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On a Boundary CFT Description of Nonperturbative $\mathcal{N} = 2$ Yang-Mills Theory

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Abstract

We describe a simple method for determining the strong-coupling BPS spectrum of four dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. The idea is to represent the magnetic monopoles and dyons in terms of D -brane boundary states of a non-compact $d = 2$ $\mathcal{N} = 2$ Landau-Ginzburg model. In this way the quantum truncated BPS spectrum at the origin of the moduli space can be directly mapped to the finite number of primary fields of the superconformal minimal models.

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1. Introduction

Boundary conformal field theory (BCFT) has turned out to be a very useful tool for investigating properties of D -branes, especially in the domain of strong quantum corrections. Most notably boundary $\mathcal{N} = 2$ minimal models [1] have provided important results on the spectrum of D -branes at the Fermat (“Gepner”) point of Calabi-Yau threefold compactifications [2,3].

A particularly interesting feature of four dimensional theories with $\mathcal{N} = 2$ supersymmetry is that the D -brane BPS spectrum at the Gepner point can be substantially different as compared to the large radius limit, where semi-classical geometry applies. Indeed, when interpolating between these regimes in the moduli space, one may cross certain lines of marginal stability. On these certain central charges become collinear and thus BPS states become unstable against decay into constituents with smaller charges.

This kind of phenomenon is well-known from $\mathcal{N} = 2$ gauge theory, in which context it had been discussed first [4,5]. While most investigations have been centered at gauge group $G = SU(2)$, there is only limited knowledge for general gauge groups and matter content, because the situation becomes rapidly very complex. The picture that seemed to emerge for pure gauge theories is that in a sufficiently small neighborhood of the origin of the moduli space, \mathcal{M}_G , the spectrum of stable BPS states is finite and consists only of those monopoles and dyons which become massless at the various singularities in \mathcal{M}_G . On the other hand, in the semi-classical regime at “infinity” in \mathcal{M}_G , the BPS spectrum is infinite, its most prominent members being the massive gauge fields.

The purpose of the present note is to determine the strong coupling spectrum of $\mathcal{N} = 2$ $SU(N)$ gauge theories, by relating them to boundary Landau-Ginzburg theories [6,7] and so effectively mapping them to $\mathcal{N} = 2$ $d = 2$ minimal models at levels $k = N - 2$. In this way, the BPS states at the origin of \mathcal{M}_G can be mapped via boundary states to the $N(N - 1)$ primary fields ϕ_m^ℓ , so that the quantum truncation of the spectrum may be interpreted in terms of the finite number of the primary fields in the 2d CFT. It will turn out that the stable BPS states have indeed precisely the electric and magnetic RR charges of the potentially massless monopoles and dyons.

Our setup is very simple. As is well-known, $\mathcal{N} = 2$ gauge theories can be systematically embedded into type II string compactifications on Calabi-Yau threefolds [8,9].

Specializing to the gauge sub-sector in question amounts to focusing on a (neighborhood of an) appropriate isolated singularity on the threefold. This can be modeled in terms of a non-compact CY threefold, whose compact piece supports the geometry of the relevant Seiberg-Witten curve. The monopoles and dyons then correspond to D -branes wrapped around the compact cycles, while the non-compact directions subsume the non-universal degrees of freedom which decouple in the rigid theory we are interested in.

Concretely, for $G = SU(N)$ the non-compact threefold can be written as [8]:

$$z + \frac{1}{z} + P_{A_{N-1}}(x_1, u_k) + x_2^2 + x_3^2 = 0, \quad (1)$$

where $P_{A_{N-1}}(x, u_k) = x^N - \sum_{k=2}^N u_k x^{N-k}$ is the normal form of the simple singularity [10] of type A_{N-1} ; the other simply laced gauge groups of type D, E can be treated analogously. Dropping the quadratic pieces, (1) turns precisely into the Riemann surface that underlies the BPS dynamics of $\mathcal{N} = 2$ Yang-Mills theory [4,11]. We will be interested in the ‘‘Gepner point’’ $u_k \equiv 0$ of this geometry and study the spectrum of wrapped D -branes in terms of boundary CFT.

2. Non-compact Landau-Ginzburg description

While the form (1) of the non-compact threefold is convenient for studying the geometry that underlies the Yang-Mills theory, it is not very useful for a CFT formulation, because for this we need an $\mathcal{N} = 2$ superconformal Landau-Ginzburg theory with $\widehat{c} = 3$ to start with. In order to find a suitable form, recall that the geometry described by (1) is given by the fibration of an ALE space (described here by $P_{A_{N-1}}$ besides the un-important quadratic pieces) over a \mathbb{P}^1 base (described by the z -dependent part) [8]. A good starting point is, therefore, to first focus on the ALE space.

The ALE space corresponds to a non-compact model of a type A_{N-1} singularity on a compact $K3$ surface, and type II string compactification on it can be described in terms of a CFT based on the following Landau-Ginzburg superpotential [12]:[†]

$$W_{A_{N-1}}^{ALE}(x, z, u_k) = x^N - \sum_{k=2}^N u_k x^{N-k} z^{-k}, \quad (2)$$

[†] We drop quadratic terms because they are irrelevant for the LG theory.

and analogously for the other simply laced groups. The theory is singular at $u_k = 0$, and this reflects the appearance of massless gauge fields (or non-critical strings, if we start from the IIB theory instead of type IIA). The Gepner point of this theory is described by

$$W_{A_{N-1}}^{ALE}(x, z, u_N = -1) = x^N + \frac{1}{z^N}, \quad (3)$$

which indeed describes a smooth CFT with $\widehat{c} = 2$ (remember that each term x^N contributes $\widehat{c}(N) = (N-2)/N$). It can also be described in terms of a coset CFT based on $\left(\frac{SU(2)_{N-2}}{U(1)} \times \frac{SL(2)_{N+2}}{U(1)}\right)/\mathbb{Z}_N$ [12,13]. The idea is that the non-compact part of the theory is a placeholder that encodes the non-universal, but decoupled dynamics that is irrelevant to our problem. The quantities we are interested in reside in the compact sub-sector, and do not depend on the details of the non-compact CFT.

We now return to describing the geometry we are really after, namely a fibration of the ALE space over \mathbb{P}^1 . As is familiar from compact $K3$ fibrations [14], this ultimately amounts to splitting z into two coordinates with half the degree each, and so we arrive, tentatively at first, at the following LG representation of the Seiberg-Witten theory:

$$W_{A_{N-1}}^{SW}(x, z_1, z_2, u_k) = x^N + \frac{1}{z_1^{2N}} + \frac{1}{z_2^{2N}} - \sum_{k=2}^N u_k x^{N-k} (z_1 z_2)^{-k}. \quad (4)$$

This describes a CFT with $\widehat{c} = 3$ which is smooth at the origin of its moduli space, $u_k = 0$; on the other hand, $u_N \rightarrow \infty$ corresponds to the large base \mathbb{P}^1 limit where we recover the ALE space. Moreover the singularities at $u_\ell = 0, u_N \pm 2$, where the purely z -dependent piece forms a complete square, correspond to the Argyres-Douglas points [15]. More generally one can check that the discriminant locus in the moduli space is the same as for (1). We will thus take (4) as the defining superpotential for our boundary Landau-Ginzburg theory.

Our purpose is now to determine the spectrum of B -type boundary states at the Gepner point of (4), by focusing on the compact piece of

$$W_{A_{N-1}}^{SW}(x, z_1, z_2) = x^N + \frac{1}{z_1^{2N}} + \frac{1}{z_2^{2N}}. \quad (5)$$

This may also be viewed as a coset model of the form $\left(\frac{SU(2)_{N-2}}{U(1)} \times \left(\frac{SL(2)_{2N+2}}{U(1)}\right)^2\right)/\mathbb{Z}_{2N}$.

3. Boundary CFT and Intersection Index

An important quantity to compute in order to verify that (5) is the correct form of the Landau-Ginzburg potential is the topological “intersection index” $I_{a,b} \equiv \text{Tr}_{a,b}[(-1)^F]$ of boundary states a, b . Using standard BCFT technology, the index will indeed turn out to coincide with the intersection matrix of the *vanishing cycles* (and not just of some arbitrary homology cycles) of the Seiberg-Witten curve. Subsequently, we will determine the spectrum of the stable wrapped D -branes.

Since the LG potential (5) represents a tensor product of $\mathcal{N} = 2$ superconformal models, we can most conveniently focus on its components. First of all, as is well known, x^N represents an $\mathcal{N} = 2$ minimal model $\frac{SU(2)_k}{U(1)}$ at level $k = N - 2$. The primary fields are labelled by (ℓ, m, s) , with $\ell = 0, \dots, N - 2$, $m = -N + 1, \dots, N \pmod{2N}$, and in addition $s = -1, 0, 1, 2 \pmod{4}$ determines the R- or NS-sectors ($\ell + m + s = 0 \pmod{2}$). We will be interested in A-type boundary states $|\ell_i, m_i, s_i\rangle$, which are labelled by the same letters as the primary fields. As has been shown in recent papers [2,7], the intersection index can be written as the following overlap amplitude:

$$I_{\ell_1, \ell_2}(m_1, m_2, s_1, s_2) \equiv {}_{\text{RR}}\langle \ell_1, m_1, s_1 | \ell_2, m_2, s_2 \rangle_{\text{RR}} . \quad (6)$$

Using the expansion of the boundary states into Ishibashi states $|\rangle\rangle$,

$$|\ell, m, s\rangle = \sum_{(\ell', m', s')} \frac{S_{(\ell, m, s)}^{(\ell', m', s')}}{\sqrt{S_{(0,0,0)}^{(\ell', m', s')}}} |\ell', m', s'\rangle\rangle , \quad (7)$$

where $S_{(\ell, m, s)}^{(\ell', m', s')} = \frac{1}{\sqrt{2N}} \sin \left[\frac{\pi}{N} (\ell + 1)(\ell' + 1) \right] \exp \left[i \frac{\pi}{N} (mm' - \frac{N}{2} ss') \right]$ is the modular transformation matrix associated with the $\mathcal{N} = 2$ minimal model characters, the result is [2,7]:

$$(I_{\ell_1, \ell_2})_{m_1}^{m_2}(s_1, s_2) = (-1)^{\frac{s_2 - s_1}{2}} N_{\ell_1, \ell_2}^{m_2 - m_1} . \quad (8)$$

This can be considered as $2N \times 2N$ matrix for fixed ℓ_i, s_i (in the following, we will keep s_i fixed). Above,

$$N_{\ell_1, \ell_2}^{\ell_3} = \frac{2}{N} \sum_{\ell=0}^k \frac{\sin \left[\frac{\pi}{N} (\ell_1 + 1)(\ell + 1) \right] \sin \left[\frac{\pi}{N} (\ell_2 + 1)(\ell + 1) \right] \sin \left[\frac{\pi}{N} (\ell_3 + 1)(\ell + 1) \right]}{\sin \left[\frac{\pi}{N} (\ell + 1) \right]} \quad (9)$$

are the Verlinde fusion coefficients associated with $SU(2)_k$. Moreover one can extend the standard range of the upper index, by defining $N_{\ell_1, \ell_2}^{-\ell_3 - 2} \equiv -N_{\ell_1, \ell_2}^{\ell_3}$ and $N_{\ell_1, \ell_2}^{-1} =$

$N_{\ell_1, \ell_2}^{N-1} \equiv 0$. The extended fusion coefficients are then periodic and so can be compactly written in terms of the \mathbb{Z}_N step generator $g(2N) \equiv \gamma^2(2N)$, where

$$\gamma(K) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{K \times K} . \quad (10)$$

One has then explicitly for the $\ell = 0$ states [2]:

$$I_{0,0}^{A_{N-1}} = \mathbf{1} - g(2N) , \quad (11)$$

which is a matrix without definite symmetry properties. Due to the selection rule $l + m + s = 0 \pmod{2}$, only every other entry needs to be considered in a given R- or NS-sector, and this is what we will do when we write down explicit matrices further below.

In order to obtain geometrically meaningful intersection matrices associated with non-compact Calabi-Yau \widehat{c} -folds, one needs to augment (11) by a contribution of the non-compact sector which pushes the central charge up to $\widehat{c} = 2$ or $\widehat{c} = 3$. This will ensure a completely symmetric or anti-symmetric intersection matrix. Note however that the structure of the non-compact models $\frac{SL(2)_{N+2}}{U(1)}$ is much more complicated than the one of the minimal models. In particular the $SL(2)$ fusion rules, if they are well-defined at all, are not known and do not even truncate, and it would be pretty pointless to try to solve the associated boundary CFT.

Fortunately, this is not necessary and all we will need from the non-compact sector is the intersection matrix for the trivial representation, $\ell = 0$. By choosing a parafermionic representation of the $SU(2)_k$ Kac-Moody algebra ($k = N - 2$), it is easy to see that when continuing to negative k , which corresponds to going to $SL(2)_k$, the $U(1)$ current $J^3 = i\sqrt{k}\partial\phi$ switches sign and the rôles of J^+ and J^- are exchanged [16,17]. As a consequence primary fields associated with parafermions ψ_m^ℓ will now be associated with negative charges, $q = -m/N$. Thus, while we certainly do not know the exact $\frac{SL(2)_{N+2}}{U(1)}$ S -matrices in the expansion of the boundary states into Ishibashi states (7), the structure should remain simple for the $\ell = 0$ states as far as the labels m are concerned, and be related to the minimal model matrices by a sign flip of the m labels.

In other words, m_1 and m_2 are exchanged in (8) and this just amounts to the transposition of $I_{0,0}$. Imposing periodicity as before, we then conclude that for a LG theory with $W = 1/x^N$ the intersection index should be:

$$I_{0,0} = \mathbf{1} - g^{-1}(2N) . \quad (12)$$

This is the generalization to negative N of the rule that in a tensor product LG model with \mathbb{Z}_K scaling symmetry, each term x^N contributes a factor $(\mathbf{1} - g^{K/N}(2K))$ to the intersection index [2,18]. The rule can be understood also from geometry: $(\mathbf{1} - g^{K/N})$ is nothing but the “variation operator” that maps relative to absolute homology [10]. It in particular appears when evaluating period integrals in terms of non-compact, V-shaped integration contours [19].

In forming the tensor product, we still need for the GSO projection to mod out the overall R symmetry, and this identifies the charge labels of the individual factors. Thus all-in-all we obtain from (3) the following intersection index for D -brane boundary states:

$$(I_{0,0})_{A_{N-1}}^{ALE} = (\mathbf{1} - g(2N)) \cdot (\mathbf{1} - g^{-1}(2N)) . \quad (13)$$

Similarly we get from the Landau-Ginzburg potential (5) for the non-compact three-fold:

$$I_{0,0}^{SW} = (\mathbf{1} - g^2(4N)) \cdot (\mathbf{1} - g^{-1}(4N))^2 . \quad (14)$$

We will show in the next section that this is indeed the correct, fully anti-symmetric intersection matrix of vanishing cycles of the Seiberg-Witten curve.

Before doing so we like to recall that the intersection index for boundary states[†] with $\ell \neq 0$ can be obtained from $I_{0,0}$ by simple matrix multiplication. More specifically, all we need is to consider $\ell = 0, \dots, [k/2]$, because of the field identification $\phi_{m,s}^\ell = \phi_{m+k+2,s+2}^{k-\ell}$ we can map $\ell = [k/2] + 1, \dots, k$, back into the smaller range (the shifted s -label implies that these states correspond to the anti-branes; the exception is at the fixed point $\ell = k/2$ where branes and anti-branes sit in the same m -orbit.). From the fusion coefficients (9) one can then deduce [18]:

$$\begin{aligned} I_{\ell_1, \ell_2} &= t_{\ell_1} \cdot I_{0,0} \cdot t_{\ell_2}^t , \\ t_\ell &\equiv \sum_{k=-\ell/2}^{\ell/2} \gamma^{2k} , \end{aligned} \quad (15)$$

[†] Of course we mean here the higher ℓ states of the minimal model only, and restrict ourselves to the $\ell = 0$ sector of the non-compact sector.

where γ is the square-root (10) of g . It follows that if $q_{(0)}^i = \{0, 0, \dots, 1_i, 0, \dots, 0\}$ are the charges of the “basic” boundary states with $\ell = 0$, the charges of the other states are

$$\vec{q}_{(\ell)} = \vec{q}_{(0)} \cdot t_\ell . \quad (16)$$

4. D -brane spectrum

We now like to verify that the boundary states, whose intersection indices are given by (13) and (14), indeed correspond to wrapped D -branes that represent gauge fields and $\mathcal{N} = 2$ SYM dyons, respectively.

We start with the simpler case (13), where is easy to see that the submatrix obtained by extracting every other entry coincides with the (cyclically extended) symmetric rank $(N - 1)$ Cartan matrix $\mathcal{C}_{A_{N-1}}$ of $SU(N)$.[◇] Therefore, the $\ell = 0$ boundary states correspond to the simple roots α_i , which means that one can choose a basis in which the boundary states have the following charges:

$$\vec{q}_{(0)} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{N-1} \\ \alpha_N \end{pmatrix} , \quad (17)$$

where the extending root is defined by $\alpha_N \equiv -\sum_i^{N-1} \alpha_i$. This is precisely as required if we want to interpret the boundary states in terms of D -branes wrapped around the compact 2-cycles of the ALE space. It is well-known that these cycles correspond to root vectors and intersect in a Dynkin diagram-like pattern.

However, in order that the D -branes describe gauge fields in the adjoint representation, we will need not just the simple roots but all the roots. One can easily see that the remaining roots are obtained from the boundary states with higher spins, $\ell = 1, \dots, [k/2]$. A simple way to deduce this is to visualize a circle of N points in the x -plane, which can be thought of as the projection of a weight diagram. The boundary states with $\ell = 0$ then correspond to the simple roots that connect subsequent points around the circle. Moreover the $\ell = 1$ states connect every other point, and so on. Group theoretically this corresponds to decomposing the adjoint representation into orbits of the Coxeter element of the Weyl group (when $(\ell + 1)$ divides N , the orbits

◇ The deeper reason is the fact that the modular S matrices that enter in the fusion rule coefficients $N_{\ell_1, \ell_2}^{\ell_3}$ happen to be the eigenvectors of the A_{N-1} Cartan matrix.

are short). In total we obtain all the $\frac{1}{2}N(N-1)$ positively charged gauge fields in this way, whose charges are given by linear combinations of the simple roots precisely as given in (16).

We now turn to the more interesting BCFT based on the non-compact threefold (5). Geometrically, what happens when we fiber an ALE space over a \mathbb{P}^1 base is that each point in the x -plane splits into a pair of branch points, and these are precisely the branch points of the Seiberg-Witten curve [8] (this has been discussed at length in the literature, see e.g., [20]). Correspondingly each 2-cycle on the ALE space splits into two 3-cycles on the 3-fold, which correspond to 1-cycles on the SW curve; examples for such cycles are shown in Fig.1.

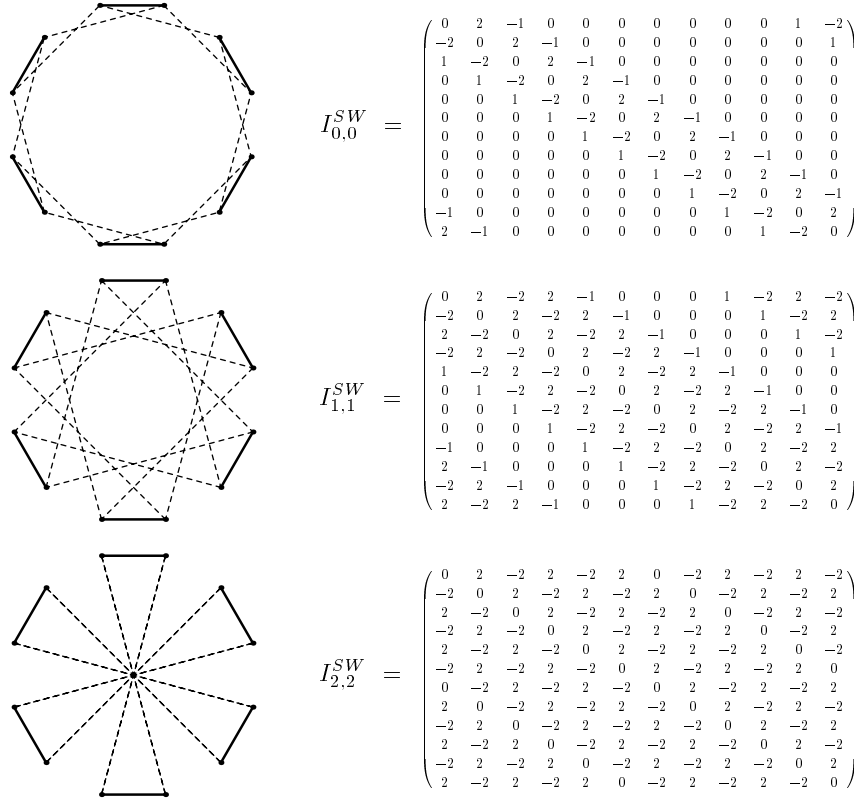


Fig.1: On the left we see a representation of the Seiberg-Witten curve for $G = SU(6)$ at the origin of its moduli space. The fat lines denote the branch cuts of a double cover of the x -plane. The dashed lines denote the vanishing cycles that correspond to the stable dyons (we show only the upper sheet). On the right we display the intersection matrices of these cycles, which coincide with the intersection indices $I_{\ell,\ell}^{SW}$ (15) of the boundary states with spin ℓ .

A suitable symplectic basis of these cycles, which exhibits the magnetic and electric charges in the form $q_i = [g_i; e_i]$, can be chosen as follows:

$$\vec{q}_{(0)} = \begin{pmatrix} \begin{bmatrix} \alpha_1 & ; & 0 \\ -\alpha_1 & ; & \alpha_1 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \alpha_i & ; & (i-1)\alpha_i \\ -\alpha_i & ; & (2-i)\alpha_i \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \alpha_N & ; & \sum(1-k)\alpha_k \\ -\alpha_N & ; & \sum(k-2)\alpha_k \end{bmatrix} \end{pmatrix}. \quad (18)$$

These are precisely the RR charge vectors of some the strong coupling dyons (namely those which are associated with the simple roots) [21,20,22]. The inner product metric of this basis takes a symplectic DZW form:

$$(I_{geom}^{SW})_i^j = q_i \circ q_j \equiv \langle g_i, e_j \rangle - \langle g_j, e_i \rangle, \quad (19)$$

where $\langle \dots, \dots \rangle$ is the inner product in weight space which given by the Cartan matrix of $SU(N)$. It is known to be the geometric intersection matrix of the vanishing cycles that correspond to the potentially massless dyons [21]. As one can easily verify, it indeed coincides with the intersection index (14) of the $\ell = 0$ boundary states:

$$I_{geom}^{SW} = I_{0,0}^{SW}.$$

Moreover, there are boundary states with spins $\ell = 1, \dots, [k/2]$ (remember that $\ell = [k/2] + 1, \dots, k$, corresponds to the anti-branes for which the orientation of the cycles is reversed). Similarly to what we have discussed for the ALE space, these correspond in the Yang-Mills theory to the dyons that originate from the non-simple roots. In total we have $2 \cdot \frac{1}{2}N(N-1)$ states whose charges are determined by (16), and one can check that these charges match the corresponding cycles on the SW curve – see again Fig.1.

Finally some remarks on the BPS nature of the boundary states, following [7]. Note that maximally $\text{rank}G = N - 1$ of the dyons can be mutually local with respect to each other, which corresponds to a choice of $N - 1$ non-intersecting cycles [15] (parallel dashed lines in Fig.1). Any given choice of such boundary states can be mapped to the set of primary chiral fields ϕ_ℓ^ℓ of the superconformal minimal model at

level $k = N - 2$. They are BPS with respect to the same linear combination of left- and right-moving supercharges. The other possible choices of mutually local states are generated by the \mathbb{Z}_{2N} symmetry, which describe states that preserve different linear combinations of the supercharges [7]. Altogether the $N(N - 1)$ states (not counting the anti-branes) exhaust the set of primary (not necessarily chiral) fields ϕ_m^ℓ in the minimal model.

Summarizing, we find that applying the rules of boundary conformal field theory to the non-compact LG potential (5), we find a complete match between the charges of the boundary states and the charges of those Yang-Mills dyons that are supposedly stable at the origin of the moduli space.

5. Discussion

The important point of the present paper is not so much that the intersection indices $I_{\ell,\ell} = \text{Tr}_{\ell,\ell}[(-1)^F]$, as computed from CFT fusion rules, coincide with the geometric intersection matrices of the vanishing cycles on the ALE space or SW curve. This is just a reflection of the universality of the simple singularities, which happen to underlie both the Yang-Mills theory [23] and the superconformal minimal models [24]. At least we may view the coincidence as a confirmation of the choice (3) and (5) of Landau-Ginzburg potentials, and as a further successful test on how BCFT techniques work in non-compact situations.

Rather, the important point is that the spectrum of the D -brane boundary states turns out to be truncated exactly such that it matches the expected dyon spectrum of the strongly coupled Yang-Mills theory, and this is more than just simple algebraic geometry and group theory. In order to appreciate this, recall that the geometry of a compact or non-compact CY manifold determines a priori only the homology lattice and thus what the *possible* RR charges of wrapped branes are. However, it does not directly tell what subset of the charge lattice corresponds to the stable, single particle quantum states, at a given point in the moduli space.[†]

As has been demonstrated in the present and in other recent papers, this kind of questions can be analyzed in terms of boundary conformal field theory, at least as far as exactly solvable models are concerned. In particular we have found that the BPS spectrum of stable D -branes on ALE spaces and SW curves, at the Gepner points of their respective moduli spaces, can be mapped one-to-one to the spectrum of primary

[†] For recent considerations about this issue, see [25].

fields of the $\mathcal{N} = 2$ superconformal minimal models. The latter is finite due to the truncation to $\ell \leq k$ of the fusion rules, which manifests itself in the sine functions in the fusion coefficients (9); this is intimately related to the truncation to integrable representations of $\widehat{SU}(2)_k$.

The fact that these two instances of quantum truncation, namely of the spectrum of wrapped D -branes on the one hand and of the spectrum of 2d primary fields on the other, can be directly mapped into each other, is what we view as the most interesting aspect of our considerations.

There are certain obvious generalizations one could think about, like considering $\mathcal{N} = 2$ $d = 4$ gauge theories with different gauge groups and matter content. As for pure Yang-Mills theories based on simply laced Lie algebras of type ADE , we expect the following Landau-Ginzburg potentials to provide a useful BCFT formulation:

$$W_{ADE}^{SW}(z_i, x_j, u_k) = \frac{1}{z_1^{2h(ADE)}} + \frac{1}{z_2^{2h(ADE)}} + P_{ADE}(x_j, u_k) . \quad (20)$$

Here, $h(ADE)$ denotes the dual Coxeter number and P_{ADE} the normal form of the simple singularity of the corresponding type [10].

Moreover it may be interesting to find a relation between our and the work of [26], where a correspondence between BPS spectra of certain two and four dimensional $\mathcal{N} = 2$ supersymmetric gauge theories has been discovered. Finally one may ask about the significance of integrable deformations [27] of the $\mathcal{N} = 2$ minimal models in the present context; these leave infinitely many two-dimensional currents conserved, and so one may wonder about enhanced integrability properties of the Yang-Mills theories on certain sub-loci on the moduli space.

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