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Hidden Virasoro Symmetry of (Soliton Solutions of) the Sine Gordon Theory

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Abstract

We present a construction of a Virasoro symmetry of the sine-Gordon (SG) theory. It is a dynamical one and has nothing to do with the space-time Virasoro symmetry of 2D CFT. Although it is clear how it can be realized directly in the SG field theory, we are rather concerned here with the corresponding N-soliton solutions. We present explicit expressions for their infinitesimal transformations and show that they are local in this case. Some preliminary stages about the quantization of the classical results presented in this paper are also given.

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1 Introduction

The 2D sine-Gordon model, defined by the action:

$$S = \frac{\pi}{\gamma}\int \mathcal{L} d^2x \quad , \quad \mathcal{L} = (\partial_\nu \phi)^2 - m^2(cos(2\phi) - 1), \tag{1.1}$$

where γ is the coupling constant and m is related to the mass scale, is one of the simplest Integrable Quantum Field Theories. It possesses an infinite number of conserved charges I_{2n+1}, $n \in \mathbb{Z}$, in involution. At present, much is known about the corresponding scattering theory. It contains soliton, antisoliton and a number of bound states called "breathers". The mass spectrum and the S-matrix are known for about 20 years [1]. Despite this on-shell information, the off-shell Quantum Field Theory is much less developed. In particular, the computation of the corresponding correlation functions is still an important open problem. Actually, some progress towards this direction has been made recently. For instance, the exact Form-Factors (FF's) of the exponential fields $< 0|\exp\phi(0)|\beta_1, ..., \beta_n >$ were computed [2]. This allows one to make predictions about the long-distance behaviour of the corresponding correlation functions. On the other hand, some efforts have been made to estimate the short distance behaviour of the theory in the context of the so-called Conformal Perturbation Theory (CPT) [3]. Also, the exact expression for the Vacuum Expectation Values (VEV's) of the exponential fields (and some descendents) were proposed in [4]. The VEV's provide a highly non-trivial non-perturbative information about the short distance expansion of the two-point correlation functions. The FF's and CPT approaches permit to make predictions about the approximate behaviour of the correlation functions of the sine-Gordon theory in the infrared and ultraviolet regions correspondingly. What remains still unclear however is the explicit form of the correlation functions, in particular their intermediate behaviour and analytic properties, a question of primary importance from the field-theoretical point of view. Some exact results exist only at the so-called free-fermion point $\gamma = \frac{\pi}{2}$ [5].

There exists another approach to the sine-Gordon theory. It consists in searchig for additional infinite-dimentional symmetries and is inspired by the success the latter had in the 2D CFT. In fact, it has been shown in [6] that the sine-Gordon theory possesses an infinite dimensional symmetry provided by the $\widehat{sl}(2)_q$ algebra. However, this symmetry connects the correlation functions of the fields in the same multiplet without giving a sufficient information about the functions themselves.

It is known to some extent that there should be another kind of symmetry present in the sine-Gordon theory. Actually, it is known that it can be obtained as a particular scaling limit of the so-called XYZ – spin chain [7]. The latter is known to possess an infinite symmetry obeying the so-called Deformed Virasoro Algebra (DVA)[8]. It is natural to suppose that in the scaling limit, represented by SG, there should be present some infinite dimensional symmtery, a particular limit of DVA. At present, a lot is known about the mathematical structure of DVA, in particular the highest weight representations and the screening charges have been constructed [9]. What remains unclear is how the corresponding symmetry is realized in the sine-Gordon field theory, for example what is the action of the corresponding generators on the exponential fields, what kind of restrictions it imposes on their correlation functions etc.

In this paper we present a construction of a Virasoro symmetry directly in the sine-Gordon theory. Although we are of course interested in the quantum theory, we restrict ourself to the classical picture in this paper. Also, though it will be clear how to implement it in the general field theory, we are mainly concerned here with the construction of this symmetry in the case of the N-soliton solutions. One of the reasons for this is that the symmetry in this case is much simpler realized (in particular it becomes *local* contrary to the field theory realization). We were also inspired by the work of Babelon,Bernard and Smirnov [10]. It was shown there that certain form-factors can be directly reconstructed by a suitable quantization of the N-soliton solutions. It was also shown in [11] that certain null-vector constraints arise in sine-Gordon theory leading to integral equations for the corresponding form-factors. It is an intriguing question of what is the symmetry structure lying behind it and in particular its relation to the Virasoro symmetry we present here.

This paper is organised as follows. In the next Section we recall the construction of the so-called additional non-isospectral Virasoro symmetry of KdV theory. It is done in the context of the so-called algebraic approach and is a generalization of the well known dressing symmetries of integrable models [12]. In Section 3 we explain how one can restrict this symmetry to the case of N-soliton solutions of KdV. It happens that it becomes *local* in this case, contrary to the field theory realization where it is *quasi-local.* In Section 4 we extend the Virasoro symmetry to the sine-Gordon soliton solutions. This is acheeved by introducing an additional *time* dynamics in the KdV theory. In such a way we obtain "negative" Virasoro flows which complete the "positive" ones of KdV to the whole Virasoro algebra in the sine-Gordon theory. Finally, in Section 5 we summarise the results obtained in the paper. We give some hints about the quantization of the classical picture presented here and discuss some important open problems.

2 Virasoro Symmetry of m-KdV theory

2.1 Dressing Symmetries

Let us recall the construction of the Virasoro symmetry in the context of (m)KdV theory[13, 14]. It was shown in [12], following the so called algebraic approach, that it appears as a generalization of the ordinary dressing transformations of integrable models. Here we briefly recall the main results of this article. Being integrable, the mKdV system admits a zero-curvature representation:

$$[\partial_t - A_t, \partial_x - A_x] = 0, \tag{2.1}$$

where the Lax connections A_x , A_t belong to $A_1^{(1)}$ loop algebra:

$$\begin{aligned} A_x &= -vh + (e_0 + e_1), \\ A_t &= \lambda^2(e_0 + e_1 - vh) - \frac{1}{2}[(v^2 - v')e_0 + (v^2 + v')e_1] - \frac{1}{2}(\frac{v''}{2} - v^3)h \end{aligned} \tag{2.2}$$

(v is connected to the mKdV field ϕ: $v = -\phi'$) and

$$e_0 = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} = \lambda E \quad , \quad e_1 = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} = \lambda F \quad , \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H \tag{2.3}$$

are the corresponding generators in the fundamental representation. The usual KdV variable u is connected to the mKdV field ϕ by the Miura transformation:

$$u = \frac{1}{2}(\phi')^2 + \frac{1}{2}\phi'' \tag{2.4}$$

(we denote by prime the derivative with respect to the *space variable* x of KdV). Of great importance in our construction is the solution $T(x,\lambda)$ to the so called associated linear problem:

$$\mathcal{L}T(x,\lambda) \equiv (\partial_x - A_x(x,\lambda))T(x,\lambda) = 0 \tag{2.5}$$

which is usually referred to (with suitable normalization) as a transfer matrix. A formal solution to (2.5) can be easily found:

$$\begin{aligned} T_{reg}(x,\lambda) = e^{H\phi(x)}\mathcal{P}\exp\left(\lambda\int_0^x dy(e^{-2\phi(y)}E + e^{2\phi(y)}F)\right) = \\ = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \end{aligned} \tag{2.6}$$

It is obvious that this solution defines $T(x,\lambda)$ as an infinite series in positive powers of λ with an infinite radius of convergence (we shall often refer to (2.6) as *regular expansion*). For further reference we present also the expansion of the corresponding matrix elements:

$$\begin{aligned} A = e^{\phi}(1 + \sum_1^{\infty}\lambda^{2n}A_{2n}) \quad , \quad B = e^{\phi}\sum_0^{\infty}\lambda^{2n+1}B_{2n+1}, \\ C[\phi] = B[-\phi] \quad , \quad D[\phi] = A[-\phi]. \end{aligned} \tag{2.7}$$

It is also clear from (2.6) that $T(x,\lambda)$ possesses an essential singularity at infinity where it is governed by the corresponding asymptotic expansion.

Obviously, the zero-curvature form (2.1) is invariant under the gauge transformation:

$$\delta_n A_x(x,\lambda) = [\theta_n(x,\lambda), \mathcal{L}] \tag{2.8}$$

for A_x, and a similar one for A_t. A suitable choice for the gauge parameter θ_n goes through the construction of the following object:

$$Z^X(x,\lambda) = T(x,\lambda)XT(x,\lambda)^{-1} \quad , \quad X = E, F, H \tag{2.9}$$

essentially the dressed generators of the underlying $A_1^{(1)}$ algebra. It is obvious by construction that it satisfies:

$$[\mathcal{L}, Z^X(x,\lambda)] = 0, \tag{2.10}$$

i.e. it is a resolvent of the Lax operator L. As we shall see, this property is important for the construction of a consistent gauge parameter. In fact, let us insert T_{reg} as defined in (2.6) in (2.9) and then construct

$$\theta^X_{-n}(x,\lambda) \equiv (\lambda^{-n} Z^X(x,\lambda))_- = \sum_{k=0}^{n-1} \lambda^{k-n} Z^X_k(x), \tag{2.11}$$

where the subscript $-$ $(+)$ means that we restrict the series only to negative (non-negative) powers of λ. One can show that due to (2.10) the r.h.s. of (2.8) is of degree zero in λ and therefore θ^X_{-n} so constructed is a good candidate for a consistent gauge parameter. There is one more consistency condition we have to impose due to the explicit form of A_x (2.2), namely δA_x should be diagonal:

$$\delta^X_{-n} A_x = H \delta^X_{-n} \phi'. \tag{2.12}$$

This implies restrictions on the indices of the transformations: it happens that one must take even ones for $X = H$ (θ^H_{-2n}) and odd ones for $X = E$ or F ($\theta^{E,F}_{-2n-1}$). The first transformations read explicitly:

$$\begin{aligned} \delta^E_{-1}\phi'(x) &= e^{2\phi(x)} \\ \delta^F_{-1}\phi'(x) &= -e^{-2\phi(x)} \\ \delta^H_{-2}\phi'(x) &= e^{2\phi(x)} \int_0^x dy e^{-2\phi(y)} + e^{-2\phi(x)} \int_0^x dy e^{2\phi(y)}. \end{aligned} \tag{2.13}$$

Note that they are essentially non-local (this is true also for the higher ones). The algebra these transformations close is the (twisted) Borel subalgebra of $A_1^{(1)}$. Therefore the remaining ones can be found from(2.13) by commutation.

At this point we want to make an important observation. Consider the KdV variable x as a *space direction* x_- of some more general system (and $\partial_- \equiv \partial_x$ as a space derivative). Introduce the *time* variable $x_+$and the corresponding evolution defining:

$$\partial_+ \equiv (\delta^E_{-1} + \delta^F_{-1}). \tag{2.14}$$

It is then obvious from (2.13) that the equation of motion for ϕ becomes:

$$\partial_+\partial_-\phi = 2\sinh(2\phi) \quad , \quad (or \quad 2sin(2\phi) \quad if \quad \phi \to i\phi) \tag{2.15}$$

i.e. the sine-Gordon equation! We consider this observation very important since it provides a *global* introduction of sine-Gordon dynamics in the KdV system - a fact which was not known before.

The construction of the gauge parameter in the asymptotic case goes along the same line as above [15]. Explicitly, the asymptotic expansion of the transfer matrix is given by:

$$T(x,\lambda)_{asy} = KG(x,\lambda)e^{-\int_0^x dyD(y)}, \tag{2.16}$$

where $K = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and

$$D(x,\lambda) = d(x,\lambda)H, \qquad d(x,\lambda) = \sum_{k=-1}^{\infty} \lambda^{-k} d_k(x) \tag{2.17}$$

(d_{2k+1} are the conserved densities). G is given by:

$$G(x,\lambda) = \mathbf{1} + \sum_{j=1}^{\infty} \lambda^{-j} G_j(x), \tag{2.18}$$

where $G_j(x)$ are off-diagonal matrices with entries $(G_j(x))_{12} = g_j(x)$ and $(G_j(x))_{21} = (-1)^{j+1} g_j(x)$ (see [15]).

The resolvent (2.9) where now $T = T_{asy}$ and $X = H$ is an infinite series in negative powers of λ:

$$Z(x,\lambda) = \sum_{k=0}^{\infty} \lambda^{-k} Z_k(x) \tag{2.19}$$

and obviously satisfies (2.10) by construction. The coefficients in (2.19) are given by:

$$Z_{2k}(x) = b_{2k}(x)E + c_{2k}(x)F \quad , \quad Z_{2k+1}(x) = a_{2k+1}(x)H. \tag{2.20}$$

A suitable gauge parameter in this case is constructed as:

$$\theta_n(x,\lambda) = (\lambda^n Z(x;\lambda))_+ = \sum_{j=0}^{n} \lambda^{n-j} Z_j(x) \tag{2.21}$$

and the additional consistency condition (2.12) implies that now the indices should be odd (θ_{2n+1}). It happens that these transformations coincide exactly with the commuting higher mKdV flows (or mKdV hierarchy):

$$\delta_{2k+1}\phi'(x) = \partial a_{2k+1}(x) \tag{2.22}$$

and are therefore local in contrast with the regular ones. It turns out that the other entries of the resolvent $b_{2n}(x)$ are exactly the conserved densities [14], namely:

$$\delta_{2k+1}\phi'(x) = \{I_{2k+1}, \phi'(x)\} \quad , \quad I_{2k-1} = \int_0^L dx b_{2k}(x). \tag{2.23}$$

They differ from d_{2k+1} (2.17) by a total derivative. For example:

$$\begin{aligned} b_2 &= -d_1 + \tfrac{1}{2}\phi'', \\ b_4 &= \frac{3}{4}d_3 + \partial(\frac{7}{32}\phi''\phi' + \frac{1}{16}(\phi')^3 + \frac{1}{16}\phi''') \quad etc. \end{aligned} \tag{2.24}$$

Finally, let us note that it can be shown that these two kind of symmetries (regular and asymptotic) commute with each other. In this sence the non-local regular transformations provide a true symmetry of the KdV hierarchy.

2.2 Generalization - Virasoro Symmetry

Now, let us explain how the Virasoro symmetry appears in the KdV system [12]. The main idea is that one can dress not only the generators of the underlying $A_1^{(1)}$ algebra but also an arbitrary differential operator in the spectral parameter. We take for example $\lambda^{m+1}\partial_\lambda$ which, as it is well known, are the vector fields of the diffeomorphisms of the unit circumference and close a Virasoro algebra. Then we proceed in the same way as above.

The analog of our basic object (2.9) now is:

$$Z^V(x,\lambda) = T(x,\lambda)\partial_\lambda T(x,\lambda)^{-1}. \tag{2.25}$$

It is clear that Z^V has again the property of being a resolvent for the Lax operator, i.e. it satisfies (2.10), which was one of the requirements for constructing a good gauge parameter. Let us consider first the regular case, i.e. take $T = T_{reg}$ in (2.25):

$$Z^V_{reg}(x,\lambda) = \sum_{n=0}^{\infty} \lambda^n Z_n(x) + \partial_\lambda. \tag{2.26}$$

It is clear that $Z^V(x,\lambda)$ is a differential operator in λ in this case. Following the same reasoning as before we construct the gauge parameter as:

$$\theta^V_{-m}(x,\lambda) = (\lambda^{-m} Z^V_{reg}(x,\lambda))_- \quad , \quad m > 0. \tag{2.27}$$

Then the additional condition (2.12) imposes that the indices of the transformation should be even $m = 2n$. The first nontrivial examples are given by:

$$\begin{aligned}
\delta^V_{-2}\phi' &= e^{2\phi(x)}\int_0^x dy e^{-2\phi(y)} - e^{-2\phi(x)}\int_0^x dy e^{2\phi(y)} = e^{2\phi(x)}B_1 - e^{-2\phi(x)}C_1 \\
\delta^V_{-4}\phi' &= e^{2\phi(x)}(3B_3(x) - A_2(x)B_1(x)) - e^{-2\phi(x)}(3C_3(x) - D_2(x)C_1(x)) \\
\delta^V_{-6}\phi' &= e^{2\phi(x)}(5B_5(x) - 3A_4(x)B_1(x) + A_2(x)B_3(x)) \\
&- e^{-2\phi(x)}(5C_5(x) - 3D_4(x)C_1(x) + D_2(x)C_3(x)),
\end{aligned} \tag{2.28}$$

where A_i, B_i, C_i, D_i were defined in (2.7). They have a form very similar to that of δ^H_{-2n} but nevertheless it can be shown [14] that they indeed close a (half) Virasoro algebra. Note that these transformations are essentially non-local, so we obtained a very nontrivial realisation of the Virasoro algebra in terms of vertex operators.

Let us now consider the asymptotic case, i.e. take $T = T_{asy}$ in (2.25):

$$Z^V_{asy}(x,\lambda) = \sum_{n=0}^{\infty} \lambda^{-n} Z_n(x) + \partial_\lambda. \tag{2.29}$$

The coefficients of the above expansion have the general form:

$$Z_{2n} = \beta_{2n}E + \gamma_{2n}F, \quad Z_{2n+1} = \alpha_{2n+1}H, \tag{2.30}$$

where for example $\beta_0 = x = \gamma_0$, $\alpha_1 = 2xg_1$, $\beta_2 = -xb_2 - g_1 + \int_0^x d_1$, $\gamma_2 = -xc_2 + g_1 + \int_0^x d_1$ etc. . Again, the suitable gauge parameter is defined by:

$$\theta_m^V(x,\lambda) = (\lambda^m Z_{asy}^V)_+ = \sum_{n=0}^{m+1} \lambda^{m+1-n} Z_n + \partial_\lambda \quad , \quad m \geq 0 \tag{2.31}$$

and the self-consistency condition implies that the indices should be even in this case too. Actually, the first transformation:

$$\delta_0^V \phi'(x) = (x\partial + 1)\phi'(x) \tag{2.32}$$

is exactly the scale transformation - it counts the dimension (or level). The first non-trivial examples are:

$$\begin{aligned} \delta_2^V \phi' &= 2xa_3' - (\phi')^3 + \frac{3}{4}\phi''' + 2a_1' \int_0^x d_1, \\ \delta_4^V \phi' &= 2xa_5' + (\phi')^5 - \frac{5}{2}\phi'''(\phi')^2 - \frac{27}{8}(\phi'')^2\phi' + \frac{5}{16}\phi^V + \\ &+ 2a_3' \int_0^x d_1 + 6a_1' \int_0^x d_3. \end{aligned} \tag{2.33}$$

We note that these depend explicitly on x and are quasi-local (they contain some indefinite integrals). For further reference we presented the integrands in (2.33) explicitly in terms of the entries of the basic objects $T(x,\lambda)$ and $Z(x,\lambda)$, defined in (2.17,2.20). Furthermore, one can find the transformation of the resolvent and therefore the transformation of the conserved densities $\delta_{2k} b_{2n}(x)$. In particular the first nontrivial transformations of the KdV variable $u = b_2$ read:

$$\begin{aligned} \delta_2^V b_2 &= \delta_2^V u = 2xb_4' + u'' - 2u^2 - \frac{1}{2}u' \int_0^x u, \\ \delta_4^V b_2 &= \delta_4^V u = 2xb_6' + 2u^3 + 3uu'' + \frac{17}{8}(u')^2 + \frac{3}{8}u^{IV} + \\ &+ u' \int_0^x b_4 + b_4' \int_0^x u. \end{aligned} \tag{2.34}$$

It can be easily shown that the asymptotic transformations also close (half) Virasoro algebra. A very non-trivial question concerns the commutation relations between the "negative" and "positive" parts so constructed in view of their completely different nature. Nevertheless it can be shown [14] that, contrary to what happened between the proper dressing transformations and the m-KdV hierarchy, in this case they close a whole Virasoro algebra:

$$[\delta_{2m}^V, \delta_{2n}^V] = (2m - 2n)\delta_{2m+2n}^V \quad , \quad m, n \in \mathbb{Z}. \tag{2.35}$$

We want to stress that the Virasoro symmetry just constructed is a dynamical one and has nothing to do with the space-time Virasoro symmetry of CFT. Actually, it is well known that one can consider the KdV system as a classical limit of the latter. So we expect that after quantization this dynamical symmetry should be present in CFT. It is interesting to investigate its significance, in particular if CFT could be solved by using this symmetry alone.

3 Soliton Solutions of (m)KdV theory

3.1 (m)KdV Solitons

We would like now to restrict the Virasoro symmetry to the soliton solutions of the (m)KdV theory. One can expect that in this case it simplifies considerably. There is also another reason for this restriction. It was shown in [10] that one can reconstruct certain form-factors of sine-Gordon theory by directly quantizing the soliton solutions. Moreover, it happens that a kind of null-vectors appear in the theory [11], leading to integral equations for the form-factors. It is intriguing to understand the rôle that the Virasoro symmetry just described is playing in all these constructions.

We start with a brief description of the well known soliton solutions of (m)KdV. They are best expressed in terms of the so-called *tau-function*. In the case of N-soliton solution of (m)KdV it has the form:

$$\tau(X_1, ..., X_N|B_1, ..., B_N) = \det(1 + V) \tag{3.1}$$

where V is a matrix:

$$V_{ij} = 2\frac{B_i X_i(x)}{B_i + B_j} \quad , \quad i, j = 1, ..., N. \tag{3.2}$$

The m-KdV field is then expressed as:

$$e^{\phi} = \frac{\tau_-}{\tau_+}, \tag{3.3}$$

where:

$$\tau_{\pm}(x) = \tau(\pm X(x)|B) \tag{3.4}$$

and $X_i(x)$ is simply given by:

$$X_i(x) = X_i \exp(2B_i x). \tag{3.5}$$

The variables B_i and X_i are the parameters describing the solitons: $\beta_i = \log B_i$ are the so-called rapidities and X_i are related to the positions. The integrals of motion, restricted to the N-soliton solutions have the form

$$I_{2n+1} = \sum_{i=1}^{N} B_i^{2n+1} \quad , \quad n \geq 0. \tag{3.6}$$

It is well known that (m)KdV admits a non-degenerate symplectic structure. One can find the corresponding Poisson brackets between the basic variables B_i and X_i [16]. The (m)KdV flows are then generated by (3.6) via

$$\delta_{2n+1} * = \{\sum_{i=1}^{N} B^{2n+1}, *\} \quad , \quad n \geq 0. \tag{3.7}$$

3.2 Analytical Variables

Our final goal is the quantization of solitons and of the Virasoro symmetry. It was argued in [10] that this is best performed in another set of variables $\{A_i, B_i\}$. The latter are the soliton limit of certain variables describing the more general quasi-periodic finite-zone solutions of (m)KdV. In that context B_i are the branch points (i.e. define the complex structure) of the hyperelliptic Riemann surface describing the solution and A_i are the zeroes of the so-called Baker-Akhiezer function defined on it. In view of the nice geometrical meaning of these variables they were called analytical variables in [10].

Explicitly, the change of variables is given by:

$$X_j \prod_{k\neq j} \frac{B_j - B_k}{B_j + B_k} = \prod_{k=1}^{N} \frac{B_j - A_k}{B_j + A_k} \quad , \quad j = 1, ..., N. \tag{3.8}$$

The non-vanishing Poisson brackets expressed in terms of these new variables take the form:

$$\{A_i, B_j\} = \frac{\prod_{k\neq i}(B_j^2 - A_k^2) \prod_{k\neq j}(A_i^2 - B_k^2)}{\prod_{k\neq i}(A_i^2 - A_k^2) \prod_{k\neq j}(B_j^2 - B_k^2)} (A_i^2 - B_j^2). \tag{3.9}$$

The corresponding tau-functions have also a very compact form in terms of the analytical variables:

$$\begin{aligned} \tau_+ &= 2^N \prod_{j=1}^{N} B_j \{ \frac{\prod_{i<j}(A_i + A_j) \prod_{i<j}(B_i + B_j)}{\prod_{i,j}(B_i + A_j)} \} \\ \tau_- &= 2^N \prod_{j=1}^{N} A_j \{ \frac{\prod_{i<j}(A_i + A_j) \prod_{i<j}(B_i + B_j)}{\prod_{i,j}(B_i + A_j)} \}. \end{aligned} \tag{3.10}$$

Therefore, from the explicit form of the m-KdV field in terms of the tau-functions (3.3) we obtain the following very simple expression:

$$e^{\phi} \equiv \frac{\tau_-}{\tau_+} = \prod_{j=1}^{N} \frac{A_j}{B_j}. \tag{3.11}$$

The equation of motion of the A_i variable is given by:

$$\partial_x A_i \equiv \delta_1 A_i = \{I_1, A_i\} = \prod_{j=1}^{N} (A_i^2 - B_j^2) \prod_{j\neq i} \frac{1}{(A_i^2 - A_j^2)}. \tag{3.12}$$

One can verify that, as a consequence, the usual KdV variable u is expressed as:

$$b_2 \equiv u = \tfrac{1}{2}(\phi')^2 + \tfrac{1}{2}\phi'' = \sum_{j=1}^{N} A_j^2 - \sum_{j=1}^{N} B_j^2. \tag{3.13}$$

One can restrict also the higher KdV flows to the soliton solutions. For example it is clear from (3.7) that

$$\delta_{2n+1} B_i = 0 \quad , \quad n \geq 0. \tag{3.14}$$

This is a reminiscence of the fact that the KdV flows do not change the complex structure of the hyperelliptic surface describing the finite-zone solution [17]. The variation of the A_i variables can be easily computed as :

$$\delta_{2n+1}A_i = \{I_{2n+1}, A_i\} \quad , \quad n \geq 0 \tag{3.15}$$

using the Poisson brackets (3.9).

3.3 Virasoro Symmetry of the Soliton Solutions

Now, we want to restrict the Virasoro symmetry of (m)KdV constructed above to the case of soliton solutions. In this section we shall be only interested in the positive part of the latter. The transformation of the rapidities can be easily deduced as a soliton limit of the Virasoro action on the finite-zone solutions described in [17]:

$$\delta_{2n}B_i = B_i^{2n+1} \quad , \quad n \geq 0, \tag{3.16}$$

i.e. the Virasoro action changes the complex structure (because of that it's often called non-isospectral symmetry). What remains is to obtain the transformations of the A_i variables. We found it quite difficult to deduce them as a soliton limit of the corresponding transformations of [17]. Instead, we propose here another approach. Namely, we use the transformation of the fields $\delta_{2n}\phi$, $\delta_{2n}\phi'$, $\delta_{2n}u$ *etc.* which we found before, restricted to the soliton solutions using (3.11,3.13). The problem is simplified by the fact that the Virasoro algebra is freely generated , i.e. we need to compute only the δ_0, δ_2 and δ_4 transformations, the remaining ones are then obtained by commutation. In practice, we perform the computation for the first few cases of $N = 1, 2, 3$ solitons and then proceed by induction.

Let us make an important observation. As we have stressed, the transformation of the basic objects in the field theory of (m)KdV are quasi-local – they contain certain indefinite integrals. It happens that the corresponding integrands become total derivatives when restricted to the soliton solutions. For example:

$$\begin{aligned}
b_2 &\equiv u = \partial_x \sum_{i=1}^{N} A_i(x), \\
b_4 &= \partial_x \sum_{i=1}^{N} A_i^3 - \tfrac{1}{2}u' \equiv \partial_x[\sum_{i=1}^{N}(A_i^3 - \tfrac{1}{2}\partial_x A_i)].
\end{aligned} \tag{3.17}$$

Therefore the Virasoro transformations become *local* when restricted to the soliton solutions! The calculation is straightforward but quite tedious so we present here only the final result:

$$\begin{aligned}
\delta_0 A_i &= (x\partial_x + 1)A_i, \\
\delta_2 A_i &= \frac{1}{3}x\delta_3 A_i + A_i^3 - (\sum_{j=1}^{N} A_j)\partial_x A_i,
\end{aligned}$$

$$\delta_4 A_i = \frac{1}{5} x \delta_5 A_i + A_i^5 - \{\sum_{j \neq i} A_i (A_i^2 - A_j^2) + \sum_{j=1}^{N} A_j \sum_{k=1}^{N} B_k^2\} \partial_x A_i, \qquad (3.18)$$

where the KdV flows read explicitly:

$$\frac{1}{3}\delta_3 A_i = (\sum_{j=1}^{N} B_j^2 - \sum_{k \neq i} A_k^2) \partial_x A_i,$$
$$\frac{1}{5}\delta_5 A_i = (\sum_{j=1}^{N} B_j^4 - \sum_{k \neq i} A_k^4) \partial_x A_i - \sum_{j \neq i} (A_i^2 - A_j^2) \partial_x A_i \partial_x A_j. \qquad (3.19)$$

As we already mentioned, the remaining transformations can be obtained by commutation, for example:

$$2\delta_6 A_i = [\delta_4, \delta_2] A_i \quad , \quad etc. \qquad (3.20)$$

4 Virasoro Symmetry of Sine-Gordon Solitons

4.1 From (m)KdV to Sine-Gordon Solitons

Now we pass to the most important part of our paper. We would like to extend the construction presented above in (m)KdV theory to the case of sine-Gordon. For this purpose one has to find a way of extending the mKdV dynamics up to the sine-Gordon one. It is to some extent known how this can be done in the case of the soliton solutions [10]. The idea is close to what we proposed before directly in the field theory of (m)KdV. Namely, let us consider the KdV variable x as a *space* variable of some more general system and call it x_- (and $\partial_- \equiv \partial_x$ correspondingly). We would like to introduce a new *time* variable x_+ and the corresponding time dynamics. In the case of the N - soliton solutions the latter is generated by the Hamiltonian:

$$I_{-1} = \sum_{i=1}^{N} B_i^{-1} \qquad (4.1)$$

(essentially the inverse power of the momentum) so that the *time* flow is given by:

$$\partial_+ * = \delta_{-1} * = \{I_{-1}, *\} \qquad (4.2)$$

using again the Poisson brackets (3.9). In particular:

$$\partial_+ A_i = \prod_{j=1}^{N} \frac{A_i^2 - B_j^2)}{B_j^2} \prod_{j \neq i} \frac{A_j^2}{(A_i^2 - A_j^2)}. \qquad (4.3)$$

One can check, using (3.11, 3.12), that with this definition the resulting equation for the field ϕ is

$$\partial_+ \partial_- \phi = 2 \sinh(2\phi) \qquad (4.4)$$

or under the change $\phi \to i\phi$:

$$\partial_+\partial_-\phi = 2\sin(2\phi) \tag{4.5}$$

i.e. the sine-Gordon equation. We were not able to establish at the moment a direct relation between the way of extending the mKdV dynamics to the sine-Gordon one directly in the field theory (2.14) and in the case of N-soliton solution just described (4.2). We hope to answer this important question elsewhere [18]. In a similar manner one can introduce the rest of the sine-Gordon Hamiltonians:

$$I_{-2n-1} = \sum_{i=1}^{N} B_i^{-2n-1} \quad , \quad n \geq 0. \tag{4.6}$$

They generate the "negative KdV flows" via the Poisson brackets (3.9):

$$\begin{aligned} \delta_{-2n-1}B_i &= 0, \\ \delta_{-2n-1}A_i &= \{I_{-2n-1}, A_1\} \quad , \quad n \geq 0. \end{aligned} \tag{4.7}$$

4.2 Negative Virasoro flows

Now we arrive at the **main conjecture** of this paper. Having in mind the symmetric role the derivatives ∂_- and ∂_+ are playing in the sine-Gordon equation *we would like to suppose that one can obtain another half Virasoro algebra by using the same construction as above but with ∂_- interchanged with ∂_+!*

So let us define as before:

$$\delta_{-2n}B_i = -B_i^{-2n+1} \quad , \quad n \geq 0 \tag{4.8}$$

(note the additional − sign in the r.h.s. which is needed for the self-consistency of the construction). Following our conjecture we construct the negative flows of the A_i variable in the same way as before but with the change $\partial_- \to \partial_+$. We have for example:

$$\begin{aligned} \delta_{-2}\phi &= x_+(2a_3^+) + b_2^+ - 2a_1^+\int_0^{x_+} b_2^+, \\ \delta_{-2}b_2^- &\equiv \delta_{-2}u = \frac{1}{3}x_+\delta_{-3}u + (\partial_+\phi - \int_0^{x_+} b_2^+)\partial_- e^{2\phi}, \end{aligned} \tag{4.9}$$

where $\delta_{-3}u \equiv \{I_{-3}, u\}$ *etc.* In (4.9) the + subscript means that we take the same objects as defined in (2.17, 2.18, 2.20) but with ∂_- changed by ∂_+. For example:

$$\begin{aligned} b_2^+ &= \tfrac{1}{2}(\partial_+\phi)^2 + \tfrac{1}{2}\partial_+^2\phi, \\ a_3^+ &= -\frac{1}{4}(\partial_+\phi)^3 + \frac{1}{8}\partial_+^3\phi \quad etc. \end{aligned} \tag{4.10}$$

At this point we want to make an important remark. Very non-trivially, it happens again that the integrands in the expressions (4.9) and similar become total derivatives when restricted to the N-soliton solutions. So that again the (negative) Virasoro symmetry is *local* in the case of solitons! We present below the first examples of this phenomenon:

$$\begin{aligned} b_2^+ &= \partial_+\{\sum_{i,j=1}^N \frac{A_iA_j}{B_i^2B_j^2}\sum_{i=1}^N A_i - \partial_- \sum_{i,j=1}^N \frac{A_iA_j}{B_i^2B_j^2}\}, \\ b_4^+ &= \partial_+\{\sum_{i,j=1}^N \frac{A_iA_j}{B_i^4B_j^4}\sum_{i=1}^N A_i^3 + b_2^-\partial_- \sum_{i,j=1}^N \frac{A_iA_j}{B_i^4B_j^4} - \\ &- \partial_-b_2^- \sum_{i,j=1}^N \frac{A_iA_j}{B_i^4B_j^4}\} \quad etc. \end{aligned} \tag{4.11}$$

We then proceed as in the case of the positive Virasoro flows, i.e. we restrict the transformations of the fields thus obtained to the soliton solutions. As we explained, it is enough to find only the first transformations $\delta_{-2}A_i$ and $\delta_{-4}A_i$ and the remaining ones are found by commutation. Following our approach we do the computation explicitly in the case of $N = 1, 2, 3$ solitons and then proceed by induction. The exact calculation will be presented in a forthcoming paper [18], here we give the final results only:

$$\begin{aligned} \delta_{-2}A_i &= x_+\delta_{-3}A_i - A_i^{-1} - (\sum_{j=1}^N A_j^{-1})\partial_+A_i, \\ \delta_{-4}A_i &= x_+\delta_{-5}A_i - A_i^{-3} - \\ &- \{\sum_{j=1}^N \frac{1}{A_i}(\frac{1}{A_i^2} - \frac{1}{A_j^2}) + \sum_{j=1}^N \frac{1}{A_j}\sum_{k=1}^N \frac{1}{B_k^2}\}\partial_+A_i, \end{aligned} \tag{4.12}$$

where as before $\delta_{-3}A_i = \{\sum_{j=1}^N B_j^{-3}, A_i\}$ etc. As stated above, we then can compute $2\delta_{-6}A_i = [\delta_{-2}, \delta_{-4}]A_i$ etc.

4.3 The Algebra

Now, we come to the important problem of the commutation relations between the two half Virasoro algebras so constructed. This is a non-trivial question in view of the different way we obtained them. In fact, it is clear that, by construction, the positive (negative) Virasoro flows commute with the corresponding ∂_- (∂_+) derivatives:

$$\begin{aligned} [\delta_{2n}, \partial_-]A_i &= 0, \\ [\delta_{-2n}, \partial_+]A_i &= 0 \quad , \quad n \geq 0. \end{aligned} \tag{4.13}$$

It is easy to see that this is not true for the "cross commutators". Actually, one finds in this case:

$$\begin{aligned} [\delta_{2n}, \partial_+]A_i &= -\delta_{2n-1}A_i, \\ [\delta_{-2n}, \partial_-]A_i &= -\delta_{-2n+1}A_i \quad , \quad n \geq 0. \end{aligned} \tag{4.14}$$

It is clear that we are interested in a *true symmetry* of the sine-Gordon theory. We must therefore obtain transformations that commute with the ∂_- and ∂_+ flows and as a consequence with the corresponding Hamiltonians. It is obvious from (4.13,4.14) that this is acheeved by a simple modification of the flows, i.e. let us define:

$$\begin{aligned} \delta'_{2n} &= \delta_{2n} - x_+ \delta_{2n-1}, \\ \delta'_{-2n} &= \delta_{-2n} - x_- \delta_{-2n+1} \quad , \quad n \geq 0. \end{aligned} \tag{4.15}$$

Then, for the modified transformation we obtain:

$$[\delta'_{2n}, \partial_\pm] A_i = 0 \quad , \quad n \in \mathbb{Z}. \tag{4.16}$$

Finally, one can show that, with this modification, the commutation relations between the positive and negative parts of the transformations close exactly the whole Virasoro algebra:

$$[\delta'_{2n}, \delta'_{2m}] A_i = (2n - 2m) \delta'_{2n+2m} A_i \quad , \quad n, m \in \mathbb{Z}. \tag{4.17}$$

5 Conclusions and discusion

To summarise, we presented in this paper a construction of a Virasoro symmetry of the sine-Gordon theory. This is acheeved by extending the corresponding symmetry of the (m)KdV theory [12]. Actually, we found it easier to work rather with the N-soliton solutions of the latter. Then, following [10], we introduced a *time* coordinate x_+ and the corresponding flow ∂_+ in addtion to the *space* variable x_- of KdV which leads to the sine-Gordon equation. The **main idea** is to make the same kind of construction as for KdV but interchanging the ∂_- with the ∂_+ derivative. This change results in what we called "negative" Virasoro flows which complete the "positive" ones coming from the original KdV to the whole Virasoro algebra. We showed also that after a certain modification, needed to obtain a true symmetry of the theory, they close the whole Virasoro algebra.

Actually, in this paper we obtained the infinitesimal transformations of the variables describing the N-soliton solutions. It is intriguing to find the corresponding conserved charges $J_{2n}(A_i, B_i)$. This is important in view of the quantization of the classical constructions presented here. Besides, in their commutation relations one can obtain a possible existence of a central extension which cannot be discovered in the case of the infinitesimal transformations.

As already mentioned, we are interested of course in the quantum sine-Gordon theory. In the case of solitons there is a standard procedure, a kind of canonical quantization of the N-soliton solutions. In fact, let us introduce, following [10], the canonically conjugated variables to the *analytical variables* A_i:

$$P_j = \prod_{k=1}^{N} \frac{B_k - A_j}{B_k + A_j} \quad , \quad j = 1..., N \tag{5.1}$$

(in the variables $\{P_j, A_j\}$ the corresponding simplectic structure is diagonal). In these variables one can perform a kind of canonical quantization of the N-soliton system introducing the deformed commutation relations between the operators A_i and P_i :

$$\begin{aligned} P_j A_j &= q A_j P_j, \\ P_k A_j &= A_j P_k \quad for \quad k \neq j, \end{aligned} \tag{5.2}$$

where q is related to the sine-Gordon coupling constant: $\exp(i\xi)$, $\xi = \frac{\pi\gamma}{\pi-\gamma}$. It is very intriguing to understand how the Virasoro symmetry is deformed after the quantization!

Another important problem is the construction of the Virasoro symmetry directly in the sine-Gordon field theory. We explained above how this can be done using the proper dressing transformations. It happens that the positive Virasoro flows commute with the *time* sine-Gordon flow x_+ introduced in this way. We expect that the negative Virasoro symmetry can be constructed following the same approach we presented in this paper for the solitons.

Finally, it will be very interesting to understand the rôle this Virasoro symmetry is playing in the sine-Gordon theory. As we mentioned, it was shown in [10] that certain form-factors can be reconstructed by a suitable quantization of the N-soliton solutions. One can expect that the Virasoro symmetry imposes some constraints leading to certain equations for the form-factors (or correlation functions in the case of field theory). We would like to note in this respect the article [11] where a kind of null-vector constraints were derived in the sine-Gordon theory. The corresponding construction is closely related to the finite-zone solutions. As we mentioned the Virasoro symmetry we presented here has a natural action on such solutions. It is natural to expect that some relation exists between this symmetry and the null-vectors of [11]. We will return to all these problems in a forthcoming paper [18].

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