

HARRY LEHMANN AND THE ANALYTICITY UNITARITY PROGRAMME¹**André MARTIN**

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ABSTRACT

I try to describe the extremely fruitful interaction I had with Harry Lehmann and the results which came out of the analyticity unitarity programme, especially the proof of the Froissart bound, which, with recent and future measurements of total cross-sections and real parts, remains topical.

Dedication

I dedicate this paper to Marie-Noëlle Fontaine,
the last of the many papers she typed so skillfully
for me, wishing her a happy retirement.

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My first meeting with Harry Lehmann was not with his person but with the famous paper of the trio Lehmann-Symanzik-Zimmermann, LSZ [1], the importance of which everybody in the Theory group of Maurice Lévy at Ecole Normale realized immediately. In spite of the fact that I did not know German (I still don't) I read it, *Nuovo Cimento* in one hand, dictionary in the other hand (I am a “corrected” left hander). Later Harry visited the Ecole Normale in person and I was immediately impressed. That was the time where there was a wave of interest into what is an unstable particle and Lehmann and Lévy were some of the people involved. I remember also quite vividly our meeting at the La Jolla Conference in 1961 which I attended, coming from CERN. It was, as I realized a posteriori, a very important conference, for physicists and for people (some of the people I met there became my very best friends). I remember that Marcel Froissart gave a talk on his famous Froissart bound [2] on the total cross-section, $\sigma_t < c \log s)^2$, s square of the centre-of-mass energy, and Harry with his very meticulous mind found out that some of the estimates of Froissart were not quite correct, though this did not affect the result (some year later, I published a sum rule on pion-nucleon scattering and Harry discovered a very well hidden mistake. I was very impressed). Anyway we were both admiring of the achievement of Froissart and for me it was a decisive turning point, since I left almost completely for many years potentials and the Schrödinger equation for the study of high-energy scattering and high-energy bounds. The Froissart bound was derived from a combination of the Mandelstam representation [3] where the scattering amplitude is the boundary value of an analytic function of two variables, which implies automatically dispersion relations proved from field theory [4] in one variable as well as the Lehmann ellipse [5] which is probably the most celebrated result of Harry, a fundamental result presented in 10 small pages of *Nuovo Cimento* (compare with the incredibly lengthy papers on what I would call “rigorous atomic physics” which appeared during the last 15 years!).

The trouble with the Mandelstam representation is that nobody was ever able to prove it even in perturbation theory (through some wrong proofs were published!). Both Harry and I were anxious to obtain high-energy bounds with minimal assumptions. A step in this direction was made by Greenberg and Low [6] who used the Lehmann Ellipse to derive a bound on the total cross-section where $(\log s)^2$ was replaced by $s(\log s)^2$. Myself, I realized that the whole Mandelstam representation was not needed to get the Froissart bound and that it was sufficient to replace the Lehmann ellipse by a larger one [7]. Later, in Princeton, Y.S. Jin (a former student of Harry) and I found a way to control the growth of the scattering amplitude for unphysical momentum transfer using positivity [8] but at the time we made no progress on the derivation of the Froissart bound. In the autumn of 1965 I was visiting IHES (Institut des Hautes Etudes Scientifiques) and Harry was there. He attracted my attention on a paper by Nakanishi which contained the claim that the Lehmann Ellipse could be enlarged by using results from perturbative field theory, leading to the obtention of the Froissart bound. As I shall explain later, we tried to make sense of the paper of Nakanishi [9] but in the end could not. Nevertheless it started again my interest in the subject, and after a visit to Cambridge where I learnt that the Nakanishi perturbative domain of analyticity [10] had been obtained independently and in a simpler way by T.T. Wu [11], I came back to CERN and finally succeeded, using positivity properties not terribly different from those I had used with Jin, to enlarge the Ellipse without using perturbation and prove the Froissart bound from first principles. Some-

thing rather rare happened: Harry sent a postcard to congratulate me, but while moving from one apartment to another one or maybe from one office to another, I lost it!

I had many occasions to meet Harry later, but the last one was in the spring of 1998 at CERN where he came to work with T.T. Wu after an operation which seemed successful. At the Ringberg Castle meeting, in the honor of Wolfhart Zimmermann, he was supposed to be the first speaker and could not come because he was ill. I became the first speaker. Then I knew I would never meet him again.

Now I believe that it is necessary to give some technical details.

In 3+1 dimensions (3 space, 1 time) the scattering amplitude depends on two variables energy and angle. For a reaction $A + B \rightarrow A + B$

$$E_{c.m.} = \sqrt{M_A^2 + k^2} + \sqrt{M_B^2 + k^2} , \quad (1)$$

k being the centre-of-mass momentum. The angle is designated by θ . There are alternative variables:

$$s = (E_{CM})^2 , \quad t = 2k^2(\cos \theta - 1) \quad (2)$$

(Notice that physical t is NEGATIVE).

We shall need later an auxiliary variable u , defined by

$$s + t + u = 2M_A^2 + 2M_B^2 \quad (3)$$

The Scattering amplitude (scalar case) can be written as a partial wave expansion, the convergence of which will be justified in a moment:

$$F(s, \cos \theta) = \frac{\sqrt{s}}{k} \sum (2\ell + 1) f_\ell(s) P_\ell(\cos \theta) \quad (4)$$

$f_\ell(s)$ is a partial wave amplitude.

The Absorptive part, which coincides for $\cos \theta$ real (i.e., physical) with the imaginary part of F , is defined as

$$A_s(s, \cos \theta) = \frac{\sqrt{s}}{k} \sum (2\ell + 1) \text{Im } f_\ell(s) P_\ell(\cos \theta) \quad (5)$$

The Unitarity condition, implies, with the normalization we have chosen

$$\text{Im } f_\ell(s) \geq |f_\ell(s)|^2 \quad (6)$$

which has, as a consequence

$$\text{Im } f_\ell(s) > 0 , \quad |f_\ell| < 1 . \quad (7)$$

The differential cross-section is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{s} |F|^2 ,$$

and the total cross-section is given by the “optical theorem”

$$\sigma_{total} = \frac{4\pi}{k\sqrt{s}} A_s(s, \cos \theta = 1) . \quad (8)$$

With these definitions, a dispersion relation can be written as:

$$F(s, t, u) = \frac{1}{\pi} \int \frac{A_s(s', t) ds'}{s' - s} + \frac{1}{\pi} \int \frac{A_u(u', t) du'}{u' - u} \quad (9)$$

with possible subtractions, i.e., for instance the replacement of $1/(s' - s)$ by $s^N/s'^N(s' - s)$ and the addition of a polynomial in s , with coefficients depending on t .

The scattering amplitude in the s channel $A + B \rightarrow A + B$ is the boundary value of F for $s + i\epsilon$, $\epsilon > 0 \rightarrow 0$, $s > (M_A + M_B)^2$. In the same way the amplitude for $A + \bar{B} \rightarrow A + \bar{B}$, \bar{B} being the antiparticle of B is given by the boundary value of F for $u + i\epsilon$, $\epsilon \rightarrow 0$ $u > (M_A + M_B)^2$. Here we understand the need for the auxiliary variable u .

The dispersion relation implies that, for fixed t the scattering amplitude can be continued in the s complex plane with two cuts. The scattering amplitude possesses the reality property, i.e., for t real it is real between the cuts and takes complex conjugate values above and below the cuts.

In the most favourable cases, like $\pi\pi \rightarrow \pi\pi$ or $\pi N \rightarrow \pi N$ scattering dispersion relations have been established for $-T < t \leq 0$, $T > 0$ [4].

In the general case, even if dispersion relations are not proved, the crossing property of Bros, Epstein and Glaser states that the scattering amplitude is analytic in a twice cut plane, minus a finite region, for any negative t [12]. So it is possible to continue the amplitude directly from $A + B \rightarrow A + B$ to the complex conjugate of $A + \bar{B} \rightarrow A + \bar{B}$. By a more subtle argument, using a path with fixed u and fixed s it is possible to continue directly from $A + B \rightarrow A + B$ to $A + \bar{B} \rightarrow A + \bar{B}$

At this point, we see already that one cannot dissociate analyticity, i.e., dispersion relations, and unitarity, since the discontinuity in the dispersion relations is given by the absorptive part. In the simple case of $t = 0$, the absorptive part is given by the total cross-section and the forward amplitude is given, as we said already for the case of Compton Scattering, by an integral over physical quantities.

It was recognized very early that the combination of analyticity and unitarity might lead to very interesting consequences and might give some hope to fulfill at least partially the S matrix Heisenberg program. This was very clearly stated already in 1956 by Murray Gell-Mann [13] at the Rochester conference. Later this idea was taken over by many people, in particular by Geff Chew. To make this program as successful as possible it seemed necessary to have an analyticity domain as large as possible. Dispersion relations are fixed t analyticity properties, in the other variable s , or u as one likes.

Another property derived from local field theory was the existence of the Lehmann ellipse [5], which states that for fixed s , physical, the scattering amplitude is analytic in $\cos \theta$ in an

ellipse with foci at $\cos\theta = \pm 1$. $\cos\theta = 1$ corresponds to $t = 0$ the ellipse therefore contains a circle

$$|t| < T_1(s) . \quad (10)$$

$T_1(s)$ is given by

$$\begin{aligned} x_0 &= 1 + \frac{T_1(s)}{2k^2} \\ x_0 &= \left[1 + \frac{(M_1^2 - M_A^2)(M_2^2 - M_B^2)}{k^2(s - (M_1 - M_2)^2)} \right]^{1/2} \end{aligned}$$

where M_A and M_B are the masses of the particles, M_1 and M_2 are the lowest intermediate states in the currents associated to the fields of the incoming particles.

Hence $T_1(s) \rightarrow 0$ for $s \rightarrow (M_A + M_B)^2$ and $s \rightarrow \infty$.

The absorptive part is analytic in the larger ellipse, the ‘‘large’’ Lehmann ellipse, containing the circle

$$|t| < T_2(s) , \quad (11)$$

T_2 is given by

$$2x_0^2 - 1 = 1 + \frac{T_2(s)}{2k^2}$$

So $T_2(s) \rightarrow c > 0$ for $s \rightarrow (M_A + M_B)^2$, $T_2(s) \rightarrow 0$ for $s \rightarrow \infty$.

It was thought by Mandelstam that these two analyticity properties, dispersion relations and Lehmann ellipses, were insufficient to carry very far the analyticity-unitarity program. he proposed the Mandelstam representation [3] which can be written schematically as

$$\begin{aligned} F = & \frac{1}{\pi^2} \int \frac{\rho(s', t') ds' dt'}{(s' - s)(t' - t)} \\ & + \text{circular permutations in } s, t, u \\ & + \text{one dimensional dispersion integrals} \\ & + \text{subtractions} \end{aligned} \quad (12)$$

This representation is nice. It gives back the ordinary dispersion relations and the Lehmann ellipse when one variable is fixed, but it was never proved nor disproved for all mass cases, even in perturbation theory,. One contributor, Jean Lascoux, refused to co-sign a ‘‘proof’’, which, in the end, turned out to be imperfect.

One very impressive consequence of Mandelstam representation was the proof, by Marcel Froissart, that the total cross-section cannot increase faster than $(\log s)^2$, the so-called ‘‘Froissart Bound’’ [2].

My own way to obtain the Froissart bound [7] was to use the fact that the Mandelstam representation implies the existence of an ellipse of analyticity in $\cos\theta$ qualitatively larger than the Lehmann ellipse, i.e., such that it contains a circle $|t| < R$, R fixed, independent of

the energy. This has a consequence that $\text{Im } f_\ell(s)$ decreases with ℓ at a certain exponential rate because of the convergence of the Legendre polynomial expansion and of the polynomial boundedness, but on the other hand the $\text{Im } f_\ell(s)$'s are bounded by unity because of unitarity [Eq. (7)]. Taking the best bound for each ℓ gives the Froissart bound.

Let me now try to recall the exchange Harry Lehmann and I had in the Autumn of 1965 in Bures sur Yvette. We had in common the same desire to find a proof of the Froissart bound without using the Mandelstam representation and to find a way to enlarge the Lehmann ellipse. Harry pointed out to me a paper published by N. Nakanishi [9] a few months earlier where he claimed that he had a proof of the Froissart bound.

Let me remind you that the largest possible ellipse of convergence of the Legendre Polynomial series for the absorptive part has necessarily a singularity at its right extremity. This is the analogue of a classical theorem on power series with positive coefficients. This means that if you succeed (take the $\pi\pi$ case, $m_\pi = 1$) in proving that the absorptive part is analytic in the neighbourhood of the segment

$$t = 0 \quad t = 4 ,$$

then it is automatically analytic in the ellipse with foci

$$t = 4 - s \quad t = 0 ,$$

and right extremity $t = 4$, and a fortiori it is analytic in the circle

$$|t| < 4 ,$$

entirely contained in the ellipse.

Nakanishi had obtained a representation valid for any Feynman diagram [10]

$$T_N(s, t) = \int \frac{d^n \alpha}{[f(\alpha) + s g(\alpha) + t h(\alpha)]^p}$$

Later on I learnt from P. Landshoff that this representation had also been obtained, independently and in a simpler way by T.T. Wu [11]. A minimal analytic domain for $T_N(s, t)$ is obtained when the denominator in the integral representation does not vanish.

This domain, for the $\pi\pi$ case for fixed complex s is a kind of strip containing the straight line going through $t = 0$ and $t = 4 - s$, (which corresponds to $\cos \theta_s$ real), and the segments $-4 < t < +4$ and $-s < t < 8 - s$. When s tends to a real value the domain shrinks to zero for $s > 4$ and for $s < -t$ (for t real).

This means that for t fixed, real $-4 < t < +4$, dispersion relations hold.

The Nakanishi-Wu representation also implies the validity of partial wave dispersion relations but this is irrelevant for our problem.

However, there is nothing like a small or a large Lehmann ellipse in this domain. The absorptive part in perturbation theory, which is defined only in the limit $s \rightarrow s_R + i\epsilon$, $\epsilon \rightarrow 0$ has a priori no analyticity in t . A priori, it is just a distribution.

In fact in perturbation theory, unitarity connects amplitudes of different orders and positivity properties of the absorptive part are completely hidden. In three-space dimensions, nobody knows if the perturbation series can be resummed (probably not!) and it is not “legal” to combine the results of axiomatic field theory and perturbation theory. Of course one can always try it as a game, which is what Harry and I tried to do, but we went nowhere.

It is only in December 1965, after a visit to Cambridge, that I found a way to enlarge the Lehmann ellipse in the framework of axiomatic field theory [14], without using at all the results of perturbation theory. I was maybe a bit unfair not to quote the Nakanishi-Wu representation because the “wrong” paper of Nakanishi was undoubtedly a source of stimulation but, on the other hand, I did not use it at all.

Our method was the following.

The positivity of $\text{Im } f_\ell$ implies, by using expansion (5),

$$\left| \left(\frac{d}{dt} \right)^n A_S(s, t) \right|_{-4k^2 \leq t \leq 0} \leq \left| \left(\frac{d}{dt} \right)^n A_S(s, t) \right|_{t=0}. \quad (13)$$

To calculate

$$F(s, t) = \frac{1}{\pi} \int_{s_0} \frac{A_s(s't) ds'}{s' - s}$$

(forget the left-hand cut and subtractions!), for s real $< s_0$ one can expand $F(s, t)$ around $t = 0$. From the property (13) one can prove that the successive derivatives can be obtained by differentiating under the integral. When one resums the series one discovers that this can be done not only for s real $< s_0$, but for any s and that the expansion has a domain of convergence in t independent of s . This means that the large Lehmann ellipse must contain a circle $|t| < R$. This is exactly what is needed to get the Froissart bound. In fact, in favourable cases, $R = 4m_\pi^2$, m_π being the pion mass. A recipe to get a lower bound for R was found by Sommer [15]

$$R \leq \sup_{s_0 < s < \infty} T_1(s) \quad (14)$$

It was already known that for $|t| < 4m_\pi^2$ the number of subtractions in the dispersion relations was at most two [8], and it lead to the more accurate bound [16]

$$\sigma_T < \frac{\pi}{m_\pi^2} (\log s)^2 \quad (15)$$

Notice that this is only a bound, not an asymptotic estimate.

In spite of many efforts the Froissart bound was never qualitatively improved, and it was shown by Kupsch [17] that if one uses only $\text{Im } f_\ell \geq |f_\ell|^2$ and full crossing symmetry one cannot do better than Froissart.

On the more theoretical side one might wonder if using crossing symmetry and analytic completion one could not prove Mandelstam representation at least for the pion-pion case using only axiomatic results. This is not the case, as I showed it in 1967 at a meeting organized

by Bob Marshak in Rochester where Harry was present [18]. One can write a representation of the scattering amplitude

$$F_\nu = \int_0^1 dx \int_{p_0 q_0}^\infty \frac{dp dq w(x,p,q)}{x(p-s)^\nu + (1-x)(q-t)^\nu}$$

+circular permutations

For $\nu = 1/2$ this is just a funny way to write the Mandelstam representation.

For $\nu = 1$, you get back to the Nakanishi-Wu representation.

For $\nu = 2/3$ you get a natural domain bigger than all you can get from axiomatic field theory and positivity.

Before 1972, rising cross-sections were a pure curiosity. Almost everybody believed that the proton-proton cross-section was approaching 40 millibarns at infinite energy. Yet, Khuri and Kinoshita [19] took seriously very early the possibility that cross-sections rise and proved, in particular, that if the scattering amplitude is dominantly crossing even, and if $\sigma_t \sim (\log s)^2$ then

$$\rho = \frac{ReF}{ImF} \sim \frac{\pi}{\log s},$$

where ReF and ImF are the real and imaginary part of the forward scattering amplitude.

As early as 1970, Cheng and Wu proposed a model in which cross-sections were rising [20] and eventually saturating the Froissart bound. However, at that time there was no experimental indication of this. It is only in 1972 that it was discovered at the ISR, at CERN, that the $p-p$ cross-section was rising by 3 millibarns from 30 GeV c.m. energy to 60 GeV c.m. energy [21]. I suggested to the experimentalists that they should measure ρ and test the Khuri-Kinoshita predictions. They did it [22] and this kind of combined measurements of σ_T and ReF are still going on. In σ_T we have now more than a 50 % increase with respect to low energy values. For an up to date review I refer to the article of Matthiae [23]. it is my strong conviction that this activity should be continued with the future LHC. A breakdown of dispersion relation might be a sign of new physics due to the presence of extra compact dimensions of space according to N.N. Khuri [24]. Future experiments, especially for ρ , will be difficult because of the necessity to go to very small angles, but not impossible [25].

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