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A GENERALIZATION OF POISSON-NIJENHUIS STRUCTURES

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Abstract

We generalize Poisson-Nijenhuis structures. We prove that on a manifold endowed with a Nijenhuis tensor and a Jacobi structure which are compatible, there is a hierarchy of pairwise compatible Jacobi structures. Furthermore, we study the homogeneous Poisson-Nijenhuis structures and their relations with Jacobi structures.

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1 Introduction

Jacobi structures, which are natural generalizations of Poisson structures, have been studied by A. Lichnerowicz and his collaborators [L], [D-L-M], [G-L], etc. A *Jacobi structure* on a manifold M is defined by a pair (Λ, E), where Λ is a bivector field, E is a vector field such that $[E, \Lambda] = 0$ and $[\Lambda, \Lambda] = 2E \wedge \Lambda$. Two Jacobi structures (Λ_1, E_1) and (Λ_2, E_2) are said to be compatible if $(\Lambda_1 + \Lambda_2, E_1 + E_2)$ is also a Jacobi structure (see [N]). Here, we give compatibility conditions between a Jacobi structure (Λ, E) and a (1,1)-tensor field J whose Nijenhuis torsion N_J vanishes (J is called a *Nijenhuis tensor*). When these compatibility conditions are satisfied, we get another Jacobi structure denoted by $(J\Lambda, JE)$, which is compatible with (Λ, E). These conditions generalize the notion of *Poisson-Nijenhuis structures* introduced by Magri in [M-M]. Recently, J. Monterde et al. (see [M-M-P]) considered *Jacobi-Nijenhuis structures.* This work contributes to further generalization of Jacobi-Nijenhuis structures.

Poisson-Nijenhuis structures play a central role in the study of integrable systems. In [V], the author defined the Poisson-Nijenhuis structures in the general algebraic framework of Gel'fand and Dorfman. Moreover, Y. Kosmann-Schwarzbach gave in [K] a characterization of Poisson-Nijenhuis structures in terms of Lie algebroids. Another one is given in [B-M].

The paper is organized as follows. In Section 2, we recall some definitions and basic results about Jacobi structures. Furthermore, inspired by the construction of Magri et al. (see [C-M-P]), we establish that certain compatible Jacobi structures define a sequence of functions in involution.

In Section 3, we give necessary and sufficient conditions for a Nijenhuis tensor J and a Jacobi structure (Λ, E) to define, in a natural way, a new Jacobi structure which is compatible with (Λ, E). Moreover, we prove that the main property of the Poisson-Nijenhuis manifolds holds for the Jacobi ones endowed with a compatible Nijenhuis tensor. Namely, they determine a sequence of Jacobi structures which are pairwise compatible (see Theorem 3.9).

Section 4 is devoted to the analysis of homogeneous Poisson structures, which are compatible with a Nijenhuis tensor. Such structures are called homogeneous Poisson-Nijenhuis structures. It is well known that homogeneous Poisson structures are related to Jacobi ones, their relations being already established in [D-L-M]. We give sufficient conditions to have homogeneous Poisson-Nijenhuis structures and deduce some consequences for Jacobi structures.

2 Preliminaries

In the sequel, all manifolds, multi-vector fields and forms are assumed to be differentiable of class C^∞.

2.1 Jacobi structures

Definition 2.1 A *Jacobi manifold* $(M, \{\ ,\ \})$ is a manifold M equipped with a $\mathbb{R}$-bilinear and skew-symmetric map $\{\ ,\ \} : C^\infty(M,\mathbb{R}) \times C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$, called a *Jacobi bracket*, which satisfies the following properties:

1) the Jacobi identity:

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0, \quad \forall\ f,g,h \in C^\infty(M,\mathbb{R});$$

2) the bracket is local (i.e. the support of $\{f,g\}$ is a subset of the intersection of the supports of f and g).

The definition of a Jacobi structure is equivalent to giving a pair (Λ, E) formed by a bivector field Λ and a vector field E such that

$$[E,\Lambda] = 0 \quad \text{and} \quad [\Lambda,\Lambda] = 2E \wedge \Lambda,$$

where $[\ ,\ ]$ is the Schouten-Nijenhuis bracket on the space of multivector fields (see [Kz]). The Jacobi bracket is then given by

$$\{f,g\} = \Lambda(df, dg) + \langle f dg - g df, E\rangle.$$

When E is zero, we obtain a Poisson structure. In other words, a *Poisson structure* on a manifold M is given by a bivector field Λ such that the Schouten-Nijenhuis bracket $[\Lambda,\Lambda]$ vanishes. Then (M,Λ) is called a *Poisson manifold*. In [L], Lichnerowicz has shown that to any Jacobi structure (Λ, E) on a manifold M, one may associate a Poisson structure π on $M \times \mathbb{R}$ defined by

$$\pi(x,t) = e^{-t}\Big(\Lambda(x) + \frac{\partial}{\partial t} \wedge E\Big).$$

Then, π is called the *Poissonization* of (Λ, E). Let us recall other examples of Jacobi structures (see [L] for example).

Example 1: locally conformal symplectic manifolds. Let M be a $2n$-dimensional manifold. A *locally conformal symplectic structure* on M is given by a pair (F,ω), where F is a nondegenerate 2-form and ω is 1-form such that

$$d\omega = 0 \quad \text{and} \quad dF + \omega \wedge F = 0.$$

We define a bivector field Λ and a vector field E by:

$$i_E F = \omega \quad \text{and} \quad i_{\Lambda\alpha} F = -\alpha.$$

Then (Λ, E) defines a Jacobi structure. In fact, for any $x \in M$, there exist a neighborhood U_x and a function f defined on U_x such that $\omega = df$ and $\Omega = e^f F$ is symplectic.

Example 2: contact manifolds. Let M be a $(2n+1)$-dimensional manifold. A differential 1-form η on M defines a *contact structure* if $\eta \wedge (d\eta)^n$ does not vanish at any point of M. So, the map $\flat : \chi(M) \to \Omega^1(M)$ defined by $\flat(X) = i_X d\eta + \eta(X)\eta$ is an isomorphism of $C^\infty(M, \mathbb{R})$-modules, where $\chi(M)$ is the space of vector fields and $\Omega^1(M)$ is the space of differential 1-forms on M. Consider the vector field E and the bivector field Λ such that

$$\Lambda(\alpha, \beta) = d\eta(\flat^{-1}(\alpha), \flat^{-1}(\beta)) \quad \text{and} \quad E = \flat^{-1}(\eta).$$

The pair (Λ, E) defines a Jacobi structure on M.

2.2 Characteristic distribution of a Jacobi manifold

Let (M, Λ, E) be a Jacobi manifold. For any $f \in C^\infty(M, \mathbb{R})$, the vector field given by

$$X_f = \Lambda(df) + fE$$

is called *Hamiltonian vector field* associated with f. We have the following proposition (see [G-L]):

Proposition 2.2 *The pair (Λ, E) defines a Jacobi structure on M if and only if*

$$X_{\{f,g\}} = [X_f, X_g], \quad \forall\, f, g \in C^\infty(M, \mathbb{R}),$$

where $\{f, g\} = \Lambda(f, g) + fE(dg) - gE(df)$. Moreover,

$$X_f = 0 \iff \{f, g\} = 0, \quad \forall g \in C^\infty(M, \mathbb{R}).$$

The *characteristic distribution* of a Jacobi manifold (M, Λ, E) is the subbundle C of TM spanned by all the Hamiltonian vectors fields. Thus, $C_x = Span\{E(x), (\Lambda\alpha)(x), \alpha \text{ is a 1-form}\}$ is the fiber at the point x. The characteristic distribution of (M, Λ, E) is completely integrable in the sense of Stefan-Sussmann (see [St] [Su]); it defines a singular foliation on M. The leaves of this foliation are contact manifolds or locally conformal symplectic manifolds, according to their dimension.

A Jacobi structure is said to be *transitive* if $C = TM$. It is known (see [L], [G-L]) that a transitive Jacobi manifold is either a contact manifold (when its dimension is odd) or a locally conformal symplectic manifold (when its dimension is even).

2.3 Jacobi pencils

A manifold M is said to be a *bihamiltonian manifold* if M is endowed with two Poisson tensors π_1 and π_2 such that $\pi_1 - \lambda\pi_2$ is a Poisson tensor for any $\lambda \in \mathbb{R}$. Then $\pi_1 - \lambda\pi_2$ is called a *Poisson pencil*. By anology, if $\{.,.\}_1$ and $\{.,.\}_2$ are two Jacobi structures such that $\{.,.\}_\lambda = \{.,.\}_1 - \lambda\{.,.\}_2$ defines a Jacobi structure for any λ in $\mathbb{R}$, then $\{.,.\}_\lambda$ will be called *Jacobi pencil*. In this case, the two Jacobi structures are said to be *compatible*.

Proposition 2.3 *(see [N]) Let (Λ_1, E_1) and (Λ_2, E_2) be two Jacobi stuctures on M. Denote by $\pi_i = e^{-t}(\Lambda_i + \partial/\partial t \wedge E_i)$, with $i = 1, 2$, the associated Poisson tensors on $M \times \mathbb{R}$. Then the following assertions are equivalent:*

(1) (Λ_1, E_1) and (Λ_2, E_2) define a Jacobi pencil on M.

(2) $[\Lambda_1, \Lambda_2] = E_1 \wedge \Lambda_2 + E_2 \wedge \Lambda_1$ and $[E_1, \Lambda_2] + [E_2, \Lambda_1] = 0$.

(3) The pair (π_1, π_2) defines a Poisson pencil on $M \times \mathbb{R}$.

From the classical Liouville theory, it follows that the integrability of a Hamiltonian system is related to the number and the independence of its first integrals *in involution* (i.e. commuting first integrals). Therefore, the methods of construction of functions in involution play an important role in integrable systems. We shall see that the one given in [C-M-P] using the Casimir of a Poisson pencil holds for Jacobi structures. Denoting by $N[[\lambda]] = C^\infty(M, \mathbb{R}) \otimes \mathbb{R}[[\lambda]]$ the space of formal power series in λ over $C^\infty(M, \mathbb{R})$, we may extend a Jacobi bracket $\{.,.\}$ defined on $C^\infty(M, \mathbb{R})$ to $N[[\lambda]]$ by

$$\Big\{\sum_{i=0}^{\infty} \lambda^i f_i \,, \sum_{j=0}^{\infty} \lambda^j g_j\Big\} := \sum_{r=0}^{\infty} \lambda^r \Big(\sum_{p+q=r} \{f_p, g_q\}\Big).$$

Now, assume that $\{.,.\}_\lambda = \{.,.\}_1 - \lambda\{\ ,\ \}_2$ is a Jacobi pencil. If (Λ_j, E_j), with $j = 1, 2$, are the tensors associated to the Jacobi brackets $\{.,.\}_j$, we consider the mapping σ_λ defined by

$$\sigma_\lambda f = (\Lambda_1 - \lambda\Lambda_2)df + f(E_1 - \lambda E_2),$$

which can be extended to $N[[\lambda]]$. If $h = \sum_{i=0}^{\infty} \lambda^i h_i \in N[[\lambda]]$ is such that $\sigma_\lambda(h) = 0$, then for any $f \in C^\infty(M, \mathbb{R})$ we have

$$\{h_{i+1}, f\}_1 = \{h_i, f\}_2.$$

We deduce that

$$\{h_i, h_{i+j}\}_1 = \{h_i, h_{i+j}\}_2 = 0, \quad \forall\ i, j.$$

So this gives a sequence of functions in involution for the Jacobi brackets $\{.,.\}_\ell$, with $\ell = 1, 2$.

2.4 The Lie algebroid of a Jacobi manifold

It was proven in [Ke-SB] that there is a Lie algebroid associated with an arbitrary Jacobi manifold (M, Λ, E). Let us recall that a vector bundle A over a differentiable manifold M is said to be a Lie algebroid if there is a Lie bracket $[\ ,\ ]_A$ on the space $\Gamma(A)$ of smooth sections of A and a bundl e map $\varrho : A \to TM$, extended to a map between sections of these bundles, such that

1) $\varrho([X, Y]_A) = [\varrho(X), \varrho(Y)]$,

2) $[X, fY]_A = f[X, Y]_A + (\varrho(X)f)Y$,

for any X, Y smooth sections of A and for any smooth function f on M. Then ϱ is called the *anchor* of the Lie algebroid.

Consider the vector bundle $T^*M \oplus \mathbb{R}$. The space $\Gamma(T^*M \oplus \mathbb{R})$ of smooth sections may be identified with $\Omega^1(M) \times C^\infty(M, \mathbb{R})$. The Lie algebroid associated with a Jacobi manifold (M, Λ, E) is $T^*M \oplus \mathbb{R}$ with the Lie bracket $\{\ ,\ \}_{(\Lambda,E)}$ on $\Gamma(T^*M \oplus \mathbb{R})$, which is defined by

$$\begin{aligned}\{(\alpha, f), (\beta, g)\}_{(\Lambda,E)} &= \Big(L_{\Lambda\alpha}\beta - L_{\Lambda\beta}\alpha - d(\Lambda(\alpha, \beta)) + fL_E\beta - gL_E\alpha - i_E(\alpha \wedge \beta), \\ &\quad -\Lambda(\alpha, \beta) + \Lambda(\alpha, dg) - \Lambda(\beta, df) + fE(dg) - gE(df)\Big),\end{aligned}$$

where d is the exterior derivative and $L_X = di_X + i_X d$ is the Lie derivation by X, for any vector field X. The anchor is given by the map $\#_{(\Lambda,E)}$ such that

$$\#_{(\Lambda,E)}(\alpha, f) = \Lambda\alpha + fE.$$

Notice that we have $\#_{(\Lambda,E)}(df, f) = X_f$.

Proposition 2.4 *The pair (Λ, E) defines a Jacobi structure on M if and only if*

$$[\#_{(\Lambda,E)}(\alpha, f), \#_{(\Lambda,E)}(\beta, g)] = \#_{(\Lambda,E)}\Big(\{(\alpha, f), (\beta, g)\}_{(\Lambda,E)}\Big).$$

Sketch of proof: The operation $\#_{(\Lambda,E)}\{.,.\}_{(\Lambda,E)}$ is the unique $\mathbb{R}$-bilinear map which satisfies

(R_1) $\#_{(\Lambda,E)}\Big(\{(df, f), (dg, g)\}_{(\Lambda,E)}\Big) = [\#_{(\Lambda,E)}(df, f), \#_{(\Lambda,E)}(dg, g)]$,

(R_2) $\#_{(\Lambda,E)}\Big(\{(\alpha, f), h(\beta, g)\}_{(\Lambda,E)}\Big) = h\Big(\#_{(\Lambda,E)}\{(\alpha, f), (\beta, g)\}_{(\Lambda,E)}\Big) + \Big(\#_{(\Lambda,E)}(\alpha, f)h\Big)\#_{(\Lambda,E)}(\beta, g)$,

for any $\alpha, \beta \in \Omega^1(M)$ and for any smooth functions f, g. Since the map

$$((\alpha, f), (\beta, g)) \longmapsto [\#_{(\Lambda,E)}(\alpha, f), \#_{(\Lambda,E)}(\beta, g)]$$

also satisfies these rules (R_1) and (R_2), they are equal.

3 Compatibility between Jacobi and Nijenhuis structures

Let J be a $(1,1)$-tensor field of M. The *Nijenhuis torsion* N_J of J with respect to the Lie bracket $[.,.]$ on the space $\chi(M)$ of vector fields is defined by

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X,Y], \quad \forall\, X, Y \in \chi(M).$$

Definition 3.1 *J is called a Nijenhuis tensor if its Nijenhuis torsion vanishes.*

Notations. To any bivector field Λ on M, we may associate the skew-symmetric linear map denoted also by $\Lambda : \Omega^1(M) \to \chi(M)$ and defined by:

$$\langle \beta, \Lambda\alpha \rangle = \langle \alpha \wedge \beta, \Lambda \rangle = \Lambda(\alpha, \beta).$$

Conversely, a linear map $\Lambda : \Omega^1(M) \to \chi(M)$ defines a bivector field on M if and only if

$$\langle \alpha, \Lambda\beta \rangle + \langle \beta, \Lambda\alpha \rangle = 0.$$

In particular, when J is a $(1,1)$-tensor field on M and $\Lambda : \Omega^1(M) \to \chi(M)$ is a linear map, then $J \circ \Lambda$ defines a bivector field if and only if $J \circ \Lambda = \Lambda \circ {}^tJ$. In this case, the associated bivector field is denoted by $J\Lambda$.

Furthermore, any bivector field Λ gives a bracket defined on the differential 1-forms by

$$\{\alpha, \beta\}_\Lambda = L_{\Lambda\alpha}\beta - L_{\Lambda\beta}\alpha - d(\Lambda(\alpha, \beta)), \quad \forall \alpha, \beta \in \Omega^1(M), \tag{1}$$

where L_X is the Lie derivation by X, for any vector field X.

Whenever $J \circ \Lambda = \Lambda \circ {}^tJ$, we denote by $C(\Lambda, J)$ the $\mathbb{R}$-bilinear map given by

$$C(\Lambda, J)(\alpha, \beta) = \{\alpha, \beta\}_{J\Lambda} - \Big(\{{}^tJ\alpha, \beta\}_\Lambda + \{\alpha, {}^tJ\beta\}_\Lambda - {}^tJ\{\alpha, \beta\}_\Lambda\Big).$$

Definition 3.2 *(see [K-M]) A Poisson-Nijenhuis structure on a manifold M is defined by a Poisson tensor π and a Nijenhuis tensor J on M such that*

(a) $J \circ \pi = \pi \circ {}^tJ$,

(b) $C(\pi, J) = 0$.

In this case, we say that π and J are compatible.

To extend this definition to Jacobi structures, it is natural to think about the Poissonization method but the latter gives a weak generalization (see subsection 3.2). We propose the following definition.

Definition 3.3 *Let (M, Λ, E) be a Jacobi manifold. A Nijenhuis tensor J on M is said to be compatible with the Jacobi structure (Λ, E) if* *(i)* $J\circ\Lambda=\Lambda\circ\ {}^tJ$,

(ii) $\Lambda(\alpha,\beta)JE-\Lambda(\alpha,\ {}^tJ\beta)E=\Lambda\Big(C(\Lambda,J)(\alpha,\beta)\Big),\quad\forall\alpha,\beta\in\Omega^1(M)$;

(iii) $[J^kE,\Lambda]+[E,J^k\Lambda]=0$ *for any* $k\in\mathbb{N}^*$.

When the property (iii) *holds only for* $k\leq p$, *and the other properties are satisfied, we will say that* (Λ,E) *and* J *are compatible up to the order* p.

When $E=0$ (i.e. Λ defines a Poisson structure), the pair (Λ,J) is said to be a *weak Poisson-Nijenhuis structure* (see [M-M-P]). In such a case, the compatibility conditions are reduced to (ii), which includes the one given in [M-M]. In other words, a Poisson-Nijenhuis structure is always a weak Poisson-Nijenhuis structure but the converse is false.

Theorem 3.4 *Let* (Λ,E) *be a Jacobi structure on* M. *Assume that* J *is a* $(1,1)$*-tensor field such that* $J\circ\Lambda=\Lambda\circ\ {}^tJ$ *and*

$$N_J(\Lambda\alpha+fE,\Lambda\beta+gE)=0,\quad\forall\alpha,\beta\in\Omega^1(M)\quad\text{and}\quad\forall f,g\in C^\infty(M,\mathbb{R}),$$

where N_J *is the Nijenhuis torsion of* J. *Then* $(J\Lambda,JE)$ *is a Jacobi structure on* M *if and only if the following properties are satisfied for all* $\alpha,\beta,\gamma\in\Omega^1(M)$:

(a) $J([JE,\Lambda]\alpha+[E,J\Lambda]\alpha)=0$,

(b) $\langle\ {}^tJ\gamma,\ \Lambda\Big(C(\Lambda,J)(\alpha,\beta)\Big)-\Lambda(\alpha,\beta)JE+\Lambda(\alpha,\ {}^tJ\beta)E\rangle=0$.

In particular, if J *is a Nijenhuis tensor compatible with* (Λ,E), *then* $(J\Lambda,JE)$ *is a Jacobi structure on* M.

The proof of Theorem 3.4 is based on the following three lemmas.

Lemma 3.5 *For any bivector field* Λ, *we have:*

$$\langle\gamma,\Lambda\{\alpha,\beta\}_\Lambda\rangle=\langle\gamma,[\Lambda\alpha,\Lambda\beta]\rangle+\frac{1}{2}[\Lambda,\Lambda](\alpha,\beta,\gamma),\quad\forall\ \alpha,\beta,\gamma\in\Omega^1(M).\tag{2}$$

This formula is proven in [G-D] and [K-M].

Lemma 3.6 *Consider a couple* (Λ,E) *formed by a bivector field* Λ *and a vector field* E *on* M *such that* $[\Lambda,\Lambda]=2E\wedge\Lambda$. *Then, for any linear map* J *on* $\chi(M)$ *satisfying* $J\circ\Lambda=\Lambda\circ^tJ$, *the following formula holds:*

$$\begin{aligned}\frac{1}{2}[J\Lambda,J\Lambda](\alpha,\beta,\gamma)\quad=\quad&(JE\wedge J\Lambda)(\alpha,\beta,\gamma)+\langle{}^tJ\gamma,\ \Lambda\Big(C(\Lambda,J)(\alpha,\beta)\Big)\rangle\\&+E({}^tJ\gamma)\Lambda(\alpha,\ {}^tJ\beta)-JE({}^tJ\gamma)\Lambda(\alpha,\beta)\\&-\langle\gamma,\ N_J(\Lambda\alpha,\Lambda\beta)\rangle.\end{aligned}$$

Proof: We use Lemma 3.5 which gives:

$$\frac{1}{2}[J\Lambda, J\Lambda](\alpha,\beta,\gamma) = \langle {}^tJ\gamma,\ \Lambda\{\alpha,\beta\}_{J\Lambda}\rangle - \langle [J\Lambda\alpha, J\Lambda\beta],\ \gamma\rangle.$$

Next, we add and withdraw the following quantity:

$$\langle {}^tJ\gamma,\ \Lambda\{{}^tJ\alpha,\beta\}_\Lambda + \Lambda\{\alpha,\ {}^tJ\beta\}_\Lambda - J\Lambda\{\alpha,\beta\}_\Lambda\rangle.$$

Using again the relation (2), we obtain

$$\begin{aligned}\frac{1}{2}[J\Lambda, J\Lambda](\alpha,\beta,\gamma) &= \frac{1}{2}\Big([\Lambda,\Lambda]({}^tJ\alpha,\beta,\ {}^tJ\gamma) + [\Lambda,\Lambda](\alpha,\ {}^tJ\beta,\ {}^tJ\gamma)\\ &\quad -[\Lambda,\Lambda](\alpha,\beta,\ {}^tJ^2\gamma)\Big) + \langle {}^tJ\gamma, \Lambda\Big(C(\Lambda,J)(\alpha,\beta)\Big)\rangle\\ &\quad -\langle\gamma, N_J(\Lambda\alpha,\Lambda\beta)\rangle. \qquad (3)\end{aligned}$$

Since $[\Lambda,\Lambda] = 2E\wedge\Lambda$ it turns out that:

$$\begin{aligned}\frac{1}{2}[J\Lambda, J\Lambda](\alpha,\beta,\gamma) &= (JE\wedge J\Lambda)(\alpha,\beta,\gamma) + E({}^tJ\gamma)\Lambda(\alpha,\ {}^tJ\beta) - JE({}^tJ\gamma)\ \Lambda(\alpha,\beta)\\ &\quad +\langle {}^tJ\gamma,\ \Lambda(C(\Lambda,J)(\alpha,\beta))\rangle - \langle\gamma, N_J(\Lambda\alpha,\Lambda\beta)\rangle.\end{aligned}$$

This is the formula wanted. ■

Lemma 3.7 *Let Λ and E be respectively a bivector field and a vector field on M. Then the following relation holds for any linear map J on $\chi(M)$:*

$$[JE, J\Lambda](\alpha,\beta) = \langle\beta,\ N_J(E,\Lambda\alpha)\rangle + \langle\beta, J[JE,\Lambda]\alpha + J[E,J\Lambda]\alpha - J^2[E,\Lambda]\alpha\rangle.$$

Proof: For any bivector field Λ and for all $\alpha,\beta\in\Omega^1(M)$, we have:

$$[E,\Lambda](\alpha,\beta) = L_E(\Lambda(\alpha,\beta)) - \Lambda(L_E\alpha,\beta) - \Lambda(\alpha, L_E\beta).$$

This is equivalent to the relation

$$[E,\Lambda]\alpha = [E,\Lambda\alpha] - \Lambda L_E\alpha, \quad \forall\alpha\in\Omega^1(M). \qquad (4)$$

Using (4), we obtain for any $\alpha\in\Omega^1(M)$:

$$\begin{aligned}[JE, J\Lambda]\alpha &= [JE, J\Lambda\alpha] - J\Lambda L_{JE}\alpha\\ &= N_J(E,\Lambda\alpha) + J[JE,\Lambda\alpha] + J[E,J\Lambda\alpha] - J^2[E,\Lambda\alpha] - J\Lambda L_{JE}\alpha.\end{aligned}$$

Replacing $[E,\Lambda\alpha]$ by $[E,\Lambda]\alpha + \Lambda L_E\alpha$, we deduce that

$$\begin{aligned}[JE, J\Lambda]\alpha &= N_J(E,\Lambda\alpha) + J([JE,\Lambda\alpha] - \Lambda L_{JE}\alpha)\\ &\quad + J([E,J\Lambda\alpha] - J\Lambda L_E\alpha) - J^2[E,\Lambda]\alpha)\\ &= N_J(E,\Lambda\alpha) + J([JE,\Lambda] + [E,J\Lambda] - J[E,\Lambda])\alpha.\end{aligned}$$

■

Proof of Theorem 3.4: Lemma 3.7 ensures that $[JE, J\Lambda] = 0$ is equivalent to (a). While Lemma 3.6 says that $[J\Lambda, J\Lambda] = 2JE \wedge J\Lambda$ if and only if property (b) is satisfied. So the theorem is proved. ■

Now, let us express the properties (a) and (b) of Theorem 3.4 using the Lie algebroid associated with the Jacobi structure (see Proposition 2.4).

Proposition 3.8 *Let (Λ, E) be a Jacobi structure on M and let J be a $(1,1)$-tensor field such that*

$$J \circ \Lambda = \Lambda \circ {}^tJ \quad \text{and} \quad N_J(\Lambda\alpha + fE, \Lambda\beta + gE) = 0, \quad \forall \alpha, \beta \in \Omega^1(M), \ \forall f, g \in C^\infty(M, \mathbb{R}).$$

Then we have the following equivalences:

$$\begin{array}{lcl} (a) \text{ is satisfied} & \Longleftrightarrow & [J\Lambda\alpha + fJE, \ gJE] = \#_{(J\Lambda, JE)}\Big(\{(\alpha, f), (0, g)\}_{(J\Lambda, JE)}\Big) \\ (b) \text{ is satisfied} & \Longleftrightarrow & [J\Lambda\alpha, \ J\Lambda\beta] = \#_{(J\Lambda, JE)}\Big(\{(\alpha, 0), (\beta, 0)\}_{(J\Lambda, JE)}\Big) \end{array}$$

Proof: we have

$$[J\Lambda\alpha + fJE, \ gJE] = g[J\Lambda\alpha, JE] + (J\Lambda(\alpha, dg) + \langle fdg - gdf, JE\rangle)JE.$$

On the other hand, we have

$$\#_{(J\Lambda, JE)}\{(\alpha, f), (0, g)\}_{(J\Lambda, JE)} = -gJ\Lambda L_{JE}\alpha + (J\Lambda(\alpha, dg) + \langle fdg - gdf, \ JE\rangle)JE.$$

We deduce that

$$\begin{aligned} [J\Lambda\alpha + fJE, gJE] - \#_{(J\Lambda, JE)}\{(\alpha, f), (0, g)\}_{(J\Lambda, JE)} &= g([J\Lambda\alpha, JE] + J\Lambda L_{JE}\alpha) \\ &= g[J\Lambda, JE]\alpha. \end{aligned}$$

But Lemma 3.7 says that

$$[J\Lambda, JE]\alpha = 0 \iff J([JE, \Lambda]\alpha + [E, J\Lambda]\alpha) = 0.$$

Hence we obtain the first equivalence. In the same way, we prove the second equivalence using Lemma 3.6.

3.1 Hierarchy of Jacobi structures

The following theorem is a generalization of a result proved in [M-M] and [K-M]:

Theorem 3.9 *For any Jacobi structure (Λ, E) compatible with a Nijenhuis tensor J on M and for each $k \in \mathbb{N}^*$, the pair $(J^k\Lambda, J^kE)$ is a Jacobi structure on M. Furthermore for $k_1,\ k_2 \in \mathbb{N}^*$, $(J^{k_1}\Lambda, J^{k_1}E)$ and $(J^{k_2}\Lambda, J^{k_2}E)$ define a Jacobi pencil.*

Lemma 3.10 *Let J be a $(1,1)$-tensor field. Then, we have:*

$$\begin{aligned} N_{J^{k+1}}(X,Y) &= N_{J^k}(JX,JY) + J^k\Big(N_J(J^kX,Y) + N_J(X,J^kY)\Big) \\ &\quad -J^2\Big(N_{J^{k-1}}(JX,JY) - N_{J^k}(X,Y)\Big), \quad \forall X,Y \in \chi(M). \end{aligned}$$

The proof of this lemma is straightforward.

Proof of Theorem 3.9: assume that $[J^\ell\Lambda, J^\ell\Lambda] = 2J^\ell E \wedge J^\ell\Lambda$, for any $\ell \leq k$. It follows from Lemma 3.6 that

$$\begin{aligned} \frac{1}{2}[J^{k+1}\Lambda, J^{k+1}\Lambda](\alpha,\beta,\gamma) &= (J^{k+1}E \wedge J^{k+1}\Lambda)(\alpha,\beta,\gamma) + \langle {}^tJ\gamma,\ J^k\Lambda C(J^k\Lambda, J)(\alpha,\beta)\rangle \\ &\quad + J^kE({}^tJ\gamma)J^k\Lambda(\alpha,\ {}^tJ\beta) - J^{k+1}E({}^tJ\gamma)J^k\Lambda(\alpha,\beta). \end{aligned}$$

We shall prove that

$$J^k\Lambda C(J^k\Lambda, J)(\alpha,\beta) + J^k\Lambda(\alpha,\ {}^tJ\beta)J^kE - J^k\Lambda(\alpha,\beta)J^{k+1}E = 0.$$

In fact, for any bivector field Λ and for any linear map J on $\chi(M)$ such that $J \circ \Lambda = \Lambda \circ\ {}^tJ$, the following relation holds (see [M-M]):

$$\langle C(J\Lambda, J)(\alpha,\beta),\ X\rangle = \langle C(\Lambda, J)({}^tJ\alpha,\beta),\ X\rangle + \langle \alpha,\ N_J(\Lambda\beta, X)\rangle. \tag{5}$$

Hence, we obtain by induction that for any $k \geq 1$,

$$C(J^k\Lambda, J)(\alpha,\beta) = C(\Lambda, J)({}^tJ^k\alpha,\beta). \tag{6}$$

Since J is compatible with (Λ, E), we have

$$\Lambda\Big(C(\Lambda, J)(\alpha,\beta)\Big) = \Lambda(\alpha,\beta)JE - \Lambda(\alpha,\ {}^tJ\beta)E. \tag{7}$$

We deduce that

$$\begin{aligned} J^k\Lambda C(J^k\Lambda, J)(\alpha,\beta) &= J^kC(\Lambda, J)({}^tJ^k\alpha,\beta) \\ &= J^k\Big(\Lambda({}^tJ^k\alpha,\beta)JE - \Lambda({}^tJ^k\alpha,{}^tJ\beta)E\Big) \\ &= J^k\Lambda(\alpha,\beta)J^{k+1}E - J^k\Lambda(\alpha,\ {}^tJ\beta)J^kE. \end{aligned}$$

So, we obtain the relation wanted. The latter implies that

$$[J^k\Lambda, J^k\Lambda] = 2J^kE\wedge J^k\Lambda \quad \text{for any} \quad k\geq 1.$$

Moreover, replacing J by J^k in Lemma 3.7, we obtain:

$$[J^kE, J^k\Lambda](\alpha,\beta) = \langle\beta,\ N_{J^k}(E,\Lambda\alpha)\rangle + \langle\ {}^tJ^k\beta, [J^kE,\Lambda]\alpha + [E, J^k\Lambda]\alpha\rangle.$$

From Lemma 3.10, we obtain by induction that the Nijenhuis torsion of J^k vanishes for any $k\geq 1$. Therefore,

$$[J^kE, J^k\Lambda] = 0 \quad \text{for any} \quad k\geq 1.$$

Thus, $(J^k\Lambda, J^kE)$ defines a Jacobi structure for any $k\geq 1$.

Now take two different pairs $(J^{k_1}\Lambda, J^{k_1}E)$ and $(J^{k_2}\Lambda, J^{k_2}E)$. We shall prove that they determine a Jacobi pencil. For any $\lambda\in\mathbb{R}$, we have to prove that

$$[J^{k_1}\Lambda - \lambda J^{k_2}\Lambda, J^{k_1}\Lambda - \lambda J^{k_2}\Lambda] = 2(J^{k_1}E - \lambda J^{k_2}E)\wedge(J^{k_1}\Lambda - \lambda J^{k_2}\Lambda).$$

Since we have

$$[J^{k_i}\Lambda, J^{k_i}\Lambda] = 2J^{k_i}E\wedge J^{k_i}\Lambda, \quad \forall i=1,2,$$

thus we have only to prove that

$$[J^{k_1}\Lambda, J^{k_2}\Lambda] = J^{k_1}E\wedge J^{k_2}\Lambda + J^{k_2}E\wedge J^{k_1}\Lambda.$$

Assume that $k_1 = k_2+\ell$, then we apply ℓ times the result saying that, for arbitrary bivector fields Λ and π on M, for any linear map J on $\chi(M)$ the following formula holds (see [M-M]):

$$\begin{aligned}[J\Lambda,\pi](\alpha,\beta,\gamma) &= [\Lambda,\pi](\alpha,\beta,\ {}^tJ\gamma) + \langle C(\pi,J)(\alpha,\gamma),\ \Lambda\beta\rangle \\ &\quad -\langle C(\pi,J)(\beta,\gamma),\ \Lambda\alpha\rangle - \langle C(\Lambda,J)(\alpha,\beta),\ \pi\gamma\rangle.\end{aligned}$$

We apply this last relation and we calculate by recursion the ℓ quantities $[J^{k_2+\ell}\Lambda, J^{k_2}\Lambda]$,...,$[J^{k_2+1}\Lambda, J^{k_2}\Lambda]$. It follows that:

$$\begin{aligned}[J^{k_1}\Lambda, J^{k_2}\Lambda](\alpha,\beta,\gamma) &= [J^{k_2}\Lambda, J^{k_2}\Lambda](\alpha,\beta,\ {}^tJ^\ell\gamma) \\ &\quad +\sum_{r=1}^{\ell}\langle C(J^{k_2}\Lambda,J)(\alpha,\ {}^tJ^{r-1}\gamma),\ J^{k_1-r}\Lambda\beta\rangle \\ &\quad -\sum_{r=1}^{\ell}\langle C(J^{k_2}\Lambda,J)(\beta,\ {}^tJ^{r-1}\gamma),\ J^{k_1-r}\Lambda\alpha\rangle \\ &\quad -\sum_{r=1}^{l}\langle C(J^{k_1-r}\Lambda,J)(\alpha,\beta),\ J^{k_2+r-1}\Lambda\gamma\rangle\end{aligned}$$

Now, we use the relation (6) as well as (7) and the fact $[J^{k_2}\Lambda, J^{k_2}\Lambda] = 2J^{k_2}E\wedge J^{k_2}\Lambda$, we obtain after computations:

$$[J^{k_1}\Lambda, J^{k_2}\Lambda] = J^{k_1}E \wedge J^{k_2}\Lambda + J^{k_2}E \wedge J^{k_1}\Lambda.$$

The last step is to show that:

$$[J^{k_1}E - \lambda J^{k_2}E, J^{k_1}\Lambda - \lambda J^{k_2}\Lambda] = 0.$$

This is equivalent to showing that $[J^{k_1}E, J^{k_2}\Lambda] + [J^{k_2}E, J^{k_1}\Lambda] = 0$. By hypothesis this relation is true when $k_2 = 1$ and using Lemma 3.7, we can easily show by induction that this formula holds for any k_1 and k_2. ■

Example 3. Let ω be a closed 1-form and let F_1, F_2 be two nondegenerate 2-forms on M. Assume that (F_1, ω) and (F_2, ω) are locally conformal symplectic structures on M. Let (Λ_i, E_i) denote the Jacobi structures associated with (F_i, ω), where $i = 1, 2$. Assume that these two Jacobi structures are compatible. Define the isomorphism of $C^\infty(M, \mathbb{R})$-modules $\flat_i : \chi(M) \to \Omega^1(M)$ by

$$\flat_i(X) = -i_X F_i.$$

We have

$$E_i = -\flat_i^{-1}(\omega) \quad \text{and} \quad \Lambda_i\alpha = \flat_i^{-1}(\alpha), \quad \forall \alpha \in \Omega^1(M).$$

Then, the $(1,1)$-tensor field $J = \flat_2^{-1} \circ \flat_1$ is compatible with (Λ_1, E_1) at any order. Indeed, for any $x \in M$, there exist a neighborhood U_x and a function f defined on U_x such that $\omega = df$. The 2-forms $\Omega_1 = e^f F_1$ and $\Omega_2 = e^f F_2$ are symplectic and the Poisson tensors associated with Ω_1, Ω_2 are respectively $\pi_1 = e^{-f}\Lambda_1$, $\pi_2 = e^{-f}\Lambda_2$.

We claim that the Jacobi structures (Λ_1, E_1) and (Λ_2, E_2) are compatible if and only if π_1 and π_2 are compatible. Let us prove this claim. Using the properties of the Schouten-Nijenhuis bracket, we get

$$[\pi_1, \pi_2] = e^{-2f}\Big([\Lambda_1, \Lambda_2] - [\Lambda_1, f] \wedge \Lambda_2 - [\Lambda_2, f] \wedge \Lambda_1\Big).$$

Since $E_i = [\Lambda_i, f] = -\Lambda_i(df)$, we have

$$[\pi_1, \pi_2] = e^{-2f}([\Lambda_1, \Lambda_2] - E_1 \wedge \Lambda_2 - E_2 \wedge \Lambda_1).$$

Therefore, $[\pi_1, \pi_2] = 0$ if and only if $[\Lambda_1, \Lambda_2] = E_1 \wedge \Lambda_2 + E_2 \wedge \Lambda_1$. Moreover, we may remark that the Jacobi identity of the Schouten-Nijenhuis bracket gives

$$\begin{aligned} [[\pi_1, \pi_2], e^f] &= -[[\pi_2, e^f], \pi_1] - [[e^f, \pi_1], \pi_2] \\ &= -[[\Lambda_2, f], e^{-f}\Lambda_1] - [[f, \Lambda_1], e^{-f}\Lambda_2]. \end{aligned}$$

The fact that $E_i = [\Lambda_i, f]$ implies

$$[[\pi_1, \pi_2], e^f] = -e^{-f}([E_2, \Lambda_1] + [E_1, \Lambda_2]).$$

Thus, $(\Lambda_i, E_i)_{i=1,2}$ form a Jocobi pencil if and only if the tensors $(\pi_i)_{i=1,2}$ define a Poisson pencil. So, we may deduce that the Nijenhuis torsion of J vanishes. Furthermore, the sequence $(J^k\pi_1)$ is formed by pairwise compatible Poisson tensors, while $(J^k\Lambda_1, J^kE_1)$ is a sequence of pairwise compatible Jacobi structures.

3.2 Compatibility and Poissonization

First, let us see why the method of Poissonization gives a weak generalization. Let (Λ, E) be a Jacobi structure and let J be a $(1,1)$-tensor field on M. Denote by π the corresponding Poisson tensor on $M \times \mathbb{R}$. Consider $\tilde{J} : \chi(M \times \mathbb{R}) \to \chi(M \times \mathbb{R})$ an extension of J of the form

$$\tilde{J} = J + \alpha_0 \otimes \frac{\partial}{\partial t} + f_0 \; dt \otimes \frac{\partial}{\partial t},$$

where $\alpha_0 \in \Omega^1(M)$ and f_0 is a smooth function on M. On the one hand, the relation $\tilde{J} \circ \pi = \pi \circ {}^t\tilde{J}$ gives a strong condition, which is the following:

$$JE = \Lambda\alpha_0 + f_0 E.$$

For instance when Λ is zero, we must have $JE = f_0E$. Moreover, if we express the fact that the Nijenhuis torsion of $\tilde{J}$ vanishes, we have other conditions on α_0 and f_0. On the other hand, when $(\pi, \tilde{J})$ is a Poisson-Nijenhuis structure on $M \times \mathbb{R}$, we have necessarily the conditions of compatibility (i), (ii) and (iii) of Definition 3.3. Indeed, in such a case, the hierarchy of pairwise comp atible Poisson structures $(\tilde{J}^k\pi)$ is given by

$$\tilde{J}^k\pi = e^{-t}(J^k\Lambda + \frac{\partial}{\partial t} \wedge J^kE).$$

We know that $\tilde{J}\pi$ is a Poisson tensor on $M \times \mathbb{R}$ if and only if the pair $(J\Lambda, JE)$ is a Jacobi structure on M. Hence, by Theorem 3.4 we have (i) and (ii). Furthermore, (iii) is obtained by using the fact that the Poisson tensors $\tilde{J}^k\pi$ are compatible with π.

Now, let us make precise why many Jacobi-Nijenhuis structures are particular cases of Jacobi structures compatible with a Nijenhuis tensor (see Definition 3.3). For any bivector field Λ (resp. vector field E), we may define a mapping $\widetilde{\#}_{(\Lambda,E)} : \Omega^1(M) \times C^\infty(M, \mathbb{R}) \to \chi(M) \times C^\infty(M, \mathbb{R})$ by

$$\widetilde{\#}_{(\Lambda,E)}(\beta, g) = (\Lambda\beta + gE, \langle\beta, E\rangle).$$

Definition 3.11 *(see [M-M-P]) Let $\tilde{\mathcal{J}} : \chi(M) \times C^\infty(M, \mathbb{R}) \to \chi(M) \times C^\infty(M, \mathbb{R})$ be a $C^\infty(M, \mathbb{R})$-linear map and let (Λ, E) be a Jacobi structure on M. The triple $(\Lambda, E, \tilde{\mathcal{J}})$ is said to be a*

Jacobi-Nijenhuis structure on M if we have $\tilde{\mathcal{J}} \circ \widetilde{\#}_{(\Lambda,E)} = \widetilde{\#}_{(\Lambda,E)} \circ {}^t\tilde{\mathcal{J}}$ and (Λ_1, E_1) is a Jacobi structure compatible with (Λ, E), where Λ_1 and E_1 are characterized by the relation

$$\widetilde{\#}_{(\Lambda_1,E_1)} = \tilde{\mathcal{J}} \circ \widetilde{\#}_{(\Lambda,E)}.$$

An extension $\tilde{J}$ of an endomorphism J of $\chi(M)$ to $\chi(M \times \mathbb{R})$ is equivalent to giving a $C^\infty(M,\mathbb{R})$-linear map $\tilde{\mathcal{J}} : \chi(M) \times C^\infty(M,\mathbb{R}) \to \chi(M) \times C^\infty(M,\mathbb{R})$. When $\tilde{\mathcal{J}}$ sends $\{0\} \times C^\infty(M,\mathbb{R})$ to itself, then we may set

$$\tilde{\mathcal{J}}(X,0) = (JX, \langle \alpha_0, X\rangle) \quad \text{and} \quad \tilde{\mathcal{J}}(0,1) = (0, f_0).$$

We get

$$\tilde{J} = J + \alpha_0 \otimes \frac{\partial}{\partial t} + f_0 \; dt \otimes \frac{\partial}{\partial t},$$

If (Λ, E) is a Jacobi structure on M and π denotes the corresponding Poisson tensor on $M \times \mathbb{R}$, then

$$\tilde{J} \circ \pi_1 = \pi_1 \circ {}^t\tilde{J} \iff \tilde{\mathcal{J}} \circ \widetilde{\#}_{(\Lambda,E)} = \widetilde{\#}_{(\Lambda,E)} \circ {}^t\tilde{\mathcal{J}},$$

Moreover $(\Lambda, E, \tilde{\mathcal{J}})$ is a Jacobi-Nijenhuis structure on M iff $(\pi, \tilde{J})$ is a Poisson-Nijenhuis structure on $M \times \mathbb{R}$.

Suppose $(\Lambda, E, \tilde{\mathcal{J}})$ is a Jacobi-Nijenhuis structure on M such that $\tilde{\mathcal{J}}(0,1) = (0, f_0)$. Then, from what we have seen above, we may deduce that the $(1,1)$-tensor field on M, which corresponds to $\tilde{\mathcal{J}}$, is compatible with (Λ, E).

4 Nijenhuis tensors and homogeneous Poisson structures

Definition 4.1 *A homogeneous Poisson manifold (M, π, Z) is a Poisson manifold (M, π) with a vector field Z over M such that*

$$[Z, \pi] = -\pi.$$

Theorem 4.2 *Assume that (M, π, Z) is a homogeneous Poisson manifold. Let J be Nijenhuis tensor compatible with π. Then $(M, J\pi, Z)$ is a homogeneous Poisson manifold if and only if we have the following property*

$$\pi \circ (L_Z \circ^t J -^t J \circ L_Z) = 0, \tag{8}$$

where $L_Z = i_Z d + d i_Z$ is the Lie derivation by Z. When this property holds, $J\pi - \lambda\pi$ defines a Poisson pencil which is homogeneous with respect to Z.

Proof: Taking into account Theorem 3.9, we have only to prove that $[Z, J\pi] = -J\pi$. Let us compute $[Z, J\pi]$. We obtain

$$\begin{aligned}
[Z, J\pi](df, dg) &= L_Z d(J\pi(df, dg)) - J\pi(L_Z df, dg) - J\pi(df, L_Z dg) \\
&= L_Z d(\pi({}^t J df, dg)) - \pi({}^t J L_Z df, dg) - \pi({}^t J df, L_Z dg)
\end{aligned}$$

Since

$$L_Z d(\pi({}^t J df, dg)) = [Z, \pi]({}^t J df, dg) + \pi(L_Z\ {}^t J df, dg) + \pi({}^t J df, L_Z dg),$$

We obtain

$$\begin{aligned}
[Z, J\pi](df, dg) &= [Z, \pi]({}^t J df, dg) + \pi(L_Z\ {}^t J df, dg) - J\pi(L_Z df, dg) \\
&= -\pi({}^t J df, dg) + \pi(L_Z\ {}^t J df, dg) - J\pi(L_Z df, dg)
\end{aligned}$$

Hence, the relation $[Z, J\pi] = -J\pi$ is equivalent to the following one:

$$\pi \circ L_Z \circ\ {}^t J = \pi \circ\ {}^t J \circ L_Z.$$

This proves the theorem. ■

Definition 4.3 *A homogeneous Poisson manifold (M, π, Z) equipped with a Nijenhuis tensor J which is compatible with π and satisfies equation (8) is said to be a homogeneous Poisson-Nijenhuis manifold.*

Corollary 4.4 *Let (M, π, J) be a Poisson-Nijenhuis manifold. If π is homogeneous with respect to a vector field Z and if the following property holds*

$$[Z, JX] = J[Z, X], \qquad \forall X \in \chi(M), \tag{9}$$

then the triple (M, π, J) is a homogeneous Poisson-Nijenhuis manifold with respect to Z.

Proof: We obtain the corollary using the above theorem and the fact that

$$[Z, JX] = J[Z, X], \quad \forall X \in \chi(M) \iff \quad L_Z \circ^t J =^t J \circ L_Z.$$

Definition 4.5 *A map $\psi : (M_1, \Lambda_1, E_1) \to (M_2, \Lambda_2, E_2)$ between two Jacobi manifolds is said to be a conformal Jacobi morphism if there exists a function $a \in C^\infty(M_1, \mathbb{R})$ which vanishes nowhere such that for any $f, g \in C^\infty(M_2, \mathbb{R})$ we have:*

$$\{a(f \circ \psi), a(g \circ \psi)\}_1 = a(\{f, g\}_2 \circ \psi),$$

where the brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ are the Jacobi brackets associated with (Λ_1, E_1) and (Λ_2, E_2) respectively.

Homogeneous Poisson manifolds are closely related to Jacobi manifolds and their relations were established in [D-L-M]. In terms of Poisson pencils, we have the following results.

Proposition 4.6 *Let* $\{.,.\}_\lambda$ *be a Jacobi pencil on* M*, then there exists a Poisson pencil on* $M\times\mathbb{R}$ *such that the projection* $P : M\times\mathbb{R}\to M$ *is a conformal Jacobi morphism.*

Proof: If (Λ_i, E_i) denotes the Jacobi structure on M associated to $\{.,.\}_i$, with $i=1,2$; then the Poisson pencil on $M\times\mathbb{R}$ is given by $\pi_1-\lambda\pi_2$ where

$$\pi_i(x,t)=e^{-t}\Big(\Lambda_i(x)+\frac{\partial}{\partial t}\wedge E_i\Big).$$

One may easily verify that $P:(M\times\mathbb{R},\pi_\lambda)\to(M,\{.,.\}_\lambda)$ is a conformal Jacobi morphism. ■

Conversely, we may prove that homogeneous Poisson pencils give Jacobi pencils by using a proof done in [D-L-M]. Precisely we have:

Proposition 4.7 *Let* π_λ *be a homogeneous Poisson pencil on* M *with respect to the vector field* Z*, and let* N *be a submanifold of* M *of codimension 1 which is transverse to* Z*. Then there exists a Jacobi pencil on* N *such that for any pair of functions* (f,g) *defined on an open set* U *of* M*, satisfying* $<Z,df>=f$ *and* $<Z,dg>=g$*, we have*

$$\{f_{|_{N\cap U}},g_{|_{N\cap U}}\}_\lambda=\pi_\lambda(df,dg)_{|_{N\cap U}}.$$

Corollary 4.8 *Let* (M,Λ,E) *be a Jacobi manifold and let* J *be a Nijenhuis tensor on* M*, which is compatible with* (Λ,E)*. Then there exists a sequence of Poisson-Nijenhuis structures* (π_k) *on* $M\times\mathbb{R}$ *that the projection* $P_k:(M\times\mathbb{R},\pi_k)\to(M,\Lambda,E)$ *is a conformal Jacobi morphism, for each* $k\geq 1$ *.*

Conversely, if (M,π,J) *is a homogeneous Poisson-Nijenhuis manifold with respect to the vector field* Z *and if* N *is a submanifold of* M *of codimension 1, which is transverse to* Z*, then there exists a sequence of pairwise compatible Jacobi structures on* N *determined by* π*,* Z *and* J*.*

This corollary is a direct consequence of Theorem 3.9 as well as Propositions 4.6 and 4.7.

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References

[B-M] J.-V. Beltran, J. Monterde, *The Poisson-Nijenhuis structures and the Vinogradov bracket*, Ann. Global Anal Geom., **12** (1994) no. 1, 65-78.

[C-M-P] P Casati, F. Magri, M. Pedroni, *Bihamiltonian manifolds and Sato's equations*, Integrable Systems, The Verdier Memorial Conference. Actes du Colloque International de Luminy, Progress in Math., **115**, 251-272.

[D-L-M] P. Dazord, A. Lichnerowicz, C-M. Marle, *Structures locales des variétés de Jacobi*, J. Math. pures et appl., **70** (1991) 101-152.

[G-D] I. M. Gel'fand, I. Ya Dorfman, *Hamiltonian operators and the classical Yang-Baxter equation*, Funct. Anal. Appl., **16** (1983), 241-248.

[G-L] F. Guédira, A. Lichnerowicz, *Géométrie des algèbres de Lie locales de Kirillov*, J. Math. pures et appl., **63** (1984), 407-484.

[Ke-SB] Y. Kerbrat, Z. Souici-Benhammadi, *Variétes de Jacobi et groupoïdes de contact*, C. R. Acad. Sci Paris, **317** Série 1 (1993), 81-86.

[K-M] Y. Kosmann-Schwarzbach, F. Magri, *Poisson-Nijenhuis structures*, Ann. Inst. Henry Poincaré, **53** (1990), 35-81.

[K] Y. Kosmann-Schwarzbach, *The Lie bialgebroid of a Poisson-Nijenhuis manifold*, Lett. in Math. Phys, **38** (1996), 421-428.

[Kz] J. L. Koszul, *Crochet de Schouten-Nijenhuis et cohomologie*, Astérisque numéro hors série " Elie Cartan et mathématiques d'aujourd'hui" (1985), 257-271.

[L] A. Lichnerowicz, *Les variétés de Jacobi et leurs algèbres de Lie associés*, J. Math. pures et appl., **57** (1978) 453-488.

[M-M] F. Magri, C. Morosi, *A geometrical characterization of integrable hamiltonian through the theory of Poisson-Nijenhuis manifolds*, Quaderno S **19**, 1984, University of Milan.

[M-M-P] J. C. Marrero, J. Monterde and E. Padron, *Jacobi-Nijenhuis manifolds and compatible Jacobi structures*, to appear in C. R. Acad. Sci Paris.

[N] J.M. Nunes da Costa, *Compatible Jacobi manifolds: geometry, reduction*, J. Phys. A: Math Gen, **31** (1998), 1025-1033.

[St] P. Stefan, *Acessibility and foliations with singularities*, Bull. Amer. Math. Soc., vol **80** (1974), 1142-1145.

[Su] H.J. Sussmann, *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc., vol **180** (1973), 171-188.

[V] I. Vaisman, *A Lecture on Poisson-Nijenhuis Structures*, Integrable Systems, and Foliations, Progress in Math., **115**, (C. Albert, R. Brouzet and J.-P. Dufour, eds), 169-184.