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A NOTE ON HERMITIAN-EINSTEIN METRICS ON PARABOLIC STABLE BUNDLES

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Abstract

Let $\overline{M}$ be a compact complex manifold of complex dimension two with a smooth Kähler metric and D a smooth divisor on $\overline{M}$. If E is a rank 2 holomorphic vector bundle on $\overline{M}$ with a stable parabolic structure along D, we prove that there exists a Hermitian-Einstein metric on $E' = E|_{\overline{M}\setminus D}$ compatible with the parabolic structure, and whose curvature is square integrable.

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1 Introduction

Let $\overline{M}$ be a compact Kähler manifold of complex dimension 2, let ω be a Kähler metric on $\overline{M}$. Let D be a smooth irreducible divisor in $\overline{M}$, and let $M = \overline{M} \setminus D$. The restriction of ω to M gives a Kähler metric on M. For simplicity, we assume in this paper that E is a rank 2 holomorphic vector bundle over $\overline{M}$ and let $E' = E|_M$ be the restriction of the bundle E to M.

We define [LN] the notion of a stable parabolic structure on E (along D) and the notion of a Hermitian-Einstein metric on E' with respect to the restriction of the Kähler metric ω to M. We proved in [LN] that there exists a Hermitian-Einstein metric on E' compatible with the parabolic structure. We prove in this paper that there exists in fact a Hermitian-Einstein metric on E' (compatible with the parabolic structure) with the property that the curvature of the metric is square integrable (Theorem 2.2). In the case of a projective surface, the square integrability was proved by Biquard [B (4.2)] using a result of Simpson, while our proof is valid with the Kähler case also.

Once we know the curvature of the H-E metric is in L^2, it is in fact in L^p for $p > 2$ (Remark 2.4), and hence the metric defines a parabolic bundle on $\overline{M}$ as in [B, Theorem 1.1]. Since the metric is also compatible with the given parabolic structure, both parabolic structures are the same. Therefore proving the result that the curvature form of the H-E metric is in L^2 completes our earlier paper and this is the motivation for this note.

2 The existence of a H-E metric

In this section we shall prove our main theorem. See [LN] for the definitions, such as Hermitian-Einstein metrics, parabolic bundles, etc.

We need the following result proved in [LN], regarding the initial metric K_0 on E'.

Lemma 2.1 *([LN] Lemma 5.2 and Proposition 6.6) Let $(E, D, \alpha_1, \alpha_2)$ be a parabolic bundle. Then there exists a Hermitian metric K_0 on $E' = E|_M$ such that*

a) the curvature form of K_0, F_{K_0} satisfies that $|F_{K_0}|_{K_0} \in L^p(M)$ for any $1 \leq p < p_0$ where $p_0 = \min\{\frac{2}{1-(\alpha_2-\alpha_1)}, \frac{2}{\alpha_2-\alpha_1}\}$ and $|trF_{K_0}| \in L^\infty(M)$.

b) $par \deg E_* =$ the analytic degree $d(E, K_0)$ and $(E, D, \alpha_1, \alpha_2)$ is parabolic stable if and only if (E, K_0) is analytic stable.

Theorem 2.2 *Let $\overline{M}$ be a compact Kähler manifold of complex dimension 2 and D a smooth irreducible divisor of $\overline{M}$. Let E be a rank 2 holomorphic vector bundle on $\overline{M}$ with a parabolic structure $E_* = (E, D, \alpha_1, \alpha_2)$. If E_* is parabolic stable there exists a Hermitian-Einstein metric H on E' compatible with the parabolic structure and whose curvature form is square integrable over M.*

We shall modify Proposition 7.2 in [LN] and its proof to prove the theorem. The main additional point is the derivation of an L^2 estimate for the curvature of the metrics arising in the heat flow.

Proposition 2.3 *Let $(E, D, \alpha_1, \alpha_2)$ be a parabolic bundle. Then there exists a Hermitian metric $K \in \mathcal{A}_{K_0}$ on $E' = E|_M$ satisfying the heat equation*

$$\begin{cases} K^{-1}\frac{dK}{dt} = -\sqrt{-1}\Lambda F_K^{\perp} \\ K|_{t=0} = K_0 \quad \text{and} \quad \det K = \det K_0 \end{cases}$$

on M, with $\||F_K|_K\|_{L^2(M)} \leq C$, $\||\Lambda F_K^{\perp}|_K\|_{L^p(M)} \leq \||\Lambda F_{K_0}^{\perp}|_{K_0}\|_{L^p(M)}$, and $|\Lambda F_K|_K \in L^\infty(M)$ for any $t > 0$, $2 < p < p_0$, where K_0 is the metric constructed in Lemma 2.1, p_0 is the constant in Lemma 2.1, $C > 0$ is a constant depending only on K_0.

Proof: Let $M_\beta = \{x \in M \mid |\sigma(x)| > \beta\}$, where $0 < \beta < 1$, and consider the above heat equation on M_β with Neumann boundary condition. More precisely, we consider, writing $h = K_0^{-1}K$, the equation

$$(\triangle_{K_0} - \frac{\partial}{\partial t})h = \sqrt{-1}h\Lambda F_{K_0}^{\perp} - \sqrt{-1}\Lambda\bar{\partial}hh^{-1}\partial_{K_0}h$$

on M_β with $h|_{t=0} = I$, $\det h = 1$, and $\frac{\partial}{\partial n}h|_{\partial M_\beta} = 0$, where $\triangle_{K_0} = -\sqrt{-1}\Lambda\bar{\partial}\partial_{K_0}$, $\frac{\partial}{\partial n}$ denotes the differentiation in the direction perpendicular to the boundary using the operator ∂_{K_0}.

In [LN] we used the Dirichlet boundary condition. We use Neumann boundary condition here so that we have the fact that $\frac{\partial}{\partial n}\Lambda F_K^{\perp}|_{\partial M_\beta} = 0$, obtained by applying $\frac{\partial}{\partial n}$ to both sides of the heat equation; this fact will enable us to apply the Stokes theorem for deriving the relation (2) below.

It was proved in [S] that this heat equation with Neumann boundary condition has a solution for all time. We denote the solution by K_β for each β. Let $h_\beta = K_0^{-1}K_\beta$.

By an argument similar to the one used in the proof of Proposition 7.2 in [LN], we can show that, there exist a sequence $\beta_i \to 0$ and a Hermitian metric $K \in \mathcal{A}_{K_0}$ such that for any relatively compact open subset Z, any $\delta > 0$, and any $R > 0$, $K_{\beta_i} \rightharpoonup K$ in $L^p_{2/1}(Z \times [\delta, R])$ for any $1 < p < \infty$. By the Sobolev embedding, we have $K_{\beta_i} \longrightarrow K$ in $C^{1/0}(Z \times [\delta, R])$. Therefore the limit K satisfies the heat equation on $M \times (0, \infty)$ and thus belongs to $C^\infty(M \times (0, \infty))$. We can also show that $\||\Lambda F_K^{\perp}|_K\|_{L^p(M)} \leq \||\Lambda F_{K_0}^{\perp}|_{K_0}\|_{L^p(M)}$ and $|\Lambda F_K|_K \in L^\infty(M)$ for any $t > 0$, $2 < p < p_0$, as we did in [LN].

Now we derive the L^2 bound of the curvature.

By the formula $\partial_{K_\beta} = \partial_{K_0} + h_\beta^{-1}\partial_{K_0}h_\beta$ and the fact that $\frac{\partial}{\partial n}\Lambda F_K^{\perp}|_{\partial M_\beta} = 0$ we can see that

$$\frac{\partial}{\partial n_\beta}\Lambda F_K^{\perp}|_{\partial M_\beta} = 0 \tag{1}$$

where $\frac{\partial}{\partial n_\beta}$ denotes the differentiation in the direction perpendicular to the boundary using the operator ∂_{K_β}.

Because $\det h_\beta = 1$, we have $trF_{K_\beta} = trF_{K_0}$ for all t, and

$$\frac{d}{dt}F^\perp_{K_\beta} = \frac{d}{dt}F_{K_\beta} = \sqrt{-1}\overline{\partial}\partial_{K_\beta}K_\beta^{-1}\frac{d}{dt}K_\beta = \overline{\partial}\partial_{K_\beta}\Lambda F^\perp_{K_\beta}.$$

Using the above identity we get

$$\begin{aligned}
&\frac{d}{dt}\int_{M_\beta}|F^\perp_{K_\beta}|^2_{K_\beta}dV \\
&= 2Re\int_{M_\beta}(\frac{d}{dt}F^\perp_{K_\beta}, F^\perp_{K_\beta})_{K_\beta}dV \\
&= 2Re\int_{M_\beta}(\overline{\partial}\partial_{K_\beta}\Lambda F^\perp_{K_\beta}, F^\perp_{K_\beta})_{K_\beta}dV \\
&= 2Re\int_{M_\beta}\nabla_{\overline{k}}\nabla_l\,(F^\perp_{K_\beta})_\gamma{}^\delta{}_{i\overline{i}}\cdot(F^\perp_{K_\beta})_\delta{}^\gamma{}_{k\overline{l}}dV \\
&= -2Re\int_{M_\beta}\nabla_l(F^\perp_{K_\beta})_\gamma{}^\delta{}_{i\overline{i}}\cdot\nabla_{\overline{k}}(F^\perp_{K_\beta})_\delta{}^\gamma{}_{k\overline{l}}dV \text{ (by (1) and Stokes theorem)} \\
&= -2Re\int_{M_\beta}\nabla_i(F^\perp_{K_\beta})_\gamma{}^\delta{}_{l\overline{i}}\cdot\nabla_{\overline{k}}(F^\perp_{K_\beta})_\delta{}^\gamma{}_{k\overline{l}}dV \text{ (by Bianchi identity)} \\
&\le 0,
\end{aligned} \tag{2}$$

Letting $\beta \to \infty$, using Fatou's lemma, we get

$$\int_M |F^\perp_K|^2_K dV \le \int_M |F^\perp_{K_0}|^2_{K_0}dV.$$

Since $|trF_{K_0}| \in L^\infty(M)$ and $trF_K = trF_{K_0}$, we have

$$\int_M |F_K|^2_K dV \le C.$$

This completes the proof of the proposition.

Proof of the main theorem: As in the proof of Theorem 7.3 in [LN], the metric $K(t)$ converges to a Hermitian-Einstein metric H (as $t \to \infty$) compatible with the parabolic structure. On the other hand, by Proposition 2.3 we have $\||F_K|_K\|_{L^2(M)} \le C$. It follows from Fatou's lemma that $|F_H|_H \in L^2(M)$.

Remark 2.4 *Once we know that the curvature form of the H-E metric is in L^2, then it belongs in fact to L^p, for $p \ge 2$, as implied by the result of Sibner-Sibner [SS Theorem 5.1 and Theorem 5.2] (see [B (4.2)]).*

Remark 2.5 *Conversely, if E' is a holomorphic vector bundle over M and admits a Hermitian-Einstein metric H with $\||F_H|_H\|_{L^p(M)} < \infty$, for some $p > 2$, one can show (cf. [B], Theorem 1.1) that E' can be extended to a holomorphic vector bundle E over $\overline{M}$ with a parabolic structure along D and such that H is compatible with the parabolic structure. Moreover E is parabolic polystable (cf. [B] and [LN]).*

Remark 2.6 *We can use our existence theorem to derive a Bogomolov Chern number inequality for parabolic bundles (cf. [L]). For the case of projective varieties see Biswas [Bs].*

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