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Compressible Potential Flows with Free Boundaries. Part I: Vibrocapillary Equilibria

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Abstract: Various variational formulations describing nonstationary compressible fluid flows are considered. In particular, for high-frequency exitations a variationally based approximating frame is deduced which may explain experimentally observed phenomena.
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Introduction

This paper centers around Hamilton's principle applied to nonstationary compressible fluid flows with partially free boundaries in different exterior fields and under surface tension. In particular some of its consequences for systems subject to high-frequency exterior exitations are analyzed.

L. Lichtenstein, in his classical textbook [14], seemingly was the first, who gave a general formulation of Hamilton's principle for the motion of compressible fluids under various boundary conditions. Later on, K.O. Friedrichs in [7] recognized the variational characterization of free boundaries in steady potential flows. Since then, in a large number of contributions [1], [2], [3], [17], [18], [22], [27] further types of variational principles were employed to characterize the nonstationary motion of a free boundary flow. Despite of the success in the mathematical treatment of compressible flows with fixed boundaries, see [15], strong mathematical results on the fluid motion (e.g. long-time existence, blow-up) remained rare as long as free boundaries are involved. Even for the local initial value problem until now a mathematically satisfactory theory has reached only limited success; for diverse existence results compare e.g. [4], [23], [25], [26] and the literature cited there. The arising difficulties originate in the nonlinearities and the complicated coupling between boundary conditions and differential equations. Thus standard techniques from PDE fail to apply in most situations. On the other hand, due to their practical relevance, free boundary problems as considered here continue to challenge the numerically interested mathematician to test new algorithms and software, see e.g. [5]. In particular, numerical schemes based on variational methods have proved useful in approximating unsteady motion. A related numerical approach to nonstationary incompressible

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flows with partially free boundaries has been pursued in the textbook [18]; ordinary differential equations including the geometric aspects of the underlying variational principle are treated in [29].

Our paper, in Section 1, is aimed at formulating diverse variational principles governing the evolution of a compressible potential flow driven by volume forces as well as by surface tension and acoustic loading along the free and fixed (container-) boundary parts, respectively. The computation of the Hamilton action in Theorem 1.1 requires the solution of a time-dependent family of Neumann problems for an elliptic equation. On the other hand, this preliminary step may be avoided at the expense of introducing additional state variables. This is performed in detail in Theorems 1.2, 1.3.

When acoustic loading is included, the behaviour of the system may change in an unexpected way. During high-frequency exitation the time-averaged free surface can take a position far from the corresponding capillary shape, cf. [8], [16], [30] for a review on experimental work on this subject. We draw attention also to [13] where the related problem of acoustic flattening of a rotating liquid drop has been treated. Having in mind the mechanical analogy in nonlinear pendulum theory [11], in [28] the last author raised the question whether the mean surface position is determined by some kind of “vibrocapillary” force and a corresponding principle of minimal potential energy which, at the same time, would allow to distinguish between stability and instability of an averaged surface shape.

Section 2 adresses the mathematical background of this question. By transformation to nondimensional variables and introduction of a small parameter characterizing the high-frequency contributions of the exitation we construct via truncating the Hamilton action a class of oscillating solutions. Their time-averaged free boundaries turn out to be critical points of a time-independent “quasi-potential”. This is outlined in Theorems 2.2, 2.3. To get further information about the principal symbol and the mapping properties of the corresponding Jacobi operator, in Section 3, Theorem 3.3 we compute the second variation of that potential.

In a forthcoming second part of this paper we apply the results developed below to a numerical study of the experimentally observed vibrocapillary phenomena.

1 Variational principles

In the following let $x = (x_1, x_2, x_3)$ euclidean coordinates in $\mathbb{R}^3$ and let t denote the time. We consider the unsteady motion of a compressible fluid occupying a time-dependent bounded domain $Q(t)$ which is part of a rigid container $\widetilde{Q} = \{x \in \mathbb{R}^3 \mid \eta(x) < 0\}$ with a smooth function η. Let $\partial Q(t) = S_1(t) \cup S_2 \cup \Sigma(t)$ with $S_1, S_2 \subseteq \partial\widetilde{Q}$, where S_2 denotes the (time-independent) location of an acoustic source and $\Sigma(t) = \{x \in \widetilde{Q} \mid \xi(x,t) = 0\}$ is the moving free boundary. In the following $\nabla\xi$ is assumed to point to the exterior of $Q(t)$ always. If $\varphi = \varphi(x,t)$, $p = p(x,t)$ and $\rho = \rho(x,t)$ are velocity potential, pressure and density of the liquid, then the free boundary problem considered here reads as:

$$\rho\nabla\left(\dot\varphi+\frac{1}{2}|\nabla\varphi|^2+U\right)=-\nabla p \quad \text{in} \quad Q(t), \tag{1.1}$$

$$\dot\rho+\operatorname{div}(\rho\nabla\varphi)=0 \quad \text{in} \quad Q(t) \tag{1.2}$$

subject to the boundary conditions

$$\partial_n\varphi=0 \ \text{ on } \ S_1(t), \quad \partial_n\varphi=-|\nabla\xi|^{-1}\dot\xi \ \text{ on } \ \Sigma(t), \quad \rho\partial_n\varphi=V \ \text{ on } \ S_2. \tag{1.3}$$

Here, $U=U(x,t)$ is the potential of volume forces and $V=V(x,t)$ measures the normal component of velocity of the acoustic source. Throughout the paper ∂_n is the derivative relative to the outer normal $n=(n_1,n_2,n_3)$ of $\partial Q(t)$ and a dot denotes differentiation with respect to time. On $\Sigma(t)$ and $\partial\Sigma(t)$, respectively, the free boundary conditions

$$p-2\sigma H=p_0 \quad \text{on} \quad \Sigma(t), \tag{1.4}$$

$$-\nabla\eta\nabla\xi=\beta|\nabla\eta||\nabla\xi| \quad \text{on} \quad \partial\Sigma(t) \tag{1.5}$$

have to be fulfilled. Here H denotes the mean curvature of Σ and p_0 is the outer atmospheric pressure which we assume to be constant. σ is the coefficient of surface tension, β the relative adhesion coefficient between the fluid and the bounding walls. In our setting the system above is completed by a barotropic pressure-density relation

$$\rho=\rho(p). \tag{1.6}$$

Additionally, to guarantee mass conservation we impose

$$\int_{S_2} V\,dS=0 \tag{1.7}$$

as an constraint on V.

To establish Hamilton's principle for (1.1)-(1.6) we introduce the following notation: For given ξ and ρ let φ be the solution (defined up to a constant) of the Neumann problem

$$\operatorname{div}(\rho\nabla\varphi)=-\dot\rho \quad \text{in} \quad Q(t), \tag{1.8}$$

$$\partial_n\varphi=0 \ \text{ on } \ S_1(t), \quad \rho\partial_n\varphi=V \ \text{ on } \ S_2, \quad \partial_n\varphi=-|\nabla\xi|^{-1}\dot\xi \ \text{ on } \ \Sigma(t). \tag{1.9}$$

Equation (1.8) is uniformly elliptic as long as ρ is bounded positively against zero. To guarantee the solvability of the Neumann problem (1.8), (1.9), in addition to (1.7), we have to restrict ξ and ρ, which we choose as state variables in the following, to satisfy the constraint

$$\int_{Q(t)}\dot\rho\,dQ-\int_{\Sigma(t)}\rho|\nabla\xi|^{-1}\dot\xi\,d\Sigma=0.$$

This implies

$$\int_{Q(t)}\rho\,dQ=const., \tag{1.10}$$

i.e. conservation of total mass. Letting $W = W(\rho)$ the inner energy density of the fluid, then

$$p = \rho^2 W'(\rho) \tag{1.11}$$

gives the inverse funtion to (1.6). With this notation

$$\begin{aligned} L(t,\xi,\rho,\dot\xi,\dot\rho) = \int_{Q(t)} \rho \left(\frac{1}{2}|\nabla\varphi|^2 - W(\rho) - U\right) dQ \\ - \sigma\Big(|\Sigma(t)| - \beta|S_1(t)|\Big) - p_0|Q(t)| \end{aligned} \tag{1.12}$$

defines the Lagrangian of (1.1)-(1.6). In (1.12) the term $\sigma|\Sigma(t)|$ corresponds to the free surface energy and $\beta|S_1(t)|$ measures the wetting energy. Due to compressibility we have to include the work $p_0|Q(t)|$ of the outer pressure.

Theorem 1.1. *For a fixed time intervall* $[t_1, t_2]$ *let*

$$A(\xi,\rho) = \int_{t_1}^{t_2} L\,dt \tag{1.13}$$

denote the action corresonding to (1.12)*, considered under the restriction* (1.10)*, and subject to*

$$\delta\xi|_{t_1,t_2} = 0, \quad \delta\rho|_{t_1,t_2} = 0. \tag{1.14}$$

Then any sufficiently regular solution ξ, ρ *of the variational equations*

$$\delta_\xi A(\xi,\rho)\{\delta\xi\} + \delta_\rho A(\xi,\rho)\{\delta\rho\} = 0 \tag{1.15}$$

– for all variations $\delta\xi, \delta\rho$ *compatible with* (1.10), (1.14) *– satisfies the equations of motion* (1.1)-(1.6) *in* $[t_1, t_2]$ *(with velocity potential* φ *computed from* (1.8), (1.9) *and pressure given by* (1.11)*).*

Proof. Our starting point is the weak formulation of the Neumann problem (1.8) defining φ in dependence of ξ, ρ:

$$\int_{Q(t)} \dot\rho\psi\,dQ = \int_{Q(t)} \rho\nabla\varphi\nabla\psi\,dQ - \int_{S_2} V\psi\,dS + \int_{\Sigma(t)} \rho\dot\xi\psi|\nabla\xi|^{-1}\,d\Sigma$$

for all sufficiently smooth functions $\psi = \psi(\cdot, t)$ on $Q(t)$. Remembering the general differentiation rule for integrals over a time-dependent domain

$$\frac{d}{dt}\int_{Q(t)} f\,dQ = -\int_{\Sigma(t)} f\dot\xi|\nabla\xi|^{-1}\,d\Sigma + \int_{Q(t)} \dot f\,dQ, \tag{1.16}$$

(note that $\nabla\xi$ is directed towards the exterior of $Q(t)$) we obtain

$$\frac{d}{dt}\int_{Q(t)} \rho\psi\,dQ - \int_{Q(t)} \rho\dot\psi\,dQ = \int_{Q(t)} \rho\nabla\varphi\nabla\psi\,dQ - \int_{S_2} V\psi\,dS. \tag{1.17}$$

The functional (1.12) is defined under the constraint (1.10) only. Therefore, computing its derivative requires

$$\int_{Q(t)} \delta\rho \, dQ - \int_{\Sigma(t)} \rho\delta\xi|\nabla\xi|^{-1}\, d\Sigma = 0,$$

for the variations $\delta\xi$, $\delta\rho$. In the following we compute the partial derivatives $\delta_\xi A$, $\delta_\rho A$ under that assumption. We start with the variation of A with respect to ρ. From the definition of L and A we get immediately

$$\delta_\rho A = \int_{t_1}^{t_2}\int_{Q(t)} \delta\rho\left(\frac{1}{2}|\nabla\varphi|^2 - \frac{d}{d\rho}(\rho W) - U\right) + \rho\nabla\varphi\nabla\delta\varphi\, dQdt. \tag{1.18}$$

On the other hand, setting $\psi = \varphi$ in the ρ-variation of equation (1.17) and using (1.14) yields

$$\int_{t_1}^{t_2}\int_{Q(t)} \rho\nabla\varphi\nabla\delta\varphi\, dQdt = -\int_{t_1}^{t_2}\int_{Q(t)} \delta\rho\big(\dot\varphi + |\nabla\varphi|^2\big)\, dQ\, dt.$$

Substituting this into (1.18) finally gives

$$\delta_\rho A = -\int_{t_1}^{t_2}\int_{Q(t)} \delta\rho\left(\dot\varphi + \frac{1}{2}|\nabla\varphi|^2 + \frac{d}{d\rho}(\rho W) + U\right) dQ.$$

Similarly, by computing the variation of A with respect to ξ we get

$$\begin{aligned}\delta_\xi A = \int_{t_1}^{t_2}\Bigg\{&\int_{Q(t)} \rho\nabla\varphi\nabla\delta\varphi\, dQ \\ &- \int_{\Sigma(t)} \delta\xi|\nabla\xi|^{-1}\left(\frac{\rho}{2}|\nabla\varphi|^2 - \rho W - \rho U - p_0 - 2\sigma H\right) d\Sigma \\ &+ \sigma\int_{\partial\Sigma} \delta\xi\big(|\nabla\xi|^{-1}|\nabla\eta|^{-1}\nabla\eta\nabla\xi + \beta\big)\, dl\Bigg\}\, dt,\end{aligned} \tag{1.19}$$

(concerning the variation of the surface area see e.g. [9]). Furthermore, variation of (1.17) with respect to ξ implies

$$\int_{t_1}^{t_2}\int_{Q(t)} \rho\nabla\varphi\nabla\delta\varphi\, dQdt = -\int_{t_1}^{t_2}\int_{\Sigma(t)} \rho\big(\dot\varphi + |\nabla\varphi|^2\big)\delta\xi|\nabla\xi|^{-1}\, d\Sigma dt,$$

hence

$$\begin{aligned}\delta_\xi A = \int_{t_1}^{t_2}\Bigg\{&\int_{\Sigma(t)}\bigg(\rho\Big(\dot\varphi + \frac{1}{2}|\nabla\varphi|^2 + W + U\Big) + p_0 + 2\sigma H\bigg)\delta\xi|\nabla\xi|^{-1}\, d\Sigma \\ &+ \sigma\int_{\partial\Sigma} \delta\xi\big(|\nabla\eta|^{-1}|\nabla\xi|^{-1}\nabla\eta\nabla\xi + \beta\big)\, dl\Bigg\}\, dt.\end{aligned} \tag{1.20}$$

Obviously, the integrals on the right-hand sides of (1.19), (1.20) can be thought of as linear functionals also without any restriction on the variables $\delta\xi$, $\delta\rho$. Adopting this point of view comparision of (1.19), (1.20) with (1.15) via the Lagrange multiplier rule leads to

$$\delta_\rho A\{\delta\rho\} = -\int_{t_1}^{t_2} \lambda \int_{Q(t)} \delta\rho \, dQ \, dt,$$
$$\delta_\xi A\{\delta\xi\} = \int_{t_1}^{t_2} \lambda \int_{\Sigma(t)} \rho\delta\xi|\nabla\xi|^{-1} \, d\Sigma \, dt$$

for all $\delta\xi$, $\delta\rho$ with a time-dependent Lagrange multiplier $\lambda = \lambda(t)$, i.e.

$$\dot\varphi + \frac{1}{2}|\nabla\varphi|^2 + \frac{d}{d\rho}(\rho W) + U = \lambda \quad \text{in} \quad Q(t), \tag{1.21}$$
$$\rho\left(\dot\varphi + \frac{1}{2}|\nabla\varphi|^2 + U + W\right) + p_0 + 2\sigma H = \lambda\rho \quad \text{on} \quad \Sigma(t),$$

and, as a result of variation along $\partial\Sigma$:

$$|\nabla\eta|^{-1}|\nabla\xi|^{-1}\nabla\eta\nabla\xi + \beta = 0.$$

Computing the pressure p from (1.11) this implies (1.1)-(1.6). □

Remark. From (1.11) and (1.21) it follows

$$\rho\left(\dot\varphi + \frac{1}{2}|\nabla\varphi|^2 + U + W\right) + p = \lambda\rho$$

along any extremal. By adding a suitable time-dependent constant to φ we can assume $\lambda = 0$ without loss of generality. With this normalization of φ we get

$$A = \int_{t_1}^{t_2} \left\{ \int_{Q(t)} (p - p_0) \, dQ - \sigma\big(|\Sigma(t)| - \beta|S_1(t)|\big) \right\} dt$$
$$+ \int_{t_1}^{t_2} \int_{S_2} V\varphi \, dS \, dt + \int_{Q(t)} \rho\varphi \, dQ \Big|_{t_1}^{t_2}.$$

for the action along an extremal. Here the kinetic energy is expressed as an integral over the pressure, cf. also [10].

In Theorem 1.1 the computation of the velocity potential requires the solution of a Neumann problem. In particular for numerical purposes it is desirable to avoid this preliminary step. As in [2], [17] and [20] we introduce φ as an additional independent variable. In this case (1.8) as well as the boundary conditions (1.9) turn into natural optimality conditions. Our starting point is the observation that the φ-variation of the functional

$$J(\xi, \rho, \varphi) = \int_{t_1}^{t_2} \left\{ \int_{Q(t)} -\rho\Big(\dot\varphi + \frac{1}{2}|\nabla\varphi|^2\Big) \, dQ + \int_{S_2} V\varphi \, dS \right\} dt$$

leads to (1.8), (1.9). In fact,

$$\delta_\varphi J = \int_{t_1}^{t_2} \left\{ \int_{Q(t)} -\rho(\delta\dot\varphi + \nabla\varphi\nabla\delta\varphi)\, dQ + \int_{S_2} V\,\delta\varphi\, dS \right\} dt$$

and, after integration by parts

$$\begin{aligned}\delta_\varphi J = \int_{t_1}^{t_2} \Big\{ &\int_{Q(t)} \big(\dot\rho + \operatorname{div}(\rho\nabla\varphi)\big)\delta\varphi\, dQ \\ &- \int_{S_1(t)} \rho\partial_n\varphi\delta\varphi\, dS - \int_{S_2} (\rho\partial_n\varphi - V)\delta\varphi\, dS \\ &- \int_{\Sigma(t)} \rho(\partial_n\varphi - \dot\xi|\nabla\xi|^{-1})\delta\varphi\, dS \Big\} dt - \int_{Q(t)} \rho\delta\varphi\, dQ \Big|_{t_1}^{t_2}\end{aligned}$$

in view of (1.16). Hence, $\delta_\varphi J(\xi,\rho,\varphi)\{\delta\varphi\} = 0$ for all $\delta\varphi$ with $\delta\varphi|_{t_1,t_2} = 0$ implies φ to be a solution of (1.8), (1.9); note that in this situation the solvability condition for (1.8), (1.9) is met automatically. With this velocity potential φ we obtain

$$J = \int_{t_1}^{t_2} \int_{Q(t)} \frac{1}{2}\rho|\nabla\varphi|^2\, dQ\, dt + \int_{Q(t)} \rho\varphi\, dQ \Big|_{t_1}^{t_2}.$$

The ξ,ρ-variations of the second term on the right-hand side vanishes if $\delta\xi$, $\delta\rho$ satisfies (1.14). Thus we get after comparision with (1.12) and Theorem 1.1:

Theorem 1.2. *Any sufficiently regular critical point* (ξ,ρ,φ) *of the functional*

$$\begin{aligned}B(\xi,\rho,\varphi) = \int_{t_1}^{t_2} \Big\{ &\int_{Q(t)} -\rho\Big(\dot\varphi + \frac{1}{2}|\nabla\varphi|^2 + U + W(\rho)\Big)\, dQ \\ &- \sigma\big(|\Sigma(t)| - \beta|S_1(t)|\big) + \int_{S_2} V\varphi\, dS - p_0|Q(t)| \Big\} dt\end{aligned}$$

subject to

$$\delta\xi|_{t_1,t_2} = 0, \quad \delta\rho|_{t_1,t_2} = 0, \quad \delta\varphi|_{t_1,t_2} = 0$$

satisfies (1.1)-(1.6). □

The following Theorem 1.3, where ξ and φ have been introduced as independent variables, can be viewed as a counterpart to Theorem 1.2. Let P be a primitive of $1/\rho$:

$$P(\tau) = \int \frac{d\tau}{\rho(\tau)} + const.,$$

where the constant is choosen such that

$$\frac{d}{d\rho}\big(\rho W(\rho)\big) = P\big(\rho^2 W'(\rho)\big). \tag{1.22}$$

Since P is strictly monotone the inverse function P^{-1} exists. With this function we have

Theorem 1.3. *Under the constraints*

$$\delta\xi|_{t_1,t_2} = 0, \quad \delta\varphi|_{t_1,t_2} = 0$$

any sufficiently regular critical point (ξ,φ) *of the functional* $C = C(\xi,\varphi)$ *with*

$$C(\xi,\varphi) = \int_{t_1}^{t_2} \Big\{ \int_{Q(t)} P^{-1}\Big(-\dot\varphi - \frac{1}{2}|\nabla\varphi|^2 - U\Big)\, dQ \\ - \sigma\big(|\Sigma(t)| - \beta|S_1(t)|\big) + \int_{S_2} V\varphi\, dS - p_0|Q(t)| \Big\}\, dt$$

satisfies (1.1)-(1.6).

Proof. For given (ξ,φ) we determine $\rho = \rho[\xi,\varphi]$ with

$$\dot\varphi + \frac{1}{2}|\nabla\varphi|^2 + U + W(\rho) + \rho W'(\rho) = 0 \quad \text{in} \quad Q(t).$$

This is equivalent to

$$\delta_\rho B(\xi,\rho,\varphi)\{\delta\rho\} = 0 \quad \text{for all} \quad \delta\rho \tag{1.23}$$

with the functional $B = B(\xi,\rho,\varphi)$ from Theorem 1.2. In view of (1.22) this means

$$-\Big(\dot\varphi + \frac{1}{2}|\nabla\varphi|^2 + U\Big) = \frac{d}{d\rho}\big(\rho W(\rho)\big) = P\big(\rho^2 W'(\rho)\big) \\ = P\Big(-\rho\Big(\dot\varphi + \frac{1}{2}|\nabla\varphi|^2 + U + W\Big)\Big),$$

hence $C(\xi,\varphi) = B(\xi,\rho,\varphi[\xi,\varphi])$ and Theorem 1.2 gives the assertion. □

2 High-frequency exitations

In this section the free boundary problem (1.1)-(1.6) is considered with a time-dependent high-frequency potential

$$U(x,t) = -gx_3 - \omega^2 a_i x_i \sin(\omega t).$$

Here the usual summation convention over repeated indices is used. In the following we study the behaviour of the system if

$$\omega \to \infty, \qquad \omega|a| = const., \tag{2.1}$$

i.e. under the influence of a time-periodic volume force with an amplitude increasing proportionally to the frequency. We disregard any additional acoustic source at the boundary, hence we may set $V = 0$ and $S(t) = S_1(t) + S_2$. In addition, (1.6) is specified to the adiabatic pressure-density relation

$$\rho = \rho_0(p/p_0)^{1/\gamma} \qquad (\gamma > 1).$$

For our purposes it is advantageous to rewrite the system (1.1)-(1.6) in nondimensional form. Letting l be a representative length, we replace the original domains and variables according to

$$Q_{new}(t) = l^{-1}Q(t), \quad \Sigma_{new}(t) = l^{-1}\Sigma(t), \quad x_{new} = l^{-1}x, \quad t_{new} = \omega t,$$

as well as

$$\varphi_{new} = \varphi/l^2\omega, \quad p_{new} = p/\rho_0 l^2\omega^2, \quad \rho_{new} = \rho/\rho_0,$$
$$p_{0,new} = p_0/\rho_0 l^2\omega^2, \quad a_{new} = a/|a|.$$

Then, introducing the nondimensional parameters

$$\varepsilon = |a_{orig}|/l, \quad \mu = \sigma/\omega^2|a_{orig}|^2 l\rho_0, \quad b = gl^2\rho_0/\sigma \text{ (“Bond number”)},$$

and retaining the original notation, the system (1.1)-(1.6) takes the form

$$\rho\nabla\Big(\dot\varphi + \frac{1}{2}|\nabla\varphi|^2 + \mu\varepsilon^2 b x_3 + \varepsilon a_i x_i \sin t\Big) = -\nabla p, \tag{2.2}$$
$$\dot\rho + \operatorname{div}(\rho\nabla\varphi) = 0 \quad \text{in} \quad Q(t), \tag{2.3}$$

subject to the boundary conditions

$$\partial_n\varphi = 0 \text{ on } S(t), \quad \partial_n\varphi = -|\nabla\xi|^{-1}\dot\xi \text{ on } \Sigma(t), \tag{2.4}$$
$$p - 2\mu\varepsilon^2 H = p_0 \text{ on } \Sigma(t), \quad -\nabla\eta\nabla\xi = \beta|\nabla\eta||\nabla\xi| \text{ on } \partial\Sigma(t). \tag{2.5}$$

With respect to the new variables the pressure-density relation reads as

$$\rho = (p/p_0)^{1/\gamma}. \tag{2.6}$$

According to (2.1) we consider (2.2)-(2.6) under the hypothesis $\varepsilon \ll 1$ and μ, b fixed. We restrict out attention to a cylindrical container $\widetilde{Q} = B \times [0,\infty)$ over a fixed bottom $B \subset \mathbb{R}^2$. The free surface is assumed to be a graph over B:

$$\Sigma(t) = \{x \in \mathbb{R}^3 \mid x_3 = \zeta(x_1, x_2, t), (x_1, x_2) \in B\},$$

i.e. $\xi = x_3 - \zeta$. In this situation, according to Theorem 1.1, which is referred to in the following exclusively, we obtain (2.2)-(2.6) as Euler-Lagrange equations of the action-functional

$$A(\zeta,\rho;\varepsilon) = \int_{t_1}^{t_2}\Big\{\int_{Q(t)}\rho\Big(\frac{1}{2}|\nabla\varphi|^2 - W - \mu\varepsilon^2 b x_3 - \varepsilon a_i x_i \sin t\Big)\,dQ$$
$$- \mu\varepsilon^2\big(|\Sigma(t)| - \beta|S(t)|\big) - p_0|Q(t)|\Big\}\,dt. \tag{2.7}$$

under the constraint of mass conservation. As a consequence of the adiabatic pressure-density relation the inner energy density is given by

$$W(\rho) = const. + p_0\rho^{\gamma-1}/(\gamma-1).$$

In the following we construct approximate solutions (in the sense explained below) to the variational equation

$$\delta A(\zeta, \rho; \varepsilon) = 0 \quad \text{subject to} \quad \delta \int_{Q(t)} \rho \, dx = 0 \tag{2.8}$$

within the class of 2π-periodic functions in time. Accordingly, time varies in $S^1 = \mathbb{R}/2\pi$. Choosing $t_1 = 0, t_2 = 2\pi$, we have to replace (1.14) by the periodicity conditions

$$\zeta(\cdot, 0) = \zeta(\cdot, 2\pi), \quad \rho(\cdot, 0) = \rho(\cdot, 2\pi).$$

If $\varepsilon = 0$, then any pair $(\zeta_0, 1)$ with a time-independent shape $\zeta_0 = \zeta_0(x_1, x_2)$ of the free surface is a solution of (2.8), which simply reflects the fact that an isolated fluid at rest is in neutral equilibrium. Let

$$Q_0 = \{x \in \mathbb{R}^3 \mid (x_1, x_2) \in B, 0 < x_3 < \zeta_0(x_1, x_2)\},$$

then

$$A(\zeta_0, 1; 0) = -2\pi\big(p_0 + W(1)\big)|Q_0| = 0, \tag{2.9}$$

if the inner energy density W is suitably normalized: $W(1) + W'(1) = 0$. A closer look at (2.7) shows, that

$$\delta A(\zeta_0, 1; 0) = 0 \quad \text{for arbitrary variations } \delta\zeta, \delta\rho \tag{2.10}$$

under this normalization. Therefore it is reasonable to choose the ansatz

$$\zeta_\varepsilon = \zeta_0(x_1, x_2) + \varepsilon\zeta_1(x_1, x_2, t), \quad \rho_\varepsilon = 1 + \varepsilon\rho_1(x, t) \tag{2.11}$$

as the starting point for our construction. Here ζ_1 is normalized by mean value zero in time:

$$\int_0^{2\pi} \zeta_1(x_1, x_2, t) \, dt = 0. \tag{2.12}$$

The side condition in (2.8) requires

$$|Q_0| = \int_B \zeta_0(x_1, x_2) \, dB = const., \tag{2.13}$$

$$\int_{Q_0} \rho_1(x, t) \, dQ + \int_B \zeta_1(x_1, x_2, t) \, dB = 0. \tag{2.14}$$

In view of (2.9), (2.10) we get by inserting (2.11) into (2.7)

$$A(\zeta_\varepsilon, \rho_\varepsilon, \varepsilon) = \varepsilon^2 \widetilde{A}(\zeta_0, \zeta_1, \rho_1) + O(\varepsilon^3) \tag{2.15}$$

with

$$\widetilde{A} = \int_0^{2\pi} \left\{ \int_{Q_0} \frac{1}{2}|\nabla\varphi_1|^2 - \frac{\rho_1^2}{2k^2} - \mu b x_3 - a_i x_i \rho_1 \sin t \, dQ \right. \\ \left. - \int_{\Sigma_0} a_i x_i \zeta_1 \sin t \, dB \right\} dt - 2\pi\mu\big(|\Sigma_0| - \beta|S_0|\big). \tag{2.16}$$

Here, the "wave number" $k = (\gamma p_0)^{-1/2}$ has been introduced, Σ_0, S_0 denote the free and wetting parts of ∂Q_0, respectively, and φ_1 is the first order term in the expansion

$$\varphi(\zeta_\varepsilon, \rho_\varepsilon) = \varepsilon\varphi_1(\zeta_0, \zeta_1, \rho_1) + O(\varepsilon^2).$$

As may be read off (2.3), (2.4) the potential φ_1 solves the Neumann problem

$$\Delta\varphi_1 = -\dot{\rho_1} \text{ in } Q_0; \quad \partial_n\varphi_1 = 0 \text{ on } S_0, \quad \partial_n\varphi_1 = (1 + |\nabla\zeta_0|^2)^{-1/2}\dot{\zeta}_0 \text{ on } \Sigma_0. \tag{2.17}$$

The expansion (2.15) motivates the definition to call the pair $(\zeta_\varepsilon, \rho_\varepsilon)$ an *ε-approximate solution* of (2.8) if $(\zeta_0, \zeta_1, \rho_1)$ is a critical point of the truncated action, i.e.

$$\delta\widetilde{A}(\zeta_0, \zeta_1, \rho_1) = 0 \tag{2.18}$$

for all variations $\delta\zeta_0, \delta\zeta_1, \delta\rho_1$ compatible with (2.12)-(2.14). To determine solutions of (2.18), firstly, we have to compute the ζ_1, ρ_1-variations of $\widetilde{A}$.

Proposition 2.1. *For fixed ζ_0 the solution of the Euler-Lagrange equations*

$$\delta_{\zeta_1}\widetilde{A}(\zeta_0, \zeta_1, \rho_1)\{\delta\zeta_1\} + \delta_{\rho_1}\widetilde{A}(\zeta_0, \zeta_1, \rho_1)\{\delta\rho_1\} = 0$$

– for all variations $\delta\zeta_1, \delta\rho_1$ compatible with (2.12), (2.14) *– leads to a time-periodic boundary value problem for an inhomogeneous wave equation:*

$$\ddot{\varphi}_1 - k^{-2}\Delta\varphi_1 = -a_i x_i \cos t \quad \textit{in} \quad Q_0 \times S^1, \tag{2.19}$$

$$\partial_n\varphi_1 = 0 \;\; \textit{on} \;\; S_0 \times S^1, \qquad \dot{\varphi}_1 = -a_i x_i \sin t \;\; \textit{on} \;\; \Sigma_0 \times S^1, \tag{2.20}$$

$$\int_0^{2\pi} \partial_n\varphi_1(\cdot, t)\, dt = 0 \quad \textit{on} \quad \Sigma_0. \tag{2.21}$$

After solving (2.19)-(2.21) *we get ζ_1, ρ_1 from*

$$\dot{\zeta}_1 = (1 + |\nabla\zeta_0|^2)^{1/2}\partial_n\varphi_1 \quad \textit{on} \quad \Sigma_0 \times S^1, \tag{2.22}$$

$$\rho_1 = -k^2(\dot{\varphi}_1 + a_i x_i \sin t) \quad \textit{in} \quad Q_0 \times S^1. \tag{2.23}$$

In view of (2.12) *ζ_1 is determined uniquely by* (2.22).

Proof. Any stationary point of $\delta\widetilde{A} = 0$ subject to (2.12), (2.14) satisfies

$$\dot{\varphi}_1 + k^{-2}\rho_1 = -a_i x_i \sin t + \lambda(t) \quad \text{in} \quad Q_0 \times S^1, \tag{2.24}$$

$$\dot{\varphi}_1 = -a_i x_i \sin t - \lambda(t) + c(x) \quad \text{on} \quad \Sigma_0 \times S^1, \tag{2.25}$$

with two Lagrangian multipliers λ and c. Since

$$\int_0^{2\pi} \lambda(t)\, dt = 2\pi c(x)$$

by (2.25), it follows $c(x) = c = const.$. After normalizing φ_1 suitably we may assume $\lambda = const. = c$. In this case integration of (2.24) yields

$$2k^2\pi c|Q_0| = \int_0^{2\pi}\int_{Q_0} \rho_1\, dQdt = -\int_0^{2\pi}\int_B \zeta_1\, dBdt = 0$$

because of (2.12), hence $c = 0$. Now, after differentiation with respect to t, (2.24), (2.25) imply (2.19), (2.20). □

To outline the solvability of the boundary value problem (2.19)-(2.21), let $\Lambda(\zeta_0)$ denote the spectrum of the Neumann-Dirichlet problem for the Laplace equation:

$$\Delta u + \lambda u = 0 \text{ in } Q_0; \quad \partial_n u = 0 \text{ on } S_0, \quad u = 0 \text{ on } \Sigma_0. \tag{2.26}$$

Under mild regularity assumptions on Σ_0 and ∂B the embedding of the Sobolev space $H^1(Q_0)$ into $L^2(Q_0)$ is compact and the trace operator $u \mapsto u|_{\Sigma_0}$ maps $H^1(Q_0)$ into $H^{1/2}(\Sigma_0)$ continuously, see e.g. [24]. Then the set $\Lambda(\zeta_0)$ consists of a countable number of positive reals with the unique limit point $+\infty$. For $k^2 \notin \Lambda(\zeta_0)$

$$\varphi_1(x,t) = \psi(x)\cos t \tag{2.27}$$

is a solution of (2.19)-(2.20) if ψ is chosen according to

$$\Delta\psi + k^2\psi = k^2 a_i x_i \text{ in } Q_0; \quad \partial_n\psi = 0 \text{ on } S_0, \quad \psi = a_i x_i \text{ on } \Sigma_0. \tag{2.28}$$

Then, in view of (2.27) we get via (2.22), (2.23)

$$\zeta_1(x_1,x_2,t) = \zeta_1^\star(x_1,x_2)\sin t, \qquad \rho_1(x,t) = \rho_1^\star(x)\sin t, \tag{2.29}$$

with

$$\zeta_1^\star = (1+|\nabla\zeta_0|^2)^{1/2}\partial_n\psi, \qquad \rho_1^\star = k^2(\psi - a_i x_i). \tag{2.30}$$

Moreover, if $n^2k^2 \notin \Lambda(\zeta_0)$ for all $n \in \mathbb{Z}$ then (2.27) is the unique solution up to a constant, as is easily seen by the Fourier separation method, see e.g. [12].

Concerning the remaining derivative of $\widetilde{A}$ with respect to ζ_0 we get, after a calculation along the lines followed in the proof of Theorem 1.1,

$$\begin{aligned}\delta_{\zeta_0}\widetilde{A} = \int_0^{2\pi}\int_{\Sigma_0} \Big\{ -\frac{1}{2}|\nabla\varphi_1|^2 + \dot\zeta_1\partial_3\varphi_1 - \dot\rho_1\varphi_1 - \frac{\rho_1^2}{2k^2} \\ - \mu\varepsilon^2 b x_3 - (a_i x_i\rho_1 + a_3\zeta_1)\sin t \Big\}\, \delta\zeta_0\, dB\, dt \\ + 2\pi\mu\int_B \operatorname{div} \mathsf{T}\zeta_0\delta\zeta_0\, dB - 2\pi\mu\int_{\partial B}(\nu\cdot\mathsf{T}\,\zeta_0 - \beta)\delta\zeta_0\, dl\end{aligned}$$

if the variation $\delta\zeta_0$ is compatible with the side conditions (2.13), (2.14). To shorten the notation, the nonlinear operator

$$\mathsf{T}\zeta_0 = (1+|\nabla\zeta_0|^2)^{-1/2}\nabla\zeta_0$$

and the outer normal ν to ∂B have been introduced. Evaluating $\delta_{\zeta_0}\widetilde{A}$ at the extremal ζ_1, ρ_1 of Proposition 2.1 leads to

Theorem 2.2. *If $k^2 \notin \Lambda(\zeta_0)$ and ψ, $\zeta_1^\star$, $\rho_1^\star$ are taken from* (2.28), (2.30) *then the pair*

$$(\zeta_\varepsilon, \rho_\varepsilon) = (\zeta_0 + \varepsilon\zeta_1^\star \sin t, 1 + \varepsilon\rho_1^\star \sin t)$$

defines an ε-approximate solution of $\delta A = 0$ if

$$4\mu H + \frac{1}{2}|\nabla(\psi - a_i x_i)|^2 - 2\mu b x_3 = \lambda \quad on \quad \Sigma_0, \tag{2.31}$$

$$\nu \cdot \mathsf{T}\zeta_0 = \beta \quad on \quad \partial B \tag{2.32}$$

with a Lagrange multiplier $\lambda = const.$ □

The principal part $H = \frac{1}{2} \operatorname{div} \mathsf{T}\zeta_0$ of (2.31) is the mean curvature of the surface Σ_0; hence (2.31) generalizes the equilibrium condition for a fluid-air interface in a vertical gravity field known from capillary theory, cf. [6]. In our setting, due to vibration, an additional nonlinear first-order pseudo-differential operator is to be included in the equilibrium condition. Alternatively (2.31) can be viewed as a counterpart to Bernoulli's equation for incompressible fluid flows. In the sequel we call any surface Σ_0 given by a solution ζ_0 of (2.31), (2.32) a *vibrocapillary equilibrium shape.*

An integration of (2.31) by parts yields

$$2\mu\big(\beta\,|\partial B| - b\,|Q_0|\big) \leq |B|\lambda,$$

which means that the Lagrange multiplier λ in (2.31) is bounded from below in terms of the given data. In the pure capillary case there holds equality.

As is clear from the above reasoning equations (2.31), (2.32) must appear as variational equations.

Theorem 2.3. *Under the assumptions of Theorem 2.2 any solution ζ_0 of the equilibrium conditions* (2.31), (2.32) *is a critical point of the time-independent functional*

$$\Pi(\zeta_0) = -\frac{1}{\pi}\widetilde{A}(\zeta_0, \zeta_1^\star \sin t, \rho_1^\star \sin t)$$

under volume conserving variations. The explicit expression of Π reads as

$$\Pi(\zeta_0) = 2\mu\big(|\Sigma_0| - \beta|S_0|\big) + \frac{1}{2}\int_{Q_0} \big(|\nabla\psi|^2 - k^2(\psi - a_i x_i)^2 + 4\mu b x_3\big)\, dQ. \square$$

Π is henceforth referred to as the *quasi-potential of vibrocapillarity.*

3 The Jacobi operator

In this Section, to get further insight into diverse mapping properties of the quasi-potential Π, we study its second variation $\delta^2\Pi$. This may be of particular interest in stability considerations as well as in various numerical approaches. In the following considerations we identify functions originally defined on Σ_0 by constant continuation along x_3-direction with functions on B.

Obviously the second variation of the capillary term $\Pi_0(\zeta_0) = |\Sigma_0| - \beta|S_0|$ in Π reads as:

$$\delta^2\Pi_0(\zeta_0)\{h,h\} = \int_B \big(|\nabla h|^2 - (\mathsf{T}\zeta_0 \cdot \nabla h)^2\big) n_3 \, dB.$$

Since admissible variations h must have mean value zero, this implies

$$\delta^2\Pi_0(\zeta_0)\{h,h\} \geq pos.C^{te}\|h\|_1^2$$

in view of Friedrich's inequality and $|\mathsf{T}\zeta_0| < 1$. Here and in the following $\|\cdot\|_s$ denote the norms in the Sobolev spaces H^s. Introducing the tangential gradient $\nabla_{\Sigma_0} = (D_1, D_2, D_3)$ and the Laplace-Beltrami operator Δ_{Σ_0}:

$$\nabla_{\Sigma_0} = \nabla - n\partial_n, \qquad \Delta_{\Sigma_0} = D_i D_i$$

there holds $|\nabla h|^2 - (\mathsf{T}\zeta_0 \cdot \nabla h)^2 = |\nabla_{\Sigma_0} h|^2$ and an integration by parts implies

$$\delta^2\Pi_0(\zeta_0)\{h,h\} = -\int_{\Sigma_0} h \operatorname{div}_{\Sigma_0}\big(n_3^2 \nabla_{\Sigma_0} h\big)\, d\Sigma$$

if $h = 0$ on ∂B. Remembering the relation

$$\Delta_{\Sigma_0} n_j = -c^2 n_j - 2D_j H,$$

where c^2 is the sum of the squares of the principal curvatures of Σ_0, see e.g. [9], we infer

$$\begin{aligned}\operatorname{div}_{\Sigma_0}\big(n_3^2\nabla_{\Sigma_0}\big) &= \operatorname{div}_{\Sigma_0}\big(n_3\nabla_{\Sigma_0}(n_3 h)\big) - \operatorname{div}_{\Sigma_0}\big(n_3 h \nabla_{\Sigma_0}(n_3)\big)\\ &= n_3\big(\Delta_{\Sigma_0}(n_3 h) + (2D_3 H + c^2 n_3)h\big).\end{aligned}$$

Hence

$$Lh = -n_3\Delta_{\Sigma_0}(n_3 h) - n_3(2D_3 H + c^2 n_3)h$$

gives the Euler-Lagrange operator to $\delta^2\Pi_0$.

According to Theorem 2.2 the first variation of the nonlocal part

$$\Pi_1(\zeta_0) = \frac{1}{2}\int_{Q_0} \big(|\nabla\psi|^2 - k^2(\psi - a_i x_i)^2\big)\, dQ.$$

of Π reads as

$$\delta\Pi_1(\zeta_0)\{h\} = -\frac{1}{2}\int_{\Sigma_0} |\nabla(\psi - a_i x_i)|^2 h \, dB.$$

This implies

$$\delta^2\Pi_1(\zeta_0)\{h,h\} = -\int_{\Sigma_0}\Big(\nabla(\psi - a_i x_i)\nabla\delta\psi + h\nabla(\psi - a_i x_i)\nabla(\psi_{x_3} - a_3)\Big) h\, dB.$$

Here $\delta\psi$ has to satisfy the Dirichlet-Neumann Problem

$$\Delta\delta\psi + k^2\delta\psi = 0 \text{ in } Q_0; \quad \partial_n\delta\psi = 0 \text{ on } S, \quad \delta\psi = (a_3 - \psi_{x_3})h \text{ on } \Sigma_0.$$

Considering the expression for the Laplacian relative to normal and tangential derivatives along Σ_0

$$\Delta = \partial_n^2 - 2H\partial_n + \Delta_{\Sigma_0}, \qquad \partial_n^2 = n_i n_j \partial_i \partial_j$$

and $\psi - a_i x_i|_{\Sigma_0} = 0$, $\Delta\psi|_{\Sigma_0} = 0$ we get

$$\partial_n^2(\psi - a_i x_i) = 2H\partial_n(\psi - a_i x_i).$$

In view of $\partial_3 = n_3\partial_n + D_3$ there holds

$$\partial_n\partial_3 = n_3\partial_n^2 + n_i D_3 \partial_i = n_3\partial_n^2 + D_3\partial_n - (D_3 n_i)\partial_i$$

and consequently

$$\partial_n(\psi_{x_3} - a_3) = (2n_3 H + D_3)\partial_n(\psi - a_i x_i).$$

We have proved

Proposition 3.1. *Under the assumption $k^2 \notin \Lambda(\zeta_0)$ let*

$$C_{\Sigma_0} : H^{1/2}(\Sigma_0) \to H^{-1/2}(\Sigma_0), \quad C_{\Sigma_0}(u) = \partial_n \widetilde{u}|_{\Sigma_0},$$

be the "capacity operator" where $\widetilde{u}$ denotes the solution of

$$\Delta\widetilde{u} + k^2\widetilde{u} = 0 \text{ in } Q_0; \quad \partial_n\widetilde{u} = 0 \text{ on } S, \quad \widetilde{u} = u \text{ on } \Sigma_0.$$

Then there holds

$$\delta^2\Pi_1(\zeta_0)\{h,h\} = \int_{\Sigma_0} (\psi_{x_3} - a_3)hC_{\Sigma_0}\big((\psi_{x_3} - a_3)h\big)\, d\Sigma - \frac{1}{2}\int_{\Sigma_0} h^2(4n_3H + D_3)|\nabla(\psi - a_i x_i)|^2\, dB$$

for variations h with mean value zero. □

In view of

$$\int_{\Sigma_0} uC_{\Sigma_0}u\, d\Sigma_0 = \int_{Q_0} |\nabla\widetilde{u}|^2 - k^2\widetilde{u}^2\, dQ$$

the principal part of the capacity operator is positive, hence Proposition 3.1 implies

Corollary 3.2. *If the data are sufficiently regular then $\delta^2\Pi_1(\zeta_0)$ is bounded on the Sobolev-space $H^{1/2}(B)$:*

$$-C^{te}\|h\|_0^2 \le \Pi_1(\zeta_0)\{h,h\} \le C^{te}\|h\|_{\frac{1}{2}}^2.$$

If additionally

$$|\psi_{x_3} - a_3| \ge c > 0 \quad \text{on} \quad \Sigma_0,$$

then $\delta^2\Pi_1(\zeta_0)$ satisfies the Garding-type inequality

$$\Pi_1(\zeta_0)\{h,h\} \ge pos.C^{te}\|h\|_{\frac{1}{2}}^2 - C^{te}\|h\|_0^2. \square$$

To collect the results of this section in a general statement we introduce the Jacobi operator $J(\zeta_0)$ of Π via

$$\delta^2\Pi(\zeta_0)\{g,h\} = \int_{\Sigma_0} gJ(\zeta_0)\{h\}\, d\Sigma_0 \quad \text{if} \quad g|_{\partial B} = h|_{\partial B} = 0.$$

Theorem 3.3. *Assuming the data to be sufficiently regular, then: (i) The bilinear form $\delta^2\Pi(\zeta_0)$ satisfies the Garding-type inequality*

$$\Pi(\zeta_0)\{h,h\} \geq pos.C^{te}\|h\|_1^2 - C^{te}\|h\|_0^2.$$

(ii) $J(\zeta_0)$ reads as

$$J(\zeta_0)h = -2\mu n_3 \Delta_{\Sigma_0}(n_3 h) + (\psi_{x_3} - a_3)C_{\Sigma_0}\big((\psi_{x_3} - a_3)h\big) + mn_3 h$$

with

$$\begin{aligned} m = &- 2\mu c^2 n_3 - 2n_3 H|\nabla(\psi - a_i x_i)|^2 + 2b\mu n_3^2 \\ &- D_3\Big(4\mu H + \frac{1}{2}|\nabla(\psi - a_i x_i)|^2 - 2b\mu x_3\Big). \end{aligned}$$

In particular

$$m = -2\mu c^2 n_3 - 2n_3 H|\nabla(\psi - a_i x_i)|^2 + 2b\mu n_3^2, \tag{3.1}$$

if ζ_0 is a critical point of Π. □

Note, that the nonlocal term in (3.1) may be replaced by

$$2n_3 H|\nabla(\psi - a_i x_i)|^2 = 2n_3 H(2\lambda + 4\mu b x_3 - 8\mu H).$$

Finally we draw attention to a simple stability criterium. Obviously, for sufficiently small wave numbers k the capacity operator is nonnegative. Therefore, if $H \leq 0$ and $bn_3 \geq c^2$, then a vibrocapillary equilibrium is stable in the sense that

$$\delta^2\Pi(\zeta_0)\{h,h\} \geq pos.C^{te}\|h\|_1^2.$$

for all "interior" variations h with mean value zero. This indicates that for convex equilibrium shapes a "vibro-force" may cause a stabilizing effect.

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