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On the Krein Space Structure of Fock Spaces Induced by Infrared Divergent Integrals

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Abstract

The GNS representation using hermitian linear functionals on (possibly indefinite) inner product spaces and the regularization of divergent integrals are preliminarily reviewed. For the collection of all possible regularizations of algebraic singularities, the corresponding free-field-like functionals are considered and the Krein space structure of the respective representation spaces is investigated in detail.

1 Introduction

It is now natural to consider (local) quantum field theories (QFT) in which not all Wightman axioms are satisfied, since there are examples of QFT in which it is impossible that the Wightman functionals W under consideration satisfy as well locality as positivity. Guided by the investigations of some models ([23], [26]), it seems to be better to keep locality and to give up positivity. Furthermore, considering QFT having two-point functions with infrared-singularities (e.g., free field of mass $m = 0$ in two space-time dimensions [22]), the regularization yields Wightman functionals not being positive. The present paper is concerned with that case. Along these lines one is led to the modified Wightman axioms of indefinite metric QFT due to Morchio and Strocchi ([21], [5, ch. 10]).

Following the Borchers-Uhlmann approach to axiomatic QFT ([6], [28]), such non-positive Wightman functionals W have to be considered now. As a first step, one has to generalize the well-known GNS-Theorem to the case of hermitian linear functionals in order to reconstruct the field operators $\pi(.)$, their domain $\mathfrak{D}$, the state space $\mathfrak{H}$, and the vacuum ψ_0. Such a generalization is due to Scheibe [24] (cf. Proposition 2.1). However, $\mathfrak{D}$ then is an inner product space rather than a unitary space, where the inner product $(.,.)$ is defined by W. For as well physically as mathematically motivated reasons, one likes to define a

Hilbert space structure on $\mathfrak{D}$, i.e., the inner product space $\mathfrak{D}$, $(.,.)$ is densely embedded in some Krein space [4] (cf. formulae (1), (2) below).

Having in mind that the "whole theory" is encoded in W, one is led to the task of establishing conditions upon W such that a Hilbert space structure exists on $\mathfrak{D}$. Such conditions were given in [30], [18], [1], [21], [15]. In the following condition (H) taken from [15] will be considered. It is then of interest whether or not the Gram operator J connecting the inner product $(.,.)$ with the positive definite (scalar) product of the Krein space leaves the domain $\mathfrak{D}$ invariant. Remember also that such a Hilbert space structure, if it exists, is not uniquely defined in general. For an explicite example of an inner product space which allows to define a whole family of non-equivalent Hilbert space structures, the reader is referred to [2] (cf. questions (Q1), ..., (Q3)).

In [11], [13], those questions were answered for hermitian linear functionals of the structure of (generalized) free fields on tensor algebras (cf. Prop. 3.1, 3.4, Cor. 3.3, below). The present paper is aimed at an investigation of the above questions for the case of hermitian linear functionals again of the structure of free fields, where the two-point functional W_2 is now defined by the regularization of a divergent integral. Considering QFT with infrared singularities, one meets such W_2 ([22],[5, section 11.1.]). In contrast to [22], it is stated that the Gram operator J leaves the domain $\mathfrak{D}$ invariant (Corollary 3.3). Furthermore all possible regularizations are considered and it is shown that they may in general lead to non-equivalent Krein space structures.

The pattern of the present paper is as follows. While in Section 2 the GNS representation with hermitian linear functionals and questions related to the Hilbert space structure are recalled, Section 3 is devoted to the structure of the representation space $\mathfrak{D}$ for the special case of functionals of the free-field type. In Section 4 some facts about the regularization of divergent integrals are recalled, and then, in Section 5, applied to the GNS representation using an hermitian linear functional W_2 which is obtained by the regularization of a divergent integral.

2 On the GNS Representation with Hermitian Linear Functionals

Let $\mathfrak{A}$ denote an (associative) *-algebra with unity $\mathbf{1}$ and W a linear and hermitian functional on $\mathfrak{A}$ satisfying $W(\mathbf{1}) = 1$. Recall the following GNS-like reconstruction theorem due to Scheibe.

Proposition 2.1. *Under the above assumptions there are*

(i) a vector space $\mathfrak{D}$ *with an inner product* $(.,.)$,

(ii) a vector $\psi_0 \in \mathfrak{D}$ *satisfying* $(\psi_0, \psi_0) = 1$,

(iii) a representation $f \mapsto \pi_W(f)$ of $\mathfrak{A}$ by linear operators on $\mathfrak{D}$ such that

$$\begin{aligned} W(f) &= (\psi_0, \pi_W(f)\psi_0), \\ \mathfrak{D} &= \text{span}\,\{\pi_W(f)\psi_0;\ f \in \mathfrak{A}\}, \quad \textit{cyclicity of } \psi_0, \\ (\varphi, \pi_W(f)\psi) &= (\pi_W(f^*)\varphi, \psi), \end{aligned}$$

$f \in \mathfrak{A}$, $\varphi, \psi \in \mathfrak{D}$.

Furthermore, $\mathfrak{D}$, ψ_0, and $\pi_W(.)$ are uniquely defined by (i),...,(iii) up to isometric linear isomorphisms.

Proof. See [24, 30]. □

Remark 2.2. Noticing that an inner product $(f, g) := W(f^*g)$, $f, g \in \mathfrak{A}$, is defined on $\mathfrak{A}$, consider the isotropic part

$$\mathfrak{A}^0 = \{f \in \mathfrak{A};\ (f, g) = 0 \text{ for all } g \in \mathfrak{A}\},$$

and the quotient space $\mathfrak{D} = \mathfrak{A}/\mathfrak{A}^0$ endowed with the inner product $(\hat{f}, \hat{g})^\sim = (f, g)$, where $\hat{\cdot} = \{\cdot + \mathfrak{A}^0\}$ denotes the residue class containing $\cdot$ (cf. [3, Ch. I, §1.8]). In the following it is written $(.,.)$ instead of $(.,.)^\sim$.

In order to make the theory mathematically manageable one has to define a *Hilbert space structure* on $\mathfrak{D}$, i.e., there is a positive definite inner product $[.,.]$ and a linear operator $J = J^*$ (Gram operator) on $\mathfrak{H} = \widetilde{\mathfrak{D}}^{\|.\|}$, $\|.\| = \sqrt{[.,.]}$, such that

$$(\varphi, \psi) = [\varphi, J\psi], \tag{1}$$

$$\|J\| := \sup_{\|\psi\| \leq 1} \|J\psi\| < \infty, \tag{2}$$

$\varphi, \psi \in \mathfrak{H}$, where the continuously extended inner products onto $\mathfrak{H}$ are also denoted by $(.,.)$, $[.,.]$. (Here and in the following, let $\widetilde{\cdot}^{\|.\|}$ (resp. $\widetilde{\cdot}^{\tau}$) denote the completed hull of a set $\cdot$ relative to the (locally convex) topology defined by $\|.\|$ (resp. τ).) Let the above Hilbert space structure be denoted by $(\mathfrak{H}, J)$.

Recall further that two Hilbert space structures $(\mathfrak{H}_j, J_j)$, $j = 1, 2$, are called equivalent, if the two positive definite inner products $[.,.]_j$ satisfy

$$(\varphi, \psi) = [\varphi, J_1\psi]_1 = [\varphi, J_2\psi]_2,$$

and both norms $\|.\|_j = \sqrt{[.,.]_j}$ ($j = 1, 2$) are equivalent on $\mathfrak{D}$, thus $\mathfrak{H}_1 = \mathfrak{H}_2$. Remember also that if the inverse operator J^{-1} exists as a bounded linear operator on $\mathfrak{H}$, then there is an equivalent Hilbert space structure $(\mathfrak{H}', J')$ given by $[\varphi, \psi]' = [\varphi, |J|\psi]$, $J' = \frac{J}{|J|}$, $\varphi, \psi \in \mathfrak{H}$ ([8, §2.7(3)]). Noticing that J' is a symmetry on $\mathfrak{H}'$ (i.e., $J' = J'^* = J'^{-1}$), $(\mathfrak{H}', J')$ is called a *Krein space structure.*

The existence of a Hilbert space structure has the following consequences:

1) there is a maximal Hilbert space structure,

2) the theory of unbounded representations applies, and consequently, the theory is well-understood and mathematically manageable (see [25]).

ad 1): A Hilbert space structure $(\mathfrak{H}, J)$ on $\mathfrak{D}$ is called maximal if there is no other one $(\mathfrak{H}_1, J_1)$ satisfying $\mathfrak{H} \subsetneqq \mathfrak{H}_1$. Recall the following.

Proposition 2.3. *The following are equivalent.*

(i) $(\mathfrak{H}, J)$ *is a maximal Hilbert space structure,*

(ii) J^{-1} *is a bounded operator on* $\mathfrak{H}$*,*

(iii) $\mathfrak{H}$ *is a Krein space.*

Proof. $(i) \Leftrightarrow (ii) :$ [21, Theorem 5], $(ii) \Leftrightarrow (iii) :$ [4, V.1.3]. □

Along these lines, the following questions arise.

Questions

(Q1) Which conditions must the functional W satisfy such that a Hilbert space structure exists on $\mathfrak{D}$?

(Q2) Under which conditions does the Gram operator J satisfy $J : \mathfrak{D} \to \mathfrak{D}$?

(Q3) Under which conditions does exactly one maximal Hilbert space structure exist on $\mathfrak{D}$?

For answering (Q1), the following Hilbert-space structure condition is introduced.

(H) *There is a quadratic seminorm* p *on* $\mathfrak{A}$ *such that for each* $g \in \mathfrak{A}$ *there is a constant* $C_g \geq 0$ *and* $|W(g^*f)| \leq C_g p(f)$ *is satisfied for all* $f \in \mathfrak{A}$.

(Remember that a seminorm p is called quadratic (or Hilbertian), if a semi-scalar product $[.,.]$ exists such that $p(f) = \sqrt{[f,f]}$, $f \in \mathfrak{A}$.) Noticing that

$$(f, g) = W(f^*g) \tag{3}$$

defines an inner product on $\mathfrak{A}$, recall that a locally convex topology τ on $\mathfrak{A}$ is called a partial majorant (resp. majorant), if the inner product $(.,.)$ is separately continuous (resp. continuous) relative to τ. (H) means then that p defines a quadratic and partial majorant on $\mathfrak{A}$.

Theorem 2.4. *The following are equivalent:*

(i) *Condition (H) is satisfied,*

(ii) *a quadratic majorant exists on* $\mathfrak{D}$*,*

(iii) *a Hilbert-space structure* $(\mathfrak{H}, J)$ *exists on* $\mathfrak{D}$*,*

(iv) *a Krein-space structure* $(\mathfrak{H}', J')$ *exists on* $\mathfrak{D}$*.*

Proof. See [15, Theorem 3]. □

An answer to (Q2) is given next. Recall that the non-degenerate inner product space $\mathfrak{D}, (.,.)$ is called *decomposable*, if a *fundamental decomposition*

$$\mathfrak{D} = \mathfrak{D}^{(+)}(\dot{+})\mathfrak{D}^{(-)}$$

exists, where $\mathfrak{D}^{(\pm)}$ are positive / negative definite subspaces of $\mathfrak{D}$, and $(\dot{+})$ denotes the orthogonal direct sum relative to the inner product $(.,.)$.

Theorem 2.5. *The following are equivalent:*

(i) *there is a Krein-space structure* $(\mathfrak{H}, J)$ *on* $\mathfrak{D}$ *such that* $J : \mathfrak{D} \to \mathfrak{D}$*,*

(ii) $\mathfrak{D}$, $(.,.)$ *is a decomposable, non-degenerate inner product space.*

Proof. See [15, Theorem 4]. □

In order to discuss (Q3), the special case that Theorem 2.5 (ii) applies is considered now. For every fundamental decomposition

$$\mathfrak{D} = \mathfrak{D}^{(+)}(\dot{+})\mathfrak{D}^{(-)}, \tag{4}$$

a norm $x \to \|x\|_J = \sqrt{(x, Jx)}$, $x \in \mathfrak{D}$, is defined on $\mathfrak{D}$, where $J = P^+ \Leftrightarrow P^-$ is called *fundamental symmetry*, and $P^\pm : \mathfrak{D} \to \mathfrak{D}^{(\pm)}$ are the fundamental projections defined by the decomposition (4). Recall that there is a one-to-one correspondence between the fundamental symmetries and the fundamental decompositions of a decomposable inner product space. Remember further that the locally convex topology τ_J defined by $\|.\|_J$ is called decomposition majorant (belonging to the above fundamental decomposition), and that there are inner product spaces having non-equivalent decomposition majorants ([4, Ex. IV.4.4]).

An answer to (Q3) is given now.

Proposition 2.6. *If* $\mathfrak{D}$, $(.,.)$ *is a non-degenerate and decomposable inner product space, the following are equivalent:*

(i) *Exactly one Krein-space structure exists on* $\mathfrak{D}$, $(.,.)$,

(ii) *for every fundamental decomposition* $\mathfrak{D} = \mathfrak{D}^{(+)}(\dot{+})\mathfrak{D}^{(-)}$, *it follows that* $\mathfrak{D}^{(+)}[\tau_J \upharpoonright_{\mathfrak{D}^{(+)}}]$ *or* $\mathfrak{D}^{(-)}[\tau_J \upharpoonright_{\mathfrak{D}^{(-)}}]$ *is complete, and thus a Hilbert space,*

(iii) *there exists a fundamental decomposition* $\mathfrak{D} = \mathfrak{D}_1^{(+)}(\dot{+})\mathfrak{D}_1^{(-)}$ *such that* $\mathfrak{D}_1^{(+)}[\tau_{J_1} \upharpoonright_{\mathfrak{D}_1^{(+)}}]$ *or* $\mathfrak{D}_1^{(-)}[\tau_{J_1} \upharpoonright_{\mathfrak{D}_1^{(-)}}]$ *is complete, and thus a Hilbert space.*

Proof. See [16, Proposition 3], [29, Satz 9]. □

For the important class of quasi-positive (resp. quasi-negative) inner product spaces, i.e., $\mathfrak{D}$ does not contain any negative definite (resp. positive definite) subspace of infinite dimension, the following gives an answer to (Q3).

Corollary 2.7. *If* $\mathfrak{D}$ *is quasi-positive or quasi-negative, then there exists exactly one Krein-space structure on* $\mathfrak{D}$.

Proof. Proposition 2.6 ($(ii) \Rightarrow (i)$) readily yields the statement under consideration. See also [15, Corollary 2]. □

3 On the GNS Representation with Free-Field-Like Functionals

3.1 Free-Field-Like Functionals on Tensor Algebras

Let E be a vector space with an involution " $*$ ", and let

$$E_{\otimes} = \bigoplus_{n=0}^{\infty} E_n$$

denote the tensor algebra (Borchers-Uhlmann algebra) over the basic space E, where $E_0 = \mathbb{C}$ (field of complex numbers), $E_n = E \otimes \cdots \otimes E$ (n copies), $n = 1, 2, 3, \ldots$. Recall that $E_{\otimes}$ becomes a *-algebra with unity $\mathbf{1} = (1, 0, 0, \ldots) \in E_{\otimes}$, where the algebraic operations are defined as usual (e.g., see [9], [10]).

Noticing that an hermitian linear functional W_2 on E_2 defines an inner product on E by

$$(x, y) = W_2(x^* \otimes y)\,,$$

$x, y \in E$, let the following assumption about as well the structure of the inner product space $E, (.,.)$ as its involution "*" be made.

Assumption I. *Let an hermitian linear functional* W_2 *on* E_2 *be given such that* $E, (.,.)$ *is decomposable with fundamental decomposition*

$$E = E^{(+)}(\dot{+})E^{(-)}(\dot{+})E^{(0)} \tag{5}$$

satisfying $(E^{(\#)})^* = E^{(\#)}$, $\# \in \{+, \Leftrightarrow, 0\}$, *where* $F^* = \{f^*; f \in F\}$.

Recalling that $E^{(+)}$, $E^{(-)}$ are pre-Hilbert spaces with respect to

$$[x, x] = W_2(x^* \otimes x),\ [y, y] = \Leftrightarrow W_2(y^* \otimes y),$$

choose orthonormal basises $\{e^{(\alpha)}\}_{\alpha \in A^+}$ and $\{e^{(\alpha')}\}_{\alpha' \in A^-}$ of $E^{(+)}$ and $E^{(-)}$, respectively, such that $e^{(\alpha)} = e^{(\alpha)*}$, $\alpha \in A^+ \cup A^-$. Let further $\{e^{(\alpha'')}\}_{\alpha'' \in A^0}$ be a basis of $E^{(0)}$. Then $\{e^{(\alpha)}\}_{\alpha \in A}$ constitutes a basis of E (A^0, A^+, A^- are sets of indices with $A = A^+ \cup A^- \cup A^0$, $A^+ \cap A^- = A^+ \cap A^0 = A^- \cap A^0 = \emptyset$), and

$$W_2(e^{(\alpha)} \otimes e^{(\alpha')}) = \begin{cases} \delta_{\alpha,\alpha'} & for \quad \alpha \in A^+ \\ \Leftrightarrow \delta_{\alpha,\alpha'} & for \quad \alpha \in A^- \\ 0 & for \quad \alpha \in A^0 \end{cases}$$

$\alpha' \in A$, is satisfied.

Consider then the hermitian linear functional (pseudo-Wightman functional)

$$W = (1, 0, W_2, 0, W_4, \dots)$$

of the type of *generalized free fields* on $E_\otimes$, where

$$W_{2n}(e^{(\alpha_1)} \otimes \cdots \otimes e^{(\alpha_{2n})}) = \sum_{(i,j)} \prod_{r=1}^{n} W_2(e^{(\alpha_{i^r})} \otimes e^{(\alpha_{j^r})}),$$

$\alpha_j \in A$ ($j = 1, 2, \dots, 2n$), and the sum $\sum_{(i,j)} \cdot$ is over all the $\frac{(2n)!}{2^n n!}$ partitions of $\{1, 2, \dots, 2n\}$ into pairs $\{i^1, j^1\}, \{i^2, j^2\}, \dots, \{i^n, j^n\}$ such that $i^1 < i^2 < \cdots < i^n$, $i^1 < j^1$, $i^2 < j^2, \dots, i^n < j^n$, $n \in \mathbb{N}$.

Let us introduce the following notation. For every $\pm e^{(\alpha_1)} \otimes \cdots \otimes e^{(\alpha_n)}$, $\alpha_j \in A$ ($j = 1, 2, \dots, n$), $n \in \mathbb{N}$, let

$$\eta(\pm e^{(\alpha_1)} \otimes \cdots \otimes e^{(\alpha_n)}) \equiv \eta(\alpha_1, \dots, \alpha_n)$$

denote the number of α_j satisfying $\alpha_j \in A^-$.

3.2 The Structure of the Representation Space

Notice that the sesqui-linear form $(f, g) := W(f^* g)$, $f, g \in E_\otimes$, defines an (indefinite) inner product on $E_\otimes$. Consider the set of isotropic vectors $E_\otimes^{(0)} = E_\otimes \cap E_\otimes^{[\perp]} = \{f \in E_\otimes; W(g^* f) = 0 \text{ for all } g \in E_\otimes\}$, and the quotient space

$$\mathfrak{D} := E_\otimes / E_\otimes^{(0)}.$$

Let the non-degenerate sesqui-linear form induced by $(.,.)$ on $\mathfrak{D}$ also be denoted by $(.,.)$.

In order to describe $\mathfrak{D}$ more explicitely, the symmetrization operators $S_k : E_k \to E_k$ defined by

$$S_k(f^{(1)} \otimes \cdots \otimes f^{(k)}) = \frac{1}{k!} \sum_{\pi \in \Pi_k} f^{(\pi(1))} \otimes \cdots \otimes f^{(\pi(k))} \tag{6}$$

is introduced, where $f_k = f^{(1)} \otimes \cdots \otimes f^{(k)} \in E_k$, and Π_k denotes the set of all the permutations of $\{1, 2, ..., k\}$, $k \in \mathbb{N}$. Consider the kernels of the symmetrization operators

$$\begin{aligned} \ker(S_m) &= \operatorname{span}\{e^{(\alpha_1)} \otimes \cdots \otimes e^{(\alpha_m)} \Leftrightarrow e^{(\alpha_{\pi(1)})} \otimes \cdots \otimes e^{(\alpha_{\pi(m)})}; \\ &\qquad \pi \in \Pi_m,\ \alpha_j \in A\,(j = 1, 2, \ldots, m)\}, \end{aligned} \tag{7}$$

$m = 2, 3, \ldots$, $\ker(S_1) = \{0\}$.

Proposition 3.1. *Letting Assumption I be satisfied, it follows:*

a) It holds $\mathfrak{D} = (\dot{+})_{m=0}^{\infty} \mathfrak{D}_m$, *where* $\mathfrak{D}_0 = \mathbb{C}$,

$$\begin{aligned} \mathfrak{D}_1 &\cong E^{(+)} (\dot{+}) E^{(-)}, \qquad (8) \\ \mathfrak{D}_m &\cong E_m / \ker(S_m) \\ &= \mathit{span}\{S_m(e^{(\alpha_1)} \otimes \ldots e^{(\alpha_m)});\ \alpha_j \in A^+ \cup A^- \ (j = 1, 2, \ldots, m)\}. \end{aligned}$$

b) $\mathfrak{D}$ *is a decomposable, nondegenerate inner product space, i.e., there are subspaces* $\mathfrak{D}^{(+)}, \mathfrak{D}^{(-)} \subset \mathfrak{D}$ *such that there is a fundamental decomposition*

$$\mathfrak{D} = \mathfrak{D}^{(+)} (\dot{+}) \mathfrak{D}^{(-)}. \tag{9}$$

c) (9) applies especially for $\mathfrak{D}^{(+)} = \mathbb{C} (\dot{+}) \mathfrak{D}_1^{(+)} (\dot{+}) \mathfrak{D}_2^{(+)} (\dot{+}) \ldots$, $\mathfrak{D}^{(-)} = \mathfrak{D}_1^{(-)} (\dot{+}) \mathfrak{D}_2^{(-)} (\dot{+}) \ldots$, *where*

$$\begin{aligned} \mathfrak{D}_n^{(+)} &\cong \mathit{span}\{S_n(e^{(\alpha_1)} \otimes \cdots \otimes e^{(\alpha_n)});\ \eta(\alpha_1, \ldots, \alpha_n) \ \mathit{is\ even}\} \\ \mathfrak{D}_n^{(-)} &\cong \mathit{span}\{S_n(e^{(\alpha_1)} \otimes \cdots \otimes e^{(\alpha_n)});\ \eta(\alpha_1, \ldots, \alpha_n) \ \mathit{is\ odd}\} \end{aligned}$$

d) It holds $\mathfrak{D}_n^{(+)} (\perp) \mathfrak{D}_n^{(-)}$, $n = 1, 2, \ldots$, *where* $(\perp)$ *denotes orthogonality relative to the inner product* $(.,.)$, *and*

$$\mathfrak{D}_n = \mathfrak{D}_n^{(+)} (\dot{+}) \mathfrak{D}_n^{(-)}. \tag{10}$$

Proof. See [13, Theorem 3.3, Lemma 3.4a)]. □

Remark 3.2.

a) Proposition 3.1 states that there are linear isomorphisms $\kappa_m : \mathfrak{D}_m \to E_m / \ker(S_m)$, and further, $\mathfrak{D}_n$ and $\mathfrak{D}_m$ are orthogonal to each other relative to the inner product $(.,.)$, $n \neq m$, $m, n = 0, 1, 2, \ldots$.

b) If $J_1 : \mathfrak{D}_1 \to \mathfrak{D}_1$ denotes the fundamental symmetry belonging to the fundamental decomposition (8), then $J_n = J_1 \otimes \cdots \otimes J_1 : \mathfrak{D}_n \to \mathfrak{D}_n$ is the fundamental symmetry belonging to (10). Furthermore, the fundamental symmetry $J : \mathfrak{D} \to \mathfrak{D}$ belonging to (9) is given by $J = (J_0, J_1, J_2, \dots)$, $J_0 = I$ (identity mapping).

Corollary 3.3. *If Assumption I applies, then a Krein-space structure $(\mathfrak{H}, J)$ exists on $\mathfrak{D}$ such that $J : \mathfrak{D} \to \mathfrak{D}$.*

Proof. Proposition 3.1b) and Theorem 2.5 ($(ii) \Rightarrow (i)$) yield the statement under consideration. □

The remainder of this section is devoted to question (Q3). Consider first the "one-particle" space $\mathfrak{D}_1$ endowed with the inner product $(.,.)_1$ defined by W_2. Due to Proposition 3.1a) and Theorem 2.5, a Krein-space structure exists on $\mathfrak{D}_1, (.,.)_1$. In general, however, Proposition 2.6 implies that this Krein-space structure is not uniquely defined. On the other hand, if $\mathfrak{D}_1, (.,.)_1$ is quasi-positive, then there is exactly one Krein-space structure on $\mathfrak{D}_1$ by Corollary 2.7.

Consider

$$\mathfrak{D}^{(n)} = \bigoplus_{m=0}^{n} \mathfrak{D}_m$$

endowed with the inner product $(.,.)^{(n)} = (.,.) \restriction_{\mathfrak{D}^{(n)} \times \mathfrak{D}^{(n)}}$, $n \in \mathbb{N}$.

Proposition 3.4. *Assuming that the "one-particle" space $\mathfrak{D}_1$, $(x,y)_1$ is quasi-positive, it follows*

a) there is exactly one Krein-space structur on $\mathfrak{D}^{(n)}, (.,.)^{(n)}$, $n \in \mathbb{N}$,

b) on $\mathfrak{D}, (.,.)$, there are non-equivalent Krein-space structures.

Proof. *a*) It follows from Proposition 3.1c) that a fundamental decomposition of $\mathfrak{D}^{(n)}$ is given by

$$\mathfrak{D}^{(n)} = \left(\bigoplus_{m=0}^{n} \mathfrak{D}_m^{(+)} \right) \; (\dot{+}) \; \left(\bigoplus_{m=0}^{n} \mathfrak{D}_m^{(-)} \right).$$

The assumption that $\mathfrak{D}_1, (x,y)_1$ is quasi-positive, then yields that $\mathfrak{D}^{(n)}, (.,.)^{(n)}$ is quasi-positive, too. The statement under consideration now follows from Corollary 2.7.

b) By Proposition 2.6 it is enough to verify that both $\mathfrak{D}^{(+)}[\tau_J \restriction_{\mathfrak{D}^{(+)}}]$ and $\mathfrak{D}^{(-)}[\tau_J \restriction_{\mathfrak{D}^{(-)}}]$ are not complete. Considering any $y = (y_0, y_1, \dots) \in \mathfrak{D}, y_j \in \mathfrak{D}_j$, notice that $\|y\|_J = \sqrt{\sum_{j=0}^{\infty} \|y_j\|_j^2}$, where $\|y_j\|_j := \|y\|_{J_j} = \sqrt{(y_j, J_j y_j)}$, and J_j, J are taken from Remark 3.2b). Take then two elements $x_1^{\pm} \in \mathfrak{D}_1^{(\pm)}$ satisfying $\|x_1^{\pm}\|_1 = 2^{-1}$. Considering $x_n^{\pm} = x_1^{\pm} \otimes \cdots \otimes x_1^{\pm}$ (n factors), it follows $\|x_n^{\pm}\|_n = 2^{-n}$. Putting $x^{+(n)} = (0, x_1^+, \dots, x_n^+, 0, 0, \dots) \in \mathfrak{D}^{(+)}$, $x^{-(n)} = (0, x_1^-, 0, x_3^-, \dots, x_{2n+1}^-, 0, 0, \dots) \in \mathfrak{D}^{(-)}, n \in \mathbb{N} \cup \{\infty\}$, notice $\|x^{\pm(\infty)}\|_J < \infty$ and $x^{\pm(n)} \to x^{\pm(\infty)}$ relative to τ_J as $n \to \infty$. Since $x^{\pm(\infty)} \notin \mathfrak{D}$, the proof is complete. □

4 On the Regularization of Integrals with Algebraic Singularities

In order to apply the results of the preceeding section to some special hermitian linear functionals, some notions and facts from the theory of divergent integrals and their regularization are recalled (see [7]; ch.I,1.7.3).

Define

$$\mathfrak{S}_\Omega = \{\varphi \in \mathfrak{S}(\mathbb{R}^d) |\, \mathrm{supp}\varphi \subset \Omega\},\ \Omega \subset \mathbb{R}^d,\ d \in \mathbb{N}.$$

Here $\mathfrak{S}(\mathbb{R}^d)$ denotes the Schwartz space of test functions of rapid decrease.

Let some (real) function $f : \Omega \to \mathbb{R},\ \Omega \subset \mathbb{R}^d,\ d \in \mathbb{N}$ be given. The function f is then called *tempered on* Ω if the mapping

$$\varphi \mapsto \int\limits_\Omega f(x)\varphi(x)dx \tag{11}$$

is a continuous linear functional on $\mathfrak{S}_\Omega$.

If f is a tempered function on $\Omega = \mathbb{R}^d \setminus \{x_0\}$, then we say that f has a *singularity* at x_0.

A singularity x_0 is an *algebraic singularity*, if there is an $m \geq 0$ such that

$$|f(x)|\ \|x - x_0\|^m \tag{12}$$

is bounded on some neighbourhood $\mathfrak{U}(x_0)$ of x_0.

The *degree* of the singularity x_0 is defined by

$$\deg_{x_0}(f) := \inf\{m \geq 0 \,|\, \exists\, \mathfrak{U}(x_0) : (12) \text{ is bounded on } \mathfrak{U}(x_0)\}.$$

Assume now that f is a tempered function on $\Omega = \mathbb{R}^d \setminus \{x_0\}$. Every distribution $T_f \in \mathfrak{S}'(\mathbb{R}^d)$ is called a *regularization of* f, if it has the following form

$$T_f(\varphi) = \int\limits_{\mathbb{R}^d} dx f(x)\varphi(x)$$

for every test function $\varphi \in \mathfrak{S}_{\mathbb{R}^d \setminus \{x_0\}}$. If furthermore the above T_f is hermitian, then it is called an *hermitian regularization* of f.

Recall the following basic facts from regularization theory which is well developed in [7]. For the sake of simplicity, $x_0 = 0$ is assumed.

Proposition 4.1. *a) Let f be a tempered function on $\mathbb{R}^d \setminus \{0\}$. If 0 is an algebraic singularity of f, then there is a regularization of f. In this case one can give a regularization by the following formula:*

$$T_f(\varphi) = \int\limits_{\mathbb{R}^d} dx\, f(x) \left[\varphi(x) \Leftrightarrow (P^{(l-1)}\varphi)(x)\, \theta(1 \Leftrightarrow |x|)\right], \tag{13}$$

where l is the integral part of $\deg_0(f)$ and $P^{(l-1)}\varphi$ denotes the Taylor series of order $l \Leftrightarrow 1$ for the test function φ about the point 0. θ is the Heaviside step function:

$$\theta(1 \Leftrightarrow |x|) = \begin{cases} 1 \ : \ |x| \leq 1 \\ 0 \ : \ |x| > 1 \end{cases}.$$

(In (13) one subtracts from the test function $\varphi(x) \in \mathfrak{S}(\mathbb{R}^d)$ so many members of the Taylor series at $x_0 = 0$, that the order of the remainder is equal to l.)

b) If T_1 and T_2 are any two different regularizations of f, then it holds $\mathrm{supp}(T_1 \Leftrightarrow T_2) = \{0\}$.

Remembering that every distribution with support in $\{0\}$ is a finite sum of the delta function and derivatives of the delta function, the general form of the regularization can be given.

Corollary 4.2. *Let us use the following description: $(c_0, c_1, \cdots, c_n, 0, \cdots) = c \in \mathrm{d}(\mathbb{R})$. Suppose that f is a tempered function on $\Omega = \mathbb{R}^d \setminus \{0\}$. Let 0 be an algebraic singularity of f. Then every regularization of f is of the form*

$$T_{f,c}(\varphi) = T_f(\varphi) + \sum_{j=0}^{\infty} c_j \varphi^{(j)}(0)\,. \tag{14}$$

Remark 4.3. If the singularity at x_0 is not an algebraic singularity, then there exists in general no regularization relative to $\mathfrak{S}$.

5 The Decomposition of $\mathfrak{S}(\mathbb{R}^+)$ with Respect to Inner Products Induced by Regularized Integrals

5.1 The Regularized $(1/x)_+$

In this section those inner products on the test function space $\mathfrak{S}(\mathbb{R}^+) = \mathfrak{S}(\mathbb{R}) / \, \{\varphi \in \mathfrak{S}(\mathbb{R}) \,|\, \varphi(x) = 0 \quad \forall x \geq 0\}$ are considered, which are derived from the regularizations of $\int_0^\infty 1/x\, \varphi(x)\, \mathrm{d}x$. They arise for example in the theory of the quantized massless scalar field in two-dimensional spacetime (see e.g. [22]).

It is convenient to define the "signature" of a test function by the linear map $\sigma : \mathfrak{S}(\mathbb{R}^+) \to \omega$, $\sigma_\varphi = (\sigma_\varphi^{(0)}, \sigma_\varphi^{(1)}, \sigma_\varphi^{(2)}, \dots) = (\varphi(0), \varphi'(0), \varphi''(0), \dots)$.

Now, define for every finite sequence of real numbers $c \in \mathrm{d}(\mathbb{R})$ and for any $\varphi, \psi \in \mathfrak{S}(\mathbb{R}^+)$ (cf. (14),(13)):

$$\begin{aligned} (\varphi,\psi)_c &= T_{(1/x)_+,c}(\overline{\varphi}\psi) = \int_0^1 \frac{\mathrm{d}x}{x}\left(\overline{\varphi(x)}\,\psi(x) - \overline{\varphi(0)}\,\psi(0)\right) \\ &\quad + \int_1^\infty \frac{\mathrm{d}x}{x}\,\overline{\varphi(x)}\,\psi(x) + \sum_{n=0}^\infty c_n \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\overline{\varphi(x)}\,\psi(x)\right)\Big|_{x=0} \\ &= (\varphi,\psi)_0 + \sum_{n=0}^\infty c_n \sum_{j=0}^n \binom{n}{j}\,\overline{\sigma_\varphi^{(j)}}\,\sigma_\psi^{(n-j)}\,. \end{aligned} \tag{15}$$

This gives a well-defined sesquilinear and hermitian inner product on $\mathfrak{S}(\mathbb{R}^+)$.

Some special functions in $\mathfrak{S}(\mathbb{R}^+)$

The following linearly independent family of test functions will be used to construct the negative definite subspaces of $\mathfrak{S}(\mathbb{R}^+)$ endowed with the above inner product:

$$\chi_{a,k,l}(x) = \frac{x^k}{k!}\,\mathrm{e}^{-(ax)^{2l}} \qquad a > 0\,,\ k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}\,,\ l \in \mathbb{N}\,.$$

The first entries of the signature of $\chi_{a,k,l}$ can be calculated using Leibniz' rule together with the facts

$$\frac{\mathrm{d}^j}{\mathrm{d}x^j}\frac{x^k}{k!}\bigg|_{x=0} = \delta_{jk}\ \ (j,k\in\mathbb{N}_0) \quad \text{and} \quad \frac{\mathrm{d}^{n-k}}{\mathrm{d}x^{n-k}}\,\mathrm{e}^{-(ax)^{2l}}\bigg|_{x=0} = \begin{cases} 1\,,\ n-k=0 \\ 0\,,\ 0<n-k<2l \end{cases}.$$

This yields:

$$\sigma_{\chi_{a,k,l}} = (\underset{\substack{\uparrow\\0}}{0},\ldots,0,\underset{\substack{\uparrow\\k}}{1},0,\ldots,0,\underset{\substack{\uparrow\\k+2l}}{?},?,\ldots)\,. \tag{16}$$

Here and in the following the question mark indicates that the corresponding entry is irrelevant (and in general not just a simple constant).

In order to calculate the inner product (15) for χ's with different values of k, it is necessary, in addition to the signatures, to know $(\chi_{a,k,l},\chi_{a,k',l})_{c=0}$:

In the case $k+k'>0$ the substitution $2(ax)^{2l}\to x$ gives

$$\begin{aligned} (\chi_{a,k,l},\chi_{a,k',l})_0 &= \int_0^\infty \frac{\mathrm{d}x}{x}\,\frac{x^{k+k'}}{k!\,k'!}\,\mathrm{e}^{-2(ax)^{2l}} \\ &= \frac{\Gamma\left(\frac{k+k'}{2l}\right)}{k!\,k'!\,2l\,2^{\frac{k+k'}{2l}}}\,a^{-(k+k')} = \Lambda_{kk'l}\,a^{-(k+k')} \qquad (k+k'>0), \end{aligned} \tag{17}$$

where Γ is Euler's function.

For $k = k' = 0$, first consider the difference $(a, b > 0)$

$$
\begin{aligned}
&(\chi_{a,0,l}, \chi_{a,0,l})_0 \Leftrightarrow (\chi_{b,0,l}, \chi_{b,0,l})_0 = \\
&\quad \int_0^1 \frac{\mathrm{d}x}{x} \left(\mathrm{e}^{-2(ax)^{2l}} \Leftrightarrow 1\right) + \int_1^\infty \frac{\mathrm{d}x}{x}\, \mathrm{e}^{-2(ax)^{2l}} \Leftrightarrow \int_0^1 \frac{\mathrm{d}x}{x} \left(\mathrm{e}^{-2(bx)^{2l}} \Leftrightarrow 1\right) \\
&\quad \Leftrightarrow \int_1^\infty \frac{\mathrm{d}x}{x}\, \mathrm{e}^{-2(bx)^{2l}} = \int_0^a \frac{\mathrm{d}x}{x} \left(\mathrm{e}^{-2x^{2l}} \Leftrightarrow 1\right) + \int_a^\infty \frac{\mathrm{d}x}{x}\, \mathrm{e}^{-2x^{2l}} \\
&\quad \Leftrightarrow \int_0^b \frac{\mathrm{d}x}{x} \left(\mathrm{e}^{-2x^{2l}} \Leftrightarrow 1\right) \Leftrightarrow \int_b^\infty \frac{\mathrm{d}x}{x}\, \mathrm{e}^{-2x^{2l}} = \int_a^b \frac{1}{x}\, \mathrm{d}x = \ln b \Leftrightarrow \ln a\,.
\end{aligned}
$$

Hence, the inner product is given by

$$
(\chi_{a,0,l}, \chi_{a,0,l})_0 = \Omega_l \Leftrightarrow \ln a\,, \tag{18}
$$

where $\Omega_l = (\chi_{1,0,l}, \chi_{1,0,l})_0 = \frac{1}{2l}\left(\int_1^\infty \frac{\mathrm{e}^{-x}}{x}\,\mathrm{d}x \Leftrightarrow \int_0^1 \frac{1-\mathrm{e}^{-x}}{x}\,\mathrm{d}x \Leftrightarrow \ln 2\right) \approx \Leftrightarrow \frac{0.635181}{l}$.

Now, using these results, the desired decomposition of $\mathfrak{S}(\mathbb{R}^+)$ into a positive and a negative definite part can be described. The general scheme for this decomposition is as follows.

General description of the decomposition

Two subspaces $\mathfrak{S}^{(+)}$ and $\mathfrak{S}^{(-)}$ are defined as

$$
\begin{aligned}
\mathfrak{S}^{(+)} &= \{\varphi \in \mathfrak{S}(\mathbb{R}^+) \,|\, \varphi(0) = 0,\ \varphi'(0) = 0,\ \ldots,\ \varphi^{(D)}(0) = 0\}\,, && (19)\\
\mathfrak{S}^{(-)} &= \mathrm{span}\{\chi_0, \ldots, \chi_D\}\,, && (20)
\end{aligned}
$$

for some $D \in \mathbb{N}_0$. Here $\chi_0, \ldots, \chi_D$ are elements of $\mathfrak{S}(\mathbb{R}^+)$ with the property $\chi_i^{(j)}(0) = \delta_{ij}$ for $i, j = 0, \ldots, D$, i.e. the corresponding signatures are

$$
\begin{array}{lcl}
\chi_0 & : & (1, 0, 0, \ldots, 0, ?, ?, \ldots) \\
\chi_1 & : & (0, 1, 0, \ldots, 0, ?, ?, \ldots) \\
\vdots & & \qquad\qquad \vdots \\
\chi_D & : & (0, 0, 0, \ldots, 1, ?, ?, \ldots).
\end{array} \tag{21}
$$

It follows that $\mathfrak{S}(\mathbb{R}^+)$ can be written as an algebraically direct sum

$$
\mathfrak{S}(\mathbb{R}^+) = \mathfrak{S}^{(+)} \dot{+} \mathfrak{S}^{(-)} \quad \text{with} \quad \dim(\mathfrak{S}^{(-)}) = D + 1\,, \tag{22}
$$

where the decomposition of $\varphi \in \mathfrak{S}(\mathbb{R}^+)$ is given by $\varphi(x) = [\,\varphi(x) \Leftrightarrow \varphi(0)\chi_0(x) \Leftrightarrow \varphi'(0)\chi_1(x) \Leftrightarrow \cdots \Leftrightarrow \varphi^{(D)}(0)\chi_D(x)\,] + [\,\varphi(0)\chi_0(x) + \varphi'(0)\chi_1(x) + \cdots + \varphi^{(D)}(0)\chi_D(x)\,]$.

Now, given a sequence c, i.e. a specific inner product of the form (15), the problem is to choose the number D and the functions $\chi_0, \ldots, \chi_D$ in such a way, that (i) the signatures are as in (21), (ii) the inner product is positive definite on $\mathfrak{S}^{(+)}$ and (iii) it is negative definite on $\mathfrak{S}^{(-)}$.

Notice that the decomposition yielded by this procedure is not a fundamental one, because it is not orthogonal. It can be made orthogonal using the following Proposition.

Proposition 5.1. *Let $\mathfrak{D}$, $(\cdot,\cdot)$ be an inner product space, which admits a decomposition $\mathfrak{D} = \mathfrak{D}^{(+)} \dot{+} \mathfrak{D}^{(-)}$, where $\mathfrak{D}^{(\pm)}$ is positive/negative definite and* $\dim(\mathfrak{D}^{(-)}) = n < \infty$.

Then there is another positive definite subspace $\mathfrak{D}_1^{(+)}$ so that $\mathfrak{D} = \mathfrak{D}_1^{(+)}(\dot{+})\,\mathfrak{D}^{(-)}$.

Proof. Observe first, that $\mathfrak{D}$ is non-degenerate.

Since $\mathfrak{D}^{(-)}$ is finite-dimensional, it is possible to construct an orthonormal basis $b_1, \ldots, b_n$ with $(b_\mu, b_\nu) = \Leftrightarrow\delta_{\mu\nu}$. Use this to define a projection $P : \mathfrak{D} \to \mathfrak{D}^{(-)}$ by $Px = \Leftrightarrow\sum_\mu (b_\mu, x) b_\mu$. Now set $\mathfrak{D}_1^{(+)} = \{x \Leftrightarrow Px \mid x \in \mathfrak{D}\}$.

It follows that $\mathfrak{D}_1^{(+)}$ and $\mathfrak{D}^{(-)}$ are orthogonal to each other and that they together span the full space $\mathfrak{D}$. Because of the non-degeneracy of $\mathfrak{D}$ their intersection is equal to $\{0\}$. These properties prove the above assertion $\mathfrak{D} = \mathfrak{D}_1^{(+)}(\dot{+})\,\mathfrak{D}^{(-)}$.

It remains to show that $\mathfrak{D}_1^{(+)}$ is positive definite. To this end suppose first the existence of an $x \in \mathfrak{D}_1^{(+)}$ satisfying $(x, x) < 0$. Then $\operatorname{span}\{x, \mathfrak{D}^{(-)}\}$ would be a negative definite subspace (since $x \perp \mathfrak{D}^{(-)}$) with dimension $n + 1$ in contradiction to the fact that there cannot be a negative definite subspace of dimension greater than n (the codimension of the positive definite $\mathfrak{D}^{(+)}$). So $\mathfrak{D}_1^{(+)}$ must be positive semidefinite.

Now suppose $x \in \mathfrak{D}_1^{(+)}$ and $(x, x) = 0$. Cauchy-Schwarz' inequality on the semidefinite $\mathfrak{D}_1^{(+)}$ implies $x \perp \mathfrak{D}_1^{(+)}$. But also $x \perp \mathfrak{D}^{(-)}$. Thus $x = 0$, again because of the non-degeneracy of $\mathfrak{D}$. This completes the proof. $\square$

Remark 5.2. In the situation of Proposition 5.1 it is in general impossible to leave the infinite-dimensional part unchanged and to find an appropriate finite-dimensional complement. A given positive definite subspace can only be used to construct a fundamental decomposition, if it is even *uniformly positive* (see [4, Thm. V.7.1.] for details, and cf. also the conclusion in the next section).

Investigation of the possible cases

a) $c = (0, 0, \ldots)$:

Set $D = 0$, so that $\mathfrak{S}^{(+)}$ is positive definite.

As the χ choose $\chi_{a,0,1}$, the signature of which is $(1, 0, ?, \ldots)$, and which thus has the right properties (independently of a).

If, finally, a is chosen large enough, then $(\chi_{a,0,1}, \chi_{a,0,1})_c = \Omega_1 \Leftrightarrow \ln a$ will be negative, i.e., with this a the inner product is negative definite on $\mathfrak{S}^{(-)}$.

b) $c = (c_0, 0, 0, \ldots)$, $c_0 \neq 0$:

$D = 0 \Rightarrow \mathfrak{S}^{(+)}$ is positive definite. The χ: $\chi_{a,0,1}$, $\sigma = (1, 0, ?, \ldots)$.

$(\chi_{a,0,1}, \chi_{a,0,1})_c = \Omega_1 \Leftrightarrow \ln a + c_0$ (see (15) together with (18) and the above signature). Again, a can be chosen such that $\mathfrak{S}^{(-)}$ becomes negative definite.

c) $c = (c_0, c_1, 0, 0, \dots)$, $c_1 \neq 0$:

Literaly like in case b).

d) $c = (c_0, c_1, c_2, 0, 0, \dots)$, $c_2 \neq 0$:

Here, two cases have to be distinguished:

(i) $c_2 > 0$:

$D = 0 \Rightarrow \mathfrak{S}^{(+)}$ is positive definite. The χ: $\chi_{a,0,2}$, $\sigma = (1, 0, 0, 0, ?, \dots)$. $(\chi_{a,0,2}, \chi_{a,0,2})_c = \Omega_2 \Leftrightarrow \ln a + c_0$.

(ii) $c_2 < 0$:

$D = 1 \Rightarrow \mathfrak{S}^{(+)}$ is positive definite.

The χ's: $\chi_{a,0,2}$, $\sigma = (1, 0, 0, 0, ?, \dots)$,
$\chi_{a,1,2}$, $\sigma = (0, 1, 0, 0, 0, ?, \dots)$.

Now, a has to be chosen so that the following matrix (the representation of the inner product in the basis of $\mathfrak{S}^{(-)}$) becomes negative definite:

$$\begin{pmatrix} (\chi_{a,0,2}, \chi_{a,0,2})_c & (\chi_{a,0,2}, \chi_{a,1,2})_c \\ (\chi_{a,1,2}, \chi_{a,0,2})_c & (\chi_{a,1,2}, \chi_{a,1,2})_c \end{pmatrix} = \begin{pmatrix} \Omega_2 \Leftrightarrow \ln a + c_0 & \Lambda_{012}\, a^{-1} + c_1 \\ \Lambda_{012}\, a^{-1} + c_1 & \Lambda_{112}\, a^{-2} + 2c_2 \end{pmatrix}.$$

According to Hurwitz' theorem, this can be done by ensuring that the main subdeterminants have alternating signs:

$$\Omega_2 \Leftrightarrow \ln a + c_0 < 0 \qquad \text{and}$$

$$(\Omega_2 \Leftrightarrow \ln a + c_0)\,(\Lambda_{112}\, a^{-2} + 2c_2) \Leftrightarrow (\Lambda_{012}\, a^{-1} + c_1)^2 > 0.$$

These inequalities become true for sufficiently large a, because the first expression tends to $\Leftrightarrow\infty$, the second one to $+\infty$ as $a \to \infty$ (since $c_2 < 0$).

e) $c = (c_0, c_1, c_2, c_3, 0, 0, \dots)$, $c_3 \neq 0$:

$D = 1 \Rightarrow \mathfrak{S}^{(+)}$ is positive definite.

The χ's: $\chi_{a,0,2}$, $\sigma = (1, 0, 0, 0, ?, \dots)$,
$\chi_{a,1,2} + \gamma\, \chi_{a,2,2}$ $(\gamma \in \mathbb{R})$, $\sigma = (0, 1, \gamma, 0, 0, ?, \dots)$.

The matrix representation of the inner product on the space $\mathfrak{S}^{(-)}$ is here:

$$\begin{pmatrix} \Omega_2 \Leftrightarrow \ln a + c_0 & \Lambda_{012}\, a^{-1} + \gamma\Lambda_{022}\, a^{-2} + c_1 + \gamma c_2 \\ \Lambda_{012}\, a^{-1} + \gamma\Lambda_{022}\, a^{-2} + c_1 + \gamma c_2 & \begin{matrix} \Lambda_{112}\, a^{-2} + \gamma^2\Lambda_{222}\, a^{-4} \\ +2\gamma\Lambda_{122}\, a^{-3} + 2c_2 + 6\gamma c_3 \end{matrix} \end{pmatrix}.$$

So, $\mathfrak{S}^{(-)}$ is negative definite, if

$$\Omega_2 \Leftrightarrow \ln a + c_0 < 0 \qquad \text{and}$$

$$(\Omega_2 \Leftrightarrow \ln a + c_0)\,(2c_2 + 6\gamma c_3 + O(a^{-2})) \Leftrightarrow (c_1 + \gamma c_2 + O(a^{-1}))^2 > 0.$$

To ensure this, choose γ, which serves here as an additional freedom, so that $2c_2 + 6\gamma c_3 < 0$ (this is possible since $c_3 \neq 0$), and then a large enough.

f) $c = (c_0, c_1, \ldots, c_{2n+1}, 0, 0, \ldots)$, $c_{2n+1} \neq 0$ $(n = 2, 3, 4, \ldots)$:

$D = n \Rightarrow \mathfrak{S}^{(+)}$ is positive definite.

$$\begin{array}{rl}
\text{The } \chi\text{'s:} & {\scriptstyle 0\ \ 1\ \ 2\ \ \ldots\ \ n\ \ n+1\ \ \ldots\ \ 2n-1\ \ 2n\ \ 2n+1\ \ \ \ldots} \\
 & \downarrow\downarrow\downarrow\quad\ \downarrow\ \downarrow\qquad\ \downarrow\ \ \downarrow\ \ \downarrow \\
\chi_{a,0,n+1} & (1,0,0,\ldots,0,\ 0\ ,\ldots,\ 0\ ,0\ ,\ 0\ ,?,\ldots) \\
\chi_{a,1,n+1} + \gamma_1\, \chi_{a,2n,n+1} & (0,1,0,\ldots,0,\ 0\ ,\ldots,\ 0\ ,\gamma_1,\ 0\ ,?,\ldots) \\
\chi_{a,2,n+1} + \gamma_2\, \chi_{a,2n-1,n+1} & (0,0,1,\ldots,0,\ 0\ ,\ldots,\ \gamma_2\ ,0\ ,\ 0\ ,?,\ldots) \\
\vdots & \ \vdots\,\vdots\,\vdots\qquad\vdots\ \ \vdots\qquad\ \ \vdots\quad\vdots\quad\vdots\qquad\vdots \\
\chi_{a,n,n+1} + \gamma_n\, \chi_{a,n+1,n+1} & (0,0,0,\ldots,1,\gamma_n,\ldots,\ 0\ ,0\ ,\ 0\ ,?,\ldots).
\end{array}$$

For brevity, the corresponding matrix representation is not written out here in detail. It can be made negative definite similarly as in part e) by choosing appropriate values for a, $\gamma_1, \ldots, \gamma_n$.

g) $c = (c_0, c_1, \ldots, c_{2n}, 0, 0, \ldots)$, $c_{2n} \neq 0$ $(n = 2, 3, 4, \ldots)$:

(i) $c_{2n} > 0$:

$D = n \Leftrightarrow 1 \Rightarrow \mathfrak{S}^{(+)}$ is positive definite.

$$\begin{array}{rl}
\text{The } \chi\text{'s:} & {\scriptstyle 0\ \ 1\ \ 2\ \ \ldots\ \ n-1\ \ n\ \ \ n+1\ \ \ \ldots\ \ 2n-2\ \ 2n-1\ \ 2n\ \ 2n+1\ \ \ \ldots} \\
 & \downarrow\downarrow\downarrow\quad\ \downarrow\ \downarrow\ \ \downarrow\qquad\ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow \\
\chi_{a,0,n+1} & (1,0,0,\ldots,\ 0\ ,0,\ \ 0\ \ ,\ldots,\ 0\ ,\ 0\ ,0,\ 0\ ,?,\ldots) \\
\chi_{a,1,n+1} + \gamma_1\, \chi_{a,2n-1,n+1} & (0,1,0,\ldots,\ 0\ ,0,\ \ 0\ \ ,\ldots,\ 0\ ,\ \gamma_1\ ,0,\ 0\ ,?,\ldots) \\
\chi_{a,2,n+1} + \gamma_2\, \chi_{a,2n-2,n+1} & (0,0,1,\ldots,\ 0\ ,0,\ \ 0\ \ ,\ldots,\ \gamma_2\ ,\ 0\ ,0,\ 0\ ,?,\ldots) \\
\vdots & \ \vdots\,\vdots\,\vdots\qquad\vdots\ \ \vdots\ \ \vdots\qquad\ \ \vdots\quad\vdots\quad\vdots\quad\vdots\qquad\vdots \\
\chi_{a,n-1,n+1} + & (0,0,0,\ldots,\ 1\ ,0,\gamma_{n-1},\ldots,\ 0\ ,\ 0\ ,0,\ 0\ ,?,\ldots) \\
\gamma_{n-1}\, \chi_{a,n+1,n+1} & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad .
\end{array}$$

(ii) $c_{2n} < 0$:

$D = n \Rightarrow \mathfrak{S}^{(+)}$ is positive definite.

$$\begin{array}{rl}
\text{The } \chi\text{'s:} & {\scriptstyle 0\ \ 1\ \ \ldots\ \ n-1\ \ n\ \ \ n+1\ \ \ \ldots\ \ 2n-1\ \ 2n\ \ 2n+1\ \ \ \ldots} \\
 & \downarrow\downarrow\quad\ \downarrow\ \downarrow\ \ \downarrow\qquad\ \downarrow\ \ \downarrow\ \ \downarrow \\
\chi_{a,0,n+1} & (1,0,\ldots,\ 0\ ,0,\ \ 0\ \ ,\ldots,\ 0\ ,0,\ 0\ ,?,\ldots) \\
\chi_{a,1,n+1} + \gamma_1\, \chi_{a,2n-1,n+1} & (0,1,\ldots,\ 0\ ,0,\ \ 0\ \ ,\ldots,\ \gamma_1\ ,0,\ 0\ ,?,\ldots) \\
\vdots & \ \vdots\,\vdots\qquad\vdots\ \ \vdots\ \ \vdots\qquad\ \ \vdots\quad\vdots\quad\vdots\qquad\vdots \\
\chi_{a,n-1,n+1} + \gamma_{n-1}\, \chi_{a,n+1,n+1} & (0,0,\ldots,\ 1\ ,0,\gamma_{n-1},\ldots,\ 0\ ,0,\ 0\ ,?,\ldots) \\
\chi_{a,n,n+1} & (0,0,\ldots,\ 0\ ,1,\ \ 0\ \ ,\ldots,\ 0\ ,0,\ 0\ ,?,\ldots).
\end{array}$$

5.2 Conclusion

It has been shown that in each of the cases the inner product space $\mathfrak{S}(\mathbb{R}^+)$ admits a decomposition according to (22). Then Proposition 5.1 shows that there even exists a fundamental decomposition in which the negative part equals the initial $\mathfrak{S}^{(-)}$ and thus $\mathfrak{S}(\mathbb{R}^+)$ is quasi-positive.

Setting $E = \mathfrak{S}(\mathbb{R}^+)$ in section 3 and defining W_2 by the above inner product, one obtains a free-field-like structure for which Assumption I. is satisfied, so that all the results of this section apply, including Proposition 3.4.

Looking at the structure of $\mathfrak{S}^{(+)}$ with $(\varphi, \psi)_c \restriction_{\mathfrak{S}^{(+)}} = \int_0^\infty 1/x\, \overline{\varphi(x)} \psi(x)\, \mathrm{d}x$ (in each of the cases but d)(i) and g)(i)), one is lead to conjecture, that in all of these cases a Krein space containing $\mathfrak{S}(\mathbb{R}^+)$ could be constructed, the positive part of which is always $\mathrm{L}^2(\mathbb{R}^+, \mathrm{d}x/x)$, the completion of $\mathfrak{S}^{(+)}$ with respect to the inner product. Then, choosing different regularizations would only result in different dimensions for the negative part. But, as was stressed in Remark 5.2, for $\mathfrak{S}^{(+)}$ to be part of a fundamental decomposition, it is nessecary and sufficient that it is uniformly positive with respect to some Krein space structure of $\mathfrak{S}(\mathbb{R}^+)$, for which we can take $\widetilde{\mathfrak{S}}_1^{(+)} (\dot{+})\, \mathfrak{S}^{(-)}$ from Proposition 5.1. Now, uniform positivity means that there is a constant $\eta > 1$ so that $(\varphi \Leftrightarrow P\varphi, \varphi \Leftrightarrow P\varphi)_c + \eta (P\varphi, P\varphi)_c \geq 0$ holds for every $\varphi \in \mathfrak{S}^{(+)}$, where P is the orthogonal projection onto $\mathfrak{S}^{(-)}$ (cf. the proof of Proposition 5.1). Consider case a), i.e., $\mathfrak{S}^{(-)}$ is spanned by $\chi_{a,0,1}$ with $(\chi_{a,0,1}, \chi_{a,0,1})_0 = \Leftrightarrow 1$. Then the condition of uniform positivity is equivalent to $|(\chi_{a,0,1}, \varphi)_0|^2 \leq C(\varphi, \varphi)_0$ ($\varphi \in \mathfrak{S}^{(+)}$) for some $C > 0$, or

$$\left| \int_0^\infty \frac{\mathrm{d}x}{x}\, e^{-(ax)^2} \varphi(x) \right|^2 \leq C \int_0^\infty \frac{\mathrm{d}x}{x}\, |\varphi(x)|^2 \quad \varphi \in \mathfrak{S}(\mathbb{R}^+),\ \varphi(0) = 0.$$

This condition is not fulfilled, as can be seen by considering a sequence φ_n of positive test functions that approximates the characteristic function of the interval $(0, 1)$. So, there is no fundamental decomposition of $\mathfrak{S}(\mathbb{R}^+)$ with $\mathfrak{S}^{(+)}$ as its positive part.

The section is closed by a summary of the results concerning the dimensions of the negative definite subspaces: $\dim(\mathfrak{S}^{(-)}) = D + 1$.

c	$\dim(\mathfrak{S}^{(-)})$	see cases
$(0, 0, \dots)$	1	a)
$(c_0, 0, 0, \dots) \quad c_0 \neq 0$	1	b)
$(c_0, \dots, c_{2n+1}, 0, 0, \dots) \quad (n \in \mathbb{N}_0)$		
$\quad c_{2n+1} \neq 0$	$n + 1$	c), e), f)
$(c_0, \dots, c_{2n}, 0, 0, \dots) \quad (n \in \mathbb{N})$		
$\quad c_{2n} > 0$	n	d)(i), g)(i)
$\quad c_{2n} < 0$	$n + 1$	d)(ii), g)(ii).

5.3 The Regularized $(1/x)_+^k$

In this section the more general case of the regularized $\int_0^\infty 1/x^k\,\varphi(x)\,\mathrm{d}x$ ($k \in \mathbb{N}$) is briefly considered.

$$
\begin{aligned}
(\varphi,\psi)_c \quad &= \quad \int_0^1 \frac{\mathrm{d}x}{x^k}\left[\overline{\varphi(x)}\,\psi(x) \Leftrightarrow \sum_{j=0}^{k-1}\frac{x^j}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}x^j}\Big(\overline{\varphi(x)}\,\psi(x)\Big)\Big|_{x=0}\right] \\
&\quad + \int_1^\infty \frac{\mathrm{d}x}{x^k}\,\overline{\varphi(x)}\,\psi(x) + \sum_{j=0}^{\infty} c_j\,\frac{\mathrm{d}^j}{\mathrm{d}x^j}\Big(\overline{\varphi(x)}\,\psi(x)\Big)\Big|_{x=0},
\end{aligned}
$$

where $c \in \mathrm{d}(\mathbb{R})$ and $\varphi,\psi \in \mathfrak{S}(\mathbb{R}^+)$.

Set again

$$
\mathfrak{S}^{(+)} = \{\varphi \in \mathfrak{S}(\mathbb{R}^+)\,|\,\varphi(0)=0,\,\varphi'(0)=0,\,\ldots,\,\varphi^{(D)}(0)=0\,\},
$$

using the following choice for D ($[x]$ denotes the integral part of x):

c	D
$(c_0,\ldots,c_{k-1},0,0,\ldots)$	$\left[\frac{k-1}{2}\right]$
$(c_0,\ldots,c_{2n+1},0,0,\ldots)$ $\quad(2n+1>k\Leftrightarrow 1)$	
$c_{2n+1}\neq 0$	n
$(c_0,\ldots,c_{2n},0,0,\ldots)$ $\quad(2n>k\Leftrightarrow 1)$	
$c_{2n}>0$	$n\Leftrightarrow 1$
$c_{2n}<0$	n .

$\mathfrak{S}^{(+)}$ is a positive definite subspace of finite codimension $D+1$ (cf. (22)). This means that there can be no negative definite subspace of greater dimension than $D+1$. So, $\mathfrak{S}(\mathbb{R}^+)$ equipped with the given inner product is quasi-positive and thus decomposable. The dimension of the negative part of the fundamental decomposition must be less or equal to $D+1$. Again, the assumptions of section 3 are satisfied.

Here only one-dimensional integrals were considered. If an inner product on $\mathfrak{S}(\mathbb{R}^d)$ is defined by a regularization of some d-dimensional function with a finite number of algebraic singularities, then analogous arguments show that the space becomes quasi-positive and thereby decomposable.

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