An $\mathcal{N} = 2$ gauge theory and its supergravity dual

A. Brandhuber[1] and **K. Sfetsos**[2]

[1]*Theory Division, CERN*
CH-1211 Geneva 23, Switzerland
brandhu@mail.cern.ch

[2]*Institut de Physique, Université de Neuchâtel*
Breguet 1, CH-2000 Neuchâtel, Switzerland
sfetsos@mail.cern.ch

Abstract

We study flows on the scalar manifold of $\mathcal{N} = 8$ gauged supergravity in five dimensions which are dual to certain mass deformations of $\mathcal{N} = 4$ super Yang–Mill theory. In particular, we consider a perturbation of the gauge theory by a mass term for the adjoint hyper-multiplet, giving rise to an $\mathcal{N} = 2$ theory. The exact solution of the 5-dim gauged supergravity equations of motion is found and the metric is uplifted to a ten-dimensional background of type-IIB supergravity. Using these geometric data and the AdS/CFT correspondence we analyze the spectra of certain operators as well as Wilson loops on the dual gauge theory side. The physical flows are parametrized by a single non-positive constant and describe part of the Coulomb branch of the $\mathcal{N} = 2$ theory at strong coupling.

1   Introduction

The AdS/CFT correspondence [1, 2, 3] provides a powerful tool for studying $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in four dimensions at large N and large 't Hooft coupling. In particular, there exist precise prescriptions to calculate correlation functions, spectra of gauge invariant operators, Wilson loops and c-functions in supergravity. These data can be compared with field theory or provide non-trivial predictions for strongly coupled field theories. A natural question is whether this correspondence can be extended to theories with spontaneously or manifestly broken superconformal symmetry. Such theories arise either by giving vacuum expectation values to scalar fields [1], [4]-[12] or by deformations of the conformal theory with relevant operators [13]-[20].

These issues can all be treated efficiently in the context of five-dimensional gauged supergravity [21, 22] and the resulting backgrounds are kink-solutions with four-dimensional Poincaré invariance which approach AdS asymptotically. A related question concerns the uplifting of these solutions to solutions of type-IIB supergravity/string theory, which can be quite involved, as we will see in this work. It is also of interest for the program of consistent truncations [23] which, as yet, has not been completed in the case of $\mathcal{N} = 8$ gauged supergravity in five dimensions.

In this note we will study a supergravity dual of a particular deformation of the $\mathcal{N} = 4$ SYM theory by a mass term that preserves $\mathcal{N} = 2$ supersymmetry [24, 25]. This is simply $\mathcal{N} = 2$ supersymmetric gauge theory with gauge group $SU(N)$ coupled to a massive hypermultiplet in the adjoint representation of the gauge group. One can think of this model also as $\mathcal{N} = 1$ supersymmetric QCD with three chiral multiplets $\Phi_{i=1,2,3}$ in the adjoint, out of which, one is massless and the other two have equal masses. Other choices of the massterms are of course possible and lead to models with $\mathcal{N} = 1$ supersymmetry. Such models have been studied previously in the context of the AdS/CFT correspondence [17, 26, 27].

The outline of this paper is as follows: in section 2 we present the background of gauged supergravity that is dual to $\mathcal{N} = 4$ SYM deformed by a mass term for one hypermultiplet. We find a family of solutions that is parametrized by one real constant c, whose value determines the physics and study fluctuation in this background. For $c \leq 0$ it turns out that we describe the strong coupling regime of part of the Coulomb branch of an $\mathcal{N} = 2$ theory, whereas flows with $c > 0$ are unphysical. In section 3, we compute the uplifted metric in ten dimensions. With the help of this type-IIB background we calculate expectation values of Wilson loops that correspond to the potential of an external heavy quark-antiquark pair. Finally, in section 4 we present some concluding remarks and comment on the singularities that are typical for backgrounds of non-conformal theories.

Note added

In the final stages of our work, the paper [28] appeared that has considerable overlap with ours. In addition to the uplifted ten-dimensional metric, that we also computed independently, these authors presented the axion/dilaton and the complex two-form in

ten dimensions. A numerical investigation has also appeared before in [29].

2   A dual of $\mathcal{N} = 2$ supersymmetric gauge theories

Our starting point is the action of five-dimensional gauged supergravity

$$S = \int d^5x \left\{ \frac{1}{4}\mathcal{R} - \frac{1}{2}\sum_I \partial_\mu \alpha_I \partial^\mu \alpha_I - P(\alpha) \right\} , \tag{1}$$

where we have chosen canonical kinetic terms for the scalars. This is possible only in certain sub-sectors of the full $E_{6(6)}/USp(8)$ coset space sigma-model parameterizing the 42 scalars of the theory. We have also set to zero all other tensor fields.

For the applications we have in mind we need the scalars that correspond to dimension 2 and 3 operators, which are in the **20** and **10** representation of $SO(6) \sim SU(4)$, respectively. The massless 5-dim dilaton and axion will not play a rôle and are constant. They correspond to exactly marginal deformations in the gauge theory. Furthermore, we are interested in solutions of (1) that preserve part of the supersymmetries. For such supersymmetric flows it is known that the scalar potential P can be written in terms of a *superpotential*, which we will denote as $W(\alpha)$:

$$P(\alpha) = \frac{1}{8}\sum_I \left(\frac{\partial W}{\partial \alpha_I} \right)^2 - \frac{1}{3}W^2 . \tag{2}$$

Furthermore, we demand that the solutions of (1) preserve four-dimensional Poincaré invariance along the brane directions, and that the metric asymptotes the AdS_5 spacetime near the boundary which we take to be at $r \to \infty$. For the metric we make the ansatz

$$ds_5^2 = e^{2A(r)}dx_{||}^2 + dr^2 , \tag{3}$$

or, if we need the metric in its conformally flat version then

$$ds_5^2 = e^{2A(z)}\left(dx_{||}^2 + dz^2 \right) , \tag{4}$$

where the relation between the different coordinate choices is $dr = -e^A dz$. The boundary at $r = \infty$ corresponds to $z = 0$.

On the supergravity side two scalars, denoted by α_2 and α_3, are involved and belong to the **20** and to the $\mathbf{10} + \overline{\mathbf{10}}$ representation of $SO(6)$, respectively. They are dual to operator-bilinears in scalars and fermions as:

$$\alpha_2 : \qquad \mathcal{O}_2 = \text{Tr}\left(\bar{Z}_1 Z_1 + \bar{Z}_2 Z_2 - 2\bar{Z}_3 Z_3 \right) ,$$

$$\alpha_3 : \qquad \mathcal{O}_3 = \text{Tr}\left(\lambda_1 \lambda_1 + \lambda_2 \lambda_2 + \ldots + h.c. \right) , \tag{5}$$

where the $Z_{i=1,2,3}$ denote the complex scalar components of the chiral superfields Φ_i, the $\lambda_{i=1,2,3}$ are the corresponding fermionic components, and the ... denote scalar trilinear

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terms. The non-vanishing scalar α_2 reduces the gauge group to $SU(2) \times SU(2) \times U(1) \subset SU(4)$ under which the scalars in the **6** and the fermions in the **4** decompose as:

$$\mathbf{6} \;\rightarrow\; (1,1)_2 + (1,1)_{-2} + (2,2)_0 \; ,$$

$$\mathbf{4} \;\rightarrow\; (2,1)_1 + (1,2)_{-1} \; . \tag{6}$$

We see that the $U(1)$ is to be identified with the $U(1)_R$ symmetry of the field theory, since the two scalars in the vector multiplet correspond to the two $SU(2)$ singlets in the **6** which have charge ± 2 under the $U(1)$ as in field theory, whereas the scalars of the hypermultiplet are singlets. The scalar α_3 breaks one of the $SU(2)$'s to $U(1)$, and the unbroken one can be identified with the $SU(2)_R$ symmetry of the gauge theory under which the scalars in the vector multiplet are singlets and the scalars in the hypermultiplet are doublets. Furthermore, the two fermions in the hypermultiplet are singlets under $SU(2)_R$ and have $U(1)_R$ charge -1, the two fermions in the vector multiplet are $SU(2)_R$ doublets and their $U(1)_R$ charge is $+1$. So in our decomposition the second $SU(2)$ is broken to $U(1)$.

The potential in the scalar field space is obtained by a truncation of the four scalar potential that has been computed in [15] and reads

$$P \;=\; \frac{1}{16} e^{-\frac{4}{\sqrt{6}}\alpha_2} \left(-4 - 8 e^{\sqrt{6}\alpha_2} \cosh \sqrt{2}\alpha_3 + e^{2\sqrt{6}\alpha_2} \sinh^2 \sqrt{2}\alpha_3 \right) \; ,$$

$$W \;=\; -e^{-2\alpha_2/\sqrt{6}} - \frac{1}{2} e^{4\alpha_2/\sqrt{6}} \cosh \sqrt{2}\alpha_3 \; , \tag{7}$$

where W is the corresponding superpotential (cf. (2)). For a supersymmetric flow the equations of motion of (1) reduce to a set of first order equations:

$$\dot{\alpha}_2 \;=\; \frac{1}{2}\partial_{\alpha_2} W = \frac{1}{\sqrt{6}} e^{-2\alpha_2/\sqrt{6}} - \frac{1}{\sqrt{6}} e^{4a_2/\sqrt{6}} \cosh \sqrt{2}\alpha_3 \; ,$$

$$\dot{\alpha}_3 \;=\; \frac{1}{2}\partial_{\alpha_3} W = -\frac{1}{2\sqrt{2}} e^{4a_2/\sqrt{6}} \sinh \sqrt{2}\alpha_3 \; , \tag{8}$$

$$\dot{A} \;=\; -\frac{1}{3} W \; ,$$

where the dot denotes the derivative with respect to the variable r introduced in (3). As a consistency check, we note that we may set the scalar α_3 to zero, as it is obvious from (8). Then, the remaining scalar describes the part of the Coulomb branch of the $\mathcal{N} = 4$ SYM theory corresponding to the background of D3-branes distributed on a disc and $SO(4) \times U(1)$ symmetry.[1]

The systems of equations in (8) form a rather complicated set of coupled first order differential equations, which, at first sight, do not have a solution that can be written

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[1]There should be a generalization of our discussion so far that includes a third scalar that further breaks $SU(2) \times U(1) \times U(1)$ to $SU(2) \times U(1)$. When the scalar in the **10** is turned off this should be the part of the Coulomb branch of the $\mathcal{N} = 4$ SYM theory corresponding to the background of D3-branes distributed on a ellipsoid, which has an $SO(4)$ symmetry. This solution was constructed in [10].

down in a closed form. This is true for the coordinate r, but if we take the scalar α_3 as a new radial coordinate[2] a solution can be given in terms of elementary functions. Note that the equations in (8) have a $\mathbf{Z}_2$ symmetry which transforms $\alpha_3 \to -\alpha_3$, so that we may restrict our analysis to $\alpha_3 \geq 0$ without loss of generality. For the scalar field α_3 we find

$$e^{\sqrt{6}\alpha_2} = \sinh^2 \sqrt{2}\alpha_3 \left[c + \ln \tanh \left(\frac{\alpha_3}{\sqrt{2}} \right) \right] + \cosh \sqrt{2}\alpha_3 \;, \tag{9}$$

where c is an integration constant. As we will see the physical picture depends crucially on whether c is positive, negative or zero.
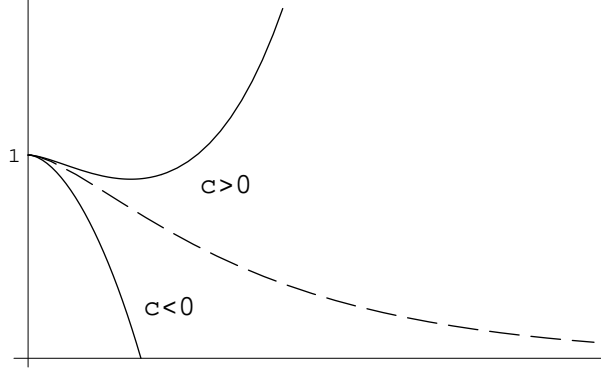

Figure 1: This figure shows the different behaviours of the function $e^{\sqrt{6}\alpha_2}$ as a function of α_3 using (9). For $c < 0$, it decreases monotonously to zero at a finite value $\alpha_3^{\max}$. For $c = 0$ (the dashed curve) the same minimum at zero is attained, but for $\alpha_3 \to \infty$. In contrast, for $c > 0$ the function first reaches a minimum, at a finite value $\alpha_3^{\min}$, before it runs off to infinity at $\alpha_3 \to \infty$.

For the 5-dim metric we find

$$ds_5^2 = \frac{1}{\sinh^2 \sqrt{2}\alpha_3} \left(2^{2/3} e^{4\alpha_2/\sqrt{6}} dx_{||}^2 + 8 e^{-8\alpha_2/\sqrt{6}} d\alpha_3^2 \right) \;. \tag{10}$$

The transition to the conformally flat metric (4) with conformal factor

$$e^{2A(z)} = \frac{2^{2/3} e^{4\alpha_2/\sqrt{6}}}{\sinh^2 \sqrt{2}\alpha_3} \;, \tag{11}$$

is given by the relation of the differentials

$$dz = 2^{7/6} e^{-\sqrt{6}\alpha_2} d\alpha_3 \;. \tag{12}$$

We note that, for generic values of the constant c, the explicit dependence of α_3 on z cannot be obtained from (12), since the corresponding integral cannot be explicitly evaluated.

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[2] From (12) and the remark after (4) it is clear that a_3 is a monotonously decreasing function of r.

Let us consider the behaviour near the boundary where the background becomes AdS_5. The boundary is approached as $\alpha_3 \to 0$ which in turn, using (9), forces $\alpha_2 \to 0$ as well. In this limit we find $z \simeq 2^{7/6}\alpha_3$. Furthermore, we want to check that the scalars behave as expected from the AdS/CFT correspondence. The scalar α_3 is dual to the fermion bilinear operator $\mathcal{O}_3$. Our field theory is obtained as a mass deformation of the $\mathcal{N} = 4$ theory and therefore the solution near the horizon should behave as a pure mass term, i.e. $\alpha_3 \sim z^1$ and there should be no z^3 term corresponding to a gluino condensate. On the other hand, the behaviour of the scalar α_2 is a beautiful example of the double rôle that the scalars play in the AdS/CFT correspondence — they can appear as deformations of the conformal theory, or they parameterize states in the theory. Here we expect both since we have to give mass to the scalar components of the chiral multiplets $\Phi_{1,2}$ as dictated by supersymmetry, and the gauge theory has a Coulomb branch parametrized by the vev of Z_3, i.e. α_2 should have contributions of the type z^2 and $z^2 \log z$. From our explicit solution we immediately find that

$$
\begin{aligned}
\alpha_3 &\sim z \\
\alpha_2 &\sim \frac{1 - \ln 2 + 2c}{\sqrt{6}} z^2 + \frac{2}{\sqrt{6}} z^2 \ln z \ .
\end{aligned}
\tag{13}
$$

From these expressions we see that the correct terms appear, as expected, and that there is a one parameter family of solutions labelled by c.[3]

In this paper we will also be interested in the spectrum corresponding to the massless scalar equation in a background metric of the type (4) and with a plane wave ansatz along the flat coordinates denoted by $x_{||}$. Within the AdS/CFT correspondence [1, 2, 3] the solutions and eigenvalues of the massless scalar equation have been associated, on the gauge theory side, with the spectrum of the operator $\mathrm{Tr}F^2$, whereas those of the graviton fluctuations polarized in the directions parallel to the brane, with the energy momentum tensor $T_{\mu\nu}$ [30, 2, 3]. Though a priori not to be expected, these two spectra and the corresponding eigenfunctions coincide [31] (for a more recent discussion see [32]). It is well known that the entire analysis can be cast into an equivalent Schrödinger problem with potential

$$
V = \frac{9}{4}A'^2 + \frac{3}{2}A''
\tag{14}
$$

and eigenvalue equal to the mass squared (M^2). Since, as already noted, for generic values of the constant c we cannot find the explicit dependence of α_3 on z, it is not possible to explicitly evaluate the potential using (11). However, one easily can show that it is a monotonously decreasing function of z. As in [8, 31, 10] the potential (14) has the same form as the potentials appearing in supersymmetric quantum mechanics and, therefore, the spectrum is bounded from below. Its supersymmetric partner is given by an equation similar to (14), but with a relative minus sign between the two terms.

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[3]As was noted in [29], there is a critical line in the space of solutions corresponding to $c = 0$, in our notation. In the notation of [29], this gives $b_{\mathrm{cr}} = (1 - \ln 2)/\sqrt{6} \simeq 0.125$, thus confirming the critical value found numerically in [29].

2.1 $c < 0$

In the following we will study the case $c < 0$ in more detail. In particular, we will determine the spectrum of fluctuations of a minimal scalar in this background.

First, note that $e^{\sqrt{6}\alpha_2}$ is a monotonically decreasing function of α_3 and, as it turns out, it has only a finite range (see fig. 1)

$$0 \leq \alpha_3 \leq \alpha_3^{\max} \ . \tag{15}$$

In certain limits $\alpha_3^{\max}$ can be found analytically:

$$\alpha_3^{\max} = \begin{cases} -\frac{1}{3\sqrt{2}} \ln\left(-\frac{3c}{16}\right) \ , & c \to 0^- \ , \\ \frac{1}{\sqrt{-2c}} \ , & c \to -\infty \ . \end{cases} \tag{16}$$

Using (9) and (12) we find that

$$e^{\sqrt{6}\alpha_2} \simeq \frac{2\sqrt{2}}{\sinh\sqrt{2}\alpha_3^{\max}} \left(\alpha_3^{\max} - \alpha_3\right) \ , \qquad \text{as} \quad \alpha_3 \to \alpha_3^{\max} \tag{17}$$

and that

$$z \simeq -2^{-1/3} \sinh\sqrt{2}\alpha_3^{\max} \ln\left(\alpha_3^{\max} - \alpha_3\right) \ , \qquad \text{as} \quad \alpha_3 \to \alpha_3^{\max} \ . \tag{18}$$

Therefore z takes values in the whole semi-infinite real line, i.e. $0 \leq z < \infty$. From (11), (17) and (18) we find that

$$A \simeq -\frac{2^{1/3}z}{3\sinh\sqrt{2}\alpha_3^{\max}} \ , \qquad \text{as} \quad z \to \infty \ . \tag{19}$$

From (14) one concludes that the mass spectrum is continuous with a mass gap. The value of the latter is found by evaluating the potential for $z \to \infty$. The result is

$$M_{\text{gap}}^2 = \frac{1}{2^{4/3} \sinh^2\sqrt{2}\alpha_3^{\max}} \ , \tag{20}$$

and in the two limits corresponding to (16)

$$M_{\text{gap}}^2 = \begin{cases} \frac{1}{4}(-3c)^{2/3} \ , & c \to 0^- \ , \\ 2^{-4/3}(-c) \ , & c \to -\infty \ . \end{cases} \tag{21}$$

Hence, the mass gap vanishes as c tends to zero and grows large as c approaches large negative values.

It is also worth noting that certain solutions corresponding to points on the Coulomb branch of $\mathcal{N} = 4$ SYM, which were obtained previously in the literature, can be obtained in special limits of our solution. The limit $c \to -\infty$ corresponds to a distribution of D3-branes on a disc of radius 2, in our units. In order to see that let us change variables as

$$\alpha_3 = \frac{1}{\sqrt{-2c}} \tanh\left(z/2\right) \ , \tag{22}$$

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and then send $c \to -\infty$. As we can see from (16), this effectively shrinks the range of α_3 to zero. After some elementary algebra we find

$$
\begin{aligned}
ds_5^2 &= \frac{\cosh^{2/3}(z/2)}{\sinh^2(z/2)}\left(dz^2 + 2^{2/3}(-c)dx_{||}^2\right) , \\
e^{\sqrt{6}\alpha_2} &= \frac{1}{\cosh^2(z/2)} .
\end{aligned}
\tag{23}
$$

This model has been studied on its own and also exhibits a mass gap. In fact in order to compute it using our formulae we first re-scale the energies by the factor $2^{2/3}(-c)$ as it is evident from the expression for the metric in (23). Then, the second line in (21) gives $M_{\text{gap}}^2 = 1/4$ which is in agreement with the result for the mass gap found in [8, 9], when it is expressed in our units. We finally note that, from a physical point of view a continuous spectrum with a mass gap should be associated with a complete screening of charges in a quark-antiquark pair at a finite separation distance (inversely proportional to M_{gap}). This will be further discussed in subsection 3.1.

2.2 $c > 0$

Contrary to the previous case, $e^{\sqrt{6}\alpha_2}$ is now not a monotonic function (see fig. 1). For increasing α_3 it first decreases, takes a minimum at $\alpha_3 = \alpha_3^{\text{min}}$ and then increases monotonically. The coordinate z on the other hand can easily be seen to have finite range:

$$
0 \leq z \leq z_0 = 2^{7/6} \int_0^\infty e^{-\sqrt{6}\alpha_2} d\alpha_3 .
\tag{24}
$$

In certain limits one can work out the value for α_3^{min}:

$$
a_3^{\text{min}} \simeq \begin{cases} \frac{1}{6\sqrt{2}} \ln \frac{8}{3c} , & c \to 0^+ , \\ \sqrt{2}e^{-c-1} , & c \to +\infty . \end{cases}
\tag{25}
$$

We also obtain from (9) and (12) that

$$
e^{\sqrt{6}\alpha_2} \simeq \frac{c}{4}e^{2\sqrt{2}\alpha_3} , \qquad \text{as} \quad \alpha_3 \to \infty
\tag{26}
$$

and that

$$
e^{-2\sqrt{2}\alpha_3} \simeq 2^{-5/3}c(z_0 - z) , \qquad \text{as} \quad z \to z_0^- .
\tag{27}
$$

Then from (11), (26) and (27) the warp factor of the metric becomes

$$
e^A \simeq 2^{7/18}c^{1/6}(z_0 - z)^{1/6} , \qquad \text{as} \quad z \to z_0^- ,
\tag{28}
$$

which corresponds to a naked singularity. It turns out to be a *bad* singularity, but we will return to this point later and discuss it in more detail. The spectrum of scalar fluctuations is now discrete since the range of z is finite. Knowing the warp factor we can easily determine the Schrödinger potential in the limit $z \to z_0^-$. The result is:

$$
V \simeq -\frac{3}{16}\frac{1}{(z - z_0)^2} , \qquad \text{as} \quad z \to z_0^- .
\tag{29}
$$

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A WKB calculation yields for the spectrum

$$M_n^2 = \frac{\pi^2}{z_0^2} m(m+1) + \mathcal{O}(m^0) , \qquad n = 1, 2, \ldots \tag{30}$$

For $c \to \infty$ the geometry reduces to that of a distribution of D3-branes smeared on a three-sphere of radius 2 in our units. In order to illustrate that, similarly to before, we first change variables as

$$\alpha_3 = \frac{1}{\sqrt{2c}} \tan(z/2) , \tag{31}$$

and then let $c \to \infty$ with the result

$$ds_5^2 = \frac{\cos^{2/3}(z/2)}{\sin^2(z/2)} \left(dz^2 + 2^{2/3} c \, dx_{||}^2 \right) ,$$

$$e^{\sqrt{6}\alpha_2} = \frac{1}{\cos^2(z/2)} . \tag{32}$$

In the limit $c \to \infty$ we find using (24) that the maximum value of z behaves as $z_0 \simeq \pi/(2^{1/3}c^{1/2})$. Then, after rescaling of the energies by the factor $2^{2/3}c$, the WKB mass spectrum is just $M_{\text{gap}}^2 = m(m+1)$ and is in fact the exact result (as noted in [8, 9]), in the units that we are using.

2.3 $c = 0$

In this case the ranges of z and α_3 are infinite and $e^{\sqrt{6}\alpha_2}$ is a monotonously decreasing function (see fig. 1). In particular, from (9) and (12) we find that

$$e^{\sqrt{6}\alpha_2} \simeq \frac{4}{3} e^{-\sqrt{2}\alpha_3} , \qquad \text{as} \quad \alpha_3 \to \infty \tag{33}$$

and that

$$e^{\sqrt{2}\alpha_3} \simeq \frac{2^{4/3}}{3} z , \qquad \text{as} \quad z \to \infty . \tag{34}$$

Hence,

$$e^A \sim z^{-4/3} , \qquad \text{as} \quad z \to \infty \tag{35}$$

and the Schrödinger potential becomes $V \simeq 6/z^2$ for $z \to \infty$. Therefore the spectrum is continuous and has no gap. Accordingly, we expect that the screening of charges in a quark-antiquark pair will be perfect only at an infinite separation. This will be confirmed in subsection 3.1.

3   The lift to ten dimensions

Recently, it has been shown how general solutions of 5-dim gauged supergravity can be uplifted to type-IIB supergravity. This is part of the program to prove that a consistent

truncation exists. This has not been shown for all fields, but for the metric the full non-linear KK ansatz has been conjectured [15] and it has passed several non-trivial tests.

The inverse of the deformed metric on S^5 is given in term of the 27-bein $\mathcal{V}$ which has global $E_{6(6)}$ and local $USp(8)$ indices. In the $SL(6, \mathbf{R}) \times SL(2, \mathbf{R})$ basis the vielbein is decomposed as $\mathcal{V} \to \left(\mathcal{V}^{IJab}, \mathcal{V}_{I\alpha}{}^{ab}\right)$ in terms of which the inverse metric becomes

$$\hat{g}^{mn} = \Delta^{-2/3}g^{mn} = 2K^m_{IJ}K^n_{KL}\tilde{\mathcal{V}}_{IJab}\tilde{\mathcal{V}}_{KLcd}\Omega^{ac}\Omega^{bd} \ , \tag{36}$$

with $\tilde{\mathcal{V}}$ being the inverse vielbein and $\Delta^2 = \det(g_{mn})/\det(g^{(0)}_{mn})$, where $(g^{(0)}_{mn})$ is the undeformed five-sphere metric. Consequently, the ten-dimensional metric takes the form of a warped product space

$$ds^2_{10} = \Delta^{-2/3}ds^2_5 + g_{mn}dy^m dy^n = \Delta^{-2/3}\left(ds^2_5 + \hat{g}_{mn}dy^m dy^n\right) \ . \tag{37}$$

Then using the parameterization presented in [33] we compute the 5-dim metric to be

$$
\begin{aligned}
d\hat{s}^2 &= \frac{e^{-\frac{2}{\sqrt{6}}\alpha_2}}{\cosh\sqrt{2}\alpha_3}d\theta^2 + \frac{e^{-\frac{8}{\sqrt{6}}\alpha_2}\sin^2\theta}{\Delta_1}d\phi^2_1 \\
&\quad + e^{-\frac{2}{\sqrt{6}}\alpha_2}\cos^2\theta\left(\frac{1}{\Delta_1\cosh\sqrt{2}\alpha_3}\sigma^2_3 + \frac{1}{\Delta_2}(\sigma^2_1 + \sigma^2_2)\right) \ ,
\end{aligned}
\tag{38}
$$

where

$$
\begin{aligned}
\Delta_1 &= e^{-\sqrt{6}\alpha_2}\cos^2\theta\cosh\sqrt{2}\alpha_3 + \sin^2\theta \ , \\
\Delta_2 &= e^{-\sqrt{6}\alpha_2}\cos^2\theta + \sin^2\theta\cosh\sqrt{2}\alpha_3 \ .
\end{aligned}
\tag{39}
$$

The Maurer–Cartan forms σ_i, $i = 1, 2, 3$ for $SU(2)$ are defined to obey $d\sigma_i = -\epsilon_{ijk}\sigma_j \wedge \sigma_k$. A convenient parameterization in terms of three Euler angles ϕ_2, ϕ_3 and ψ is given by

$$
\begin{aligned}
\sigma_1 &= \frac{1}{2}\left(\cos\phi_2 d\psi + \sin\phi_2\sin\psi d\phi_3\right) \ , \\
\sigma_2 &= \frac{1}{2}\left(-\sin\phi_2 d\psi + \cos\phi_2\sin\psi d\phi_3\right) \ , \\
\sigma_3 &= \frac{1}{2}\left(d\phi_2 + \cos\psi d\phi_3\right) \ .
\end{aligned}
\tag{40}
$$

Finally, the warp factor in (37) is easily computed to be

$$\Delta^{-2/3} = e^{\frac{2}{\sqrt{6}}\alpha_2}\left(\Delta_1\Delta_2\cosh\sqrt{2}\alpha_3\right)^{1/4} \ . \tag{41}$$

The metric (38) is manifestly $SU(2)$ invariant being written in terms of the corresponding Maurer–Cartan forms. In addition, there is an $U(1) \times U(1)$ invariance corresponding to the isometry with respect to the commuting Killing vectors $\partial/\partial\phi_1$ and $\partial/\partial\phi_2$.

3.1 Wilson loops and screening

We now calculate string probes in the ten-dimensional background (38) that correspond to Wilson loops in field theory associated with the potential between an external quark-antiquark pair [34, 35]. The relevant probe action is that of a fundamental string which involves only the string frame metric and the NS B-field. As in a similar computation in [9], involving a rotating D3-brane, we consider trajectories with constant values for the angles θ and ϕ_1, as well as for the Euler angles entering into the definition of the Maurer–Cartan forms. For such constant values the pull-back of the B-field, which was recently calculated [28], vanishes, so that the action consists only of the Nambu–Goto term. Since there is an explicit dependence of the metric on θ, consistency requires that the variation of this action with respect to θ is zero, in order for the equations of motion to be obeyed. It is easy to see that this procedure allows $\theta = 0$ or $\theta = \pi/2$.

The action can be written as

$$S_{NG} = \frac{T}{2\pi\alpha'} \int dx \sqrt{g(\alpha_3)\left(\frac{d\alpha_3}{dx}\right)^2 + f(\alpha_3)} , \qquad (42)$$

with T coming from the trivial integration in the Euclidean-time direction and x is one of the spatial directions along the brane. The functions $g(\alpha_3, \theta) = g_{\tau\tau}g_{\alpha_3\alpha_3}$, $f(\alpha_3, \theta) = g_{\tau\tau}g_{xx}$, where eventually only the values $\theta = 0$ and $\theta = \pi/2$ are allowed. Following standard procedures one can find integral expressions for the separation of the two sources and the potential as function of the maximal distance of the string from the boundary. For small separations we find, as expected, a Coulomb potential $V_{q\bar{q}} \sim -1/L$. However, for larger values of L we have to distinguish between the cases of negative and zero c. Let us begin with the $c < 0$ case: as we increase L the energy increases until it vanishes at a certain maximal length. Beyond that there doesn't exist a geodesic connecting the two sources, the configuration will be that of two disconnected straight strings and the potential is zero. We can interpret this as complete screening with the maximum length given by

$$L_{\max} = \begin{cases} \frac{\pi}{M_{\text{gap}}} , & \theta = 0 , \\ \frac{\pi}{2M_{\text{gap}}} , & \theta = \frac{\pi}{2} , \end{cases} \qquad (43)$$

where the mass gap M_{gap} is given by (20). This behaviour of Wilson loops is precisely the one found in the context of the Coulomb branch of theories with sixteen supercharges [9] for D3-branes distributed uniformly over a disc.

For $c = 0$ the behaviour is quite different. There doesn't exist a finite screening length and the potential vanishes only at an infinite separation. For a short separation the potential exhibits the usual Coulombic behaviour, but we can also work out the behaviour for a very large separation. The result depends on the direction on the sphere: for $\theta = 0$ we find again $V_{q\bar{q}} \sim -1/L$ which is reminiscent of a conformal theory, although the metric is not that for the AdS_5 space-time. For $\theta = \pi/2$ the potential is screened, but it still exhibits a power-law behaviour as $V_{q\bar{q}} \sim -1/L^2$. This is consistent with the fact that there is a ring-type singularity for the metric at $\theta = \pi/2$ as for the enhancon in

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[36] and for a uniform distribution of D3-branes in a ring in [6]. For the latter case the Wilson loop was analyzed in [9] and also exhibits a $1/L^2$ behaviour for large separations.

Note also that, for the qualitative features that we have presented for the Wilson loops the value of the 10-dim dilaton computed in [28] plays no rôle except for the case with $c = 0$ and $\theta = \pi/2$. Then the inclusion of the dilaton factor is indeed necessary in order to obtain the $1/L^2$ behaviour that we have mentioned.

4   Concluding remarks

The 5-dim backgrounds that we described have naked singularities in the interior (IR in field theory). Hence, the question arises whether the physics or more specifically string theory is singular or not in such backgrounds. Well known examples of singular geometries that are resolved by string theory are orbifolds, orientifolds and conifolds. More recently singular geometries appeared in the supergravity duals of non-conformal theories. Examples are dilatonic branes, backgrounds dual to theories with sixteen super-charges on the Coulomb branch, for which the singularity is the source of a distribution of branes, and duals of theories with eight supercharges where the singularity is removed by a mechanism explained in [36]. We think that our geometries are similar in nature to the latter example.

Furthermore, the distinction between good and bad singularities should be consistent with the AdS/CFT correspondence. This means that only a well-defined deformation and vacuum-state in field theory is dual to a geometry with a good singularity. We have encountered this issue in our paper. For $c > 0$ we seemingly try to give a vev to a massive scalar field in the field theory. However, since we have turned on a non-zero source (mass term) for these scalars, its vev has to be fixed uniquely. Therefore, the case $c > 0$ should correspond to a bad singularity, and $c \leq 0$, on the other hand, should be an acceptable one.

Unfortunately, we do not know enough about string theory in such backgrounds to answer the question whether the singularity is physical or not. Instead, we can ask a somewhat simpler question: is the propagation of a quantum test particle well defined in the presence of the singularity? This criterion [37, 38] is identical to finding a unique self-adjoint extension of the wave-operator. This can be quickly answered by looking at the two solutions of the wave-operator locally near the singularity. The relevant part of the wave equation is

$$\frac{d}{dr}\left(\sqrt{g}g^{rr}\frac{d\psi}{dr}\right) = 0 \tag{44}$$

and the norm is $\frac{q^2}{2}\int \sqrt{g}g^{tt}\psi^\dagger\psi + \frac{1}{2}\int \sqrt{g}g^{ij}D_i\psi^\dagger D_j\psi$. This is the Sobolev norm and it is bounded from above by a constant times the energy of the fluctuation [39]. This is a physical sensible norm because it guarantees that the backreaction of the fluctuation is small.

For $c \leq 0$ only one solution is normalizable and the non-normalizable is discarded. Therefore, there exists a unique self-adjoint extension and the singularity is wave-regular [38], in accord with our expectations. However, for $c > 0$, the metric near $z = z_0$ is approximated by

$$ds^2 = \tilde{x}^{1/3}(dx_{||}^2 + d\tilde{x}^2) , \tag{45}$$

where $\tilde{x} = z_0 - z \to 0^+$. Now both solutions $\psi \sim 1$, $\tilde{x}^{1/2}$ are normalizable and the singularity is wave-singular. This is consistent with field theory expectations and a different criterion presented in [29]. Actually, there is a subtlety in the choice of the norm of the wavefunction. As ref. [37] we use the Sobolev norm which contains derivatives of the wavefunction, whereas in [38] a norm without derivatives is used. This distinction is important for example in the case of the negative mass Schwarzschild blackhole: using the Sobolev norm it is wave-regular, but it is singular in the convention of [38]. On the other hand the criterion using the Sobolev norm is in accord with the criterion of [29] for most of the cases we studied, but we found a discrepancy for a background that corresponds to a specific state on the Coulomb branch of $\mathcal{N} = 4$ SYM. In 10 dimensions it is lifted to a distribution of D3-branes including positive and negative tension branes (case $n = 5$ in [8] and $n = 1$ in [10]). The background is unphysical by the criterion of [29], but is wave-regular using our criterion. At present we have no convincing explanation for this and we will postpone this issue for future work as this seems to be one of the few examples in the AdS/CFT correspondence where the field theory side can teach us something about the associated string theory.

Acknowledgements

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