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On the Construction of Selfpolar and Selfpolar Hilbertian Norms on Inner Product Spaces

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Abstract

Noticing that the selfpolar Hilbertian norms are the appropriate substitutes for the canonical norms when a pre-Hilbert space is replaced by a (possibly indefinite) inner product space, the Aronszajn-Schatten iteration of selfpolar norms is discussed.

Furthermore, we discuss the question of how many non-equivalent selfpolar and selfpolar Hilbertian norms exist on (non-degenerate) inner product spaces of countably infinite dimension by constructing a large subclass of such norms and deciding which of them are equivalent. As a preparation for this discussion, we give a complete and constructive classification of all selfpolar and all selfpolar Hilbertian norms on two-dimensional Minkowski space.

1 Introduction

Spaces with indefinite inner product were first applied to quantum field theory (QFT) by Dirac and Pauli about 70 years ago (cf. [6], [21]). Since the Gupta-Bleuler formalism of QED (quantum electro-dynamics) it is clear that the use of indefinite metric is indispensable. As explained by Kugo, Nakanishi, Ojima, and Strocchi, now the use of indefinite metric is quite natural as the framework of QFT, and especially, of the operator formalism of gauge theories and quantum gravity (cf. [17], [18], [19], [25]). In those theories the Hilbert space of state vectors is replaced by some Krein space ([4], [5]).

Considering a pre-Hilbert (unitary) space $E, [.,.]$ (i.e., the inner product $[.,.]$ is positive definite), a canonical norm

$$p(x) = \sqrt{[x,x]}, \tag{1.1}$$

$x \in E$, exists on E. Recall that the following apply:

(a) p is the unique selfpolar norm on $E, [.,.]$, i.e., $p(x) = \sup_{0\neq y\in E} \frac{|[x,y]|}{p(y)}$, $x \in E$ (cf. [7, pp 891 -892]).

(b) p satisfies the parallelogram identity.

(c) the completed hull $\widetilde{E}^{\tau(p)}$ endowed with the continuously extended inner product $[.,.]^{\sim}$ is a Hilbert space, where $\tau(p)$ denotes the topology defined by the norm p.

Motivated by the first paragraph, it is of interest to find the corresponding substitutes when the pre-Hilbert space is replaced by some non-degenerate inner product space $E, (.,.)$, (i.e., E is a real or complex vector space equipped with a bilinear symmetric respective sesquilinear Hermitian form $(.,.)$ which is non-degenerate; in the latter case it is assumed that the inner product $(.,.)$ is antilinear in its first variable).

If the inner product $(.,.)$ is indefinite, then (1.1) does not define a norm, and thus, no canonical norm is available. The role of the canonical norm (1.1) is now played by the selfpolar Hilbertian norms q on E, i.e., it holds

$$q(x) = \sup_{0\neq y\in E} \frac{|(x,y)|}{q(y)}, \tag{1.2}$$

and furthermore, there is a positive definite inner product $[.,.]$ on $E \times E$ such that

$$q(x) = \sqrt{[x,x]}, \tag{1.3}$$

$x \in E$. Recall also that a norm is Hilbertian (i.e., (1.3) applies) if, and only if, it satisfies the parallelogram identity (cf. [15, p. 241]). The above (a), (b) show that the selfpolar Hilbertian norms are the natural generalization of the canonical norm when the class of pre-Hilbert spaces is generalized to inner product spaces.

The significance of selfpolar and selfpolar Hilbertian norms is due to the following:

(I) If q is a selfpolar norm on a non-degenerate inner product space $E, (.,.)$, then

(i) the inner product $(.,.)$ extends $\tau(q)$-continuously onto $\widetilde{E} \times \widetilde{E}$,

(ii) the inner product space $\widetilde{E}, (.,.)^{\sim}$ is non-degenerate.

(II) If q is a selfpolar Hilbertian norm on a non-degenerate inner product space $E, (.,.)$, then $\widetilde{E}, (.,.)^{\sim}$ is a Krein space ([11, Th.1(iii), Prop. 2]), i.e., $\widetilde{E}, [.,.]$ ($q(.) = \sqrt{[.,.]}$) is a Hilbert space and the Gram operator J defined by $(x,y)^{\sim} = [x, Jy]$, $x, y \in \widetilde{E}$ is a symmetry ($J = J^* = J^{-1}$).

Notice also that (c) is generalized from the case of positive definite inner products $[.,.]$ to (possibly indefinite) inner products in the following:

(c') if q is a Hilbertian selfpolar norm on E, then the completed hull $\widetilde{E}^{\tau(q)}$ endowed with the continuously extended inner product $(.,.)^{\sim}$ is a Krein space, (cf. [12]).

In generalization of the positive definite case, where the canonical norm is uniquely defined, the following is possible in the general case of an inner product space:

(i) there is no selfpolar norm on $E, (.,.)$ (cf. [5, Example III.3.2]).

(ii) there is exactly one normed self-polar topology τ on $E, (.,.)$, i.e., a selfpolar norm p defining τ exists, and furthermore, every selfpolar norm on E defines a topology equivalent to τ (cf. [12], and references cited there).

(iii) there is a whole family of selfpolar norms defining non-equivalent topologies on $E, (.,.)$ (such a family of selfpolar norms was explicitely constructed by Araki in [2]).

Along these lines, it is known that if $E, (.,.)$ is non-degenerate and quasi-definite (i.e. quasi-positive or quasi-negative, where quasi-positive (-negative) means that E has no negative (positive) definite subspace of infinite dimension), then all selfpolar norms are equivalent to the same selfpolar Hilbertian norm [5].

While an operator description for selfpolar Hilbertian norms is developed by Hansen on the special class of inner product spaces which allow a complete Hilbertian majorant ([7], see Theorem 3.9), the present paper is aimed at explicit constructions of as well selfpolar as selfpolar Hilbertian norms on non-degenerate inner product spaces.

In section 2 the Aronszajn-Schatten iteration of such norms is given. Our results are a generalization of as well Aronszajn's iteration of selfpolar norms on inner product spaces as of Schatten's iteration of the σ-topology on the tensor product $E \otimes E$ (cf. [3], [7], [20], [24], [26]). It is shown in Theorem 2.8 that for each positively homogeneous functional p on a non-degenerate inner product space satisfying $p' \leq p$ (see Notation 2.1), a selfpolar norm p_∞ is constructed by an iteration process starting from p. This iteration can be used for a numerical calculation, and as well a priori as a posteriori estimates are obtained in Lemma 2.12. Furthermore, if p is Hilbertian, then the selfpolar norm p_∞ is Hilbertian, too (see Corollary 2.11).

While the Aronszajn-Schatten iteration yields the existence of selfpolar norms, it does not answer the question whether or not all selfpolar norms are equivalent. Furthermore it is of interest whether a given selfpolar norm is equivalent to a Hilbertian selfpolar norm. Section 3 is devoted to these questions. For the class of inner product spaces of countable algebraic dimension and a huge family of selfpolar norms on them an answer is given in Theorem 3.12. These inner product spaces are of interest for the investigation of some special models of gauge-field theories ([1], [8], [9], [10], [13], [14],).

The analysis of section 3 relies on a complete classification of the selfpolar and selfpolar Hilbertian norms on $\mathcal{M}^2$ – the real Minkowski space of dimension two – only the results of which are given there. The proofs of these results are postponed until section 4, where all selfpolar and selfpolar Hilbertian norms on $\mathcal{M}^2$ are explicitely constructed (section 4 is to some extent independent of the rest of the paper and may be omitted, if the reader is not interested in the details of the two-dimensional case). It turns out that in contrast to the positive definite case, where exactly one selfpolar Hilbertian norm exists, on $\mathcal{M}^2$ there are as well non-denumerably many selfpolar as non-denumerably many selfpolar Hilbertian norms. The idea of the construction is to describe the norms p on $\mathcal{M}^2$ by the boundaries ∂U_p of their unit balls U_p, where ∂U_p are convex curves. Polarization of p in $\mathcal{M}^2$ then transforms to a Legendre-like transformation "*" of convex curves, which is introduced in section 4.1. In order to construct convex curves which are invariant under the *-operation, and thus describe selfpolar norms, it is shown that every arc connecting the two mass shells $h_1 := \{x = (x^0, x^1) \in \mathcal{M}^2; x^2 = 1, x^0 > 0\}$, $h_2 := \{x = (x^0, x^1) \in \mathcal{M}^2; x^2 = -1, x^1 > 0\}$, and lying completely between them (see Figure 2), completes to such a *-invariant curve on $\mathcal{M}^2$. Furthermore, it is shown that every convex curve related to a selfpolar norm is obtained by that construction (see Theorem 4.17). The special case of selfpolar Hilbertian norms on $\mathcal{M}^2$ is considered in Proposition 3.10.

2 Construction of selfpolar norms using the Aronszajn-Schatten iteration

Letting $E, (.,.)$ be a non-degenerate inner product space, the following is aimed at defining selfpolar norms on E. Though p will mostly be a norm in the subsequent applications, it is first assumed that p only satisfies

$$p(\lambda x) = |\lambda|\, p(x), \quad 0 \leq p(x) < \infty \tag{2.1}$$

for each $x \in E$, $\lambda \in \mathbb{C}$. Notice that $U_p = \{x \in E; p(x) \leq 1\}$ is circled and absorbent, but not necessarily absolutely convex, since it is not assumed that the triangle inequality applies to p.

Setting $\mathbb{R}^+ = \{\mu \in \mathbb{R}; \mu \geq 0\}$, the following is confirmed.

2.1 Notation. For an inner product space $E, (.,.)$, let $\mathcal{P}$ (resp. $\widetilde{\mathcal{P}}$) denote the set of all norms (resp. all functionals $p : E \to \mathbb{R}^+$ satisfying (2.1)) on E. It is obvious that $\mathcal{P} \subset \widetilde{\mathcal{P}}$.

2.2 Notation. Let $p, q \in \widetilde{\mathcal{P}}$.

a) Setting $p^{(0)} = p$, define

$$p^{(n)}(x) := \sup_{p^{(n-1)}(y) \leq 1} |(x, y)| = \sup_{0 \neq y \in E} \frac{|(x, y)|}{p^{(n-1)}(y)}\,, \tag{2.2}$$

$x \in E,\ n = 1, 2, \ldots$, where for the last expression "0/0" is defined to be zero. Here $p^{(n)}(x_0) = \infty$ is possible for certain $x_0 \in E$. For $p^{(1)}$ and $p^{(2)}$, it will also be written p' and p'', respectively.

b) Let $p \prec q$ denote that there is a constant $0 < c < \infty$ such that

$$p(x) \leq c\, q(x) \tag{2.3}$$

for each $x \in E$. Furthermore, let $p \precneqq q$ denote that (2.3) applies, and additionally, there is a sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in E$, such that $p(x_n) = 1$ and $q(x_n) \to \infty$ as $n \to \infty$.

c) For a norm p let $\tau(p)$ denote the locally convex topology defined py p. For two l.c. topologies τ_j $(j = 1, 2)$, let $\tau_1 \prec \tau_2$ (resp. $\tau_1 \precneqq \tau_2$) denote that τ_2 is finer (resp. strictly finer) than τ_1.

Considering the dual pair (E, E), where the pairing is given by the inner product $(.,.)$, the weak topology σ is defined by the following set of seminorms

$$x \to q_y(x) = |(x, y)|\,,$$

$x, y \in E$.

Some properties of $p^{(n)}(.)$ are collected in the following. Since the proofs are straightforward, they are omitted.

2.3 Lemma. *Let $p^{(0)}(.) = p(.)$, $q(.) \in \widetilde{\mathcal{P}}$ on a non-degenerate inner product space E, $(.,.)$. Then,*

a) $p^{(n)}$ satisfy the triangle inequality and homogenity, i.e., $p^{(n)}(x + y) \leq p^{(n)}(x) + p^{(n)}(y)$, $p^{(n)}(\lambda x) = |\lambda|\, p^{(n)}(x)$ for each $x, y \in E$, $\lambda \in \mathbb{C}$, $n \in \mathbb{N}$,

b) for $0 < \lambda \in \mathbb{R}$, it follows $(\lambda p)'(x) = \lambda^{-1} p'(x)$, $x \in E$,

c) if $p(x) \leq q(x)$ for each $x \in E$, then $p'(y) \geq q'(y)$ for each $y \in E$,

c') if $p \precneqq q$, then $q' \precneqq p'$,

d) it holds

$$|(x, y)| \leq p^{(n)}(x)\, p^{(n-1)}(y) \tag{2.4}$$

for each $x, y \in E$, $n \in \mathbb{N}$, where the product on the right-hand side of (2.4) is defined to be infinite whenever one factor is infinite.

e) it holds

$$\begin{array}{rcl} p(x) & \geq & p^{(2)}(x) = p^{(4)}(x) = \ldots, \\ p^{(1)}(x) & = & p^{(3)}(x) = p^{(5)}(x) = \ldots \end{array}$$

for each $x \in E$,

f) the following are equivalent:

(i) $p'(x) < \infty$ *for each* $x \in E$,

(ii) $0 < p^{(n)}(z) < \infty$ *for each* $0 \neq z \in E$ *and* $n = 1, 2, \ldots$.

If furthermore $p \in \mathcal{P}$, *then (i) (and consequently also (ii)) is equivalent to*

(iii) $\tau(p) \succ \sigma$.

For Hilbertian norms, the following holds.

2.4 Lemma. *If* p *is a Hilbertian norm on a non-degenerate inner product space satisfying* $\sigma \prec \tau(p)$, *then all* $p^{(n)}$, $n = 1, 2, \ldots$, *are Hilbertian norms, too.*

Proof. Notice first that Lemma 2.3f) yields that all $p^{(n)}$ are norms on E, $n \in \mathbb{N}$. Assume then that there is an $s \in \mathbb{N}$ such that $p^{(s)}$ is a Hilbertian norm. Then there is a positive definite inner product $[.,.]^{(s)}$ on E such that $p^{(s)}(x) = \sqrt{[x,x]^{(s)}}$, $x \in E$. Noticing that E, $[.,.]^{(s)}$ is a pre-Hilbert space, consider the completed hull $\widetilde{E}^{\tau(p^{(s)})}$ endowed with the continuously extented inner product, which is also denoted by $[.,.]^{(s)}$. $\widetilde{E}$, $[.,.]^{(s)}$ then is a Hilbert space. Consider now the linear functionals $\varphi_y(.) = (y,.)$, $y \in E$, defined on E, and notice that their continuity relative to $\tau(p^{(s)})$ follow from

$$|\varphi_y(x)| = |(y,x)| \leq p^{(s-1)}(y)\, p^{(s)}(x) = C_y\, p^{(s)}(x)\,,$$

where (2.4) with $n = s$ was applied, and $C_y := p^{(s-1)}(y) < \infty$ by Lemma 2.3f). Considering then the $\tau(p^{(s)})$-continuous extensions $\widetilde{\varphi_y}(.)$ of $\varphi_y(.)$ onto the Hilbert space $\widetilde{E}$, $[.,.]^{(s)}$, it follows

$$||\widetilde{\varphi_y}|| = ||\varphi_y||\,,$$

where $||.||$ denotes the norm of a linear functional. The Theorem of Fréchet-Riesz yields as well the existence of an element $\tilde{z} \in \widetilde{E}$ such that $\widetilde{\varphi_y}(\tilde{x}) = [\tilde{z}, \tilde{x}]^{(s)}$, $\tilde{x} \in \widetilde{E}$, as $||\widetilde{\varphi_y}|| = ||\tilde{z}||^{(s)}$, $||.||^{(s)} := \sqrt{[.,.]^{(s)}}$, for the norm $||\widetilde{\varphi_y}||$ of the linear functional $\widetilde{\varphi_y}(.)$. The latter now yields

$$||\tilde{z}||^{(s)} = ||\widetilde{\varphi_y}|| = ||\varphi_y|| = \sup_{0 \neq x \in E} \frac{|(y,x)|}{p^{(s)}(x)} = p^{(s+1)}(y)\,. \tag{2.5}$$

Since the above mapping $y \mapsto \tilde{z}$ is linear and $||.||^{(s)}$ is Hilbertian, (2.5) implies that the parallelogram identity applies to $p^{(s+1)}$. Hence, $p^{(s+1)}$ is Hilbertian. Recalling that $p = p^{(0)}$ is assumed to be Hilbertian, the lemma under consideration is verified. $\square$

Letting E, $(.,.)$ be a non-degenerate inner product space, the present is aimed at constructing selfpolar norms on E, $(.,.)$. This construction is based on an iteration process, where the iteration uses a function $\Theta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the properties listed below.

2.5 Definition. A function $\Theta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is called *selfpolarly norm-iterating function* if

(i) $\Theta(a,b) = \Theta(b,a)$ for each $a, b \in \mathbb{R}^+$,

(ii) $\Theta(a,c) \geq \Theta(a,b)$, if $c \geq b$ and $a, b, c \in \mathbb{R}^+$,

(iii) $\Theta(a,b) \geq \sqrt{ab}$ for each $a, b \in \mathbb{R}^+$,

(iv) $\Theta(\lambda a, \lambda b) = \lambda \Theta(a,b)$ for each $a, b, \lambda \in \mathbb{R}^+$,

(v) the equation $a = \Theta(a,b)$ is solvable in $\mathbb{R}^+$, and if $a > 0$, then there is the unique solution $b = a$.

The following examples are of some special interest for applications.

2.6 Example. Consider

a) $\Theta_1(a,b) = \frac{1}{2}(a+b)$,

b) $\Theta_2(a,b) = \sqrt{\frac{1}{2}(a^2+b^2)}$,

c) $\Theta_3(a,b) = \sqrt{ab}$,

$a, b \in \mathbb{R}^+$. It is straightforward to check that Θ_j, $j = 1,2,3$, are selfpolarly norm-iterating functions.

Recalling Notation 2.1, let us confirm the following.

2.7 Notation. For a non-degenerate inner product space $E, (.,.)$, let us put

$$\begin{aligned} \mathcal{Q} &:= \{p \in \mathcal{P};\, p'(x) \leq p(x),\, x \in E\}\,, \\ \widetilde{\mathcal{Q}} &:= \{p \in \widetilde{\mathcal{P}};\, p'(x) \leq p(x),\, x \in E\}\,. \end{aligned}$$

The Aronszajn-Schatten iteration of selfpolar norms is given in the following.

2.8 Theorem. *Letting E, $(.,.)$ be a non-degenerate inner product space, $p \in \widetilde{\mathcal{Q}}$, and $\Theta(.,.)$ a selfpolarly norm-iterating function, define recursively*

$$p_{n+1}(x) := \Theta(p_n(x), p_n'(x))\,, \tag{2.6}$$

$p_0(x) = p(x)$, $x \in E$, $n = 0,1,2,\ldots$ *Then,*

a) $p_\infty(x) := \lim_{n\to\infty} p_n(x)$, $p^{(\infty)}(x) := \lim_{n\to\infty} p_n'(x)$ *exist, and* $p_\infty(x) = p^{(\infty)}(x)$ *for each* $x \in E$,

b) $p'(x) \leq p_1'(x) \leq \cdots \leq p_n'(x) \leq p_{n+1}'(x) \leq \cdots \leq p_\infty(x) \leq \cdots \leq p_{n+1}(x) \leq p_n(x) \leq \cdots \leq p_1(x) \leq p(x)$, $x \in E$,

c) $p_\infty(.)$ *is a selfpolar norm on* E.

Proof. a), b): Assuming that for some $n \in \mathbb{N}_0$, $p_n(.) \in \widetilde{\mathcal{Q}}$. Then

$$p'_n(x) \leq p_n(x) < \infty \tag{2.7}$$

holds for each $x \in E$, and consequently,

$$p_{n+1}(x) = \Theta(p_n(x), p'_n(x)) \overset{(*)}{\leq} \Theta(p_n(x), p_n(x)) \overset{(**)}{\leq} p_n(x)\,, \tag{2.8}$$

where (*) follows from (2.7) and Definition 2.5 (ii), and (**) is a consequence of Definition 2.5 (v). Noticing that $p_{n+1}(.)$ also satisfies (2.1) due to Definition 2.5 (iv), Lemma 2.3c) applies to $p_{n+1}(.)$ and $p_n(.)$, and thus (2.8) yields

$$p'_n(x) \leq p'_{n+1}(x)\,. \tag{2.9}$$

Using (2.4), it follows as well $|(x,y)| \leq p'_n(x)\, p_n(y)$ as $|(x,y)| \leq p_n(x)\, p'_n(y)$, and then

$$|(x,y)| \leq \sqrt{p'_n(x)\, p_n(y)\, p_n(x)\, p'_n(y)}\,, \tag{2.10}$$

$x, y \in E$, where all factors on the right-hand side of (2.10) are finite due to (2.7). Then,

$$\begin{aligned} p'_{n+1}(x) \quad &= \quad \sup_{0 \neq y \in E} \left(\frac{|(x,y)|}{p_{n+1}(y)} \right) \overset{(+)}{\leq} \sup_{0 \neq y \in E} \left(\frac{|(x,y)|}{\sqrt{p'_n(y)\, p_n(y)}} \right) \\ &\overset{(++)}{\leq} \quad \frac{\sqrt{p'_n(x)\, p_n(y)\, p_n(x)\, p'_n(y)}}{\sqrt{p'_n(y)\, p_n(y)}} = \sqrt{p'_n(x)\, p_n(x)} \\ &\overset{(+)}{\leq} \quad \Theta(p_n(x), p'_n(x)) = p_{n+1}(x)\,, \end{aligned} \tag{2.11}$$

$x, y \in E$, where both inequalities (+) follow from (2.6) and Definition 2.5 (iii), and (++) is a consequence of (2.10). Since $p_0(.) = p(.) \in \widetilde{\mathcal{Q}}$ by assumption of the theorem under consideration, (2.8), (2.9) and (2.11) hold for each $n \in \mathbb{N}_0$, and thus

$$\begin{aligned} p'_0(x) \quad &\leq \quad p'_1(x) \leq \cdots \leq p'_n(x) \leq p'_{n+1}(x) \leq \ldots \\ &\leq \quad p_{n+1}(x) \leq p_n(x) \leq \cdots \leq p_1(x) \leq p_0(x)\,. \end{aligned} \tag{2.12}$$

For each $x_0 \in E$, consider now the two sequences of real numbers $(p_n(x_0))_{n=0}^{\infty}$ and $(p'_n(x_0))_{n=0}^{\infty}$. Since $(p_n(x_0))_{n=0}^{\infty}$ is monotonously decreasing and bounded from below by $p'_0(x_0)$ (resp. $(p'_n(x_0))_{n=0}^{\infty}$ is monotonously increasing and bounded from above by $p_0(x_0)$) due to (2.12), a theorem of classical analysis on the convergence of monotonous sequences yields the existence of real numbers $p_\infty(x_0)$, $p^{(\infty)}(x_0)$ such that

$$p_\infty(x_0) = \lim_{n \to \infty} p_n(x_0) \quad \text{and} \quad p^{(\infty)}(x_0) = \lim_{n \to \infty} p'_n(x_0)\,.$$

For varifying *a)* and *b)*, it remains to show that $p_\infty(x) = p^{(\infty)}(x)$, $x \in E$. If $x_0 = 0$, then obviously $p_\infty(0) = p^{(\infty)}(0) = 0$ by (2.12) and $p_0(0) = 0$. Assuming now $x_0 \neq 0$, the assumption $p(.) \geq p'(.)$ of the theorem under consideration and Lemma 2.3f) ((i) $\Rightarrow$ (ii)) yield

$$p'(x_0) > 0\,. \tag{2.13}$$

Considering the limit $n \to \infty$ in (2.6), it follows

$$0 < p'_0(x_0) \leq p_\infty(x_0) = \Theta(p_\infty(x_0), p^{(\infty)}(x_0))\,,$$

and then $p_\infty(x_0) = p^{(\infty)}(x_0)$ by (v) of Definition 2.5.
c): For showing that $p_\infty(.) = p^{(\infty)}(.)$ is a norm, notice first that for each $n \in \mathbb{N}_0$, $p'_n(x+y) \leq p'_n(x)+p'_n(y)$, $p'_n(\lambda x) = |\lambda|\, p'_n(x)$, $x, y \in E$, $\lambda \in \mathbb{C}$. Taking the limit $n \to \infty$ in the above relations, it follows that $p_\infty(.)$ is a seminorm. Taking (2.13) and (2.12) into account, it follows that $p_\infty(.)$ even is a norm. For showing that $p_\infty(.)$ is selfpolar, take $p'_n(x) \leq p_\infty(x) \leq p_n(x)$ from *b).* Using Lemma 2.3c) and Lemma 2.3e), it follows

$$p'_n(x) \leq p'_\infty(x) \leq p''_n(x) \leq p_n(x)\,, \tag{2.14}$$

$x \in E$, $n \in \mathbb{N}_0$. Considering the limit $n \to \infty$ in (2.14) und using $p_\infty(.) = p^{(\infty)}(.)$,

$$p_\infty(x) = \lim_{n\to\infty} p'_n(x) \leq p'_\infty(x) \leq \lim_{n\to\infty} p_n(x) = p_\infty(x)\,,$$

$x \in E$, are implied. The proof is complete. □

Sometimes it will be written $p_{\infty,\Theta}(.)$ instead of $p_\infty(.)$ in order to refer explicitly to the function Θ used in the iteration process (2.6).

2.9 Corollary. *Let $E, (.,.)$ be a non-degenerate inner product space and p a norm on E.*

a) If $\tau(p') \prec \tau(p)$, then there is a selfpolar norm q_∞ on $E, (.,.)$ such that $\tau(p') \prec \tau(q_\infty) \prec \tau(p)$.

b) If $\tau(p) = \tau(p')$, then a selfpolar norm q_∞ exists on $E, (.,.)$ such that $\tau(p) = \tau(q_\infty)$.

Proof. a) Assuming $\tau(p') \prec \tau(p)$, there is a constant $0 < c < \infty$ such that $p'(x) \leq c\, p(x)$, $x \in E$. Setting $q(x) := \sqrt{c}\, p(x)$, it follows $\tau(q) = \tau(p)$ and

$$q'(x) \overset{(*)}{=} (\sqrt{c})^{-1}\, p'(x) \leq \sqrt{c} p(x) = q(x)\,,$$

$x \in E$, where (*) follows from Lemma 2.3b). Applying Theorem 2.8 to the norm q, there is a selfpolar norm p_∞ satisfying $q'(x) \leq q_\infty(x) \leq q(x)$, $x \in E$. Hence, $\tau(p') = \tau(q') \prec \tau(q_\infty) \prec \tau(q) = \tau(p)$ follow. *b)* is an immediate consequence of *a).* □

2.10 Corollary. *Let $E, (.,.)$ be a non-degenerate inner product space and p, q norms on E such that $\tau(p) = \tau(q)$ and $p'(x) \leq p(x)$, $q'(x) \leq q(x)$ for each $x \in E$. For every selfpolarly norm-iterating function Θ, take the selfpolar norms $p_{\infty,\Theta}$, $q_{\infty,\Theta}$ from Theorem 2.8. It then follows $\tau(p_{\infty,\Theta}) = \tau(q_{\infty,\Theta})$.*

Proof. Letting the assumptions of the corollary under consideration be satisfied, $\tau(p) = \tau(q)$ implies the existence of constants $0 < c, d < \infty$ such that $c\,p(x) \leq q(x) \leq d\,p(x)$, $x \in E$. Setting $\lambda = \max\{c^{-1}, d\}$, it follows

$$\lambda^{-1}\, p(x) \leq q(x) \leq \lambda\, p(x)\,, \tag{2.15}$$

and then by Lemma 2.3b), c),

$$\lambda^{-1}\, p'(x) \leq q'(x) \leq \lambda\, p'(x)\,, \tag{2.16}$$

$x \in E$. Considering the first step of the Aronszajn-Schatten iteration, it follows

$$p_1(x) = \Theta(p(x), p'(x)) \overset{(*)}{\leq} \Theta(\lambda q(x), \lambda q'(x)) \overset{(**)}{=} \lambda\, \Theta(q(x), q'(x)) = \lambda q_1(x)\,,$$

and analogously, $q_1(x) \leq \lambda p_1(x)$, where (*) follows from (2.15), (2.16) and Definition 2.5 (i), (ii), and (**) is a consequence of Definition 2.5 (iv). Hence, $\lambda^{-1}\, p_1(x) \leq q_1(x) \leq \lambda\, p_1(x)$ and $\lambda^{-1}\, p_1'(x) \leq q_1'(x) \leq \lambda\, p_1'(x)$. Assuming now that there is an $n \in \mathbb{N}$ such that

$$\begin{aligned} \lambda^{-1}\, p_n(x) \leq q_n(x) \leq \lambda\, p_n(x)\,, \\ \lambda^{-1}\, p_n'(x) \leq q_n'(x) \leq \lambda\, p_n'(x)\,, \end{aligned} \tag{2.17}$$

the same reasoning as above yields $\lambda^{-1}\, p_{n+1}(x) \leq q_{n+1}(x) \leq \lambda\, p_{n+1}(x)$. Hence (2.17) holds for each $n \in \mathbb{N}$. Considering now the limit $n \to \infty$ in (2.17), it follows $\lambda^{-1}\, p_{\infty,\Theta}(x) \leq q_{\infty,\Theta}(x) \leq \lambda\, p_{\infty,\Theta}(x)$, $x \in E$. Hence, $\tau(p_{\infty,\Theta}) = \tau(q_{\infty,\Theta})$. □

2.11 Corollary. *Letting p be a Hilbertian norm on a non-degenerate inner product space $E, (.,.)$ satisfying $p'(x) \leq p(x)$, $x \in E$, and using the selfpolarly norm-iterating function Θ_2, the Aronszajn-Schatten iteration yields a Hilbertian selfpolar norm p_{∞,Θ_2} on $E, (.,.)$.*

Proof. Assume that there is an $n \in \mathbb{N}_0$ such that $p_n'(x) \leq p_n(x)$, $x \in E$, and p_n is a Hilbertian norm on E, where p_n is taken from (2.6) using Θ_2. Noticing that Lemma 2.3f) ((i) $\Rightarrow$ (iii)) yields $\tau(p_n) \succ \sigma$, it follows from Lemma 2.4 that p_n' is a Hilbertian norm on E, too. Hence the parallelogram identity applies to both norms p_n and p_n', and thus,

$$\begin{aligned} &(p_{n+1}(x+y))^2 + (p_{n+1}(x-y))^2 \\ &\quad = \Theta_2(p_n(x+y), p_n'(x+y))^2 + \Theta_2(p_n(x-y), p_n'(x-y))^2 \\ &\quad = p_n(x+y)^2 + p_n'(x+y)^2 + p_n(x-y)^2 + p_n'(x-y)^2 \\ &\quad = 2(p_n(x)^2 + p_n(y)^2 + p_n'(x)^2 + p_n'(y)^2) \\ &\quad = 2\,(\Theta(p_n(x), p_n'(x))^2 + \Theta(p_n(y), p_n'(y))^2 \\ &\quad = 2\,(p_{n+1}(x)^2 + p_{n+1}(y)^2)\,, \end{aligned} \tag{2.18}$$

$x, y \in E$, verifying that the parallelogram identity applies to p_{n+1}, too. Hence p_{n+1} is a Hilbertian norm on E, and $p'_{n+1}(x) \leq p_{n+1}(x)$, $x \in E$ by (2.11). Noticing that the assumption made at the beginning of the proof holds for $n = 0$ due to the assumptions of the corollary under consideration, (2.18) applies to each $n \in \mathbb{N}_0$. Considering the limit $n \to \infty$ in (2.18), it follows that the norm p_{∞,Θ_2} satisfies the parallelogram identity, and consequently it is Hilbertian. $\square$

Taking Θ_1 from Example 2.6, there are simple "a priori" and "a posteriori" estimates for the Aronszajn-Schatten iteration process.

2.12 Lemma. *Letting $p \in \widetilde{\mathcal{Q}}$ on a non-degenerate inner product space $E, (.,.)$, consider the Aronszajn-Schatten iteration*

$$p_n(x) = \Theta_1(p_{n-1}(x), p'_{n-1}(x)) = \frac{1}{2}(p_{n-1}(x) + p'_{n-1}(x)), \tag{2.19}$$

$p_0(x) = p(x)$, $n = 1, 2, \ldots$, $x \in E$. Then,

a) $|p_{\infty,\Theta_1}(x) - p_n(x)| \leq 2^{-n}(p(x) - p'(x))$ (a priori estimate),

b) $|p_{\infty,\Theta_1}(x) - p_n(x)| \leq p_n(x) - p_{n-1}(x)$ (a posteriori estimate) for each $x \in E$.

Proof. a) Using Theorem 2.8b) and (2.19), the a priori estimate follows from

$$\begin{aligned}
|p_{\infty,\Theta_1}(x) - p_n(x)| &\leq p_n(x) - p'_n(x) \\
&\leq \frac{1}{2}(p_{n-1}(x) + p'_{n-1}(x)) - p'_{n-1}(x) \\
&= \frac{1}{2}(p_{n-1}(x) - p'_{n-1}(x)) \\
&\leq \cdots \\
&\leq 2^{-n}(p_0(x) - p'_0(x)), \quad x \in E.
\end{aligned}$$

b) Again using Theorem 2.8b) and (2.19), the a posteriori estimate follows from

$$\begin{aligned}
|p_{\infty,\Theta_1}(x) - p_n(x)| &\leq p_n(x) - p'_n(x) \leq p_n(x) - p'_{n-1}(x) \\
&= p_n(x) - (2p_n(x) - p_{n-1}(x)) \\
&= p_{n-1}(x) - p_n(x), \quad x \in E.
\end{aligned}$$

$\square$

3 Selfpolar and selfpolar Hilbertian norms on inner product spaces of countable dimension

While the existence and construction of selfpolar and selfpolar Hilbertian norms was considered in Theorem 2.8 and Corollary 2.11, resp., the question of whether or not all selfpolar norms are equivalent is not answered by those results. The aim of the present section is to give, for a certain class of inner product spaces, an explicit construction of infinitely many different selfpolar topologies (in contrast to the definite case – cf. (a) of the introduction).

3.1 Inner product spaces of countable dimension

Let E be a real vector space of countably infinite algebraic dimension endowed with an inner product $(.,.)_E$ so that $E,(.,.)_E$ is non-degenerate.

If E is quasi-definite, then all selfpolar norms are equivalent (cf. the introduction). So, for the following E is supposed *not* to be quasi-definite.

Since there is an orthonormal basis $\{e_n\}_{n=1}^{\infty}$, $(e_n,e_m)_E = \pm\delta_{nm}$ [5, Sect.IV.3], it is obvious that $E,(.,.)_E$ is isometrically isomorphic to $\mathrm{d}(\mathbb{R})$, the space of all sequences of real numbers with a finite number of non-zero members, endowed with the inner product

$$\left((x^0,x^1,x^2,\ldots),(y^0,y^1,y^2,\ldots)\right)_{\mathrm{d}} := \sum_{k=0}^{\infty}(x^{2k}y^{2k} - x^{2k+1}y^{2k+1}). \tag{3.1}$$

Letting $\mathcal{M}^2$ denote the two dimensional Minkowski space, i.e. $\mathbb{R}^2$ equipped with the inner product $(x,y) := x^0y^0 - x^1y^1$, one can think of the above $\mathrm{d}(\mathbb{R}),(.,.)_{\mathrm{d}}$ as composed of infinitely many copies of $\mathcal{M}^2,(.,.)$. Hence, a large class of selfpolar norms on $\mathrm{d}(\mathbb{R}),(.,.)_{\mathrm{d}}$ can be constructed starting from sequences of selfpolar norms on $\mathcal{M}^2,(.,.)$:

3.1 Proposition.

a) Let $p_0,p_1,p_2,\ldots$ be a sequence of selfpolar norms on $\mathcal{M}^2,(.,.)$ and define

$$p(x) := \sqrt{\sum_{k=0}^{\infty} p_k^2(x^{2k},x^{2k+1})}\ , \qquad x=(x^0,x^1,x^2,\ldots)\in \mathrm{d}(\mathbb{R}). \tag{3.2}$$

Then p is a selfpolar norm on $\mathrm{d}(\mathbb{R}),(.,.)_{\mathrm{d}}$.

If in addition all the p_k are Hilbertian, then so is p.

b) If p is any selfpolar norm on $\mathrm{d}(\mathbb{R}),(.,.)_{\mathrm{d}}$, and U is an isometric or antiisometric automorphism of this space, then $p_U(x) := p(U^{-1}x)$ is also a selfpolar norm.

If p is Hilbertian, then so is p_U.

Proof. In order to show in a) that p is selfpolar, first observe that $|(x,y)_{\mathrm{d}}| \le p(x)\,p(y)$. Now, given any $x\in\mathrm{d}(\mathbb{R})$, choose for each k a pair (y^{2k},y^{2k+1}) such that $x^{2k}y^{2k} - x^{2k+1}y^{2k+1} = p_k^2(x^{2k},x^{2k+1})$ and $p_k(y^{2k},y^{2k+1}) = p_k(x^{2k},x^{2k+1})$. Collecting all these pairs one obtains a $y\in\mathrm{d}(\mathbb{R})$ satisfying $(x,y)_{\mathrm{d}} = p^2(x)$ and $p(y)=p(x)$. The rest of the proof is straightforward. □

Two norms p and q on $\mathrm{d}(\mathbb{R})$, constructed as in Proposition 3.1 a) from different sequences (p_k) and (q_k) might or might not be equivalent. There is a simple criterion to decide this question:

3.2 Lemma. *Let $p_0, p_1, p_2, \dots$ and $q_0, q_1, q_2, \dots$ be two sequences of selfpolar norms on $\mathcal{M}^2, (.,.)$ and denote by p resp. q the corresponding norms on* $\mathrm{d}(\mathbb{R})$ *as in (3.2).*
Set $\rho_k^+(p,q) = \sup_{0 \neq x \in \mathcal{M}^2} \frac{p_k(x)}{q_k(x)}$, $\rho_k^-(p,q) = \inf_{0 \neq x \in \mathcal{M}^2} \frac{p_k(x)}{q_k(x)}$.
Then the following are equivalent:

(i) p and q define the same topology on $\mathrm{d}(\mathbb{R})$.

(ii) $\rho_k^+(p,q) \leq C < \infty$ for each k.

(iii) $\rho_k^-(p,q) \geq c > 0$ for each k.

Proof. Obviously, (ii) applies if and only if p is weaker than q, and (iii) is equivalent to q being weaker than p. In the present case, both p and q are selfpolar. This means that if one of them is weaker than the other they must be equivalent [5, Thm. IV.4.2]. □

3.2 Selfpolar norms on $\mathcal{M}^2$

Since the aim is to use Proposition 3.1 and Lemma 3.2 in order to provide a (partial) overview of the possible selfpolar and selfpolar Hilbertian topologies on $\mathrm{d}(\mathbb{R})$, the following two sections are devoted to an investigation of the selfpolar and selfpolar Hilbertian norms on $\mathcal{M}^2$. They are described in detail and a complete classification is given. All the proofs of the assertions below and more details will be given in chapter 4.

It turns out to be usefull, not to consider a norm p itself, but the curve $\partial U_p := \{x \in \mathcal{M}^2 \,|\, p(x) = 1\}$, i.e. the boundary of the corresponding unit ball U_p. Curves in $\mathbb{R}^2$ will be described in polar coordinates, i.e. by functions $r(\varphi)$, where r gives the (Euclidean) length of a vector, and $\varphi \in S^1$ denotes the angle between the vector and the positive x^0-axis.

3.3 Observation. There is the following bijection between the norms p on $\mathbb{R}^2$ and the π-periodic, convex, closed curves $r : S^1 \to [r_1, r_2]$ $(r_1, r_2 > 0)$ in $\mathbb{R}^2$:

$$\begin{aligned} p &\mapsto r_p,\ r_p(\varphi) := p\left(\binom{\cos\varphi}{\sin\varphi}\right)^{-1},\ \varphi \in S^1; \\ r &\mapsto p_r,\ p_r(x) := \frac{\rho}{r(\varphi)},\ x = \rho\binom{\cos\varphi}{\sin\varphi} \in \mathbb{R}^2\ (\rho \geq 0). \end{aligned} \tag{3.3}$$

The homogeneity of the norm includes its reflection invariance, and thus the π-periodicity of r_p. The restriction of the image of r to some closed interval $[r_1, r_2]$ reflects both, the fact that the unit ball associated to p must be absorbing and that it must lie in some finite Euclidean ball. Convexity finally, which by definition means that for any two points of the curve the straight line between these points lies completely inside the curve, comes from homogeneity and the triangle inequality for p.

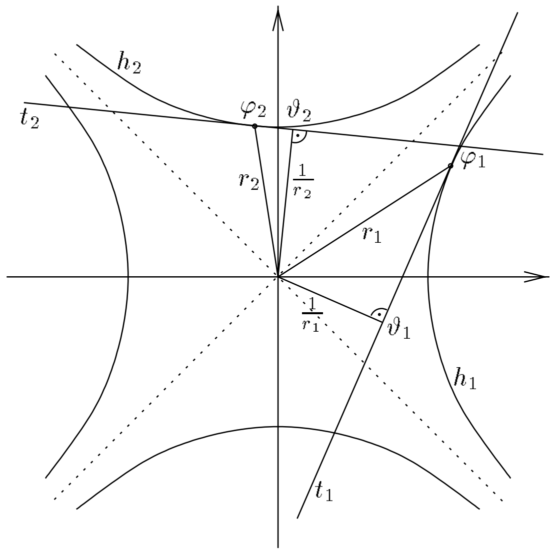

Figure 1: The choice of (r_1, φ_1) and (r_2, φ_2). The hyperbolae intersect the coordinate axes at ± 1.

3.4 Observation. The set of selfpolar norms on $\mathcal{M}^2$ is invariant under Lorentz transformations. Consider any selfpolar norm p and let Λ denote a linear isometry of $\mathcal{M}^2$. Then $p_\Lambda(x) := p(\Lambda^{-1}x)$ is a selfpolar norm as well. This remains true if Λ is chosen to be an anti-isometry, $(\Lambda x, \Lambda y) = -(x, y)$. The curve which is associated to p_Λ is the image of the curve r_p under Λ. If, e.g., p is the Euclidean norm, i.e. $r_p(\varphi) \equiv 1$ describes the unit circle, then under a Lorentz transformation, this circle turns into some ellipse, the axes of which lie along the bisectors of the first and second quadrants and for which the product of the half axes equals unity.

The construction of selfpolar norms on $\mathcal{M}^2$ which will now be described needs the following two ingredients:

I. The first ingredient is *a pair of angles* $(\varphi_1, \varphi_2) \in S^1 \times S^1$ subject to the following conditions:

$$-\tfrac{\pi}{4} < \varphi_1 < \tfrac{\pi}{4}, \quad \tfrac{\pi}{4} < \varphi_2 < \tfrac{3\pi}{4}, \tag{3.4}$$

$$\cos^2(\varphi_1 + \varphi_2) \leq \cos(2\varphi_1)\,|\cos(2\varphi_2)|. \tag{3.5}$$

Define

$$r_1 := \tfrac{1}{\sqrt{\cos 2\varphi_1}}, \quad r_2 := \tfrac{1}{\sqrt{|\cos 2\varphi_2|}}, \quad \vartheta_1 := -\varphi_1, \quad \vartheta_2 := \pi - \varphi_2.$$

The condition (3.5) is now equivalent to $r_1 r_2 |\cos(\varphi_1 + \varphi_2)| \leq 1$.

Figure 1 shows the situation: Each of the points (r_1, φ_1) and (r_2, φ_2) lies on one of the hyperbolae $(x^0)^2 - (x^1)^2 = \pm 1$. Let us call them h_1 and h_2 as shown in the picture. The tangents of h_1 and h_2 in the given points (called t_1 and t_2 resp.)

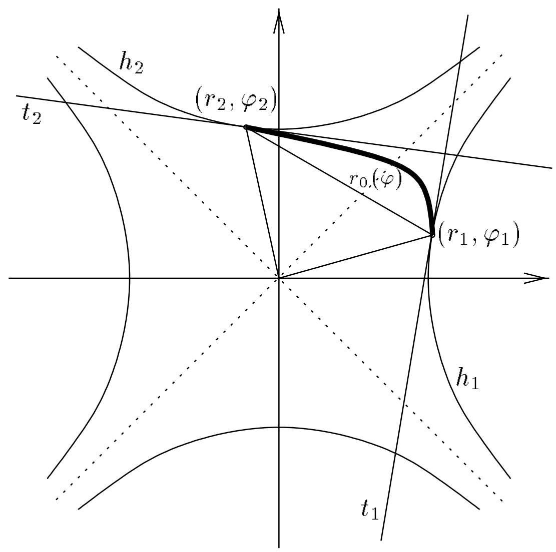

Figure 2: The choice of $r_0(\varphi)$.

are given by their respective normal directions ϑ_1 and ϑ_2. The projections of r_1 and r_2 onto the directions ϑ_1 and ϑ_2 are $1/r_1$ and $1/r_2$ respectively.

According to condition (3.5), the points on the hyperbolae have to be chosen in such a way that (r_1, φ_1) lies "below" or on t_2 and that (r_2, φ_2) lies "to the left of" or on t_1.

II. The second ingredient for the construction described below is *an arbitrary convex arc* $r_0 : [\varphi_1, \varphi_2] \to \mathbb{R}$ satisfying

$$r_0(\varphi_1) = r_1, \quad r_0(\varphi_2) = r_2, \tag{3.6}$$

$$\vartheta_1 \leq \partial^{(\mathrm{r})} r_0(\varphi_1) \leq \partial^{(\mathrm{l})} r_0(\varphi_2) \leq \vartheta_2. \tag{3.7}$$

Here, $\partial^{(\mathrm{l}/\mathrm{r})} r_0(\varphi)$ denotes the normal direction of the left resp. right derivative of the curve r_0 in the point φ (cf. Definition 4.3).

The convexity in this case is to be understood in the same way as for closed curves. See Figure 2 for an illustration of the above conditions: The arc $r_0(\varphi)$ must lie inside the triangle formed by t_1, t_2 and the line through (r_1, φ_1) and (r_2, φ_2).

The idea of the construction is, to extend the chosen arc $r_0(\varphi)$ to a complete closed curve, which will represent the desired selfpolar norm.

First, define $r_*\colon [\varphi_2 - \pi, \varphi_1] \to \mathbb{R}$ by

$$r_*(\psi) := \frac{1}{\sup_{\varphi \in [\varphi_1, \varphi_2]} r_0(\varphi) \cos(\psi + \varphi)}; \quad \psi \in [\varphi_2 - \pi, \varphi_1].$$

Now, the completed curve is given by $r : S^1 \to \mathbb{R}$

$$r(\varphi) := \begin{cases} r_0(\varphi) & ; \varphi \in [\varphi_1, \varphi_2] \\ r_*(\varphi) & ; \varphi \in [\varphi_2 - \pi, \varphi_1] \\ r_0(\varphi + \pi) & ; \varphi \in [\varphi_1 - \pi, \varphi_2 - \pi] \\ r_*(\varphi - \pi) & ; \varphi \in [\varphi_2, \varphi_1 + \pi]. \end{cases} \tag{3.8}$$

This yields the desired selfpolar norm:

3.5 Proposition (see Prop. 4.14). *The function $r(\varphi)$ is the representation in polar coordinates of a π-periodic convex closed curve, i.e., it defines a norm p_r in $\mathbb{R}^2$ (cf. (3.3)).*

3.6 Theorem (see Thm. 4.15).

$$p_r = p_r', \;\; p_r \text{ is selfpolar}$$

($'$: Polarization with respect to $\mathcal{M}^2$).

It is shown in chapter 4 that every selfpolar norm on $\mathcal{M}^2$ can be constructed in the above described manner:

3.7 Theorem (see Thm. 4.17). *Set*

$$\begin{aligned} \mathcal{P}_g := \{\, (\varphi_1, \varphi_2) \in S^1 \times S^1 \mid & -\tfrac{\pi}{4} < \varphi_1 < \tfrac{\pi}{4},\ \tfrac{\pi}{4} < \varphi_2 < \tfrac{3\pi}{4}, \\ & \cos^2(\varphi_1 + \varphi_2) < \cos(2\varphi_1)\,|\cos(2\varphi_2)|\,\} \\ \times \{\, \rho : [0, \tfrac{\pi}{2}] \to \mathbb{R} \mid & \rho \text{ describes a convex arc from } (1,0) \text{ to } (1, \tfrac{\pi}{2}), \\ & \text{completely lying inside the triangle } (1,0);\ (\sqrt{2}, \tfrac{\pi}{4});\ (1, \tfrac{\pi}{2})\,\}, \end{aligned}$$

$$\begin{aligned} \mathcal{P}_e := \{\, (\varphi_1, \varphi_2) \in S^1 \times S^1 \mid & -\tfrac{\pi}{4} < \varphi_1 < \tfrac{\pi}{4},\ \tfrac{\pi}{4} < \varphi_2 < \tfrac{3\pi}{4}, \\ & \cos^2(\varphi_1 + \varphi_2) = \cos(2\varphi_1)\,|\cos(2\varphi_2)|\,\}. \end{aligned}$$

There is a bijection from $\mathcal{P} := \mathcal{P}_g \uplus \mathcal{P}_e$ (disjoint union) into the set of selfpolar norms on $\mathcal{M}^2$ (essentially given by the above construction).

3.8 Remark (Description of the admissible angles φ_1, φ_2).
The (angle parts of the) parameter sets $\mathcal{P}_g$ and $\mathcal{P}_e$ which occur in Theorem 3.7 are depicted in Figure 3. Using the transformation $\alpha_\pm := \varphi_1 \pm (\varphi_2 - \pi/2)$ one calculates: $\cos^2(\varphi_1 + \varphi_2) = \sin^2(\alpha_+)\,;\ -\cos(2\varphi_1)\cos(2\varphi_2) = \cos^2(\alpha_-) - \sin^2(\alpha_+)$.
The condition for $\mathcal{P}_g$ reads in these coordinates: $\sin^2(\alpha_+) < 1/2\cos^2(\alpha_-)$. For $\mathcal{P}_e$: $\sin^2(\alpha_+) = 1/2\cos^2(\alpha_-)$. Hence, the points (φ_1, φ_2) belonging to $\mathcal{P}_g$ form the interior of the gray lense, the elements of $\mathcal{P}_e$ are exactly the points of its boundary.

3.3 Selfpolar Hilbertian norms on $\mathcal{M}^2$

In this section it will be determined, which of the selfpolar norms described before are in addition Hilbertian (i.e. which of them are derived from scalar

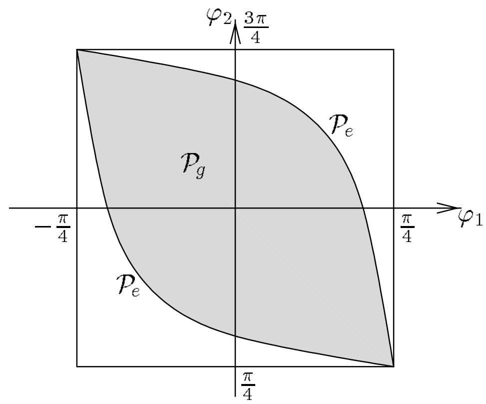

Figure 3: The admissible values for (φ_1, φ_2)

products). The easiest way to do so is to use the following theorem due to Hansen [7].

3.9 Theorem (Hansen). *Let $E, (\cdot,\cdot)$ be a non-degenerate inner product space with an additional scalar product $[\cdot,\cdot]$. The corresponding norm $\|x\| := \sqrt{[x,x]}$ is supposed to satisfy $|(x,y)| \le C\|x\|\,\|y\|$ for some $C > 0$ (i.e. $\|\cdot\|$ is to be a majorant). On the Hilbert space $\mathcal{H} := \widetilde{E}^{\|\cdot\|}$ (completion with respect to the norm topology) the extended inner product $(\cdot,\cdot)^{\sim}$ is assumed to be non-degenerate. Let p be a norm on E.*

Then p is Hilbertian and selfpolar if and only if there is a positive and bounded operator T on $\mathcal{H}, [\cdot,\cdot]$ with the following properties:

a) 0 is not an eigenvalue of T,

b) $J\mathcal{H} \subset \mathcal{D}(T^{-\frac{1}{2}})$
(the operator J connects the two inner products: $(x,y) = [x, Jy]$),

c) $\|T^{\frac{1}{2}}x\| = \|T^{-\frac{1}{2}}Jx\|$, $x \in E$,

d) $p(x) = \|T^{\frac{1}{2}}x\|$, $x \in E$.

For two dimensional Minkowski space, $E = \mathcal{M}^2$, $[x,y] = x^0y^0 + x^1y^1$, $(x,y) = x^0y^0 - x^1y^1$, $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, the theorem reads:
The Hilbertian selfpolar norms p have the form $p(x) = \|T^{\frac{1}{2}}x\|$, where T is any positive definite matrix satisfying $\|T^{\frac{1}{2}}x\| = \|T^{-\frac{1}{2}}Jx\|$. The condition for T is in this case equivalent to $T = J\,T^{-1}J$.

3.10 Proposition. *The Hilbertian selfpolar norms on $\mathcal{M}^2$ are given by the formula*

$$p(x) = \sqrt{\alpha\,\xi_+^2 + \frac{1}{\alpha}\,\xi_-^2}\,, \quad \xi_\pm := \frac{1}{\sqrt{2}}(x^0 \pm x^1) \quad (x \in \mathcal{M}^2),$$

for any $\alpha > 0$.

3.11 Remark. The curves $p(x) = 1$ corresponding to the Hilbertian selfpolar norms p on $\mathcal{M}^2$ are exactly the ellipses which were described in Observation 3.4 (and which can be derived from the unit circle by Lorentz transformation).

Proof of the Proposition. p must have the form $p^2(x) = [x, Tx] = x^\mathsf{T} Tx$, where T satisfies $T = JT^{-1}J$. Let x_1, x_2 denote an orthonormal basis of $\mathbb{R}^2$, $[\cdot,\cdot]$, with $Tx_1 = t_1 x_1$, $Tx_2 = t_2 x_2$ $(t_1, t_2 > 0)$. Then $TJx_1 = t_1^{-1} Jx_1$, so t_1^{-1} is also an eigenvalue.

Now, if $t_1 = t_2$ then $t_1^{-1} = t_2 = t_1$, and so $T = \mathbb{1}$. If $t_1 \neq t_2$ then Jx_1, since it is an eigenvector, must be either parallel or perpendicular to x_1. In the former case, x_2 must also be an eigenvector of J (because it is perpendicular to x_1) and so $t_1^{-1} = t_1$ and $t_2^{-1} = t_2$, yielding $t_1 = t_2 = 1$. In the latter case, $(x_1, x_1) = [x_1, Jx_1] = 0$, i.e., x_1 (and with it x_2) is lightlike and thus proportional to $(1,1)$ or $(1,-1)$. The construction of T from its eigenvectors and the eigenvalues t_1, $t_2 = t_1^{-1}$ yields the given form for $p(x)$. $\square$

3.4 Results for non-quasi-definite spaces of countable dimension

Collecting the results of the previous sections, one finds the following properties of $\mathrm{d}(\mathbb{R}), (.,.)_\mathrm{d}$ which carry over to $E, (.,.)_E$ (as explained at the beginning of section 3.1).

3.12 Theorem.

a) There is a non-denumerable set of different selfpolar topologies.

b) Even more, there are non-denumerably many non-equivalent selfpolar Hilbertian norms.

c) Every selfpolar norm which is constructed as in Proposition 3.1 a) and b) is equivalent to some non-selfpolar norm.

d) Every selfpolar norm which is constructed as in Proposition 3.1 a) and b) is equivalent to some selfpolar Hilbertian norm.

Proof. It suffices to consider norms of the form (3.2), because all the properties are invariant under the transformations in 3.1 b). These norms are given by sequences (p_k) of norms on $\mathcal{M}^2$, i.e. sequences of curves (r_{p_k}) (cf. Observation 3.3). Given two such sequences (p_k) and (q_k), the possible equivalence of the corresponding norms on $\mathrm{d}(\mathbb{R})$, p and q, is determined by $\rho_k^+(p,q) = \sup_{\varphi \in S^1} \frac{r_{q_k}(\varphi)}{r_{p_k}(\varphi)}$ (or alternatively $\rho_k^-(p,q) = \inf_{\varphi \in S^1} \frac{r_{q_k}(\varphi)}{r_{p_k}(\varphi)}$) – cf. Lemma 3.2.

Now, it is easy to construct sequences of "selfpolar" curves (using the construction of section 3.2) in such a way that the corresponding selfpolar norms on $\mathrm{d}(\mathbb{R})$ become mutually non-equivalent. It is also possible to do this using only

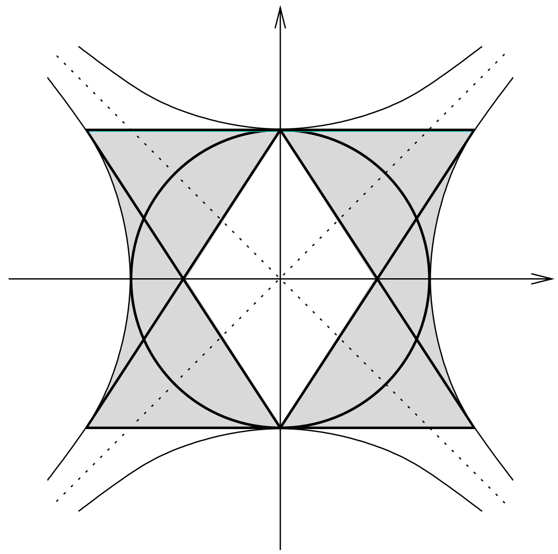

Figure 4: The allowed sector for the Lorentz transformed r_{p_k} (see the proof of Thm. 3.12 d)). The diagonal lines (tangents of the hyperbolae in $(\pm\sqrt{2}, \pm 1)$) are given by the points $(0, \pm 1)$, $(\pm\frac{1}{\sqrt{2}}, 0)$.

the ellipses which correspond to selfpolar Hilbertian norms (Remark 3.11). This proves a) and b).

c) For any given norm of the form (3.2) it is possible to modify one or more of the curves r_{p_k} and thereby destroy the property of being selfpolar, e.g. by leaving the part of the curve between φ_1 and φ_2 unchanged and deforming the part in $[\varphi_2 - \pi, \varphi_1]$ (cf. the construction in section 3.2). If this is done in such a way that (ii) and (iii) of Lemma 3.2 hold, then the given norm and the modified one are equivalent.

d) Let p be any selfpolar norm of the form (3.2). Denote for every k by $(r_2(k), \varphi_2(k))$ the point on h_2 which is touched by r_{p_k} (since p_k is selfpolar there is exactly one such point, because r_{p_k} can be recovered by the construction of section 3.2) – cf. e.g. Figure 2. Define q_k as that selfpolar Hilbertian norm on $\mathcal{M}^2$ which is described by the ellipse which touches $(r_2(k), \varphi_2(k))$. The q_k combine to a selfpolar Hilbertian norm q and it remains to show, that p and q are equivalent. To this end $\rho_k^+(p, q)$ (in the above form) has to be considered. Apply for each k to p_k and q_k that Lorentz transformation which maps the respective ellipse r_{q_k} to the unit circle. Since the transformation is linear, this does not change the ρ_k^+, which thus can be calculated using the circle and the transformed r_{p_k}. The latter passes through $(0, 1)$ and, since it is again selfpolar, it can be described by the construction of section 3.2. Now, the conditions of this construction strongly limit the possible course of the curve. It must lie inside the shaded area shown in Figure 4. This is valid for every k, so there are bounds on ρ_k^+ which are independent of k. Hence, the equivalence of p and q follows by Lemma 3.2. $\square$

3.13 Remark.

a) It is known that there are selfpolar norms on $\mathrm{d}(\mathbb{R}), (.,.)_{\mathrm{d}}$ which are not equivalent to any Hilbertian norm.

   An example of such a norm can be derived from the Araki-Hansen example [7, Ex. 2.10]. More precisely, on $\mathrm{d}(\mathbb{R})$ endowed with the inner product $\langle .|. \rangle$ given in [7, Eq. (2.35)] the ℓ_1-norm (restricted to d) is selfpolar, but it is not equivalent to any Hilbertian norm. Remember that $\mathrm{d}(\mathbb{R}), \langle .|. \rangle$ is isometrically isomorphic to $\mathrm{d}(\mathbb{R}), (.,.)_{\mathrm{d}}$.

b) There are selfpolar norms on $\mathrm{d}(\mathbb{R}), (.,.)_{\mathrm{d}}$ which are not equivalent to any of the norms of Proposition 3.1 a) and b), due to part a) and Theorem 3.12 d).

4 Selfpolar norms on $\mathcal{M}^2$

In this section the selfpolar norms on two dimensional Minkowski space $\mathcal{M}^2$, i.e. $\mathbb{R}^2$ equipped with the inner product $(x, y) := x^0 y^0 - x^1 y^1$, are discussed in detail and a complete classification is given.

As pointed out in section 3.2 the idea is not to consider the norms on $\mathbb{R}^2$ themselves but the boundaries of their unit balls – cf. Observation 3.3.

In order to exploit the correspondence between norms and curves, a few general notions and facts about curves with the properties listed in Observation 3.3 are needed:

4.1 Convex curves

Using periodicity would be no advantage here, so let for this entire section $r : S^1 \to [r_1, r_2]$ $(r_1, r_2 > 0)$ be the representation in polar coordinates of a convex, closed curve in $\mathbb{R}^2$.

For simplicity of notation the following conventions with respect to S^1 will be used:

a) A statement $\varphi_1 \leq \varphi_2 \leq \varphi_3 \dots$ involving three or more points $\varphi_i \in S^1$ means that these points are lying in the given order anti-clockwise around the circle. The sign $<$ instead of $\leq$ means that the points are in addition different from each other.

b) Intervals of S^1 are denoted as follows: $(\varphi_1, \varphi_2) := \{\varphi \in S^1 \mid \varphi_1 < \varphi < \varphi_2\}$ and $[\varphi_1, \varphi_2] := \{\varphi \in S^1 \mid \varphi_1 \leq \varphi \leq \varphi_2\}$ if $\varphi_1 \neq \varphi_2$. $[\varphi, \varphi] := \{\varphi\}$.

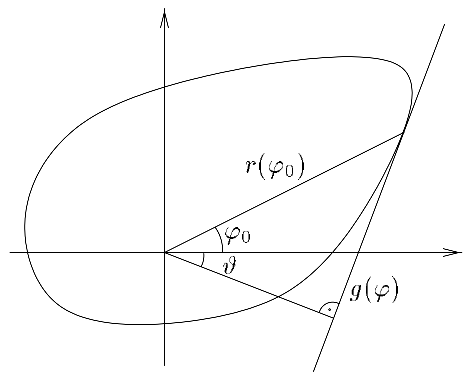

Figure 5: The definition of a support line (cf. Definition 4.1).

c) For a given vector $x \in \mathbb{R}^2$, $\Phi(x) \in S^1$ denotes the polar angle of x (with respect to the positive x^0-axis).

Now, the convexity of r (in the sense of the above definition) is equivalent to the following property of $\kappa(\varphi) := r(\varphi)^{-1}$:

$$\kappa(\varphi_1)\sin(\varphi_2 - \varphi_3) + \kappa(\varphi_2)\sin(\varphi_3 - \varphi_1) + \kappa(\varphi_3)\sin(\varphi_1 - \varphi_2) \leq 0$$
$$\text{for } \varphi_1 < \varphi_2 < \varphi_3, \ \ 0 < \varphi_3 - \varphi_1 < \pi,$$

as can be seen by an elementary calculation in polar coordinates. Thus, κ is a so-called trigonometrically convex function. In addition it is bounded from above and from below by positive constants. This has – similarly as in the case of convex functions in the usual sense – some very strong implications (cf. [16]) which directly carry over from κ to r :

a) The function r is continuous,

b) At every point, r possesses a left and a right derivative,

c) The function r is everywhere differentiable, except on a countable set of points.

The following definitions and lemmas are merely adaptations of the discussion of usual convex functions to the present case (cf. [22, ch.I] or [23, ch.V]). So, most of the proofs are only sketched or completely omitted. The reader is, however, invited to make the following statements clear to himself by simply drawing two dimensional sketches.

4.1 Definition. The straight line

$$g(\varphi) = \frac{r(\varphi_0)\cos(\varphi_0 - \vartheta)}{\cos(\varphi - \vartheta)} \qquad \varphi \in (\vartheta - \tfrac{\pi}{2}, \vartheta + \tfrac{\pi}{2}), \tag{4.1}$$

which touches the curve in $(r(\varphi_0), \varphi_0)$, is called a *support line* of $r(\varphi)$ in φ_0, if its defining angle $\vartheta \in S^1$ (its normal direction) satisfies

$$r(\varphi)\cos(\varphi - \vartheta) \le r(\varphi_0)\cos(\varphi_0 - \vartheta) \quad \forall \varphi \in S^1,$$

i.e., if the entire curve $r(\varphi)$ lies on one side of the line (cf. Figure 5).

4.2 Lemma. *The support lines of $r(\varphi)$ in φ_0 are exactly those lines given by (4.1), for which ϑ satisfies*

$$\Phi\left[\frac{\mathrm{d}^{(\mathrm{l})}}{\mathrm{d}\varphi}\begin{pmatrix} r(\varphi)\cos\varphi \\ r(\varphi)\sin\varphi \end{pmatrix}_{\varphi=\varphi_0}\right] - \frac{\pi}{2} \le \vartheta \le \Phi\left[\frac{\mathrm{d}^{(\mathrm{r})}}{\mathrm{d}\varphi}\begin{pmatrix} r(\varphi)\cos\varphi \\ r(\varphi)\sin\varphi \end{pmatrix}_{\varphi=\varphi_0}\right] - \frac{\pi}{2},$$

where $\frac{\mathrm{d}^{(\mathrm{l})}}{\mathrm{d}\varphi}$ and $\frac{\mathrm{d}^{(\mathrm{r})}}{\mathrm{d}\varphi}$ denote the left and the right derivative respectively.

The subtraction of $\pi/2$ is necessary here, in order to transform the directions of the extremal support lines given by the derivatives into their normal directions (cf. Definition 4.1).

This Lemma motivates the following definition.

4.3 Definition.

a) Define $\partial^{(\mathrm{l})} r : S^1 \to S^1$ and $\partial^{(\mathrm{r})} r : S^1 \to S^1$ by $\partial^{(\mathrm{l}/\mathrm{r})} r(\varphi_0) := \Phi\left[\frac{\mathrm{d}^{(\mathrm{l}/\mathrm{r})}}{\mathrm{d}\varphi}\begin{pmatrix} r(\varphi)\cos\varphi \\ r(\varphi)\sin\varphi \end{pmatrix}_{\varphi=\varphi_0}\right] - \frac{\pi}{2}$.

b) Define $\partial r : S^1 \to \{\text{intervals of } S^1\}$ by $\partial r(\varphi_0) := [\partial^{(\mathrm{l})} r(\varphi_0), \partial^{(\mathrm{r})} r(\varphi_0)]$.

4.4 Lemma.

a) Let $\varphi_1 < \varphi_2 < \varphi_3$.
Then $\partial^{(\mathrm{l})} r(\varphi_1) \le \partial^{(\mathrm{r})} r(\varphi_1) \le \partial^{(\mathrm{l})} r(\varphi_2) \le \partial^{(\mathrm{r})} r(\varphi_2) \le \partial^{(\mathrm{l})} r(\varphi_3) \le \partial^{(\mathrm{r})} r(\varphi_3)$.

b) Let $\varphi_1, \varphi_2 \in S^1$. Then $\bigcup_{\varphi \in [\varphi_1, \varphi_2]} \partial r(\varphi) = [\partial^{(\mathrm{l})} r(\varphi_1), \partial^{(\mathrm{r})} r(\varphi_2)]$.

Sketch of the proof.

a) Consider the (support) lines given by the normal directions $\partial^{(\mathrm{l}/\mathrm{r})} r(\varphi_i)$ and add the lines which pass through the points $(r(\varphi_1), \varphi_1)$, $(r(\varphi_2), \varphi_2)$ or $(r(\varphi_2), \varphi_2)$, $(r(\varphi_3), \varphi_3)$ respectively. By definition of convexity, the arcs between the points must lie "beyond" the latter lines. This yields the assertion. Cf. also [23, Thm. 24.1].

b) follows from a). □

Most important for the later application is the notion of the *conjugate convex curve* r^*:

4.5 Definition.

$$r^*(\vartheta) := \frac{1}{\sup_\varphi r(\varphi)\cos(\varphi-\vartheta)}\,; \quad \vartheta \in S^1.$$

4.6 Proposition. *The function $r^* : S^1 \to [r_1^*, r_2^*]$ $(r_1^*, r_2^* > 0)$ is the representation in polar coordinates of a convex, closed curve in $\mathbb{R}^2$.*

Sketch of the proof. In this instance it is appropriate to write r^* in the obviously equivalent form $r^*(\vartheta) = \inf_{\varphi\in(\vartheta-\pi/2,\vartheta+\pi/2)}[r(\varphi)\cos(\varphi-\vartheta)]^{-1}$. Consider the family of lines $\{g_\varphi(\vartheta) = [r(\varphi)\cos(\varphi-\vartheta)]^{-1} \mid \varphi \in S^1\}$. Now, the curve $r^*(\vartheta)$ arises as the inner boundary of that part of the plane, which is covered by these lines. $\square$

4.7 Lemma.

a) $r^*(\vartheta)\, r(\varphi)\cos(\varphi-\vartheta) \leq 1 \quad \forall \varphi, \vartheta.$

b) $r^*(\vartheta)\, r(\varphi)\cos(\varphi-\vartheta) = 1 \iff \vartheta \in \partial r(\varphi).$

Sketch of the proof. a) follows directly from Definition 4.5. b) is a consequence of Lemma 4.2 and Definition 4.1. $\square$

The operation * defines an involution on the set of curves considered in this section:

4.8 Proposition.

$$r^{**}(\varphi) = r(\varphi) \quad \forall \varphi.$$

Proof. The two parts of Lemma 4.7 imply:

$$\frac{1}{r(\varphi)} = \sup_\vartheta r^*(\vartheta)\cos(\vartheta-\varphi) = \frac{1}{r^{**}(\varphi)}.$$ $\square$

4.9 Remark (Geometrical interpretation of the involution $r \leftrightarrow r^*$).
The notion of the conjugate convex curve defined above is an adaptation of the Legendre transformation for usual convex functions (cf. [22, 23]). This is illustrated by Figure 6. Compare also the formula in Definition 4.5, which reads in terms of the trigonometrically convex functions $\kappa := r^{-1}$, $\kappa^* := (r^*)^{-1}$:

$$\kappa^*(\vartheta) = \sup_\varphi \frac{\cos(\vartheta-\varphi)}{\kappa(\varphi)},$$

with the usual Legendre transformation

$$f^\sharp(x) = \sup_y\,(xy - f(y)).$$

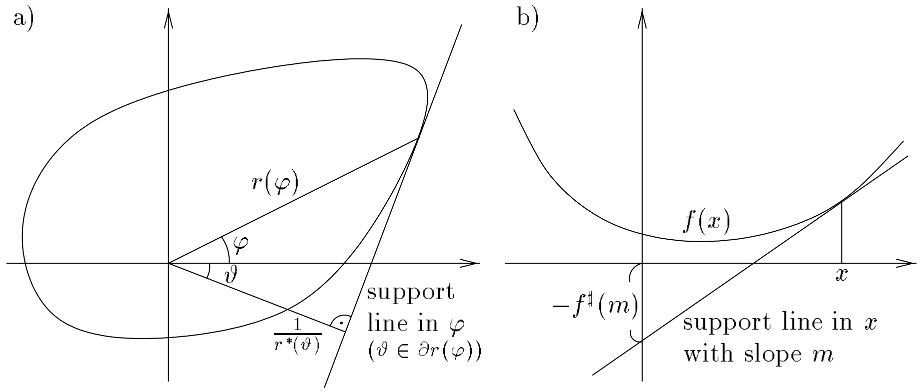

Figure 6: a) The *-operation described in this section, b) the usual Legendre transformation $\sharp$ (cf. Remark 4.9).

4.10 Lemma (Connection to the polarization of norms).

a) Let $'$ denote the polarization with respect to the Euclidean scalar product in $\mathbb{R}^2$.
Then $r_{p'}(\varphi) = (r_p)^*(\varphi)$ for all φ and any norm p.

b) Now, let $'$ denote the polarization with respect to the Minkowski scalar product $(x, y) = x^0y^0 - x^1y^1$.
Then $r_{p'}(\varphi) = (r_p)^*(-\varphi)$ for all φ and any norm p.

Proof.
b)
$$\frac{1}{r_{p'}(\varphi)} = p'\left(\binom{\cos\varphi}{\sin\varphi}\right) = \sup_\psi p\left(\binom{\cos\psi}{\sin\psi}\right)^{-1} \left|\left(\binom{\cos\varphi}{\sin\varphi}, \binom{\cos\psi}{\sin\psi}\right)\right|$$
$$= \sup_\psi r_p(\psi)\,|\cos(\varphi+\psi)| = \frac{1}{(r_p)^*(-\varphi)}$$
(cf. equation (3.3) and Definition 4.5). Part a) can be proven analogously. □

4.2 Construction of selfpolar norms on $\mathcal{M}^2$

As already mentioned in section 3.2, the construction which is to be described now, starts from two ingredients (called I. and II. there). Let, for the present section, these objects be given. Remember also the definition of the quantities $r_1, r_2, \vartheta_1, \vartheta_2$ which are derived from the ingredients.

The existence of an arc r_0 with the properties listed in II. is ensured by part c) of the following Lemma:

4.11 Lemma.

a) $\varphi_1 \le \vartheta_1 + \frac{\pi}{2} \le \varphi_1 + \pi,\ \ \varphi_2 \le \vartheta_2 + \frac{\pi}{2} \le \varphi_2 + \pi.$

b) $\vartheta_1 \le \vartheta_2 \le \vartheta_1 + \pi \le \vartheta_2 + \pi$.

c) Let ϑ_{12} *denote the normal direction of the line through* (r_1, φ_1) *and* (r_2, φ_2). *Then* $\vartheta_1 \le \vartheta_{12} \le \vartheta_2$.

Proof. The restrictions (3.4) are imposed in order to insure a) and b). Condition (3.5) implies c). □

The idea of the construction is, to extend the chosen arc $r_0(\varphi)$ to a complete closed curve, which will represent the desired selfpolar norm.

First, define $r_* : [\varphi_2 - \pi, \varphi_1] \to \mathbb{R}$ by

$$r_*(\psi) := \frac{1}{\sup_{\varphi \in [\varphi_1, \varphi_2]} r_0(\varphi) \cos(\psi + \varphi)}\,; \quad \psi \in [\varphi_2 - \pi, \varphi_1].$$

4.12 Lemma.

a) $r_*(\psi)$ *describes a convex arc.*

b) $r_*(\psi)\, r_0(\varphi) \cos(\varphi + \psi) \le 1 \quad \forall \varphi \in [\varphi_1, \varphi_2],\ \psi \in [\varphi_2 - \pi, \varphi_1]$.

c) $r_*(\psi)\, r_0(\varphi) \cos(\varphi + \psi) = 1 \iff -\psi \in \partial r_0(\varphi)$
$(\,\partial r_0(\varphi_1) := [\vartheta_1, \partial^{(\mathrm{r})} r_0(\varphi_1)]\,;\ \partial r_0(\varphi_2) := [\partial^{(\mathrm{l})} r_0(\varphi_2), \vartheta_2]\,)$.

Sketch of the proof.

a) Write r_* as $r_*(\psi) = \inf_{\varphi \in (-\psi - \pi/2, -\psi + \pi/2) \cap [\varphi_1, \varphi_2]} [r_0(\varphi) \cos(\varphi + \psi)]^{-1}$. Note that due to (3.4) the intersection in this formula is non-empty! Analogous to the situation in Proposition 4.6, the arc r_* is defined by a family of lines.

b) By definition.

c) Like in the proof of Lemma 4.7 b). For the limit cases $\varphi = \varphi_1$, $\varphi = \varphi_2$ use the fact that $-\psi \in [\vartheta_1, \vartheta_2]$. □

4.13 Lemma.

$$r_*(\varphi_1) = r_0(\varphi_1) = r_1\,; \quad r_*(\varphi_2 - \pi) = r_0(\varphi_2) = r_2.$$

Proof.
$-\varphi_1 = \vartheta_1 \in \partial r_0(\varphi_1) \Rightarrow r_*(\varphi_1)\, r_0(\varphi_1) \cos(2\varphi_1) = 1$ (Lemma 4.12 c)) $\Rightarrow r_*(\varphi_1) = r_1$ (def. of r_1).
$-(\varphi_2 - \pi) = \vartheta_2 \in \partial r_0(\varphi_2) \Rightarrow r_*(\varphi_2 - \pi)\, r_0(\varphi_2)\, |\cos(2\varphi_2)| = 1$ (Lemma 4.12 c)) $\Rightarrow r_*(\varphi_2 - \pi) = r_2$ (def. of r_2). □

Now, the completed curve is given by $r : S^1 \to \mathbb{R}$

$$r(\varphi) := \begin{cases} r_0(\varphi) & ; \varphi \in [\varphi_1, \varphi_2] \\ r_*(\varphi) & ; \varphi \in [\varphi_2 - \pi, \varphi_1] \\ r_0(\varphi + \pi) & ; \varphi \in [\varphi_1 - \pi, \varphi_2 - \pi] \\ r_*(\varphi - \pi) & ; \varphi \in [\varphi_2, \varphi_1 + \pi]. \end{cases} \tag{4.2}$$

4.14 Proposition. *The function $r(\varphi)$ is the representation in polar coordinates of a π-periodic convex closed curve, i.e., it defines a norm p_r in $\mathbb{R}^2$ (cf. (3.3)).*

Proof. Periodicity and closedness follow directly from the construction (cf. Lemma 4.13). Convexity: The four arcs constituting the curve $r(\varphi)$ are convex (Lemma 4.12 a)). Now, the composed curve is convex, because the lines t_1 and t_2 are support lines of $r(\varphi)$ in φ_1 and φ_2 respectively (Lemma 4.12 b)). $\square$

4.15 Theorem.
$$r(\varphi) = r^*(-\varphi) \quad \forall \varphi \in S^1 , \quad \textit{i.e., } p_r = p_r', \; p_r \textit{ is selfpolar}$$
($'$: Polarization with respect to $\mathcal{M}^2$, cf. Lemma 4.10 b)).

Proof. Since r and r^* are π-periodic, it is enough to consider $\varphi \in [\varphi_2 - \pi, \varphi_2]$.

Case 1: $\varphi \in [\varphi_2 - \pi, \varphi_1]$
In this case $-\varphi \in [\vartheta_1, \vartheta_2]$. Consequently there is a $\psi \in [\varphi_1, \varphi_2]$ such that $-\varphi \in \partial r(\psi)$ (Lemma 4.4 b)). Thus, Lemma 4.7 b) implies: $r^*(-\varphi)\, r(\psi) \cos(\psi + \varphi) = 1$. On the other hand $-\varphi \in \partial r_0(\psi)$. Lemma 4.12 c) yields: $1 = r_*(\varphi)\, r_0(\psi) \cos(\psi + \varphi) = r(\varphi)\, r(\psi) \cos(\psi + \varphi)$. Thus, $r^*(-\varphi) = r(\varphi)$.

Case 2: $\varphi \in [\varphi_1, \varphi_2]$
Here $-\varphi \in [\vartheta_2 - \pi, \vartheta_1]$. So, there must be a $\psi \in [\varphi_2 - \pi, \varphi_1]$ such that $-\varphi \in \partial r(\psi)$ (Lemma 4.4 b)). Lemma 4.7 b) implies again: $r^*(-\varphi)\, r(\psi) \cos(\psi + \varphi) = 1$.

On the other hand $\varphi \in \partial r^*(-\psi)$.

Indeed, if $\psi \in (\varphi_2 - \pi, \varphi_1)$, then $\partial r^*(-\psi) = -\partial r(\psi)$ due to case 1. If $\psi = \varphi_1$, then $-\varphi \in \partial r(\varphi_1) \cap [\vartheta_2 - \pi, \vartheta_1] = [\partial^{(\mathrm{l})} r(\varphi_1), \vartheta_1]$. So, $\varphi \in [\varphi_1, -\partial^{(\mathrm{l})} r(\varphi_1)] = [\varphi_1, \partial^{(\mathrm{r})} r^*(-\psi)]$ (cf. case 1). Now, the inequality $r^*(\vartheta) \cos(\vartheta - \varphi_1) \leq 1/r_1 = r^*(-\psi) \cos(-\psi - \varphi_1)\; \forall \vartheta$ (Lemma 4.7 a)) shows, using Lemma 4.2, $\varphi_1 \in \partial r^*(-\psi)$, or: $\varphi \in [\varphi_1, \partial^{(\mathrm{r})} r^*(-\psi)] \subset \partial r^*(-\psi)$. The final case $\psi = \varphi_2 - \pi$ can be treated analogously.

Finally apply Lemma 4.7 b) to r^* and use Proposition 4.8: $r(\varphi)\, r^*(-\psi) \cos(\psi + \varphi) = 1$. An application of case 1 yields $r(\varphi)\, r(\psi) \cos(\psi + \varphi) = 1$, and thus $r^*(-\varphi) = r(\varphi)$. $\square$

4.3 Uniqueness and completeness of the construction

The aim of the present section is to show 1) that different choices of the ingredients I, II for the construction lead to different selfpolar norms and 2) that every selfpolar norm can be obtained by the above construction (Thm. 4.17).

4.16 Lemma. *Let p denote any selfpolar norm on $\mathcal{M}^2$.*
Then the curve $r_p(\varphi)$ from eq. (3.3) touches each of the four hyperbolae $(x^0)^2 - (x^1)^2 = \pm 1$ in exactly one point. The tangents of the hyperbolae in these points are support lines of $r_p(\varphi)$.

Proof.
Uniqueness:
The hypothesis means: $r_p^*(-\varphi) = r_p(\varphi)$ for every φ. By Lemma 4.7 a) it follows that $[r_p(\varphi)]^2 \leq |\cos(2\varphi)|^{-1}$ for all φ. Thus, the entire curve $r_p(\varphi)$ must lie between the four hyperbolae. On the other hand it has to be convex, so there cannot be more than one point of intersection with each of the hyperbolae.

Existence of points of intersection:
Consider the hyperbola $h := h_1$ in the quadrant $-\frac{\pi}{4} < \varphi < \frac{\pi}{4}$. Set, similarly to Definition 4.3:

$$\partial h(\varphi_0) := \Phi\Big[\tfrac{\mathrm{d}}{\mathrm{d}\varphi}\binom{h(\varphi)\cos\varphi}{h(\varphi)\sin\varphi}_{\varphi=\varphi_0}\Big] - \tfrac{\pi}{2} = -\varphi_0\,, \quad \varphi_0 \in [-\tfrac{\pi}{4}, \tfrac{\pi}{4}].$$

Now define $\Delta : [-\frac{\pi}{4}, \frac{\pi}{4}] \to \{\text{intervals of } S^1\}$ by $\Delta(\varphi) := \partial r_p(\varphi) - \partial h(\varphi)$. By Lemma 4.4 b): $\bigcup_{\varphi\in(-\pi/4\,,\,\pi/4)} \Delta(\varphi) = (\partial^{(\mathrm{r})} r_p(-\frac{\pi}{4}) - \partial h(-\frac{\pi}{4})\,, \partial^{(\mathrm{l})} r_p(\frac{\pi}{4}) - \partial h(\frac{\pi}{4})) = (\partial^{(\mathrm{r})} r_p(-\frac{\pi}{4}) - \frac{\pi}{4}\,, \partial^{(\mathrm{l})} r_p(\frac{\pi}{4}) + \frac{\pi}{4})$. This latter interval contains the zero angle, since $\partial^{(\mathrm{r})} r_p(-\frac{\pi}{4}) \in (-\frac{3\pi}{4}, \frac{\pi}{4})$ and $\partial^{(\mathrm{l})} r_p(\frac{\pi}{4}) \in (-\frac{\pi}{4}, \frac{3\pi}{4})$. Hence, there must be some $\varphi_0 \in (-\frac{\pi}{4}, \frac{\pi}{4})$, for which $0 \in \Delta(\varphi_0)$, or: $\partial h(\varphi_0) \in \partial r_p(\varphi_0)$. The hyperbola considered here has the property $\partial h(\varphi_0) = -\varphi_0$. Using this, Lemma 4.7 b) implies: $[r_p(\varphi_0)]^2\cos(2\varphi_0) = r_p^*(-\varphi_0)\, r_p(\varphi_0)\cos(2\varphi_0) = 1$. Since the equation of the hyperbola is $[h(\varphi)]^2\cos(2\varphi) = 1$, the above identity shows that $r_p(\varphi)$ intersects $h(\varphi)$ in the direction φ_0.
The property $\partial h(\varphi_0) \in \partial r_p(\varphi_0)$ means that the tangent of h in this point of intersection is a support line of r_p.
The remaining three hyperbolae can be treated in the same way because of symmetry. $\square$

At this stage the main theorem of the present section can be proven:

4.17 Theorem. *Set*

$$\begin{aligned}\mathcal{P}_g := \{\, &(\varphi_1,\varphi_2) \in S^1 \times S^1 \,|\, -\tfrac{\pi}{4} < \varphi_1 < \tfrac{\pi}{4},\ \tfrac{\pi}{4} < \varphi_2 < \tfrac{3\pi}{4}, \\ &\qquad \cos^2(\varphi_1+\varphi_2) < \cos(2\varphi_1)\,|\cos(2\varphi_2)|\,\} \\ \times \{\, &\rho : [0, \tfrac{\pi}{2}] \to \mathbb{R} \,|\, \rho \text{ describes a convex arc from } (1,0) \text{ to } (1, \tfrac{\pi}{2}), \\ &\qquad \text{completely lying inside the triangle } (1,0);\ (\sqrt{2}, \tfrac{\pi}{4});\ (1, \tfrac{\pi}{2})\,\},\end{aligned}$$

$$\begin{aligned}\mathcal{P}_e := \{\, &(\varphi_1,\varphi_2) \in S^1 \times S^1 \,|\, -\tfrac{\pi}{4} < \varphi_1 < \tfrac{\pi}{4},\ \tfrac{\pi}{4} < \varphi_2 < \tfrac{3\pi}{4}, \\ &\qquad \cos^2(\varphi_1+\varphi_2) = \cos(2\varphi_1)\,|\cos(2\varphi_2)|\,\}.\end{aligned}$$

There is a bijection from $\mathcal{P} := \mathcal{P}_g \uplus \mathcal{P}_e$ (disjoint union) into the set of selfpolar norms on $\mathcal{M}^2$ (essentially given by the construction described in section 4.2).

Proof.

Description of the mapping:

Let $(\varphi_1, \varphi_2, \rho) \in \mathcal{P}_g$. Set $r_1 := (\cos 2\varphi_1)^{-1/2}$, $r_2 := |\cos 2\varphi_2|^{-1/2}$. As a consequence of the third inequality satisfied by φ_1 and φ_2, the points (r_1, φ_1) and (r_2, φ_2) form a non-degenerate triangle with the intersection point of t_1 and t_2, the tangents of the hyperbolae in these points. Denote by A the (unique) affine mapping on $\mathbb{R}^2$, which transforms the triangle $(1, 0)$; $(1, \frac{\pi}{2})$; $(\sqrt{2}, \frac{\pi}{4})$ into the former triangle. To be precise, A is supposed to send $(1, 0)$ to (r_1, φ_1), $(1, \frac{\pi}{2})$ to (r_2, φ_2) and $(\sqrt{2}, \frac{\pi}{4})$ to the tangents' intersection point.

Let now $r_0 : [\varphi_1, \varphi_2] \to \mathbb{R}$ be the description in polar coordinates of the image of the curve ρ under the mapping A. Then φ_1, φ_2 and r_0 meet the requirements of the construction in section 4.2. Hence, the construction yields a selfpolar norm p_r on $\mathcal{M}^2$ by Theorem 4.15.

For $(\varphi_1, \varphi_2) \in \mathcal{P}_e$, on the other hand, let $r_0 : [\varphi_1, \varphi_2] \to \mathbb{R}$ describe the straight line from (r_1, φ_1) to (r_2, φ_2). Again, the conditions for the construction are satisfied, and it produces a selfpolar norm p_r.

Injectivity:

If two elements of the parameter set $\mathcal{P}$ produce the same selfpolar norm p, then the curve r_p must intersect the hyperbolae in $-\frac{\pi}{4} < \varphi < \frac{\pi}{4}$ or $\frac{\pi}{4} < \varphi < \frac{3\pi}{4}$ respectively in exactly one point (Lemma 4.16). The initial data used in the constructions which yield p must in both cases be equal to this pair of intersection points. Hence, the two parameters have to be equal.

Surjectivity:

Let p denote an arbitrary selfpolar norm on $\mathcal{M}^2$. Lemma 4.16 insures the existence of two points (r_1, φ_1), (r_2, φ_2) on the curve r_p with the properties $-\frac{\pi}{4} < \varphi_1 < \frac{\pi}{4}$, $\frac{\pi}{4} < \varphi_2 < \frac{3\pi}{4}$, $r_1 = (\cos 2\varphi_1)^{-1/2}$, $r_2 = |\cos 2\varphi_2|^{-1/2}$. Also, (3.5) is valid due to the convexity of r_p. Since in the points of intersection the tangents of the hyperbolae are support lines of r_p (Lemma 4.16), the angles $\vartheta_1 := -\varphi_1$, $\vartheta_2 := \pi - \varphi_2$ satisfy $\vartheta_1 \le \partial^{(\mathrm{r})} r_p(\varphi_1) \le \partial^{(\mathrm{l})} r_p(\varphi_2) \le \vartheta_2$. If, finally, $r_0 : [\varphi_1, \varphi_2] \to \mathbb{R}$ is defined as the corresponding restriction of r_p, then all the prerequisites of the construction are fulfilled. The curve r produced by the construction coincides with r_p on $[\varphi_1, \varphi_2]$ and on $[\varphi_1 - \pi, \varphi_2 - \pi]$.

If one can prove equality of r and r_p also on $[\varphi_2 - \pi, \varphi_1]$ (and thus on $[\varphi_2, \varphi_1 - \pi]$ as well), then, after transformation by A^{-1}, a parameter in $\mathcal{P}$ is found which is mapped on the given norm p.

So, let now $\varphi \in [\varphi_2 - \pi, \varphi_1]$. Then $-\varphi \in [\vartheta_1, \vartheta_2]$. Due to Lemma 4.4 b) there is a $\psi \in [\varphi_1, \varphi_2]$ such that $-\varphi \in \partial r(\psi)$ and (in the same point ψ!) $-\varphi \in \partial r_p(\psi)$. Lemma 4.7 b) then implies: $r_p(\varphi)\, r_p(\psi) \cos(\varphi + \psi) = r_p^*(-\varphi)\, r_p(\psi) \cos(\varphi + \psi) = 1 = r^*(-\varphi)\, r(\psi) \cos(\varphi + \psi) = r(\varphi)\, r_p(\psi) \cos(\varphi + \psi)$. It follows: $r(\varphi) = r_p(\varphi)$. $\square$

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