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Superconformal interpretation of BPS states in AdS geometries

Sergio Ferrara[†] and Emery Sokatchev[‡]

[†] CERN Theoretical Division, CH 1211 Geneva 23, Switzerland
and Department of Physics and Astronomy, University of California,
Los Angeles, CA 90095, USA

[‡] Laboratoire d'Annecy-le-Vieux de Physique Théorique[1] LAPTH, Chemin de Bellevue - BP 110 - F-74941 Annecy-le-Vieux Cedex, France

Abstract

We carry out a general analysis of the representations of the superconformal algebras $\mathrm{SU}(2,2/N)$, $\mathrm{OSp}(8/4,\mathbb{R})$ and $\mathrm{OSp}(8^*/4)$ and give their realization in superspace. We present a construction of their UIR's by multiplication of the different types of massless superfields ("supersingletons").
Particular attention is paid to the so-called "short multiplets". Representations undergoing shortening have "protected dimension" and correspond to BPS states in the dual supergravity theory in anti-de Sitter space.
These results are relevant for the classification of multitrace operators in boundary conformally invariant theories as well as for the classification of AdS black holes preserving different fractions of supersymmetry.

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1 Introduction

The study of superconformal algebras has recently become of central importance because of their dual rôle in describing the gauge symmetries of supergravity in anti-de Sitter bulk and the global symmetries of the boundary field theory [1, 2, 3].

A special class of configurations which are particularly relevant are the so-called BPS states, i.e. dynamical objects corresponding to representations which undergo "shortening".

These representations can only occur when the conformal dimension of a (super)primary operator is "quantized" in terms of the R symmetry quantum numbers and they are at the basis of the so-called "non-renormalization" theorems of supersymmetric quantum theories [4].

There exist different methods of classifying the UIR's of superconformal algebras. One is the so-called oscillator construction of the Hilbert space in which a given UIR acts[5]-[9]. Another one, more appropriate to describe field theories, is the realization of such representations on superfields defined in superspaces [10, 11]. The latter are "supermanifolds" which can be regarded as the quotient of the conformal supergroup by some of its subgroups.

In the case of ordinary superspace the subgroup in question is the supergroup obtained by exponentiating a non-semisimple superalgebra which is the semidirect product of a super-Poincaré graded Lie algebra with dilatation ($\mathrm{SO}(1,1)$) and the R symmetry algebra. This is the superspace appropriate for non-BPS states. Such states correspond to bulk massive states which can have "continuous spectrum" of the AdS mass (or, equivalently, of the conformal dimension of the primary fields).

BPS states are naturally associated to superspaces with lower number of "odd" coordinates and, in most cases, with some internal coordinates of a coset space G/H. Here G is the R symmetry group of the superconformal algebra, i.e. the subalgebra of the even part which commutes with the conformal algebra of space-time and H is some subgroup of G having the same rank as G.

Such superspaces are called "harmonic" [12] and they are characterized by having a subset of the initial odd coordinates θ. The complementary number of odd variables determines the fraction of supersymmetry preserved by the BPS state. If a BPS state preserves K supersymmetries then the θ's of the associated harmonic superspace will transform under some UIR of H_K.

For 1/2 BPS states, i.e. states with maximal supersymmetry, the superspace involves the minimal number of odd coordinates (half of the original one) and H_K is then a maximal subgroup of G. On the other hand, for states with the minimal fraction of supersymmetry H_K reduces to the "maximal torus" whose Lie algebra is the Cartan subalgebra of G.

It is the aim of the present paper to give a comprehensive treatment of BPS states related to "short representations" of superconformal algebras for the cases which are most relevant in the context of the AdS/CFT correspondence, i.e. the $d = 3$ ($N = 8$), $d = 6$ ($N = (2, 0)$) and $d = 4$ (for arbitrary N). The underlying conformal field theories correspond to world-volume theories of N_c copies of M_2, M_5 and D_3 branes in the large N_c limit [13]-[19] which are "dual" to AdS supergravities describing the horizon geometry of the branes [20].

Some of the results presented in this paper have already appeared elsewhere [21]-[24]. [1] Here we give a systematic and unified treatment of the BPS states corresponding to the three superconformal algebras above. The method we use is developed in full detail in the case of the $d = 4$ superconformal algebra $\mathrm{SU}(2, 2/N)$ in Sections 2-5. In Section 2 we carry out an abstract analysis of the conditions for Grassmann (G-)analyticity [25] (the generalization of the familiar concept of chirality [11]) in a superconformal context. We find the constraints on the conformal dimension and R symmetry quantum numbers of a superfield following from the requirement that it do not depend on one or more Grassmann variables. Introducing G-analyticity in a traditional superspace cannot be done without breaking the R symmetry. The latter can be restored by extending the superspace by harmonic variables [26, 12, 27, 28, 29] parametrizing the coset G/H_K. In Section 3 the (N, p, q) harmonic superspaces [29, 30] relevant to the description of BPS states preserving $p + q/2N$ supersymmetries are reviewed. In Section 4 the massless UIR's ("supersingleton" multiplets) [31, 32, 33] of $\mathrm{SU}(2, 2/N)$ are considered, first as constrained superfields in ordinary superspace [34, 35] and then, for a part of them, as $(N, p, N - p)$ G-analytic harmonic superfields [12, 30]. In Section 5 we use supersingleton multiplication to construct UIR's of $\mathrm{SU}(2, 2/N)$. We show that in this way one can reproduce the complete classification of UIR's of ref. [36]. We give the full list of BPS states obtained by multiplying chiral and G-analytic supersingletons as well as the restricted

[1] The new results were reported by one of us at the Workshop on "Strings, Branes and M-theory" at the CIT-USC Center for Theoretical Physics, Los Angeles, California on April 5 and 7, 2000.

classes of BPS states obtained from one type of G-analytic supersingleton alone. We also discuss different kinds of shortening which certain superfields (not of the BPS type) may undergo. In Sections 6 and 7 we apply the same method to extend these results to $d = 6$ and $d = 3$ for the superalgebras of the maximal supersymmetries, i.e. $\mathrm{OSp}(8^*/4)$ and $\mathrm{OSp}(8/4, \mathbb{R})$. We conclude the paper by listing the various BPS states in the physically relevant cases of D3, M_2 and M_5 branes horizon geometry where only one type of supersingletons appears.

Applications of the present results are found [37, 21] in the classification of multitrace operators in four-dimensional $N = 4$ $\mathrm{SU}(N_c)$ Yang-Mills theory [38]-[42], dual to type IIB supergravity on $AdS_5 \times S^5$ [1].

Another area of interest is the classification of AdS black holes [44]-[47], according to the fraction of supersymmetry preserved by the black hole background.

In a parallel analysis with black holes in asymptotically flat background [48], the AdS/CFT correspondence predicts that such BPS states should be dual to superconformal states undergoing "shortening" of the type discussed here.

2 Grassmann analyticity and conformal supersymmetry

In this section we shall study the realizations of $D = 4$ N-extended conformal supersymmetry $\mathrm{SU}(2, 2/N)$ on superfields depending on a subset of the $4N$ odd variables. Such superfields will be called Grassmann (G-)analytic.

The non-vanishing (anti)commutation relations involving the odd generators of the superalgebra $\mathrm{SU}(2, 2/N)$ are given below:

$$
\begin{aligned}
\{Q^i_\alpha, \bar{Q}_{\dot\alpha j}\} &= 2\delta^i_j(\sigma^\mu)_{\alpha\dot\alpha}P_\mu \ , \\
\{S_{\alpha j}, \bar{S}^i_{\dot\alpha}\} &= 2\delta^i_j\ (\sigma^\mu)_{\alpha\dot\alpha}K_\mu \ , \\
\{Q^i_\alpha, S^\beta_j\} &= -\delta^i_j(\sigma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu} - 4\delta^\beta_\alpha T^i_j - 2\delta^\beta_\alpha\delta^i_j(R + iD) \ , \\
[Q^i_\alpha, K_\mu] &= -(\sigma_\mu)_{\alpha\dot\alpha}\bar{S}^{\dot\alpha i}, \qquad [\bar{Q}_{\dot\alpha i}, K_\mu] = (\sigma_\mu)_{\alpha\dot\alpha}S^\alpha_i, \\
[S_{\alpha i}, P_\mu] &= -(\sigma_\mu)_{\alpha\dot\alpha}\bar{Q}^{\dot\alpha}_i, \qquad [\bar{S}^i_{\dot\alpha}, P_\mu] = (\sigma_\mu)_{\alpha\dot\alpha}Q^{\alpha i} \ ,
\end{aligned}
\tag{2.1}
$$

Here the odd generators are [2]: Q^i_α, $\bar{Q}_{\dot\alpha i} = (Q^i_\alpha)^\dagger$ of Poincaré supersymmetry

[2]Two-component spinor indices are raised and lowered with the help of the Levi-Civita

and $S_{\alpha i}$, $\bar{S}^i_{\dot\alpha} = (S_{\alpha i})^\dagger$ of special conformal supersymmetry. The even generators are: P_μ of translations, K_μ of conformal boosts, $M_{\mu\nu} = -M_{\nu\mu}$ of the Lorentz group, D of dilatations, T^i_j of SU(N) and R of $U(1)$ ("R charge").

Further, the Lorentz and SU(N) generators commute with Q as follows:

$$[M_{\mu\nu}, Q_\alpha] = -\frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta, \qquad [M_{\mu\nu}, \bar{Q}_{\dot\alpha}] = \frac{1}{2}(\tilde\sigma_{\mu\nu})^{\dot\beta}{}_{\dot\alpha}\bar{Q}_{\dot\beta}\ , \tag{2.2}$$

$$[T^i_j, Q^k] = \delta^k_j Q^i - \frac{1}{N}\delta^i_j Q^k\ , \qquad [T^i_j, \bar{Q}_k] = -\delta^i_k \bar{Q}_j + \frac{1}{N}\delta^i_j \bar{Q}_k\ , \tag{2.3}$$

and similarly for S. Next, the commutators of Q and S with the dilatation and R charge generators are given below:

$$\begin{aligned}
&[D,Q] = \frac{i}{2}Q\ , \quad [D,\bar{Q}] = \frac{i}{2}\bar{Q}\ ; \\
&[D,S] = -\frac{i}{2}S\ , \quad [D,\bar{S}] = -\frac{i}{2}\bar{S}\ ; \qquad (2.4)\\
&[R,Q] = \frac{4-N}{2N}Q\ , \quad [R,\bar{Q}] = -\frac{4-N}{2N}\bar{Q}\ ; \\
&[R,S] = -\frac{4-N}{2N}S\ , \quad [R,\bar{S}] = \frac{4-N}{2N}\bar{S}\ . \qquad (2.5)
\end{aligned}$$

Finally, the SU(N) generators T^i_j, $(T^i_j)^\dagger = T^j_i$, $\sum_{i=1}^N T^i_i = 0$ form the algebra

$$[T^i_j, T^k_l] = \delta^k_j T^i_l - \delta^i_l T^k_j\ . \tag{2.6}$$

The rest of the superalgebra SU$(2,2/N)$ is the conformal algebra of M, P, K, D which will not be needed here.

The superspace traditionally used for the realization of SU$(2,2/N)$ (as well as for Poincaré supersymmetry) is given by the real coset

$$\mathbb{R}^{4|2N,2N} = \frac{\mathrm{SU}(2,2/N)}{\{K,S,\bar{S},M,D,T,R\}} = (x^\mu, \theta^\alpha_i, \bar\theta^{\dot\alpha i})\ . \tag{2.7}$$

It is parametrized by 4 even coordinates x^μ and $2N$ left-handed odd spinor coordinates θ^α_i in the fundamental of SU(N) together with the $2N$ right-handed complex conjugates $\bar\theta^{\dot\alpha i} = \overline{\theta^\alpha_i}$. The superalgebra is realized on superfields $\Phi(x,\theta,\bar\theta)$ defined as functions in the coset (2.7). The generators of the

tensor: $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$, $\bar\chi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\chi_{\dot\beta}$, $\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$, $\bar\chi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\chi^{\dot\beta}$; $\epsilon_{12} = \epsilon_{\dot1\dot2} = -\epsilon^{12} = -\epsilon^{\dot1\dot2} = 1$.

coset denominator $K, S, \bar{S}, M, D, T, R$ act on the superspace coordinates as well as on the external indices of the superfield. The latter action is given by the matrix parts of these generators, $K_\mu \to k_\mu$, $S_{\alpha i} \to s_{\alpha i}$, $\bar{S}^i_{\dot\alpha} \to \bar{s}^i_{\dot\alpha}$, $M_{\mu\nu} \to m_{\mu\nu}$, $D \to i\ell$, $T^i_j \to t^i_j$, $R \to r$. [3] A standard assumption is that the matrix parts of the transitive generators K, S vanish,

$$s_{\alpha i}\Phi = \bar{s}^i_{\dot\alpha}\Phi = k_\mu \Phi = 0 \tag{2.8}$$

(the third constraint follows from the first two, see (2.1)). The homogeneous action of the remaining ones, d, l, r, t, on the superfield and, in particular, on its lowest component $\phi(x) = \Phi|_{\theta=\bar\theta=0}$ defines the latter as an irrep of $\mathrm{SO}(1,1) \times \mathrm{SL}(2,\mathbb{C}) \times \mathrm{U}(1) \times \mathrm{SU}(N)$ with the following quantum numbers:

$$\mathcal{D}(\ell; j_1, j_2; r; a_1, \ldots, a_{N-1}) \tag{2.9}$$

where ℓ is the conformal dimension, j_1, j_2 are the two Lorentz quantum numbers ("spins"), r is the R charge and $a_1, \ldots, a_{N-1}$ are the $\mathrm{SU}(N)$ Dynkin labels.

2.1 Chiral superfields

The superalgebra $\mathrm{SU}(2,2/N)$ can be realized in a smaller superspace, called "chiral" superspace. It is obtained by adding half of the Poincaré supersymmetry generators, for instance, the right-handed ones $\bar{Q}^{\dot\alpha}_i$, to the coset denominator:

$$\mathbb{C}^{4|2N,0} = \frac{\mathrm{SU}(2,2/N)}{\{K, S, \bar{S}, M, D, T, R, \bar{Q}\}} = (x^\mu, \theta^\alpha_i)\ . \tag{2.10}$$

This means adding a new constraint to the set (2.8):

$$\bar{q}^{\dot\alpha}_i \Phi = 0 \tag{2.11}$$

where $\bar{q}$ is the matrix part of the generator $\bar{Q}$. However, in this case the superalgebra (2.1) implies restrictions on the allowed values of the quantum

[3] We assign the R charge $r_\theta = -(4-N)/2N$ to the left-handed Grassmann coordinates θ^α in order to be consistent with the convention that chiral superfields $\Phi(\theta)$ have $r = -\ell$ for any N (see (2.13)). Note that for $N = 4$, $r_\theta = 0$ and the r quantum number becomes a "central charge" [36, 33]. In this case one refers to the $\mathrm{PSU}(2,2/4)$ algebra for $r = 0$ and to the $\mathrm{PU}(2,2/4)$ algebra for $r \neq 0$.

numbers (2.9) [?]. Indeed, the constraints (2.11), (2.8) yield the compatibility condition

$$\{\bar{q}^{\dot\alpha}_i, \bar{s}^j_{\dot\beta}\}\Phi = \Big[-\delta^j_i (\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} m_{\mu\nu} - 2\delta^{\dot\alpha}_{\dot\beta}\left(2t^i_j + \delta^j_i(\ell + r)\right)\Big]\,\Phi = 0\ . \qquad (2.12)$$

This is only possible if the superfield (i.e., its first component (2.9)) carries no right-handed spin, no SU(N) indices and has R charge $r = -\ell$:

$$\mathbb{C}^{4|2N,0}\ \Rightarrow\ \mathcal{D}(\ell; j_1, 0; -\ell; 0, \dots, 0)\ . \qquad (2.13)$$

Such superfields are called (left-handed) chiral. Note that both the superspace (2.10) and the superfields defined in it are complex.

Given a general superfield $\Phi(x, \theta, \bar\theta)$, one can restrict it to the coset (2.10) by imposing the following differential "chirality" constraint [11]

$$\bar{D}^{\dot\alpha}_i \Phi(x, \theta, \bar\theta) = 0\ . \qquad (2.14)$$

Here $\bar{D}$ is the right-handed half of the "covariant spinor derivatives"

$$D^i_\alpha = \frac{\partial}{\partial\theta^\alpha_i} + i\bar\theta^{\dot\alpha i}(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu\ , \qquad \bar{D}_{\dot\alpha i} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha i}} - i\theta^\alpha_i(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu\ . \qquad (2.15)$$

Note that these derivatives are only covariant with respect to the super-Poincaré subalgebra of SU($2, 2/N$). They obey the following anticommutation relations:

$$\{D^i_\alpha, D^j_\beta\} = \{\bar{D}_{\dot\alpha i}, \bar{D}_{\dot\beta j}\} = 0\ , \quad \{D^i_\alpha, \bar{D}_{\dot\beta j}\} = -2i\delta^i_j(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu\ . \qquad (2.16)$$

A crucial observation is that the chirality constraint (2.14) can be solved by going to the "left-handed chiral" basis

$$x^\mu_L = x^\mu + i\theta_{Li}\sigma^\mu\bar\theta^i_L, \quad \theta^\alpha_{Li} = \theta^\alpha_i, \quad \bar\theta^{\dot\alpha i}_L = \bar\theta^{\dot\alpha i}\ . \qquad (2.17)$$

There $\bar{D}$ becomes just a partial derivative, $\bar{D}_{\dot\alpha i} = -\partial/\partial\bar\theta^{\dot\alpha i}_L$, so (2.14) simply implies

$$\Phi = \Phi(x^\mu_L, \theta^\alpha_{Li})\ . \qquad (2.18)$$

An important property of the chiral superfields (2.18) is that the product of two of them is still a chiral superfield, i.e. they form a "ring structure". Note the close analogy with the typical property of ordinary analytic functions. As we shall see in the next subsection, this analogy can be further developed.

2.2 Grassmann analytic superfields

A natural question is whether one can find other realizations of $SU(2,2/N)$ in superspaces involving only part of the odd coordinates. In the chiral case above we chose to add all of the right-handed generators $\bar{Q}_i^{\dot\alpha}$, which form an irrep of $SU(N)$, to the coset denominator. Now, let us assume for a moment the possibility to break $SU(N)$. [4] We can then take just one of the Q's or the $\bar{Q}$'s , e.g., Q^1_α and put it in the denominator. The resulting coset has $2N-2$ left-handed and $2N$ right-handed odd coordinates:

$$\mathbb{A}^{4|2N-2,2N} = \frac{SU(2,2/N)}{\{K,S,\bar{S},M,D,T,R,Q^1\}} = (x^\mu, \theta^\alpha_2, \ldots, \theta^\alpha_N, \bar\theta^1_{\dot\alpha}, \ldots, \bar\theta^N_{\dot\alpha}) \ . \tag{2.19}$$

This means replacing the chirality condition (2.11) by

$$q^1_\alpha \Phi = 0 \ . \tag{2.20}$$

Then, a compatibility condition analogous to (2.12) follows from the anticommutator

$$\{q^1_\alpha, s^\beta_1\}\Phi = \left[-(\sigma^{\mu\nu})_\alpha{}^\beta m_{\mu\nu} - 2\delta^\beta_\alpha \left(2t^1_1 - \ell + r\right)\right]\Phi = 0 \ . \tag{2.21}$$

It implies $(\sigma^{\mu\nu})_\alpha{}^\beta m_{\mu\nu}\Phi = 0$, i.e. no left-handed spin, as well as a relation between the eigenvalue of the $SU(N)$ generator t^1_1, the R charge and the conformal dimension:

$$j_1 = 0 \ , \qquad 2t^1_1 = \ell - r \ . \tag{2.22}$$

Further, anticommuting q^1_α with the remaining projections $s^\beta_{2,3,\ldots,N}$, we obtain

$$t^1_i = 0 \ , \quad 2 \le i \le N \ . \tag{2.23}$$

Let us now make a digression and discuss the $SU(N)$ generators t^i_j. In the Cartan decomposition of the $SU(N)$ algebra (2.6) the generators with $1 \le i < j \le N$ are associated to the positive roots ("raising operators"). Among them t^i_{i+1}, $i = 1, \ldots, N-1$ correspond to the simple roots, which means that the other raising operators are obtained by commuting the simple ones. Similarly, the generators with $N \ge i > j \ge 1$ are associated to the negative

[4] Superspaces of this type can be introduced without breaking $SU(N)$ in the framework of harmonic superspace, see Section 3.

roots ("lowering operators"), the simple ones being t_i^{i+1}, $i = 1, \ldots, N-1$. Finally, the $N-1$ independent generators t_i^i (recall that $\sum_{i=1}^N t_i^i = 0$) define the $N-1$ charges of the Cartan subalgebra of $[\mathrm{U}(1)]^{N-1} \subset \mathrm{SU}(N)$ as follows:

$$m_k = t_k^k - t_N^N = t_k^k + \frac{m}{N} \ , \qquad 1 \leq k \leq N \ , \qquad m = \sum_{i=1}^N m_i \tag{2.24}$$

where $m_N \equiv 0$. An irrep of SU(N) is generated from the highest weight state (HWS) $|a_1, \ldots, a_{N-1}\rangle$ specified, for example, by the Dynkin labels defined by

$$a_k = m_k - m_{k+1} \geq 0 \ , \qquad 1 \leq k \leq N-1 \ . \tag{2.25}$$

Correspondingly, the charges (2.24) of a HWS take eigenvalues $m_1 \geq m_2 \geq \ldots \geq m_{N-1} \geq m_N = 0$. In the language of Young tableaux m_k is just the number of boxes in the k-th row. The HWS is by definition annihilated by all the raising operators:

$$t_j^i |a_1, \ldots, a_{N-1}\rangle = 0 \ , \quad 1 \leq i < j \leq N \ . \tag{2.26}$$

In these terms conditions (2.23) are just a subset of the irreducibility conditions (2.26). From (2.22) we obtain the following restrictions on the quantum numbers:

$$\frac{2m}{N} - 2m_1 = r - \ell \ . \tag{2.27}$$

We can go on and consider a superspace of the type (2.19) where the first p θ's are missing:

$$\begin{aligned} \mathbb{A}^{4|2N-2p,2N} &= \frac{\mathrm{SU}(2,2/N)}{\{K, S, \bar{S}, M, D, T, R, Q^1, \ldots, Q^p\}} \\ &= (x^\mu, \theta^\alpha_{p+1}, \ldots, \theta^\alpha_N, \bar{\theta}^1_{\dot{\alpha}}, \ldots, \bar{\theta}^N_{\dot{\alpha}}) \ . \end{aligned} \tag{2.28}$$

As before, this means to impose

$$q^i_\alpha \Phi = 0 \ , \quad 1 \leq i \leq p \ . \tag{2.29}$$

Then, from the anticommutators $\{q^i_\alpha, s^\beta_i\} = 0$, $1 \leq i \leq p$ we obtain conditions similar to (2.27):

$$\frac{2m}{N} - 2m_i = r - \ell \ , \quad 1 \leq i \leq p \ . \tag{2.30}$$

Also, $\{q^i_\alpha, s^\beta_j\} = 0$ for $1 \leq i < j \leq p$ yields a bigger subset of the irreducibility conditions (2.26). In addition, this time we obtain a new type of condition:

$$t^i_j|a_1,\ldots,a_{N-1}\rangle = 0\ , \quad p \geq i > j \geq 1\ . \tag{2.31}$$

The generators in (2.31) are lowering operators of SU(N). In fact, these new constraints are corollaries of (2.30). Indeed, from (2.30) follows

$$a_1 = \ldots = a_{p-1} = 0 \quad \text{for } p \geq 2\ . \tag{2.32}$$

Now, the HWS $|a_1,\ldots,a_{N-1}\rangle$ has the property [5]

$$(t^{k+1}_k)^{a_k+1}|a_1,\ldots,a_{N-1}\rangle = 0\ . \tag{2.33}$$

Then it is obvious that (2.32) and (2.33) imply (2.31).

The argument above can be reversed. Take a superfield defined in the superspace $\mathbb{A}^{4|2N-2,2N}$ (2.19) whose lowest component is in the SU(N) irrep with Dynkin labels $[0,\ldots,0,a_p,\ldots,a_{N-1}]$, $p > 1$. Then (2.31) holds and combining it with the constraint (2.20), we obtain the full set of constraints (2.29). Thus, such a superfield effectively lives in a smaller superspace.

It is clear than we can repeat the same procedure in the right-handed sector. This time the starting point will be a superspace where $\bar\theta^N_{\dot\alpha}$ is absent (note that in our convention q^1 and $\bar q_N$ are the HWS's of the fundamental irrep of SU(N) and of its conjugate, respectively). From the corresponding condition $\bar q^{\dot\alpha}_N\Phi = 0$ we derive

$$j_2 = 0\ , \qquad \frac{2m}{N} = \ell + r\ . \tag{2.34}$$

Going on and removing q right-handed odd variables, $\bar\theta^N_{\dot\alpha},\ldots,\bar\theta^{N-q+1}_{\dot\alpha}$, i.e., imposing the constraints

$$\bar q^{\dot\alpha}_i\Phi = 0\ , \quad N-q+1 \leq i \leq N\ , \tag{2.35}$$

in addition to (2.34) we find

$$m_i = 0\ , \quad N-q+1 \leq i \leq N-1 \quad \text{for } q \geq 2\ . \tag{2.36}$$

[5] The explanation is as follows. The generators t^{k+1}_k, t^k_{k+1} and $t^k_k - t^{k+1}_{k+1}$ form the algebra of $\mathrm{SU}(2)_k \subset \mathrm{SU}(N)$. The state $|a_1,\ldots,a_{N-1}\rangle$ can be regarded as the HWS of an irrep of this $\mathrm{SU}(2)_k$ of U(1) charge a_k, i.e. of dimension a_k+1. Eq. (2.33) then follows from the fact that t^{k+1}_k is the lowering operator of $\mathrm{SU}(2)_k$.

As before, this implies the vanishing of the last $q-1$ Dynkin labels:

$$a_i = 0 \ , \quad N-q+1 \leq i \leq N-1 \quad \text{for } q \geq 2 \ . \tag{2.37}$$

Correspondingly, the HWS is annihilated by the lowering operators t^i_j, $N \geq i > j \geq N-q+1$.

Finally, we can combine left- and right-handed constraints and define the most general G-analytic superspace as follows:

$$\begin{aligned} \mathbb{A}^{4|2N-2p,2N-2q} &= \frac{\mathrm{SU}(2,2/N)}{\{K,S,\bar{S},M,D,T,R,Q^1,\ldots,Q^p,\bar{Q}_{N-q+1},\ldots,\bar{Q}_N\}} \\ &= (x^\mu,\theta^\alpha_{p+1},\ldots,\theta^\alpha_N,\bar{\theta}^1_{\dot\alpha},\ldots,\bar{\theta}^{N-q}_{\dot\alpha}) \ , \quad p+q \leq N \ . \end{aligned} \tag{2.38}$$

Following [30] we shall call (2.38) an "(N,p,q) superspace" [6]. It is important to realize that anticommuting the Q's and $\bar{Q}$'s in the denominator should not produce the translation generator P_μ which belongs to the coset. This explains the condition $p+q \leq N$ in (2.38). The superfields defined in this coset are annihilated by a subset of the Poincaré supersymmetry generators:

$$q^i_\alpha \Phi = \bar{q}^{\dot\alpha}_j \Phi = 0 \ , \qquad 1 \leq i \leq p \ , \quad N-q+1 \leq j \leq N \ . \tag{2.39}$$

These conditions lead to restrictions on the quantum numbers obtained by combining the ones found above:

$$\begin{aligned} & j_1 = j_2 = 0 \ ; \\ & \ell = m_1 \ ; \\ & r = \frac{2m}{N} - m_1 \ ; \\ & m_1 = m_2 = \ldots = m_p \ , \\ & m_i = 0 \ , \quad N-q+1 \leq i \leq N-1 \ , \quad q \geq 2 \ . \end{aligned} \tag{2.40}$$

Such $\mathrm{SU}(N)$ representations have the first $p-1$ and the last $q-1$ Dynkin labels vanishing:

$$[0,\ldots,0,a_p,\ldots,a_{N-q},0,\ldots,0] \ . \tag{2.41}$$

An interesting limiting case is obtained when $p+q=N$. Such superspaces contain exactly one half of the initial number of Grassmann variables (p left-handed and $N-p$ right-handed spinors). The $\mathrm{SU}(N)$ representation of the

[6] The first example of a $(3,2,1)$ superspace was given in [29].

lowest component of the superfield has only one non-vanishing Dynkin label, $a_p \neq 0$. Consequently, $\ell = a_p$ and $r = \left(\frac{2p}{N} - 1\right) a_p$. In Section 4 we shall see that in the special case $a_p = 1$ such superfields describe some of the massless superconformal multiplets.

We remark that chiral superspace can be viewed as a limiting case of the above when, e.g., $p = 0$ and $q = N$. In this case only $j_1 = 0$, the other Lorentz quantum number j_2 remains arbitrary.

3 (N, p, q) harmonic superspace

The chiral superspace introduced in Section 2.1 is naturally realized in terms of superfields satisfying a differential constraint of the type (2.14). The question arises if we can formulate similar differential constraints restricting a superfield to the G-analytic superspaces of Section 2.2. It is quite clear that one should impose constraints similar to (2.39) with the supersymmetry generators replaced by spinor covariant derivatives. The only problem is that in (2.29) we have explicitly broken the SU(N) invariance. Here the situation is the same as at the time when the concept of Grassmann analyticity was first introduced in ref. [25]. This can be repaired by extending the framework of standard superspace to the so-called harmonic superspace [12].

3.1 Harmonic variables on the coset SU$(N)/[\mathbf{U}(1)]^{N-1}$

Harmonic superspace is obtained from the ordinary one (2.7) by tensoring it with a coset of the group SU$(N)/H$ where H is a maximal subgroup of SU(N). In order to be able to describe the most general case of G-analytic superfields one has to choose the smallest such subgroup, which is the Cartan subgroup $[\mathrm{U}(1)]^{N-1}$. The resulting coset SU$(N)/[\mathrm{U}(1)]^{N-1}$ is a compact complex manifold ("flag manifold" [49, 30]) of complex dimension $N(N-1)/2$. Note, however, that (N, p, q) superfields for $p \geq 2$ and/or $q \geq 2$ effectively live in the smaller cosets SU$(N)/[\mathrm{U}(1)]^{N-p-q+1} \times$ SU$(p) \times$ SU(q), as we shall explain below (see also [30]).

3.1.1 Covariant description of the coset SU$(N)/[\mathbf{U}(1)]^{N-1}$

The harmonic variables u_i^I and their conjugates $u_I^i = (u_i^I)^*$ form an SU(N) matrix where i is an index in the fundamental representation of SU(N) and

$I = 1, \ldots, N$ are the projections of the second index onto the subgroup $[\mathrm{U}(1)]^{N-1}$. Further, we define two *independent* $\mathrm{SU}(N)$ groups, a left one acting on the index i and a right one acting on the projected index I of the harmonics:

$$(u_i^I)' = \Lambda_i^j u_j^J \Sigma_J^I \ , \qquad \Lambda \in \mathrm{SU}(N)_L \ , \quad \Sigma \in \mathrm{SU}(N)_R \ . \tag{3.1}$$

In particular, the charge operators (2.24) of $\mathrm{SU}(N)_R$ act on the harmonics as follows:

$$m_K \ u_i^I = (\delta_{KI} - \delta_{KN}) u_i^I \ , \qquad m_K \ u_I^i = -(\delta_{KI} - \delta_{KN}) u_I^i \ . \tag{3.2}$$

The harmonics satisfy the following $\mathrm{SU}(N)$ defining conditions:

$$u \in \mathrm{SU}(N) : \qquad \begin{array}{l} u_i^I u_J^i = \delta_J^I \ , \\ u_i^I u_I^j = \delta_i^j \ , \\ \varepsilon^{i_1 \ldots i_N} u_{i_1}^1 \ldots u_{i_N}^N = 1 \ . \end{array} \tag{3.3}$$

3.1.2 Harmonic functions

A basic assumption of the harmonic approach to the coset $\mathrm{SU}(N)/[\mathrm{U}(1)]^{N-1}$ is that any harmonic function is homogeneous under the action of $[\mathrm{U}(1)_R]^{N-1}$, i.e., it is an eigenfunction of the charge operators m_I,

$$m_I \ f_{L_1 \ldots L_r}^{K_1 \ldots K_q}(u) = (\delta_{K_1 I} - \delta_{K_1 N} - \delta_{L_1 I} + \delta_{L_1 N} + \ldots) f_{L_1 \ldots L_r}^{K_1 \ldots K_q}(u) \tag{3.4}$$

(note that the projections (charges) $K_1 \ldots K_q; L_1 \ldots L_r$ are not necessarily all different). Thus the harmonic function effectively depends on the $(N^2-1)-(N-1) = N(N-1)$ real coordinates of the coset $\mathrm{SU}(N)/[\mathrm{U}(1)]^{N-1}$. This description of the coset is global and coordinateless. The function (3.4) is given by its harmonic expansion on the coset (hence the term "harmonic space"). In our $\mathrm{SU}(N)$ covariant notation this expansion is $[\mathrm{U}(1)_R]^{N-1}$ *covariant* and $\mathrm{SU}(N)_L$ *invariant*. To give a simple example, consider the case $N = 2$ and the harmonic function

$$\begin{aligned} f^1(u) \ &= \ f^i u_i^1 + f^{ijk} u_i^1 u_j^1 u_k^2 + \ldots \\ &\quad + f^{i_1 \ldots i_{n+1} j_1 \ldots j_n} u_{i_1}^1 \ldots u_{i_{n+1}}^1 u_{j_1}^2 \ldots u_{j_n}^2 + \ldots \ . \end{aligned} \tag{3.5}$$

Note that each term in the expansion has the same overall $\mathrm{U}(1)_R$ charge 1. The first coefficient f^i is in the fundamental of $\mathrm{SU}(2)_L$, and the following ones are symmetric in all of their indices (either because $u^1_i u^1_j$ is symmetric in i,j or because the antisymmetrization of $u^1_i u^2_j$ reduces it to a preceding term in (3.5)), thus realizing irreps of $\mathrm{SU}(2)_L$ of isospin $n+1/2$. As a second example, consider the function

$$f^1_2(u) \equiv f^{11} = f^{ij}u^1_i u^1_j + f^{ijkl}u^1_i u^1_j u^1_k u^2_l + \dots \ . \tag{3.6}$$

This time the overall charge is even, therefore the irreps of the expansion carry integer isospin.

We remark that the irreducible products of harmonics play the rôle of the familiar spherical harmonics in the case $N=2$, where the coset $\mathrm{SU}(2)/\mathrm{U}(1) \sim S^2$ (see [12] for details).

The above $N=2$ examples are generalized to any N as follows. [7] Consider first a function of the type

$$f^{\overbrace{1\dots1}^{m_1}\overbrace{2\dots2}^{m_2}\cdots\overbrace{\text{N-1}\dots\text{N-1}}^{m_{N-1}}}(u) \ , \qquad m_1 \geq m_2 \geq \dots \geq m_{N-1} \ . \tag{3.7}$$

Note that the charges form a sequence corresponding to the canonical structure of a Young tableau. This tableau defines the smallest irrep of $\mathrm{SU}(N)_L$ that one finds in the expansion. All the remaining irreps are obtained by the following procedure. Denote the HWS of the smallest irrep by its Dynkin labels, $|a_1,\dots,a_{N-1}\rangle$ and that of any irrep present in the expansion by $|A_1,\dots,A_{N-1}\rangle$. The vector $|a_1,\dots,a_{N-1}\rangle$ appears in the multiplet generated by the HWS $|A_1,\dots,A_{N-1}\rangle$, so it can be obtained by the action of the lowering operators of $\mathrm{SU}(N)_L$:

$$|a_1,\dots,a_{N-1}\rangle = (t^2_1)^{n_1}(t^3_2)^{n_2}\dots(t^N_{N-1})^{n_{N-1}}|A_1,\dots,A_{N-1}\rangle \ . \tag{3.8}$$

Here we only use the simple roots; the ordering in (3.8) is of no importance for our argument. From the $\mathrm{SU}(N)$ algebra we easily find the following relations between the two sets of Dynkin labels:

$$A_k = a_k + 2n_k - n_{k-1} - n_{k+1} \geq 0 \ , \quad k = 1,\dots,N-1 \ . \tag{3.9}$$

Note that the coefficients in (3.9) form the Cartan matrix of $\mathrm{SU}(N)$. The total number of boxes of the Young tableaux (i.e., number of indices of the coefficients, see below) is given by

$$M = \sum_{k=1}^{N-1} kA_k = m + Nn_{N-1} \ . \tag{3.10}$$

[7] We are grateful to P. Sorba for help in developing this argument.

Thus one finds an $N-1$-parameter family of irreps where the choice of the parameters n_k is only limited by the requirements $A_k \geq 0$.

As an illustration of the above, look at the first term in the expansion of the function (3.7):

$$f^{i_1\ldots i_{m_1} j_1\ldots j_{m_2}\ldots k_1\ldots k_{m_{N-1}}}\ u^1_{i_1}\ldots u^1_{i_{m_1}} u^2_{j_1}\ldots u^2_{j_{m_2}}\ldots u^{N-1}_{k_1}\ldots u^{N-1}_{k_{m_{N-1}}}\ . \tag{3.11}$$

Unlike the simple SU(2) examples above, here the coefficients f are not necessarily irreducible under $\mathrm{SU}(N)_L$. Indeed, they only possess the symmetry associated to each type of harmonic projection but no antisymmetrization between any two different projections has been performed. Comparing the term (3.11) to the general case (3.9) we can say that in (3.11) the total number of indices (boxes in a Young tableau) is $M = m$, so what is left is the $N-2$-parameter family of irreps corresponding to $n_{N-1} = 0$.

The general term in the expansion of the function (3.7) is obtained from (3.11) by multiplying it by the chargeless harmonic monomial $u^1_{i_1}\ldots u^N_{i_N}$ (the total antisymmetrization of the indices $i_1,\ldots,i_N$ results in an $\mathrm{SU}(N)_L$ singlet, so it should be eliminated):

$$f^{\overbrace{1\ldots1}^{m_1}\overbrace{2\ldots2}^{m_2}\cdots\overbrace{\text{N-1}\ldots\text{N-1}}^{m_{N-1}}}(u) =$$

$$\sum_{n_{N-1}=0}^{\infty} f^{i_1\ldots i_M}(u^1)^{m_1+n_{N-1}}\ldots(u^{N-1})^{m_{N-1}+n_{N-1}}(u^N)^{n_{N-1}}\ . \tag{3.12}$$

We use n_{N-1} from (3.8) as the expansion parameter. Each term in (3.12) has a coefficient with a total number of indices M given by (3.10). This coefficient is decomposed into a set of $\mathrm{SU}(N)_L$ irreps according to the rule (3.9).

If the charges ($[\mathrm{U}(1)_R]^{N-1}$ projections) of the harmonic function do not appear in the canonical order (3.7), then one should reorder the indices $1,2,\ldots,N$ so that they can label a Young tableau. For instance, the $N=4$ function f^{122233} should be rewritten as f^{222331}, so it corresponds to the Young tableau $(3,2,1)$. If a complete set of N different projections is present, it can be suppressed, e.g., the $N=4$ function $f^{11234} \equiv f^1$. Finally, if the function carries lower indices (projections of the complex conjugate fundamental representation), they should be converted into sets of $N-1$ upper indices, for example, the $N=4$ function $f^1_4 \equiv f^{1123}$ or $f^{12}_1 \equiv f^{12234} \equiv f^2$.

3.1.3 Harmonic derivatives

The harmonic derivatives are operators which respect the defining relations (3.3):

$$\partial^I_J = u^I_i \frac{\partial}{\partial u^J_i} - u^i_J \frac{\partial}{\partial u^i_I} - \frac{1}{N} \sum_{K=1}^{N} \delta^I_J \left(u^K_i \frac{\partial}{\partial u^K_i} - u^i_K \frac{\partial}{\partial u^i_K} \right) . \tag{3.13}$$

They act on the harmonics as follows:

$$\partial^I_J u^K_i = \delta^K_J u^I_i - \frac{1}{N} \delta^I_J u^K_i , \qquad \partial^I_J u^i_K = -\delta^I_K u^i_J + \frac{1}{N} \delta^I_J u^i_K . \tag{3.14}$$

Note that we prefer to treat u^I_i and u^i_I as independent variables subject to the constraints (3.3).

Clearly, the derivatives ∂^I_J are the generators of the group $\mathrm{SU}(N)_R$ acting on the $[\mathrm{U}(1)_R]^{N-1}$ projected indices of the harmonics. The assumption (3.4) is then translated into the requirement that the harmonic functions $f(u)$ are eigenfunctions of the diagonal derivatives ∂^I_I which count the $\mathrm{U}(1)_R$ charges:

$$(\partial^I_I - \partial^N_N) f^{K_1 \ldots K_q}_{L_1 \ldots L_r}(u) = (\delta_{K_1 I} - \delta_{K_1 N} - \delta_{L_1 I} + \delta_{L_1 N} + \ldots) f^{K_1 \ldots K_q}_{L_1 \ldots L_r}(u) . \tag{3.15}$$

Then the independent harmonic derivatives on the coset are the $N(N-1)/2$ complex derivatives ∂^I_J, $I < J$ corresponding to the raising operators of $\mathrm{SU}(N)_R$ (or their conjugates ∂^I_J, $I > J$ corresponding to the lowering operators of $\mathrm{SU}(N)_R$).

From the above it follows that the harmonic differential conditions

$$\partial^I_J f^{K_1 \ldots K_q}_{L_1 \ldots L_r}(u) = 0 , \quad I < J \tag{3.16}$$

impose severe constraints on the harmonic function. Indeed, if the function is of the type (3.7), it is reduced to just one harmonic monomial giving rise to an $\mathrm{SU}(N)$ irrep whose HWS is labeled by the charges. Any other harmonic function subject to the condition (3.16) must vanish.

As an example, take $N = 2$ and the function $f^1(u)$ (3.5) subject to the constraint

$$\partial^1_2 f^1(u) = 0 \ \Rightarrow \ f^1(u) = f^i u^1_i \tag{3.17}$$

since this is the only term in the expansion (3.5) which automatically satisfies the condition (3.17). So, the harmonic function is reduced to a doublet of

SU(2). Similarly, for $N = 4$ the function $f^{12}(u)$ is reduced to the $\underline{6}$ of SU(4). Indeed, the constraints $\partial_3^2 f^{12}(u) = \partial_4^3 f^{12}(u) = 0$ ensure that $f^{12}(u)$ depends on u^1, u^2 only, $f^{12}(u) = f^{ij} u_i^1 u_j^2$. Then the constraint $\partial_2^1 f^{12}(u) = f^{ij} u_i^1 u_j^1 = 0$ implies $f^{ij} = -f^{ji}$. An example of a harmonic function which vanishes if subject to the constraint (3.16) is, e.g., in $N = 2$, $f_1(u) \equiv f^2(u)$, since no term in its expansion can satisfy the condition $\partial_2^1 f^2(u) = 0$.

Note that not all of the derivatives ∂_J^I , $I < J$ are independent, as follows from the SU(N) algebra. The independent ones,

$$\partial_2^1 \ , \ \partial_3^2 \ , \ \ldots , \ \partial_N^{N-1} \tag{3.18}$$

correspond to the simple roots of SU(N). Then the constraint (3.16) is equivalent to

$$\partial_{I+1}^I f_{L_1 \ldots L_r}^{K_1 \ldots K_q}(u) = 0 \ , \qquad I = 1, \ldots, N-1 \ . \tag{3.19}$$

We remark that the coset SU(N)/U(1)$^{N-1}$ can be parametrized by $N(N-1)/2$ complex coordinates. In our context this amounts to making a choice of the harmonic matrix u_i^I such that the group $[\mathrm{U}(1)_R]^{N-1}$ is identified with $[\mathrm{U}(1)_L]^{N-1} \subset \mathrm{SU}(N)_L$. Then the harmonic derivatives become Cartan's covariant derivatives on the coset. The constraints (3.16) take the form of covariant Cauchy-Riemann analyticity conditions. For this reason we can call the set of constraints (3.16) (or (3.19)) harmonic (H-)analyticity conditions. The above argument shows that H-analyticity is equivalent to defining a HWS of SU(N), i.e. it is the SU(N) irreducibility condition on the harmonic functions.

3.2 (N, p, q) harmonic superfields

The main purpose of introducing harmonic variables is to be able to define manifestly SU(N) covariant superfields living in the G-analytic superspaces (2.38). This is done following the example of the chiral superfields. There we replaced the condition (2.11) by the differential chirality constraint (2.14). In the case of (N, p, q) analyticity we have to replace conditions (2.39) by analogous differential constraints. The crucial point now is to let the superfield depend on the harmonic variables and obtain the adequate $[\mathrm{U}(1)]^{N-1}$ projections with the help of harmonic variables:

$$D_\alpha^I \Phi(x, \theta, \bar\theta, u) = \bar D_J^{\dot\alpha} \Phi(x, \theta, \bar\theta, u) = 0 \tag{3.20}$$

where

$$D^I_\alpha = D^i_\alpha u^I_i \ , \quad \bar D^{\dot\alpha}_J = \bar D^{\dot\alpha}_i u^i_J \ , \quad 1 \le I \le p \ , \ N-q+1 \le J \le N \ . \tag{3.21}$$

The derivatives appearing in (3.20) anticommute (see (2.16)), therefore there exists a G-analytic basis in superspace,

$$\begin{aligned} x^\mu_A &= x^\mu - i(\theta_1\sigma^\mu\bar\theta^1 + \ldots + \theta_p\sigma^\mu\bar\theta^p - \theta_{N-q+1}\sigma^\mu\bar\theta^{N-q+1} - \ldots - \theta_N\sigma^\mu\bar\theta^N) \ , \\ \theta^\alpha_I &= \theta^\alpha_i u^i_I \ , \quad \bar\theta^{\dot\alpha I} = \bar\theta^{\dot\alpha i} u^I_i \ . \end{aligned} \tag{3.22}$$

where these derivatives become just $D^I_\alpha = \partial/\partial\theta^\alpha_I$, $\bar D_{\dot\alpha\ J} = -\partial/\partial\bar\theta^{\dot\alpha\ J}$. Consequently, in this basis the analytic superfield (3.20) becomes an unconstrained function of $N-p$ θ's and $N-q$ $\bar\theta$'s, as well as of the harmonic variables:

$$\Phi(x_A, \theta_{p+1}, \ldots, \theta_N, \bar\theta^1, \ldots, \bar\theta^{N-q}, u) \ . \tag{3.23}$$

Let us now turn to the harmonic dependence in (3.23). In principle, each component in the θ expansion of the superfield is a harmonic function having an infinite harmonic expansion of the type (3.12). If we want to deal with a finite set of fields, we have to impose a harmonic irreducibility condition of the type (3.16) (or the equivalent subset (3.19)). However, in the G-analytic basis (3.22) the harmonic derivatives become covariant, D^I_J. In particular, the derivatives

$$D^I_J = \partial^I_J + 2i\theta_J\sigma^\mu\bar\theta^I\partial_\mu - \theta_J\partial^I + \bar\theta^I\bar\partial_J \ , \quad 1 \le I \le N-q, \ p+1 \le J \le N \tag{3.24}$$

acquire space-time derivative terms. In the next section we shall see that this has important consequences on a G-analytic superfield subject to the additional H-analyticity constraints

$$D^I_J \Phi^{[a_1,\ldots,a_{N-1}]}(x_A, \theta_{p+1}, \ldots, \theta_N, \bar\theta^1, \ldots, \bar\theta^{N-q}, u) = 0 \ , \quad 1 \le I < J \le N \ . \tag{3.25}$$

Here we have indicated the SU(N) representation carried by the superfield.

3.3 (N,p,q) conformal superfields

So far in this section we have only discussed G-analytic superfields as representations of Poincaré supersymmetry. From the analysis of Section 2 we know that superconformal invariance yields additional restrictions, in particular, on the SU(N) irrep carried by the superfield. Adapting the arguments of Section 2, one finds that (3.20) implies the following harmonic conditions (even if we do not impose the SU(N) irreducibility conditions (3.25)):

$$D^I_{I+1}\Phi^{[a_1,\ldots,a_{N-1}]} = D^{I+1}_I\Phi^{[a_1,\ldots,a_{N-1}]} = 0 \ ,$$
$$1 \leq I \leq p-1 \quad \text{and} \quad N-q+1 \leq I \leq N-1 \ . \tag{3.26}$$

These two subsets of raising and lowering operators of SU(N) generate the algebra of SU(p) × SU(q). In the spirit of the coset construction of Section 2 this means that we have added the factor SU(p) × SU(q) to the denominator of the harmonic coset. In other words, a conformally covariant (N, p, q) superfield lives not only in a smaller superspace, but also in a smaller harmonic space as compared to our initial coset SU(N)/$[\mathrm{U}(1)]^{N-1}$. From Section 2 we also know that the Dynkin labels of such a superfield are restricted (see (2.41)). To summarize, a G-analytic conformal superfield has the form

$$\Phi^{[0,\ldots,0,a_p,\ldots,a_{N-q},0,\ldots,0]}(x_A, \theta_{p+1}, \ldots, \theta_N, \bar\theta^1, \ldots, \bar\theta^{N-q}, u) \tag{3.27}$$

and lives in the harmonic coset

$$\begin{array}{ll} \dfrac{\mathrm{SU}(N)}{[\mathrm{U}(1)]^{N-p-q+1}\times \mathrm{SU}(p)\times\mathrm{SU}(q)} & \text{for } p\geq 2\ ,\ q\geq 2\ ; \\ \dfrac{\mathrm{SU}(N)}{[\mathrm{U}(1)]^{N-q}\times \mathrm{SU}(q)} & \text{for } p=0,1\ ,\ q\geq 2\ ; \\ \dfrac{\mathrm{SU}(N)}{[\mathrm{U}(1)]^{N-p}\times \mathrm{SU}(p)} & \text{for } p\geq 2\ ,\ q=0,1\ ; \\ \dfrac{\mathrm{SU}(N)}{[\mathrm{U}(1)]^{N-1}} & \text{for } p=0,1 \text{ and } q=0,1\ . \end{array} \tag{3.28}$$

This effective reduction of the harmonic coset has been pointed out in [30], although the coset proposed there only applies to superfields belonging to very special SU(N) irreps with only two non-vanishing Dynkin labels:

$$\Phi^{[0,\ldots,0,a_p,0,\ldots,0,a_{N-q},0,\ldots,0]}(x_A, \theta_{p+1}, \ldots, \theta_N, \bar\theta^1, \ldots, \bar\theta^{N-q}, u) \quad \Rightarrow$$
$$u \in \frac{\mathrm{SU}(N)}{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q)\times\mathrm{U}(N-p-q))}\ . \tag{3.29}$$

Note, however, that in the limiting cases $N = p+q$ and $N = p+q+1$ the two cosets (3.28) and (3.29) coincide.

4 Massless superconformal multiplets

Massless multiplets are a particular class of superconformal multiplets. Their components are fields carrying Lorentz spin $(j_1,0)$, $\phi_{\alpha_1\ldots\alpha_{2j_1}}(x)$ or $(0,j_2)$, $\bar\phi_{\dot\alpha_1\ldots\dot\alpha_{2j_2}}(x)$ (all indices are symmetrized). In addition, they satisfy the massless field equations

$$\partial^\mu\sigma_\mu^{\alpha\dot\alpha}\phi_{\alpha\alpha_2\ldots\alpha_{2j_1}}=0\ ,\qquad \partial^\mu\sigma_\mu^{\alpha\dot\alpha}\bar\phi_{\dot\alpha\dot\alpha_2\ldots\dot\alpha_{2j_2}}=0 \tag{4.1}$$

(or $\Box\phi=0$ in the case of spin $(0,0)$). These massless fields are known [50] to form UIR's of the conformal algebra SU$(2,2)$ if $\ell=j+1$. Consequently, the massless superconformal multiplets form UIR's of SU$(2,2/N)$ [36, 33].

In the language of AdS supersymmetry such multiplets are called "supersingletons" [51, 52].

In this section we shall formulate the massless multiplets of SU$(2,2/N)$ first in terms of ordinary superfields and then, for a subclass of them, in $(N,k,N-k)$ harmonic superspace (the simplest example is provided by the $(2,1,1)$ hypermultiplet [12]; the generalization to the case $(N,k,N-k)$ was given in [30]).

4.1 Massless multiplets as constrained superfields

There exist three types of massless N-extended superconformal multiplets. They can be described in terms of ordinary constrained superfields [34, 35].

(i). The first type is given by scalar superfields

$$W^{i_1\ldots i_k}(x^\mu,\theta^\alpha_i,\bar\theta^{\dot\alpha i})\ ,\qquad k=1,\ldots,N-1 \tag{4.2}$$

with k totally antisymmetrized indices of the fundamental representation of SU(N) (i.e., carrying Dynkin labels $[0,\ldots,0,\overset{k}{1},0,\ldots,0]$). They satisfy the following constraints:

$$D^{(j}_\alpha W^{i_1)i_2\ldots i_k}=0\ , \tag{4.3}$$
$$\bar D_{\dot\alpha\{j}W^{i_1\}i_2\ldots i_k}=0 \tag{4.4}$$

where $()$ means symmetrization and $\{\}$ means the traceless part. In the cases $N=2,3,4$ these constraints define the on-shell $N=2$ matter (hyper)multiplet [53] and the $N=3,4$ on-shell super-Yang-Mills multiplets [54].

Their generalization to arbitrary N has been given in Refs. [34, 35] where it has also been shown that they describe on-shell massless multiplets.

After rewriting the constraints (4.3), (4.4) in harmonic superspace in Section 4.2, we shall see that the above massless multiplets are superconformal if

$$\ell = 1 \ , \qquad r = \frac{2k}{N} - 1 \ . \tag{4.5}$$

We also note their SU(N) quantum numbers

$$m_1 = \ldots = m_k = 1 \ , \quad m_{k+1} = \ldots = m_{N-1} = 0 \ , \quad m = k \ . \tag{4.6}$$

(ii). The second type is given by a chiral scalar superfield

$$\bar{D}^{\dot\alpha}_i \Phi = 0 \tag{4.7}$$

satisfying the additional constraint (field equation)

$$D^{i\,\alpha} D^j_\alpha \Phi = 0 \ . \tag{4.8}$$

This superfield is an SU(N) singlet. The corresponding massless multiplet is superconformal if (see Section 2.1)

$$\ell = -r = 1 \ . \tag{4.9}$$

Similarly, one can introduce an antichiral multiplet:

$$D^i_\alpha \bar\Phi = 0 \ , \qquad \bar{D}_{i\,\dot\alpha} D^{\dot\alpha}_j \bar\Phi = 0 \tag{4.10}$$

with quantum numbers

$$\ell = r = 1 \ . \tag{4.11}$$

(iii). The third type is given by chiral superfields carrying external Lorentz spin $(j_1, 0)$:

$$\bar{D}^{\dot\alpha}_i w_{\alpha_1 \ldots \alpha_{2j_1}} = 0 \ . \tag{4.12}$$

Here the $2j_1$ spinor indices are totally symmetrized. These superfields are SU(N) singlets. They satisfy the massless field equation

$$D^{i\,\alpha} w_{\alpha\alpha_2 \ldots \alpha_{2j_1}} = 0 \ . \tag{4.13}$$

As we have seen in Section 2.1, conformal supersymmetry requires that

$$\ell = -r = j_1 + 1 \ . \tag{4.14}$$

Similarly, one can introduce antichiral superfields with Lorentz spin $(0, j_2)$:

$$D^i_\alpha \bar{w}_{\dot\alpha_1\ldots\dot\alpha_{2j_2}} = 0 \ , \qquad \bar{D}^{\dot\alpha}_i \bar{w}_{\dot\alpha\dot\alpha_2\ldots\dot\alpha_{2j_2}} = 0 \tag{4.15}$$

with

$$\ell = r = j_2 + 1 \ . \tag{4.16}$$

It is straightforward to see that such massless representations coincide with the massless supermultiplets of N-extended Poincaré supersymmetry (for an $N = 8$ example see ref. [55].).

4.2 Type (i) massless multiplets as analytic superfields

Now, let us use the harmonic variables to covariantly project all the SU(N) indices in the constraints (4.3), (4.4) onto $[\mathrm{U}(1)_R]^{N-1}$. For example, the projection

$$W^{12\ldots k} = W^{i_1 i_2\ldots i_k}(x, \theta, \bar\theta) u^1_{i_1} u^2_{i_2} \ldots u^k_{i_k} \tag{4.17}$$

satisfies the constraints

$$D^1_\alpha W^{12\ldots k} = D^2_\alpha W^{12\ldots k} = \ldots = D^k_\alpha W^{12\ldots k} = 0 \ , \tag{4.18}$$

$$\bar{D}_{\dot\alpha\; k+1} W^{12\ldots k} = \bar{D}_{\dot\alpha\; k+2} W^{12\ldots k} = \ldots = \bar{D}_{\dot\alpha\; N} W^{12\ldots k} = 0 \tag{4.19}$$

where $D^I_\alpha = D^i_\alpha u^I_i$ and $\bar{D}_{\dot\alpha\; I} = \bar{D}_{\dot\alpha\; i} u^i_I$. The first of them, eq. (4.18), is a corollary of the commuting nature of the harmonics variables, and the second one, eq. (4.19), of the defining conditions (3.3). In eqs. (4.18), (4.19) one recognizes the conditions for G-analyticity (3.20) of the type $(N, k, N-k)$. As explained in Section 3.2, in the appropriate G-analytic basis (3.22) $W^{12\ldots k}$ becomes an unconstrained function of k $\bar\theta$'s and $N-k$ θ's:

$$W^{12\ldots k} = W^{12\ldots k}(x_A, \theta_{k+1}, \ldots, \theta_N, \bar\theta^1, \ldots, \bar\theta^k, u) \ . \tag{4.20}$$

It is important to realize that the G-analytic superfield (4.20) is an SU(N) covariant object only because it depends on the harmonic variables. In order to recover the original harmonic-independent but constrained superfield $W^{i_1 i_2\ldots i_k}(x, \theta, \bar\theta)$ (4.3), (4.4) we need to impose differential constraints involving the harmonic variables. In Section 3.2 we have shown that they take the form of SU(N) irreducibility conditions, eq. (3.25). In this particular case they are

$$D^I_J W^{12\ldots k} = 0 \ , \quad 1 \le I < J \le N \tag{4.21}$$

or the equivalent set

$$D^I_{I+1} W^{12\dots k} = 0\ , \quad 1 \le I < J \le N-1\ . \tag{4.22}$$

In the initial real basis (2.7) of the full superspace $\mathbb{R}^{4|2N,2N}$ these constraints simply mean that the superfield is a polynomial in the harmonics, as in (4.17). However, in the G-analytic basis (3.22) the harmonic derivatives (3.24) contain space-time derivatives. This leads to a number of constraints on the component fields. The detailed analysis can be found in [22], here we only recall the final result:

$$\begin{aligned} W^{12\dots k} \;=\; & \phi^{12\dots k} \\ & +\bar\theta^1_{\dot\alpha}\bar\psi^{\dot\alpha\ 23\dots k} + \dots + \bar\theta^k_{\dot\alpha}\bar\psi^{\dot\alpha\ 12\dots k-1} \\ & +\theta^\alpha_{k+1}\chi^{1\dots k\ k+1}_\alpha + \dots + \theta^\alpha_N\chi^{1\dots k\ N}_\alpha \\ & +\bar\theta^1_{\dot\alpha}\bar\theta^2_{\dot\beta}\bar\psi^{(\dot\alpha\dot\beta)\ 3\dots k} + \dots + \bar\theta^{k-1}_{\dot\alpha}\bar\theta^k_{\dot\beta}\bar\psi^{(\dot\alpha\dot\beta)\ 1\dots k-2} \\ & +\theta^\alpha_{k+1}\theta^\beta_{k+2}\chi^{1\dots k\ k+1\ k+2}_{(\alpha\beta)} + \dots + \theta^\alpha_{N-1}\theta^\beta_N\chi^{1\dots k\ N-1\ N}_{(\alpha\beta)} \\ & \dots \\ & +\bar\theta^1_{\dot\alpha_1}\dots\bar\theta^k_{\dot\alpha_k}\bar\psi^{(\dot\alpha_1\dots\dot\alpha_k)} + \theta^{\alpha_1}_{k+1}\dots\theta^{\alpha_{N-k}}_N\chi_{(\alpha_1\dots\alpha_{N-k})} \\ & +\text{derivative terms}\ . \end{aligned} \tag{4.23}$$

Here all the component fields belong to totally antisymmetric irreps of SU(N), e.g., $\phi^{12\dots k}(x,u) = \phi^{[i_1 i_2 \dots i_k]}(x) u^1_{i_1} u^2_{i_2} \dots u^k_{i_k}$. Further, these fields satisfy massless field equations of the type (4.1).

We conclude this section by a remark concerning the conformal properties of the above multiplets. The $(N,k,N-k)$ analytic superfield $W^{12\dots k}$ is characterized by the SU(N) quantum numbers $m_1 = \dots = m_k = 1,\ m_{k+1} = \dots = m_{N-1} = 0$. From eqs. (2.40) we see that if

$$\ell_k = 1\ , \qquad r_k = \frac{2k}{N} - 1 \tag{4.24}$$

$W^{12\dots k}$ realizes a massless UIR of the superconformal algebra.

5 UIR's of $D = 4$ N-extended conformal supersymmetry

In this section we shall show how the complete classification of UIR's of SU$(2,2/N)$ found in [36] (see also [56]) can be obtained by multiplying the three types of massless superfields introduced in Section 4.

5.1 The three series of UIR's

The results of [36] [8] fall into three distinct series. The simplest one (called series C in [22]) is given by the following conditions:

$$\text{C)} \qquad \ell = m_1 \ , \quad r = \frac{2m}{N} - m_1 \ , \quad j_1 = j_2 = 0 \ . \tag{5.1}$$

We can construct the superfield realization of series C by multiplying massless G-analytic superfields [9] ("supersingletons") of the type (4.20):

$$W^{[a_1,\ldots,a_{N-1}]} = (W^1)^{a_1}(W^{12})^{a_2}\ldots(W^{12\ldots N-1})^{a_{N-1}} \ . \tag{5.2}$$

Since each factor in (5.2) satisfies the usual harmonic irreducibility constraints, the same is true for the product:

$$D^I_J W^{[a_1,\ldots,a_{N-1}]} = 0 \ , \qquad 1 \le J < I \le N \ . \tag{5.3}$$

As a result, the lowest component of the superfield (5.2) is an irrep of SU(N) with Dynkin labels $[a_1,\ldots,a_{N-1}]$. This is easily seen by realizing that: i) all the SU(N) indices projected with harmonics u^K_i for a given K are symmetrized; ii) their total number is $m_K = \sum_{i=K}^{N-1} a_i$; iii) the harmonic conditions (5.3) remove all symmetrizations between indices projected with different harmonics u^K_i and u^L_i. All this reproduces the structure of a Young tableau with numbers of boxes $(m_1, m_2, \ldots, m_{N-1})$, i.e. Dynkin labels $[a_1,\ldots,a_{N-1}]$.

Further, from (4.24) we find $\ell = \sum_{k=1}^{N-1} a_k\ell_k = m_1$ and $r = \sum_{k=1}^{N-1} a_k r_k = \frac{2m}{N} - m_1$, which exactly reproduces (5.1). Thus, we have proved that the complete series C is realized by the product (5.2) of massless multiplets.

We remark that for a generic choice of the Dynkin labels the superfield (5.2) is $(N,1,1)$ G-analytic. However, if the first $p-1$ or the last $q-1$ (or both) factors in (5.2) are absent, i.e., if the corresponding Dynkin labels vanish, we obtain further analyticity conditions of the type (N,p,q), in accord with (3.27). We should mention that in ref. [36] a list of the

[8] Our conventions differ from those of [36] in the following sense: $r \to -r$, $2m/N \to 2m_1 - 2m/N$.

[9] Series of operators obtained as powers of the $N = 4$ super-Yang-Mills field strength considered as a G-analytic harmonic superfield were introduced in [57]. They were identified with short multiplets of $SU(2,2/4)$ and their correspondence with the K-K spectrum of IIB supergravity was established in [58].

possible superconformal differential conditions on superfields is given. There one only finds $(N,1,1)$ G-analyticity conditions, but this can be explained by the above observation.

The second series (called B in [22]) is given by the following conditions:

$$\text{B)}\qquad \ell = -r + \frac{2m}{N} \geq 2 + 2j_1 + r + 2m_1 - \frac{2m}{N}\ , \quad j_2 = 0 \tag{5.4}$$

(or $j_1 \to j_2$, $r \to -r$, $\frac{2m}{N} \to 2m_1 - \frac{2m}{N}$). It can be obtained by multiplying the G-analytic massless superfield (5.2) by left-handed chiral ones as follows:

$$w_{\alpha_1\ldots\alpha_{2j_1}}\ \Phi^k\ W^{[a_1,\ldots,a_{N-1}]} \tag{5.5}$$

where $k \geq 0$ is an integer. The first factor in (5.5) brings in the Lorentz spin $(j_1, 0)$. The second factor adjusts the dimension and R charge of the series,

$$\ell = 1 + j_1 + m_1 + k\ , \qquad r = -1 - j_1 - k - m_1 + \frac{2m}{N}\ , \tag{5.6}$$

so that they exactly match (5.4). The conformal bound in (5.4) is obtained for $k = 0$, i.e. without employing any scalar chiral superfields. The alternative series of this type is obtained by replacing chiral by antichiral superfields.

Finally, the most general series (called A in [22]) is given by the following conditions:

$$\text{A)}\qquad \ell \geq 2 + 2j_2 - r + \frac{2m}{N} \geq 2 + 2j_1 + r + 2m_1 - \frac{2m}{N} \tag{5.7}$$

(or $j_1 \to j_2$, $r \to -r$, $\frac{2m}{N} \to 2m_1 - \frac{2m}{N}$). This series is obtained by multiplying together all possible types of massless superfields:

$$w_{\alpha_1\ldots\alpha_{2j_1}}\ \bar{w}_{\dot\alpha_1\ldots\dot\alpha_{2j_2}}\ \Phi^k\ \bar\Phi^s\ W^{[a_1,\ldots,a_{N-1}]} \tag{5.8}$$

where $k \geq s \geq 0$ are integers. This time we find

$$\ell = 2 + j_1 + j_2 + m_1 + k + s\ , \qquad r = j_2 - j_1 - k + s - m_1 + \frac{2m}{N} \tag{5.9}$$

which corresponds to (5.7). The two conformal bounds in (5.7) are saturated for $s = 0$ or $k = s = 0$, i.e. without employing one or the other type (or both) of scalar chiral superfields. These bounds correspond to superfields satisfying differential constraints, as explained in Section 5.3. The alternative series is obtained by taking $s \geq k \geq 0$.

Note that in the abstract series (5.4) and (5.7) the dimension ℓ and R charge r can be any real numbers. In order to account for this, the powers k and s in (5.5) and (5.8) will have to take non-integer values. This does not happen for series C where ℓ is always integer and r is rational.

One final remark concerns the unitarity of the above series of representations. Earlier we mentioned that the massless multiplets (supersingletons) are known to be UIR's of the superconformal algebra. Then it is clear that by multiplying them as we did above we automatically obtain series of UIR's.

5.2 Series obtained from one type of supersingleton

In Section 5.1 we used all possible G-analytic supersingletons $W^{12\ldots n}$ with $1 \leq n \leq N-1$ to reproduce the complete series C. An alternative approach is to use different realizations of the same type of supersingleton (i.e., for a fixed value of n). We presented a similar construction in [22], where we only considered the case $n = N/2$ (for even N). The generalization is straightforward. The result is a series of UIR's which is a particular case of the series B above.

The supersingleton $W^{12\ldots n}$ can be equivalently rewritten by choosing different harmonic projections of its SU(N) indices and, consequently, different sets of G-analyticity constraints. This amounts to superfields of the type

$$W^{I_1 I_2 \ldots I_n}(\theta_{J_{n+1}}, \ldots, \theta_{J_N}, \bar{\theta}^{I_1}, \ldots, \bar{\theta}^{I_n}) \tag{5.10}$$

where $I_1, \ldots, I_n$ and $J_{n+1}, \ldots, J_N$ are two complementary sets of N indices. Each of these superfields depends on $2N$ Grassmann variables, i.e. half of the total number of $4N$. This is the minimal size of a G-analytic superspace, so we can say that the W's are the "shortest" superfields (superconformal multiplets).

The idea now is to start multiplying different versions of the W's of the type (5.10) (for a fixed value of n) in order to obtain composite objects depending on various numbers of odd variables. The following choice of W's and of the order of multiplication covers all possible intermediate types of G-analyticity:

$$\begin{aligned}
&A(p_1, p_2, \ldots, p_{N-1}) \\
&= [W^{1\ldots n}(\theta_{n+1\ldots N}\bar{\theta}^{1\ldots n})]^{p_1+\ldots+p_{N-1}} \\
&\times [W^{1\ldots n-1\ n+1}(\theta_{\underline{n}\ n+2\ldots N}\bar{\theta}^{1\ldots n-1\ \underline{n+1}})]^{p_2+\ldots+p_{N-1}}
\end{aligned}$$

$$\times[W^{1\dots n-1\,n+2}(\theta_{n\,n+1\,n+3\dots N}\bar\theta^{1\dots n-1\,\underline{n+2}})]^{p_3+\dots+p_{N-1}}$$
$$\dots$$
$$\times[W^{1\dots n-1\,N-1}(\theta_{n\dots N-2\,N}\bar\theta^{1\dots n-1\,\underline{N-1}})]^{p_{N-n}+\dots+p_{N-1}}$$
$$\times[W^{1\dots n-2\,n\,n+1}(\theta_{\underline{n-1}\,n+2\dots N}\bar\theta^{1\dots n-2\,n\,n+1})]^{p_{N-n+1}+\dots+p_{N-1}}$$
$$\times[W^{1\dots n-3\,n-1\,n\,n+1}(\theta_{\underline{n-2}\,n+2\dots N}\bar\theta^{1\dots n-3\,n-1\,n\,n+1})]^{p_{N-n+2}+\dots+p_{N-1}}$$
$$\dots$$
$$\times[W^{13\dots n+1}(\theta_{\underline{2}\,n+2\dots N}\bar\theta^{13\dots n+1})]^{p_{N-2}+p_{N-1}}$$
$$\times[W^{23\dots n+1}(\theta_{\underline{1}\,n+2\dots N}\bar\theta^{23\dots n+1})]^{p_{N-1}}\ . \tag{5.11}$$

The power $\sum_{r=k}^{N-1} p_r$ of the k-th W is chosen in such a way that each new p_r corresponds to bringing in a new realization of the same supersingleton. As a result, at each step a new θ or $\bar\theta$ appears (they are underlined in (5.11)), thus adding new odd dimensions to the G-analytic superspace. The only exception of this rule is the second step at which both a new θ and a new $\bar\theta$ appear. So, the series (5.11) covers the cases $(N,n,N{-}n)$, $(N,n{-}1,N{-}n{-}1)$ and then all intermediate cases up to $(N,1,0)$.

The superfield $A(p_1,p_2,\dots,p_{N-1})$ should be submitted to the same H-analyticity constraints as one would impose on $W^{1\dots n}$ alone,

$$D^I_{I+1}A(p_1,p_2,\dots,p_{N-1})=0\ ,\qquad I=1,2,\dots,N-1\ . \tag{5.12}$$

This is clearly compatible with the G-analyticity conditions on $A(p_1,p_2,\dots,p_{N-1})$ since they form a subset of these on $W^{1\dots n}$. As before, H-analyticity makes $A(p_1,p_2,\dots,p_{N-1})$ irreducible under SU(N).

By counting the number of occurrences of each projection $1,2,\dots,N-1$ and the dimensions and R charges in (5.11), we easily find the relations

$$\ell=\sum_{k=1}^{N-1}kp_k\ ,\quad m_1=\ell-p_{N-1}\ ,\quad m=n\ell\ ,\quad r=\left(\frac{2n}{N}-1\right)\ell\ . \tag{5.13}$$

If $N=2n$ this series has no R charge. If $p_{N-1}=0$ the product (5.11) represents a G-analytic superfield and is thus a particular case of the series C. If $p_{N-1}\geq 1$ it depends on all θ's and on all $\bar\theta$'s but $\bar\theta^N$, so it is a particular case of the series B (5.6) with $j_1=0$.

Finally, the Dynkin labels of the SU(N) irrep carried by the first component of $A(p_1,p_2,\dots,p_{N-1})$ are given below:

$$a_1=p_{N-2}\ ,$$

$$
\begin{aligned}
&a_2 = p_{N-3} \ , \quad \ldots \ , \quad a_{n-2} = p_{N-n+1} \ , \\
&a_{n-1} = (N-n-2) \sum_{k=N-n+1}^{N-1} p_k + \sum_{k=2}^{N-n} (k-1) p_k \ , \\
&a_n = p_1 \ , \\
&a_{n+1} = p_2 + \sum_{k=N-n+1}^{N-1} (k-N+n) p_k \ , \\
&a_{n+2} = p_3 \ , \quad \ldots \ , \quad a_{N-2} = p_{N-n-1} \ , \\
&a_{N-1} = \sum_{k=N-n}^{N-1} p_k \ .
\end{aligned}
\tag{5.14}
$$

An interesting particular case is obtained if $a_{N-1} = 0$. This implies $p_{N-n} = \ldots = p_{N-1} = 0$, so $a_1 = \ldots = a_{n-2} = 0$. In other words, this is a G-analytic superfield of the type $(N, n-1, 2)$. The remaining Dynkin labels are $a_{n-1} = \sum_{k=2}^{N-n-1} (k-1) p_k$, $a_n = p_1$, $a_{n+1} = p_2$, $\ldots$, $a_{N-2} = p_{N-n-1}$. In general, none of these labels vanishes, therefore the harmonic coset in which this $(N, n-1, 2)$ superfield lives is not smaller than the expected one, $\mathrm{SU}(N)/[\mathrm{U}(1)]^{N-n} \times \mathrm{SU}(n-1) \times \mathrm{SU}(2)$.

5.3 Shortness conditions

In the AdS literature the term "short" applies to multiplets which do not reach their maximal spin (equal to $(j_1 + \frac{N}{2}, j_2 + \frac{N}{2})$ where (j_1, j_2) is the spin of the first component) or which contain constrained fields like, e.g., conserved vectors. Our construction of the UIR's of $\mathrm{SU}(2,2/N)$ in terms of supersingletons allows us to easily find out when and what type of "shortness" condition takes place.

To this end we recall that the building blocks w, Φ and W are all constrained superfields corresponding to the "ultrashort" supersingleton multiplets. They are either G-analytic ((4.18), (4.19)) or chiral ((4.7), (4.12)). In addition, they satisfy on-shell constraints which take the form of $\mathrm{SU}(N)$ irreducibility harmonic conditions (4.21) in the G-analytic case or are of the type (4.8) or (4.13) in the chiral case.

Now, the most general product of chiral, antichiral and G-analytic superfields as in the series A (5.8) only satisfies the harmonic constraints (4.21) (recall that w and Φ are harmonic-independent). However, there is a number of particular cases where some constraints on the θ dependence still take place.

i) The product $w_{\alpha_1\ldots\alpha_{2j_1}}\ W^{[a_1,\ldots,a_{N-1}]}$ satisfies the intersection of the constraints (4.12), (4.13) of the factor w with the G-analyticity ones of the factor W. In the generic case the latter is of the type $(N,1,1)$, so we have

$$\bar{D}^{\dot{\alpha}}_N(w_{\alpha_1\ldots\alpha_{2j_1}}\ W^{[a_1,\ldots,a_{N-1}]}) = 0\ , \tag{5.15}$$

$$D^{1\alpha}(w_{\alpha\alpha_2\ldots\alpha_{2j_1}}\ W^{[a_1,\ldots,a_{N-1}]}) = 0\ . \tag{5.16}$$

If W carries Dynkin labels like in (3.27), it is of the type (N,p,q) and, correspondingly, we obtain q equations like (5.15) and p ones like (5.16).

Similarly, the product $\Phi\ W^{[a_1,\ldots,a_{N-1}]}$ satisfies the constraints

$$\bar{D}^{\dot{\alpha}}_N(\Phi\ W^{[a_1,\ldots,a_{N-1}]}) = 0\ , \tag{5.17}$$

$$D^{1\alpha}D^1_\alpha(\Phi\ W^{[a_1,\ldots,a_{N-1}]}) = 0 \tag{5.18}$$

or more of the same type is W is (N,p,q) analytic.

ii) The bilinear products of chiral with anti-chiral superfields are current-like objects. They satisfy constraints which turn the top spin in the superfield into a conserved "current". The simplest example is the bilinear $\Phi\bar{\Phi}$:

$$D^{i\alpha}D^j_\alpha(\Phi\bar{\Phi}) = 0\ , \tag{5.19}$$

$$\bar{D}_{i\dot{\alpha}}\bar{D}^{\dot{\alpha}}_j(\Phi\bar{\Phi}) = 0\ . \tag{5.20}$$

These constraints can be weakened if we multiply $\Phi\bar{\Phi}$ by a G-analytic factor W. In this case only certain projections of (5.19) are preserved, e.g.,

$$D^{1\alpha}D^1_\alpha(\Phi\bar{\Phi}\ W^{[a_1,\ldots,a_{N-1}]}) = \bar{D}_{N\dot{\alpha}}\bar{D}^{\dot{\alpha}}_N(\Phi\bar{\Phi}\ W^{[a_1,\ldots,a_{N-1}]}) = 0\ . \tag{5.21}$$

Yet another current-like object is the bilinear $w_{\alpha_1\ldots\alpha_{2j_1}}\ \bar{w}_{\dot{\alpha}_1\ldots\dot{\alpha}_{2j_2}}$. It satisfies the constraints

$$\bar{D}^{\dot{\alpha}}_i(w_{\alpha_1\ldots\alpha_{2j_1}}\ \bar{w}_{\dot{\alpha}\dot{\alpha}_2\ldots\dot{\alpha}_{2j_2}}) = 0\ , \tag{5.22}$$

$$D^{i\alpha}(w_{\alpha\alpha_2\ldots\alpha_{2j_1}}\ \bar{w}_{\dot{\alpha}_1\ldots\dot{\alpha}_{2j_2}}) = 0\ . \tag{5.23}$$

As before, the product $w_{\alpha_1\ldots\alpha_{2j_1}}\ \bar{w}_{\dot{\alpha}_1\ldots\dot{\alpha}_{2j_2}}\ W^{[a_1,\ldots,a_{N-1}]}$ satisfies only the corresponding projections of the above.

Similarly, the bilinear $w_{\alpha_1\ldots\alpha_{2j_1}}\ \bar{\Phi}$ satisfies the constraints

$$D^{i\alpha}(w_{\alpha\alpha_2\ldots\alpha_{2j_1}}\ \bar{\Phi}) = 0\ , \tag{5.24}$$

$$\bar{D}_{i\dot{\alpha}}\bar{D}^{\dot{\alpha}}_j(w_{\alpha_1\ldots\alpha_{2j_1}}\ \bar{\Phi}) = 0\ . \tag{5.25}$$

iii) A different class of "short" objects are obtained from the most general product (5.8) of series A either by setting $s = 0$ or $j_2 = 0$ and $s = 1$. In other words, we take the current-like bilinears above and multiply them by a BPS object (i.e., product of a chiral and a G-analytic factors). The resulting objects satisfy the constraints (for a generic W):

$$\bar{D}^{\dot\alpha}_N(w_{\alpha_1\ldots\alpha_{2j_1}}\ \bar{w}_{\dot\alpha\dot\alpha_2\ldots\dot\alpha_{2j_2}}\ \Phi^k\ W^{[a_1,\ldots,a_{N-1}]}) = 0\ , \tag{5.26}$$
$$\bar{D}_{N\dot\alpha}\bar{D}^{\dot\alpha}_N(w_{\alpha_1\ldots\alpha_{2j_1}}\ \bar\Phi\ \Phi^k\ W^{[a_1,\ldots,a_{N-1}]}) = 0\ . \tag{5.27}$$

We call such objects "intermediate short". Note that they saturate the first conformal bound in (5.7). Intermediate short multiplets, as they are defined above, will also occur in $d = 6$ and $d = 3$ (see Sections 6.4 and 7.4).

5.4 BPS states of SU$(2, 2/N)$

Here we give a summary of the SU$(2, 2/N)$ multiplets which correspond to BPS states. [10] They are realized in terms of superfields which do not depend on at least one spinor coordinate. There are three distinct ways to obtain such multiplets.

5.4.1 (p, q) BPS states

Superfields which do not depend on the first p θ's and the last q $\bar\theta$'s are obtained by multiplying G-analytic objects:

$$\frac{p+q}{2N}\ \text{BPS:} \qquad W^{[0,\ldots,0,a_p,a_{p+1},\ldots,a_{N-q},0,\ldots,0]}(\theta_{p+1},\ldots,\theta_N,\bar\theta^1,\ldots,\bar\theta^{N-q})$$
$$= (W^{12\ldots p})^{a_p}(W^{12\ldots p+1})^{a_{p+1}}\ldots(W^{12\ldots N-q})^{a_{N-q}} \tag{5.28}$$

where

$$1 \le p, q \le N-1\ , \qquad p+q \le N\ . \tag{5.29}$$

Note that the fraction of supersymmetry preserved by a (p, q) BPS state ranges as follows:

$$\frac{1}{N} \le \frac{p+q}{2N} \le \frac{1}{2}\ . \tag{5.30}$$

The two end points are obtained for $p = q = 1$ and for $p + q = N$.

---

[10] Note that such BPS states have a close resemblance to BPS Poincaré multiplets in five dimensions [59], as expected by a limiting procedure.

Such states have the first $p-1$ and the last $q-1$ SU(N) Dynkin labels vanishing. The remaining quantum numbers are:

$$\ell = \sum_{k=p}^{N-q} a_k \ , \quad j_1 = j_2 = 0 \ , \quad r = \sum_{k=p}^{N-q} (\frac{2k}{N} - 1) a_k \ . \tag{5.31}$$

Generically, such superfields live in the harmonic space

$$\frac{\mathrm{SU}(N)}{[\mathrm{U}(1)]^{N-p-q+1} \times \mathrm{SU}(p) \times \mathrm{SU}(q)} \ . \tag{5.32}$$

If a subset of the Dynkin labels vanish, for instance,

$$a_{p+m} = a_{p+m+1} = \ldots = a_{N-q-n} = 0 \ , \qquad p+q+m+n \leq N \ ,$$

the coset (5.32) is further restricted to

$$\frac{\mathrm{SU}(N)}{[\mathrm{U}(1)]^{m+n} \times \mathrm{SU}(p) \times \mathrm{SU}(q) \times \mathrm{SU}(N-p-q-m-n+2)} \ . \tag{5.33}$$

5.4.2 $(0,q)$ BPS states

Superfields which do not depend on the last q $\bar\theta$'s (or, alternatively, on the first p θ's) are obtained by multiplying G-analytic objects by left- (or right-) handed chiral ones:

$$\begin{aligned} \frac{q}{2N} \text{ BPS:} \qquad & W^{[a_1,a_2,\ldots,a_{N-q},0,\ldots,0]}_{\alpha_1\ldots\alpha_{2j_1}}(\theta_1,\ldots,\theta_N,\bar\theta^1,\ldots,\bar\theta^{N-q}) \\ & = w_{\alpha_1\ldots\alpha_{2j_1}} \ \Phi^s \ (W^1)^{a_1} (W^{12})^{a_2} \ldots (W^{12\ldots N-q})^{a_{N-q}} \end{aligned} \tag{5.34}$$

where $s \geq 0$ is an integer and

$$1 \leq q \leq N-1 \ . \tag{5.35}$$

Note that the fraction of supersymmetry preserved by a $(0,q)$ BPS state ranges as follows:

$$\frac{1}{2N} \leq \frac{q}{2N} \leq \frac{N-1}{2N} \ . \tag{5.36}$$

Such states have the last $q-1$ SU(N) Dynkin labels vanishing. The remaining quantum numbers are:

$$\ell = 1 + j_1 + s + \sum_{k=p}^{N-q} a_k \ , \quad j_2 = 0 \ , \quad r = -1 - j_1 - s + \sum_{k=p}^{N-q} (\frac{2k}{N} - 1) a_k \ . \tag{5.37}$$

Generically, such superfields live in the harmonic space

$$\frac{\mathrm{SU}(N)}{[\mathrm{U}(1)]^{N-q} \times \mathrm{SU}(q)} \ . \tag{5.38}$$

If a subset of the Dynkin labels vanish, for instance,

$$a_i = 0 \ , \qquad 1 \leq n \leq N - q - 1 \ ,$$

the coset (5.32) is further restricted to

$$\frac{\mathrm{SU}(N)}{[\mathrm{U}(1)]^{N-q-n} \times \mathrm{SU}(q) \times \mathrm{SU}(n+1)} \ . \tag{5.39}$$

5.4.3 Chiral BPS states

These are described by superfields which do not depend on all of the $\bar{\theta}$'s (or, alternatively, on the θ's), i.e. which are left- (or right-) handed chiral:

$$\frac{1}{2} \ \text{BPS:} \qquad W_{\alpha_1 \ldots \alpha_{2j_1}}(\theta_1, \ldots, \theta_N) = w_{\alpha_1 \ldots \alpha_{2j_1}} \ \Phi^s \ . \tag{5.40}$$

They are SU(N) singlets. The remaining quantum numbers are:

$$\ell = 1 + j_1 + s \ , \quad j_2 = 0 \ , \quad r = -1 - j_1 - s \ . \tag{5.41}$$

The chiral superfields are harmonic-independent.

6 The six-dimensional case

The method described above can also be applied to the superconformal algebras in six dimensions. We will first examine the consequences of G-analyticity and conformal supersymmetry and find out the relation to BPS states. Then we will make a conjecture about the possible structure of the general UIR's of the superconformal algebra. We restrict ourselves to the most interesting case of $(2,0)$ conformal supersymmetry, i.e. to the superalgebra OSp($8^*/4$). Some of the results have already been presented in [24]. Our consideration could be easily extended to $(N,0)$ conformal supersymmetry with underlying superalgebra OSp($8^*, 2N$) with arbitrary N.

6.1 The conformal superalgebra OSp$(8^*/4)$ and Grassmann analyticity

The part of the conformal superalgebra OSp$(8^*/4)$ relevant to our discussion is given below:

$$\{Q^i_\alpha, Q^j_\beta\} = 2\Omega^{ij}\gamma^\mu_{\alpha\beta}P_\mu \ , \tag{6.1}$$

$$\{Q^i_\alpha, S^{\beta\, j}\} = i(\gamma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu} + 2\delta^\beta_\alpha(4T^{ij} - i\Omega^{ij}D) \ , \tag{6.2}$$

$$[T^{ij}, Q^k_\alpha] = -\frac{1}{2}(\Omega^{ki}Q^j_\alpha + \Omega^{kj}Q^i_\alpha) \ , \tag{6.3}$$

$$[T^{ij}, T^{kl}] = \frac{1}{2}(\Omega^{ik}T^{lj} + \Omega^{il}T^{kj} + \Omega^{jk}T^{li} + \Omega^{jl}T^{ki}) \tag{6.4}$$

(the commutation relations of D with Q and S are the same as in (2.4)). Here Q^i_α are the generators of Poincaré supersymmetry carrying a right-handed chiral spinor index $\alpha = 1,2,3,4$ of the Lorentz group SU$^*(4) \sim$ SO$(5,1)$ (generators $M_{\mu\nu}$) and an index $i = 1,2,3,4$ of the fundamental representation of the R symmetry group USp$(4) \sim$ SO(5) (generators $T^{ij} = T^{ji}$); $S^{\beta\, j}$ are the generators of conformal supersymmetry carrying a left-handed chiral spinor index; D is the generator of dilations and P_μ of translations. The symplectic matrix $\Omega^{ij} = -\Omega^{ji}$ has non-vanishing entries $\Omega^{14} = \Omega^{23} = -\Omega^{32} = -\Omega^{41} = 1$. The chiral spinors satisfy a pseudo-reality condition of the type $\overline{Q^i_\alpha} = \Omega^{ij}Q^\beta_j c_{\beta\alpha}$ where c is a 4×4 unitary "charge conjugation" matrix.

The standard realization of this superalgebra makes use of the superspace

$$\mathbb{R}^{6|16} = \frac{\text{OSp}(8^*/4)}{\{K,S,M,D,T\}} = (x^\mu, \theta^{\alpha\, i}) \tag{6.5}$$

where $\theta^{\alpha\, i}$ is a left-handed spinor. Unlike the four-dimensional case, here chirality is not an option but is already built in. The only way to obtain smaller superspaces is through Grassmann analyticity. We begin by imposing a single condition of G-analyticity (cf. eq. (2.20)):

$$q^1_\alpha \Phi(x,\theta) = 0 \tag{6.6}$$

which amounts to considering the coset

$$\mathbb{A}^{6|12} = \frac{\text{OSp}(8^*/4)}{\{K,S,M,D,T,Q^1\}} = (x^\mu, \theta^{\alpha\, 1,2,3}) \tag{6.7}$$

(note that with our conventions $\theta^{\alpha\,1} = \theta^\alpha_4$, $\theta^{\alpha\,2} = \theta^\alpha_3$, $\theta^{\alpha\,3} = -\theta^\alpha_2$, $\theta^{\alpha\,4} = -\theta^\alpha_1$). From the algebra (6.1)-(6.4) we obtain

$$m_{\mu\nu} = 0\ , \tag{6.8}$$
$$t^{11} = t^{12} = t^{13} = 0\ , \tag{6.9}$$
$$4t^{14} + \ell = 0\ . \tag{6.10}$$

Eq. (6.8) implies that the superfield Φ must be a Lorentz scalar. In order to interpret eqs. (6.9), (6.10), we need to split the generators of USp(4) into raising operators (positive roots), U(1) charges and lowering operators (negative roots):

$$\mathrm{USp}(4):\quad \left\{ \begin{array}{rl} \text{positive roots:} & T^{13},\ T^{22}\ \ (T^{11},\ T^{12}) \\ \mathrm{U}(1)\times\mathrm{U}(1): & H_1 = 2(T^{14} - T^{23}),\ H_2 = -2T^{23} \\ \text{negative roots:} & T^{24},\ T^{33}\ \ (T^{34},\ T^{44}) \end{array} \right. \tag{6.11}$$

(the composite positive or negative roots are in parenthesis). On the HWS of a USp(4) irrep the two U(1) charges H_1, H_2 take eigenvalues equal to the Dynkin labels a_1, a_2 of the irrep. For instance, the generator Q^1 is the HWS of the fundamental irrep $(1,0)$.

Now it becomes clear that (6.9) is part of the USp(4) irreducibility conditions whereas (6.10) relates the conformal dimension to the Dynkin labels:

$$\ell = 2(a_1 + a_2)\ . \tag{6.12}$$

Let us denote the highest-weight UIR's of the OSp($8^*/4$) algebra by

$$\mathcal{D}(\ell; J_1, J_2, J_3; a_1, a_2)$$

where ℓ is the conformal dimension, J_1, J_2, J_3 are the SU*(4) Dynkin labels and a_1, a_2 are the USp(4) Dynkin labels of the first component. Then the G-analytic superfields defined above are of the type

$$\Phi(\theta^{1,2,3})\ \Leftrightarrow\ \mathcal{D}(2(a_1+a_2); 0,0,0; a_1, a_2)\ . \tag{6.13}$$

The next step is to add the generator Q^2_α to the superspace coset denominator:

$$\mathbb{A}^{6|8} = \frac{\mathrm{OSp}(8^*/4)}{\{K, S, M, D, T, Q^1, Q^2\}} = (x^\mu, \theta^{\alpha\,1,2})\ . \tag{6.14}$$

This implies the constraints

$$4t^{23} + \ell = 0 \quad \Rightarrow \; a_1 = 0 \; , \tag{6.15}$$
$$t^{24} = 0 \; . \tag{6.16}$$

Note that the vanishing of the lowering operator t^{24} means that the subalgebra SU(2) $\subset$ USp(4) formed by t^{13}, t^{24} and $t^{14} - t^{23}$ acts trivially on the particular USp(4) irreps. This is equivalent to setting $a_1 = 0$, as in (6.15). Thus, the new G-analytic superfields are of the type

$$\Phi(\theta^{1,2}) \; \Leftrightarrow \; \mathcal{D}(2a_2; 0,0,0; 0, a_2) \; . \tag{6.17}$$

From (6.1) it is clear that we cannot have any further G-analyticity constraints. We can summarize the above discussion by saying that the superconformal algebra OSp(8^*/4) admits the following short UIR's corresponding to BPS states:

$$1/2 \text{ BPS}: \qquad \mathcal{D}(2a_2; 0,0,0; 0, a_2) \; ; \tag{6.18}$$
$$1/4 \text{ BPS}: \qquad \mathcal{D}(2(a_1 + a_2); 0,0,0; a_1, a_2) \; . \tag{6.19}$$

6.2 Supersingletons

There exist four types of massless multiplets in six dimensions corresponding to ultrashort UIR's (supersingletons) of OSp(8^*/4) [60]. All of them can be formulated in terms of constrained superfields as follows.

(i) The first type is described by a (real) superfield $W^{\{ij\}}(x,\theta)$ antisymmetric and traceless in the external USp(4) indices. It satisfies the constraint [61] (see also [62])

$$D^{(k}_\alpha W^{\{i)j\}} = 0 \qquad \Rightarrow \; \mathcal{D}(2; 0,0,0; 0,1) \tag{6.20}$$

where the spinor covariant derivatives satisfy the supersymmetry algebra

$$\{D^i_\alpha, D^j_\beta\} = -2i\Omega^{ij}\gamma^\mu_{\alpha\beta}\partial_\mu \; . \tag{6.21}$$

The components of this superfield are massless fields forming the on-shell tensor $(2,0)$ multiplet in six dimensions [61, 63].

(ii) The second type is described by a superfield $W^i(x,\theta)$ which is in the fundamental UIR of USp(4). The corresponding constraint is

$$D^{(k}_\alpha W^{i)} = 0 \qquad \Rightarrow \; \mathcal{D}(2; 0,0,0; 1,0) \; . \tag{6.22}$$

(iii) The third type is the only one described by a (real) superfield without external indices, $w(x,\theta)$. The corresponding constraint is second-order in the spinor derivatives:

$$D^{(i}_{\alpha}D^{j)}_{\beta}w = 0 \qquad \Rightarrow \; \mathcal{D}(2;0,0,0;0,0) \; . \tag{6.23}$$

(iv) Finally, there exists a series of multiplets described by superfields with n totally symmetrized external Lorentz spinor indices, $w_{(\alpha_1\ldots\alpha_n)}(x,\theta)$. These superfields can be made real in the case $n = 2k$. Now the constraint takes the form

$$D^i_{[\beta}w_{(\alpha_1]\ldots\alpha_n)} = 0 \qquad \Rightarrow \; \mathcal{D}(2+n/2;n,0,0;0,0) \; . \tag{6.24}$$

As shown in ref. [24], the six-dimensional massless conformal fields only carry reps $(J_1, 0)$ of the little group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ of a light-like particle momentum. This results applies to all $(N, 0)$ $d = 6$ superconformal algebras and is related to the analysis of conformal fields in d dimensions [64]. This fact implies that massless superconformal multiplets are classified by a single SU(2) and $\mathrm{USp}(2N)$ R-symmetry and are therefore identical to massless super-Poincaré multiplets in five dimensions. Some physical implication of the above circumstance have recently been discussed in ref. [65] where it was suggested that certain strongly coupled $d = 5$ theories effectively become six-dimensional.

6.3 Harmonic superspace

The massless multiplets *(i)*-*(iii)* admit an alternative formulation in harmonic superspace [66, 67, 68]. The advantage of this formulation is that the constraints (6.20), (6.22) become conditions for G-analyticity. We introduce harmonic variables describing the coset $\mathrm{USp}(4)/[\mathrm{U}(1)]^2$:

$$u \in \mathrm{USp}(4) : \qquad u^I_i u^i_J = \delta^I_J \; , \;\; u^I_i \Omega^{ij} u^J_j = \Omega^{IJ} \; , \;\; u^I_i = (u^i_I)^* \; . \tag{6.25}$$

Here the indices i, j belong to the fundamental representation of USp(4) and I, J are labels corresponding to the $[\mathrm{U}(1)]^2$ projections. The harmonic derivatives

$$D^{IJ} = \Omega^{K(I} u^{J)}_i \frac{\partial}{\partial u^K_i} \tag{6.26}$$

form the algebra of $\mathrm{USp}(4)_R$ (see (6.4)) realized on the indices I, J of the harmonics.

Let us now project the defining constraint (6.20) of the $(2,0)$ tensor multiplet with the harmonics $u^1_k u^1_i u^2_j$ and $u^2_k u^2_i u^1_j$:

$$D^1_\alpha W^{12} = D^2_\alpha W^{12} = 0 \tag{6.27}$$

where $D^{1,2}_\alpha = D^i_\alpha u^{1,2}_i$ and $W^{12} = W^{\{ij\}} u^1_i u^2_j$. In other words, the constraint (6.20) now takes the form of a G-analyticity condition. In the appropriate basis in superspace one obtains a short superfield depending on half of the odd coordinates:

$$W^{12}(x_A, \theta^1, \theta^2, u) \ . \tag{6.28}$$

In addition to (6.27), the projected superfield W^{12} clearly satisfies the USp(4) harmonic irreducibility conditions

$$D^{13} W^{12} = D^{22} W^{12} = 0 \tag{6.29}$$

(only the simple roots of USp(4) are shown). The equivalence between the two forms of the constraint follows from the obvious properties of the harmonic products $u^1_{[k} u^1_{i]} = u^2_{[k} u^2_{i]} = 0$ and $\Omega^{ij} u^1_i u^2_j = 0$. The harmonic constraints (6.29) make the superfield ultrashort.

One can treat the case *(ii)* in the same way. Projecting the constraint (6.22) with $u^1_k u^1_i$ we obtain the following constraint of G-analyticity:

$$D^1_\alpha W^1 = 0 \ \Rightarrow \ W^1(\theta^1, \theta^2, \theta^3) \ . \tag{6.30}$$

In addition, one has to impose the conditions of USp(4) irreducibility, $D^{13} W^1 = D^{22} W^1 = 0$, after which the superfield becomes ultrashort.

Finally, in case (iii), projecting the constraint (6.23) with $u^I_i u^I_j$ where $I = 1,2,3,4$ (no summation), we obtain the condition

$$D^I_\alpha D^I_\beta w = 0 \ . \tag{6.31}$$

It implies that the superfield w is *linear* in each projection $\theta^{\alpha I}$.

6.4 Series of UIR's of OSp$(8^*/4)$ and shortening

It is now clear that we can realize the two BPS series of UIR's (6.18) and (6.19) as products of the two types of G-analytic superfields (supersingletons) (6.27) and (6.30):

$$\text{BPS :} \qquad [W^1(\theta^1, \theta^2, \theta^3)]^{a_1} [W^{12}(\theta^1, \theta^2)]^{a_2} \ . \tag{6.32}$$

As a bonus, we also prove the unitarity of these series, since they are obtained by multiplying massless unitary multiplets (supersingletons).

We remark that our harmonic coset USp(4)/[U(1)]^2 is effectively reduced to USp(4)/U(2) if $a_1 = 0$. Indeed, such UIR's of USp(4) have the property that they are annihilated by the lowering operator T^{24} (see (6.16)). The latter, together with the raising one T^{13} and the U(1) charge H_1, form an SU(2) subalgebra of USp(4) which acts trivially on such representations. So, the 1/2 BPS states (6.18) are associated to the harmonic coset USp(4)/U(2). We note that this smaller harmonic space was used in Ref. [68] to formulate the $(2,0)$ tensor multiplet.

We are not aware of an exhaustive study of the most general UIR's of OSp(8^*/4) similar to one of Ref. [36] for the case of $SU(2,2/N)$. Yet, based on our experience from Section 5, we can make the following conjecture. The generic UIR is likely to be obtained by multiplying all four types of supersingletons above:

$$w_{\alpha_1\ldots\alpha_{m_1}} w_{\beta_1\ldots\beta_{m_2}} w_{\gamma_1\ldots\gamma_{m_3}}\ w^k\ (W^1)^{a_1}(W^{12})^{a_2} \tag{6.33}$$

where $m_1 \geq m_2 \geq m_3$ and the spinor indices are (anti)symmetrized so that they form the SU*(4) UIR with Young tableau (m_1, m_2, m_3) or Dynkin labels $[J_1, J_2, J_3]$. The corresponding UIR of OSp(8^*/4) is

$$\mathcal{D}(6 + \frac{1}{2}(J_1 + 2J_2 + 3J_3) + 2(k + a_1 + a_2); J_1, J_2, J_3; a_1, a_2)\ . \tag{6.34}$$

We can have four distinct series:

$$\begin{aligned}
&\text{A)} \qquad m_1 m_2 m_3 \neq 0\ , \qquad \ell \geq 6 + \frac{1}{2}(J_1 + 2J_2 + 3J_3) + 2(a_1 + a_2)\ ;\\
&\text{B)} \qquad m_1 m_2 \neq 0\ ,\ m_3 = 0\ , \qquad \ell \geq 4 + \frac{1}{2}(J_1 + 2J_2) + 2(a_1 + a_2)\ ;\\
&\text{C)} \qquad m_1 \neq 0\ ,\ m_2 = m_3 = 0\ , \qquad \ell \geq 2 + \frac{1}{2}J_1 + 2(a_1 + a_2)\ ;\\
&\text{D)} \qquad m_1 = m_2 = m_3 = 0\ , \qquad \ell = 2(a_1 + a_2)\ .
\end{aligned} \tag{6.35}$$

In cases A, B, C the conformal bound is saturated when $k = 0$ in (6.33), i.e., when no scalar supersingleton w is used.

In the generic case the multiplet (6.33) is "long", but in some special cases certain shortening can take place. Repeating the argument of Section 5.3, we find three such cases:

i) The product of a supersingleton of type (iii) or (iv) with a BPS object satisfies the intersection of the defining constraints (6.24), (6.23), (6.30) and (6.27) of the four building blocks, e.g.,

$$D^1_{[\beta}\left(w_{\alpha_1]\ldots\alpha_n}\ (W^1)^{a_1}(W^{12})^{a_2}\right) = 0\ , \tag{6.36}$$

$$D^1_{\alpha}D^1_{\beta}\left(w\ (W^1)^{a_1}(W^{12})^{a_2}\right) = 0\ . \tag{6.37}$$

ii) The bilinear products of supersingletons are current-like objects. Apart from the ones involving BPS objects, these are:

$$\begin{aligned}
&D^{(i}_{\alpha}D^{j}_{\beta}D^{k)}_{\gamma}(w^2) = 0\ ;\\
&D^{(i}_{[\beta}D^{j)}_{\gamma}(w\ w_{\alpha_1]\ldots\alpha_n}) = 0\ ;\\
&D^{i}_{\underline{\gamma}}(w_{\underline{\alpha_1}\ldots\alpha_{m_1}}w_{\underline{\beta_1}\ldots\beta_{m_2}}) = 0
\end{aligned} \tag{6.38}$$

(underlining the indices in the third equation means projecting out the Young tableau $(m_1, m_2, 1)$).

iii) Intermediate short objects are obtained by multiplying the currents (6.38) by a BPS object. They satisfy the corresponding projections of eqs. (6.38).

6.5 BPS states of OSp$(8^*/4)$

Here we give a summary of the SU$(2,2/N)$ multiplets which correspond to BPS states. They are realized as products of the two types of G-analytic superfields.

6.5.1

$$\frac{1}{4}\ \text{BPS}: \qquad W^{[a_1,a_2]}(\theta^1,\theta^2,\theta^3) = (W^1)^{a_1}(W^{12})^{a_2}\ . \tag{6.39}$$

These superfields can carry arbitrary USp(4) quantum numbers and have dimension $\ell = 2(a_1+a_2)$ and vanishing spin. In the generic case they live on the harmonic coset

$$\frac{\text{USp}(4)}{[\text{U}(1)]^2} \tag{6.40}$$

If $a_2 = 0$ this coset becomes USp(4)/U(2).

6.5.2

$$\frac{1}{2}\ \mathrm{BPS}: \qquad W^{[0,a_2]}(\theta^1,\theta^2) = (W^{12})^{a_2}\ . \tag{6.41}$$

This time the first USp(4) Dynkin label vanishes. The dimension is $\ell = 2a_2$ and the spin is zero. The harmonic coset is

$$\frac{\mathrm{USp}(4)}{\mathrm{U}(2)}\ . \tag{6.42}$$

7 The three-dimensional case

In this section we carry out the analysis of the $d=3$ $N=8$ superconformal algebra $\mathrm{OSp}(8/4,\mathbb{R})$ in a way similar to the above. Some of the results have already been presented in [23]. As in the previous cases, our results could easily be extended to $\mathrm{OSp}(N/4,\mathbb{R})$ superalgebras with arbitrary N.

7.1 The conformal superalgebra OSp$(8/4,\mathbb{R})$ and Grassmann analyticity

The part of the conformal superalgebra $\mathrm{OSp}(8/4,\mathbb{R})$ relevant to our discussion is given below:

$$\{Q^i_\alpha, Q^j_\beta\} = 2\delta^{ij}\gamma^\mu_{\alpha\beta}P_\mu\ , \tag{7.1}$$
$$\{Q^i_\alpha, S^j_\beta\} = \delta^{ij}M_{\alpha\beta} + 2\epsilon_{\alpha\beta}(T^{ij} + \delta^{ij}D)\ , \tag{7.2}$$
$$[T^{ij}, Q^k_\alpha] = i(\delta^{ki}Q^j_\alpha - \delta^{kj}Q^i_\alpha)\ , \tag{7.3}$$
$$[T^{ij}, T^{kl}] = i(\delta^{ik}T^{jl} + \delta^{jl}T^{ik} - \delta^{jk}T^{il} - \delta^{il}T^{jk})\ . \tag{7.4}$$

Here we find the following generators: Q^i_α of $N=8$ Poincaré supersymmetry carrying a spinor index $\alpha = 1,2$ of the $d=3$ Lorentz group $\mathrm{SL}(2,\mathbb{R}) \sim \mathrm{SO}(1,2)$ (generators $M_{\alpha\beta} = M_{\beta\alpha}$) and a vector [11] index $i = 1,\ldots,8$ of the R symmetry group SO(8) (generators $T^{ij} = -T^{ji}$); S^i_α of conformal supersymmetry; P_μ, $\mu = 0,1,2$, of translations; D of dilations.

[11]Since SO(8) has three 8-dimensional representations, 8_v, 8_s and 8_c related by triality, the choice which one to ascribe to the supersymmetry generators is purely conventional. In order to be consistent with the other N-extended $d=3$ supersymmetries where the odd generators always belong to the vector representation, we prefer to put an 8_v index i on the supercharges.

The standard realization of this superalgebra makes use of the superspace

$$\mathbb{R}^{3|16} = \frac{\mathrm{OSp}(8/4,\mathbb{R})}{\{K,S,M,D,T\}} = (x^\mu, \theta^{\alpha\, i}) \ . \tag{7.5}$$

In order to study G-analyticity we need to decompose the generators Q^i_α under $[\mathrm{U}(1)]^4 \subset \mathrm{SO}(8)$. Besides the vector representation 8_v of SO(8) we are also going to use the spinor ones, 8_s and 8_c. In this context we find it convenient to introduce the four subgroups U(1) by successive reductions: $\mathrm{SO}(8) \ \rightarrow \ \mathrm{SO}(2)\times\mathrm{SO}(6) \sim \mathrm{U}(1)\times\mathrm{SU}(4) \ \rightarrow \ [\mathrm{SO}(2)]^2\times\mathrm{SO}(4) \sim [\mathrm{U}(1)]^2\times \mathrm{SU}(2)\times\mathrm{SU}(2) \ \rightarrow \ [\mathrm{SO}(2)]^4 \sim [\mathrm{U}(1)]^4$. Denoting the four U(1) charges by $\pm$, $(\pm)$, $[\pm]$ and $\{\pm\}$, we decompose the three 8-dimensional representations as follows:

$$\begin{aligned}
8_v: &\quad Q^i \ \rightarrow \ Q^{\pm\pm},\ Q^{(\pm\pm)} \ ; \ Q^{[\pm]\{\pm\}}, && (7.6)\\
8_s: &\quad \phi^a \ \rightarrow \ \phi^{+(+)[\pm]},\ \phi^{-(-)[\pm]},\ \phi^{+(-)\{\pm\}} \ ; \ \phi^{-(+)\{\pm\}} && (7.7)\\
8_c: &\quad \sigma^{\dot a} \ \rightarrow \ \sigma^{+(+)\{\pm\}},\ \sigma^{-(-)\{\pm\}},\ \sigma^{+(-)[\pm]} \ .\ \sigma^{-(+)[\pm]} && (7.8)
\end{aligned}$$

The definition of the charge operators H_i, $i = 1,2,3,4$ can be read off from the corresponding projections of the relation (7.2):

$$\begin{aligned}
\{Q^{++}_\alpha, S^{--}_\beta\} &= \frac{1}{2}M_{\alpha\beta} + \epsilon_{\alpha\beta}(D - \frac{1}{2}H_1) \ ,\\
\{Q^{(++)}_\alpha, S^{(--)}_\beta\} &= \frac{1}{2}M_{\alpha\beta} + \epsilon_{\alpha\beta}(D - \frac{1}{2}H_2) \ ,\\
\{Q^{[+]\{+\}}_\alpha, S^{[-]\{-\}}_\beta\} &= \frac{1}{2}M_{\alpha\beta} + \epsilon_{\alpha\beta}(D - \frac{1}{2}H_3 - \frac{1}{2}H_4) \ ,\\
\{Q^{[+]\{-\}}_\alpha, S^{[-]\{+\}}_\beta\} &= -\frac{1}{2}M_{\alpha\beta} - \epsilon_{\alpha\beta}(D - \frac{1}{2}H_3 + \frac{1}{2}H_4) \ .
\end{aligned} \tag{7.9}$$

In this notation we have

$$\begin{aligned}
&[H_1, Q^{\pm\pm}_\alpha] = [H_2, Q^{(\pm\pm)}_\alpha] = \pm 2i Q^{\pm\pm}_\alpha \ ,\\
&[H_3, Q^{[\pm]\{\pm\}}] = [H_4, Q^{[\pm]\{\pm\}}] = \pm i Q^{[\pm]\{\pm\}} \ .
\end{aligned} \tag{7.10}$$

Let us denote a quasi primary superconformal field of the $\mathrm{OSp}(8/4,\mathbb{R})$ algebra by the quantum numbers of its HWS:

$$\mathcal{D}(\ell; J; a_1, a_2, a_3, a_4) \tag{7.11}$$

where ℓ is the conformal dimension, J is the Lorentz spin and a_i are the Dynkin labels (see, e.g., [69]) of the SO(8) R symmetry. In fact, in our scheme the natural labels are the four charges h_i (the eigenvalues of H_i). They are related to the Dynkin labels as follows:

$$\begin{aligned} h_1 &= 2(a_1 + a_2) + a_3 + a_4 \ , \\ h_2 &= 2a_2 + a_3 + a_4 \ , \\ h_3 &= a_3 \ , \quad h_4 = a_4 \ , \end{aligned} \tag{7.12}$$

or, inversely,

$$a_1 = \frac{1}{2}(h_1 - h_2) \ , \quad a_2 = \frac{1}{2}(h_2 - h_3 - h_4) \ , \quad a_3 = h_3 \ , \quad a_4 = h_4 \ . \tag{7.13}$$

A HWS $|a_i\rangle$ of SO(8) is by definition annihilated by the positive simple roots of the SO(8) algebra:

$$T^{[++]}|a_i\rangle = T^{\{++\}}|a_i\rangle = T^{++(--)}|a_i\rangle = T^{(++)[-]\{-\}}|a_i\rangle = 0 \ . \tag{7.14}$$

In order to build G-analytic superspaces we have to add one or more projections of Q^i_α to the coset denominator. In choosing the subset of projections we have to make sure that: i) they anticommute among themselves; ii) the subset is closed under the action of the raising operators of SO(8) (7.14). Then we have to examine the consistency of the vanishing of the chosen projections with the conformal superalgebra (7.9). Thus we find the following sequence of G-analytic superspaces corresponding to BPS states:

$$\frac{1}{8}\ \text{BPS}: \quad \begin{cases} q^{++}_\alpha \Phi = 0 \ \rightarrow \\ \Phi(\theta^{++}, \theta^{(\pm\pm)}, \theta^{[\pm]\{\pm\}}) \\ \mathcal{D}(a_1 + a_2 + \frac{1}{2}(a_3 + a_4); 0; a_1, a_2, a_3, a_4) \end{cases} \tag{7.15}$$

$$\frac{1}{4}\ \text{BPS}: \quad \begin{cases} q^{++}_\alpha \Phi = q^{(++)}_\alpha \Phi = 0 \ \rightarrow \\ \Phi(\theta^{++}, \theta^{(++)}, \theta^{[\pm]\{\pm\}}) \\ \mathcal{D}(a_2 + \frac{1}{2}(a_3 + a_4); 0; 0, a_2, a_3, a_4) \end{cases} \tag{7.16}$$

$$\frac{3}{8}\ \text{BPS}: \quad \begin{cases} q^{++}_\alpha \Phi = q^{(++)} \Phi = q^{[+]\{+\}}_\alpha \Phi = 0 \ \rightarrow \\ \Phi(\theta^{++}, \theta^{(++)}, \theta^{[+]\{\pm\}}, \theta^{[-]\{+\}}) \\ \mathcal{D}(\frac{1}{2}(a_3 + a_4); 0; 0, 0, a_3, a_4) \end{cases} \tag{7.17}$$

$$\frac{1}{2}\ \text{BPS (type I)}: \quad \begin{cases} q^{++}_\alpha \Phi = q^{(++)}_\alpha \Phi = q^{[+]\{\pm\}}_\alpha \Phi = 0 \ \rightarrow \\ \Phi(\theta^{++}, \theta^{(++)}, \theta^{[+]\{\pm\}}) \\ \mathcal{D}(\frac{1}{2}a_3; 0; 0, 0, a_3, 0) \end{cases} \tag{7.18}$$

$$\frac{1}{2}\ \text{BPS (type II)}: \qquad \left\{ \begin{array}{l} q_\alpha^{++}\Phi = q_\alpha^{(++)}\Phi = q_\alpha^{[\pm]\{+\}}\Phi = 0 \ \rightarrow \\ \Phi(\theta^{++}, \theta^{(++)}, \theta^{[\pm]\{+\}}) \\ \mathcal{D}(\frac{1}{2}a_4; 0; 0, 0, 0, a_4) \end{array} \right. \tag{7.19}$$

Note the existence of two types of 1/2 BPS states due to the two possible subsets of projections of q^i closed under the raising operators of SO(8) (7.14).

We remark that in the cases 1/4, 3/8 and 1/2 the states are annihilated by some of the lowering operators of SO(8). This means that certain subalgebras of SO(8) act trivially on them:

$$\frac{1}{4}:\ \text{SU}(2) \ \leftrightarrow \ \left\{ T^{++(--)}\ ,\ T^{--(++)}\ ,\ H_1 - H_2 \right. \tag{7.20}$$

$$\frac{3}{8}:\ \text{SU}(3) \ \leftrightarrow \ \left\{ \begin{array}{l} T^{++(--)}\ ,\ T^{--(++)}\ ,\ H_1 - H_2 \\ T^{(++)[-]\{-\}}\ ,\ T^{(--)[+]\{+\}}\ ,\ H_2 - H_3 - H_4 \end{array} \right. \tag{7.21}$$

$$\frac{1}{2}:\ \text{SU}(4)_I \ \leftrightarrow \ \left\{ \begin{array}{l} T^{++(--)}\ ,\ T^{--(++)}\ ,\ H_1 - H_2 \\ T^{(++)[-]\{-\}}\ ,\ T^{(--)[+]\{+\}}\ ,\ H_2 - H_3 - H_4 \\ T^{\{++\}}\ ,\ T^{\{--\}}\ ,\ H_4 \end{array} \right. \tag{7.22}$$

$$\frac{1}{2}:\ \text{SU}(4)_{II} \ \leftrightarrow \ \left\{ \begin{array}{l} T^{++(--)}\ ,\ T^{--(++)}\ ,\ H_1 - H_2 \\ T^{(++)[-]\{-\}}\ ,\ T^{(--)[+]\{+\}}\ ,\ H_2 - H_3 - H_4 \\ T^{[++]}\ ,\ T^{[--]}\ ,\ H_3 \end{array} \right. \tag{7.23}$$

These properties are equivalent to the restrictions on the possible values of the SO(8) Dynkin labels in (7.15)-(7.19). Note that the existence of two types of 1/2 BPS states can be equivalently explained by the two possible ways to embed SU(4) in SO(8), as shown in (7.22) and (7.23).

7.2 Supersingletons and harmonic superspace

The supersingletons are the simplest $\text{OSp}(8/4,\mathbb{R})$ representations of the type (7.18) or (7.19) and correspond to $\mathcal{D}(1/2; 0; 0, 0, 1, 0)$ or $\mathcal{D}(1/2; 0; 0, 0, 0, 1)$. The existence of two distinct types of $d = 3$ $N = 8$ supersingletons has first been noted in Ref. [70]. Each of them is just a collection of eight Dirac supermultiplets [32] made out of "Di" and "Rac" singletons [31].

In order to realize the supersingletons in superspace we note that the HWS in the two supermultiplets above has spin 0 and the Dynkin labels of the 8_s or 8_c of SO(8), correspondingly. Therefore we take a scalar superfield $\Phi_a(x^\mu, \theta^\alpha_i)$ (or $\Sigma_{\dot{a}}(x^\mu, \theta^\alpha_i)$) carrying an external 8_s index a (or an 8_c index $\dot{a}$).

These superfields are subject to the following on-shell constraints [12]:

$$\text{type I:} \qquad D^i_\alpha \Phi_a = \frac{1}{8} \gamma^i_{a\dot b} \tilde\gamma^j_{\dot b c} D^j_\alpha \Phi_c \ ; \tag{7.24}$$

$$\text{type II:} \qquad D^i_\alpha \Sigma_{\dot a} = \frac{1}{8} \tilde\gamma^i_{\dot a b} \gamma^j_{b\dot c} D^j_\alpha \Sigma_{\dot c} \ . \tag{7.25}$$

The two multiplets consist of a massless scalar in the 8_s (8_c) and spinor in the 8_c (8_s).

The harmonic superspace description of these supersingletons can be realized by taking the harmonic coset [13]

$$\frac{\mathrm{SO}(8)}{[\mathrm{SO}(2)]^4} \sim \frac{\mathrm{Spin}(8)}{[\mathrm{U}(1)]^4} \ . \tag{7.26}$$

Since $\mathrm{SO}(8) \sim \mathrm{Spin}(8)$ has three inequivalent fundamental representations, $8_s, 8_c, 8_v$, following [75] we introduce three sets of harmonic variables:

$$u^A_a \ , \ w^{\dot A}_{\dot a} \ , \ v^I_i \tag{7.27}$$

where A, $\dot A$ and I denote the decompositions of an 8_s, 8_c and 8_v index, correspondingly, into sets of four U(1) charges (see (7.6)-(7.8)). Each of the 8×8 real matrices (7.27) belongs to the corresponding representation of $\mathrm{SO}(8) \sim \mathrm{Spin}(8)$. This implies that they are orthogonal matrices (this is a peculiarity of SO(8) due to triality):

$$u^A_a u^B_a = \delta^{AB} \ , \quad w^{\dot A}_{\dot a} w^{\dot B}_{\dot a} = \delta^{\dot A\dot B} \ , \quad v^I_i v^J_i = \delta^{IJ} \ . \tag{7.28}$$

These matrices supply three copies of the group space, and we only need one to parametrize the harmonic coset. The condition which identifies the three

[12] See also [68] for the description of a supersingleton related to ours by SO(8) triality. Superfield representations of other $OSp(N/4)$ superalgebras have been considered in [71, 72].

[13] A formulation of the above multiplet in harmonic superspace has been proposed in Ref. [68] (see also [73] and [74] for a general discussion of three-dimensional harmonic superspaces). The harmonic coset used in [68] is $Spin(8)/U(4)$. Although the supersingleton itself does indeed live in this smaller coset (see Section 7.5.4), its residual symmetry $U(4)$ would not allow us to multiply different realizations of the supersingleton. For this reason we prefer from the very beginning to use the coset (7.26) with a minimal residual symmetry.

sets [14] of harmonic variables is

$$u^A_a(\gamma^I)_{A\dot{A}}w^{\dot{A}}_{\dot{a}} = v^I_i(\gamma^i)_{a\dot{a}} \ . \tag{7.29}$$

Further, we introduce harmonic derivatives (the covariant derivatives on the coset (7.26)):

$$D^{IJ} = u^A_a(\gamma^{IJ})^{AB}\frac{\partial}{\partial u^B_a} + w^{\dot{A}}_{\dot{a}}(\gamma^{IJ})^{\dot{A}\dot{B}}\frac{\partial}{\partial w^{\dot{B}}_{\dot{a}}} + v^{[I}_i\frac{\partial}{\partial v^{J]}_i} \ . \tag{7.30}$$

They respect the algebraic relations (7.28), (7.29) among the harmonic variables and form the algebra of SO(8) realized on the indices $A, \dot{A}, I$ of the harmonics.

We now use the harmonic variables for projecting the supersingleton defining constraints (7.24), (7.25). Using the relation (7.29) it is easy to show that the projections $\Phi^{+(+)[+]}$ and $\Sigma^{+(+)\{+\}}$ satisfy the following G-analyticity constraints:

$$D^{++}\Phi^{+(+)[+]} = D^{(++)}\Phi^{+(+)[+]} = D^{[+]\{\pm\}}\Phi^{+(+)[+]} = 0 \ , \tag{7.31}$$

$$D^{++}\Sigma^{+(+)\{+\}} = D^{(++)}\Sigma^{+(+)\{+\}} = D^{[+]\{\pm\}}\Sigma^{+(+)\{+\}} = 0 \tag{7.32}$$

where $D^I_\alpha = v^I_i D^i_\alpha$, $\Phi^A = u^A_a\Phi_a$ and $\Sigma^{\dot{A}} = w^{\dot{A}}_{\dot{a}}\Sigma_{\dot{a}}$. This is the superspace realization of the 1/2 BPS shortening conditions (7.18), (7.19). In the appropriate basis in superspace $\Phi^{+(+)[+]}$ and $\Sigma^{+(+)\{+\}}$ depend on different halves of the odd variables as well as on the harmonic variables:

$$\text{type I}: \qquad \Phi^{+(+)[+]}(x_A, \theta^{++}, \theta^{(++)}, \theta^{[+]\{\pm\}}, u, w) \ , \tag{7.33}$$

$$\text{type II}: \qquad \Sigma^{+(+)\{+\}}(x_A, \theta^{++}, \theta^{(++)}, \theta^{[\pm]\{+\}}, u, w) \ . \tag{7.34}$$

In addition to the G-analyticity constraints (7.31), (7.32), the on-shell superfields $\Phi^{+(+)[+]}$, $\Sigma^{+(+)\{+\}}$ are subject to the SO(8) irreducibility harmonic conditions obtained from (7.14) by replacing the SO(8) generators by the corresponding harmonic derivatives. The combination of the latter with eq. (7.31) is equivalent to the original constraint (7.24).

It should be stressed that $\Phi^{+(+)[+]}$, $\Sigma^{+(+)\{+\}}$ automatically satisfy additional harmonic constraints involving lowering operators of SO(8) (cf. (7.22) and (7.23)). As mentioned earlier, this means that the supersingleton harmonic superfields effectively live in the smaller harmonic coset Spin(8)/U(4).

[14] Although each of the three sets of harmonic variables depends on the same 28 parameters, we need at least two sets to be able to reproduce all possible representations of SO(8).

7.3 $\mathrm{OSp}(8/4,\mathbb{R})$ supersingleton composites

One way to obtain short multiplets of $\mathrm{OSp}(8/4,\mathbb{R})$ is to multiply different analytic superfields describing the type I supersingleton. The point is that above we chose a particular projection of, e.g., the defining constraint (7.24) which lead to the analytic superfield $\Phi^{+(+)[+]}$. In fact, we could have done this in a variety of ways, each time obtaining superfields depending on different halves of the total number of odd variables. Leaving out the 8_v lowest weight θ^{--}, we can have four distinct but equivalent analytic descriptions of the type I supersingleton:

$$\begin{aligned}
&\Phi^{+(+)[+]}(\theta^{++},\theta^{(++)},\theta^{[+]\{+\}},\theta^{[+]\{-\}})\ ,\\
&\Phi^{+(+)[-]}(\theta^{++},\theta^{(++)},\theta^{[-]\{+\}},\theta^{[-]\{-\}})\ ,\\
&\Phi^{+(-)\{+\}}(\theta^{++},\theta^{(--)},\theta^{[+]\{+\}},\theta^{[-]\{+\}})\ ,\\
&\Phi^{+(-)\{-\}}(\theta^{++},\theta^{(--)},\theta^{[+]\{-\}},\theta^{[-]\{-\}})\ .
\end{aligned} \tag{7.35}$$

Then we can multiply them in the following way:

$$(\Phi^{+(+)[+]})^{p+q+r+s}(\Phi^{+(+)[-]})^{q+r+s}(\Phi^{+(-)\{+\}})^{r+s}(\Phi^{+(-)\{-\}})^{s} \tag{7.36}$$

thus obtaining three series of $\mathrm{OSp}(8/4,\mathbb{R})$ UIR's exhibiting 1/8, 1/4 or 1/2 BPS shortening:

$$\begin{aligned}
&\frac{1}{8}\ \text{BPS:}\qquad \mathcal{D}(a_1+a_2+\frac{1}{2}(a_3+a_4),0;a_1,a_2,a_3,a_4)\ ,\quad a_1-a_4=2s\geq 0\ ;\\
&\frac{1}{4}\ \text{BPS:}\qquad \mathcal{D}(a_2+\frac{1}{2}a_3,0;0,a_2,a_3,0)\ ;\\
&\frac{1}{2}\ \text{BPS:}\qquad \mathcal{D}(\frac{1}{2}a_3,0;0,0,a_3,0)
\end{aligned} \tag{7.37}$$

where

$$a_1=r+2s\ ,\quad a_2=q\ ,\quad a_3=p\ ,\quad a_4=r\ . \tag{7.38}$$

We see that multiplying only one type of supersingletons cannot reproduce the general result of Section 7.1 for all possible short multiplets. Most notably, in (7.37) there is no 3/8 series. The latter can be obtained by mixing the two types of supersingletons:

$$[\Phi^{+(+)[+]}(\theta^{++},\theta^{(++)},\theta^{[+]\{\pm\}})]^{p+q}[\Sigma^{+(+)\{+\}}(\theta^{++},\theta^{(++)},\theta^{[\pm]\{+\}})]^{q}\ , \tag{7.39}$$

or the same with Φ and Σ exchanged. Counting the charges and the dimension, we find exact matching with the series (7.17) where $a_3=p+q$

and $a_4 = q$. Further, mixing two realizations of type I and one of type II supersingletons, we can construct the 1/4 series

$$[\Phi^{+(+)[+]}]^{m+k}[\Phi^{+(+)[-]}]^{k}[\Sigma^{+(+)\{+\}}]^{n} \tag{7.40}$$

which corresponds to (7.16) where $a_2 = k$, $a_3 = m$, $a_4 = n$. Finally, the full 1/8 series (7.15) (i.e., without the restriction $a_1 - a_4 = 2s \geq 0$ in (7.37)) can be obtained in a variety of ways.

In this section we have analyzed all short highest weight UIR's of the OSp$(8/4, \mathbb{R})$ superalgebra whose HWS's are annihilated by part of the super-Poincaré odd generators. The number of distinct possibilities have been shown to correspond to different BPS conditions on the HWS. When the algebra is interpreted on the AdS_4 bulk, for which the 3d superconformal field theory corresponds to the boundary M-2 brane dynamics, these states appear as BPS massive excitations, such as K-K states or AdS black holes, of M-theory on $AdS_4 \times S^7$. Since in M-theory there is only one type of supersingleton related to the M-2 brane transverse coordinates [76], according to our analysis massive states cannot be 3/8 BPS saturated, exactly as it happens in M-theory on $M^4 \times T^7$. Indeed, the missing solution was also noticed in Ref. [77] by studying AdS_4 black holes in gauged $N = 8$ supergravity. Curiously, in the ungauged theory, which is in some sense the flat limit of the former, the 3/8 BPS states are forbidden [48] by the underlying $E_{7(7)}$ symmetry of $N = 8$ supergravity [78].

7.4 Series of UIR's of OSp$(8/4, \mathbb{R})$

In addition to the short objects of the BPS type considered above, we can define two current-like objects. One of them is a bilinear of two supersingletons of, e.g., type I, $\Phi_a\Phi_a$. Using (7.24) one can show that it satisfies the constraint

$$D^i_\alpha D^j_\beta D^k_\gamma(\Phi_a\Phi_a) = \underline{56} \oplus \underline{8} \,, \tag{7.41}$$

meaning that only these SO(8) representations survive in the decomposition of the $\underline{8} \otimes \underline{8} \otimes \underline{8}$ on the left-hand side of (7.41). The other current carries SL$(2, \mathbb{R})$ spinor indices, $W_{\alpha_1 \ldots \alpha_{2J}}$, and satisfies the constraint

$$D^{i\alpha} W_{\alpha\alpha_2 \ldots \alpha_{2J}} = 0 \,. \tag{7.42}$$

Our conjecture for the generic UIR of OSp$(8/4, \mathbb{R})$ is as follows:

$$W_{\alpha_1 \ldots \alpha_{2J}} \, (\Phi_a\Phi_a)^k \, \mathrm{BPS}[a_1, a_2, a_3, a_4] \,. \tag{7.43}$$

Here we have used the factor W to obtain the spin, the factor $\Phi_a\Phi_a$ for the conformal dimension and the BPS factor for the SO(8) quantum number. The unitarity bound is given by

$$\ell \geq 1 + J + a_1 + a_2 + \frac{1}{2}(a_3 + a_4) \tag{7.44}$$

and is saturated if $k = 0$ in (7.43).

7.5 BPS states of OSp$(8/4, \mathbb{R})$

Here we give a summary of all possible OSp$(8/4, \mathbb{R})$ BPS multiplets. Denoting the UIR's by

$$\mathcal{D}(\ell; J; a_1, a_2, a_3, a_4) \tag{7.45}$$

where ℓ is the conformal dimension, J is the spin and a_1, a_2, a_3, a_4 are the SO(8) Dynkin labels, we find four BPS conditions:

7.5.1

$$\frac{1}{8}\ \text{BPS}: \qquad q_\alpha^{++} = 0\ . \tag{7.46}$$

The corresponding UIR's are:

$$\mathcal{D}(a_1 + a_2 + \frac{1}{2}(a_3 + a_4); 0; a_1, a_2, a_3, a_4) \tag{7.47}$$

and the harmonic coset is

$$\frac{\text{Spin}(8)}{[\text{U}(1)]^4}\ . \tag{7.48}$$

If $a_2 = a_3 = a_4 = 0$ this coset becomes Spin(8)/U(4).

7.5.2

$$\frac{1}{4}\ \text{BPS}: \qquad q_\alpha^{++} = q_\alpha^{(++)} = 0\ . \tag{7.49}$$

The corresponding UIR's are:

$$\mathcal{D}(a_2 + \frac{1}{2}(a_3 + a_4); 0; 0, a_2, a_3, a_4) \tag{7.50}$$

and the harmonic coset is

$$\frac{\text{Spin}(8)}{[\text{U}(1)]^2 \times \text{U}(2)}\ . \tag{7.51}$$

If $a_3 = a_4 = 0$ this coset becomes Spin(8)/U(1) $\times$ $[\text{SU}(2)]^3$.

7.5.3

$$\frac{3}{8}\ \text{BPS}: \qquad q_\alpha^{++} = q_\alpha^{(++)} = q_\alpha^{[+]\{+\}} = 0\ . \tag{7.52}$$

The corresponding UIR's are:

$$\mathcal{D}(\frac{1}{2}(a_3 + a_4); 0; 0, 0, a_3, a_4) \tag{7.53}$$

and the harmonic coset is

$$\frac{\text{Spin}(8)}{\text{U}(1) \times \text{U}(3)}\ . \tag{7.54}$$

7.5.4

$$\frac{1}{2}\ \text{BPS (type I)}: \qquad q_\alpha^{++} = q_\alpha^{(++)} = q_\alpha^{[+]\{+\}} = q_\alpha^{[+]\{\pm\}} = 0\ ; \tag{7.55}$$

$$\frac{1}{2}\ \text{BPS (type II)}: \qquad q_\alpha^{++} = q_\alpha^{(++)} = q_\alpha^{[+]\{+\}} = q_\alpha^{[\pm]\{+\}} = 0\ . \tag{7.56}$$

The corresponding UIR's are:

$$\frac{1}{2}\ \text{BPS (type I)}: \qquad \mathcal{D}(\frac{1}{2}a_3; 0; 0, 0, a_3, 0)\ ; \tag{7.57}$$

$$\frac{1}{2}\ \text{BPS (type II)}: \qquad \mathcal{D}(\frac{1}{2}a_4; 0; 0, 0, 0, a_4)\ . \tag{7.58}$$

and the harmonic coset is

$$\frac{\text{Spin}(8)}{\text{U}(4)}\ . \tag{7.59}$$

8 Conclusions

Here we give a summary of the different types of BPS states which are realized as products of supersingletons described by G-analytic harmonic superfields. We shall restrict ourselves to the physically interesting cases of D3, M_2 and M_5 branes horizon geometry where only one type of such supersingletons appears. This construction gives rise to a restricted class of the most general BPS states.

8.1 **PSU**$(2,2/4)$

The BPS states are constructed in terms of the $N=4$ $d=4$ super-Yang-Mills multiplet W^{ij} in three equivalent G-analytic realizations:

$$(W^{12}(\theta_{3,4},\bar\theta^{1,2}))^{p+q+r}(W^{13}(\theta_{2,4},\bar\theta^{1,3}))^{q+r}(W^{23}(\theta_{1,4},\bar\theta^{2,3}))^{r}\ . \tag{8.1}$$

BPS	SU(4)	Dimension	Harmonic space
$\frac{1}{2}$	(0,p,0)	p	SU(4)/S(U(2)×U(2))
$\frac{1}{4}$	(q,p,q)	p+2q	SU(4)/[U(1)]^3
	(q,p,q+2r)	p+2q+3r	SU(4)/[U(1)]^3
$\frac{1}{8}$	(0,p,2r)	p+3r	SU(4)/U(1)×U(2)
	(0,0,2r)	3r	SU(4)/U(3)

8.2 **OSp**$(8^*/4)$

The BPS states are constructed in terms of the $(2,0)$ $d=6$ tensor multiplet $W^{\{ij\}}$ in two equivalent G-analytic realizations:

$$(W^{12}(\theta^{1,2})^{p+q}(W^{13}(\theta^{1,3}))^{q}\ . \tag{8.2}$$

BPS	USp(4)	Dimension	Harmonic space
$\frac{1}{2}$	(0,p)	2p	USp(4)/U(2)
$\frac{1}{4}$	(2q,p)	2p+4q	USp(4)/[U(1)]^2
	(2q,0)	4q	USp(4)/U(2)

8.3 $\mathbf{OSp}(8/4,\mathbb{R})$

The type I BPS states are constructed in terms of the $N=8$ $d=3$ matter multiplet Φ_a carrying an external 8_s $SO(8)$ spinor index in four equivalent G-analytic realizations:

$$\begin{aligned}
&[\Phi^{+(+)[+]}(\theta^{++,(++),[+]\{\pm\}})]^{p+q+r+s}\times\\
&[\Phi^{+(+)[-]}(\theta^{++,(++),[-]\{\pm\}})]^{q+r+s}\times\\
&[\Phi^{+(-)\{+\}}(\theta^{++,(--),[\pm]\{+\}})]^{r+s}\times\\
&[\Phi^{+(-)\{-\}}(\theta^{++,(--),[\pm]\{-\}})]^{s}\ .
\end{aligned}\qquad(8.3)$$

BPS	SO(8)	Dimension	Harmonic space
$\frac{1}{2}$	(0,0,p,0)	$\frac{1}{2}p$	Spin(8)/U(4)
$\frac{1}{4}$	(0,q,p,0)	$\frac{1}{2}(p+2q)$	Spin(8)/U(2)×U(2)
$\frac{1}{8}$	(r+2s,q,p,r)	$\frac{1}{2}(p+2q+3r+4s)$	Spin(8)/$[\mathrm{U}(1)]^4$

The type II BPS states are constructed in terms of the $N=8$ $d=3$ matter multiplet $\Sigma_{\dot{a}}$ carrying an external 8_c $SO(8)$ spinor index in four equivalent G-analytic realizations:

$$\begin{aligned}
&[\Sigma^{+(+)\{+\}}(\theta^{++,(++),[\pm]\{+\}})]^{p+q+r+s}\times\\
&[\Sigma^{+(+)\{-\}}(\theta^{++,(++),[\pm]\{-\}})]^{q+r+s}\times\\
&[\Sigma^{+(-)[+]}(\theta^{++,(--),[+]\{\pm\}})]^{r+s}\times\\
&[\Sigma^{+(-)[-]}(\theta^{++,(--),[-]\{\pm\}})]^{s}\ .
\end{aligned}\qquad(8.4)$$

BPS	SO(8)	Dimension	Harmonic space
$\frac{1}{2}$	(0,0,0,p)	$\frac{1}{2}p$	Spin(8)/U(4)
$\frac{1}{4}$	(0,q,0,p)	$\frac{1}{2}(p+2q)$	Spin(8)/U(2)×U(2)
$\frac{1}{8}$	(r+2s,q,r,p)	$\frac{1}{2}(p+2q+3r+4s)$	Spin(8)/[U(1)]^4

Note added

Just before submitting this paper to the hep-th archive, we saw a new article by P. Heslop and P.S. Howe [79]. It partially overlaps with our treatment of the $d = 4$ case.

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