begin{aligned}
\psi_k^{(k-1)}(t) &\sim e^{-ib_k^2/2}\, e^{-ib_k t} \left[\prod_{\substack{j=2 \\ j\neq k}}^{N} |b_j - b_k|^{ia_j^2}\right] (-a_k) \int_0^{\infty} dx\, e^{ixt}\, x^{-1+ia_k^2}\,, \\
\psi_k^{(k)}(t) &\sim e^{-ib_k^2/2}\, e^{-ib_k t} \left[\prod_{\substack{j=2 \\ j\neq k}}^{N} |b_j - b_k|^{ia_j^2}\right] a_k \int_0^{\infty} dx\, e^{-ixt}\, x^{-1+ia_k^2}.
\end{aligned} \tag{6.6}$$

This integral can be evaluated exactly in terms of the gamma function.

6.3 Results

Fig. 1 shows which ones of the various $\psi_k^{(n)}(t)$ have non-vanishing behaviors for $t \to -\infty$ and $t \to \infty$.

These non-vanishing behaviors are:

$k \backslash n$	1	2	3	4	$\cdots$	N-1	N
1	∞						-∞
2	$\times$	$\times$					
3		$\times$	$\times$				
4			$\times$	$\times$			
$\vdots$				$\ddots$	$\ddots$		
N-1					$\ddots$	$\times$	
N						$\times$	$\times$

Figure 1: Table of non-vanishing components of $\psi_k^{(n)}(t)$ as $t \to \pm\infty$. A cross means that the component is non-vanishing both for $t \to -\infty$ and $t \to +\infty$; the symbol ∞ means for $t \to +\infty$ only, and $-\infty$ means for $t \to -\infty$ only.

For t positive and large,

$$\psi_1^{(1)}(t) \sim \sqrt{2\pi}\, e^{-i\pi/4}\, t^{i\alpha}\, e^{it^2/2}, \tag{6.7}$$

$$\psi_k^{(k-1)}(t) \sim e^{-ib_k^2/2}\, e^{-ib_k t}\left[\prod_{\substack{j=2\\ j\neq k}}^{N} |b_j - b_k|^{ia_j^2}\right](-a_k)e^{-\pi a_k^2/2}\,\Gamma(ia_k^2)t^{-ia_k^2}, \tag{6.8}$$

$$\psi_k^{(k)}(t) \sim e^{-ib_k^2/2}\, e^{-ib_k t}\left[\prod_{\substack{j=2\\ j\neq k}}^{N} |b_j - b_k|^{ia_j^2}\right]a_k\, e^{\pi a_k^2/2}\,\Gamma(ia_k^2)t^{-ia_k^2}; \tag{6.9}$$

while, for t negative and large,

$$\psi_1^{(N)}(t) \sim \sqrt{2\pi}\, e^{-i\pi/4}\, |t|^{i\alpha}\, e^{it^2/2}, \tag{6.10}$$

$$\psi_k^{(k-1)}(t) \sim e^{-ib_k^2/2}\, e^{-ib_k t}\left[\prod_{\substack{j=2\\ j\neq k}}^{N} |b_j - b_k|^{ia_j^2}\right](-a_k)e^{\pi a_k^2/2}\,\Gamma(ia_k^2)|t|^{-ia_k^2}, \tag{6.11}$$

$$\psi_k^{(k)}(t) \sim e^{-ib_k^2/2}\, e^{-ib_k t}\left[\prod_{\substack{j=2\\ j\neq k}}^{N} |b_j - b_k|^{ia_j^2}\right]a_k\, e^{-\pi a_k^2/2}\,\Gamma(ia_k^2)|t|^{-ia_k^2}. \tag{6.12}$$

All the other components approach zero as $t \to \infty$ and as $t \to -\infty$. In the asymptotic formulas (6.7) and (6.10), α is the quantity

$$\alpha = \sum_{j=2}^{N} a_j^2. \tag{6.13}$$

In the formulas (6.8), (6.9), (6.11) and (6.12),

$$2 \leq k \leq N, \tag{6.14}$$

where N as always is the number of neutrino species.

7 Discussion

When we started to investigate the MSW differential equations for three neutrino species in the case of the linear electron density, we were mostly interested in various possibilities of finding approximate solutions. Therefore, it was quite a surprise to us that these coupled differential equations can be solved exactly not only for three, but also for any number of neutrino species.

In the work of Wolfenstein, Mikheyev, Smirnov [1] and others [2] on the sun taking into account two species of neutrinos, it has been found that most of the effect takes place in a fairly narrow region around a particular value of the electron density. Because of this, it is quite accurate to use a linear approximation to the electron density.

For more than two species of neutrinos, it is no longer true in general that there is a narrow region for most of the activity. Nevertheless, there are a number of circumstances where this is true. However, the conditions for this to hold has not yet been studied systematically. This is one direction for future work.

Under the assumption of the electron density being a linear function of distance, the exact, general solution of the MSW differential equation is given by Eqs. (5.12) and (5.13). This solution is in the form of a number of single integrals. When the number of neutrino species is more than 2, these integrals cannot be evaluated in terms of known functions, and therefore their properties need to be investigated. A small step in this direction has been taken in Sec. 6, where the asymptotic behaviors of these integrals have been evaluated for large distances but with all the a's and b's held fixed. It is believed that, in so far as this case of linear electron density is applicable to the physically interesting case of solar neutrinos, the asymptotic evaluation of Sec. 6 is far from being sufficient. It is more likely that not only the distance, but also some of the parameters, the a's and b's, are large. This is a second direction for future work.

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