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Tune Shift with Amplitude induced by Quadrupole Fringe Fields

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Abstract

Using Lie algebra techniques, we derive an analytical expression for the nonlinear Hamiltonian and the linear tune shift with amplitude due to quadrupole fringe fields. Numerical examples for the FNAL muon storage ring are compared with results from the computer code COSY INFINITY [1].

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1 Introduction

Quadrupole fringe fields can severely limit the dynamic aperture in muon storage rings of future neutrino factories [2, 3]. Analytical expressions of the fringe-field effects reveal the dependence on magnet and optics parameters, and they may also be used to verify or disprove numerical results.

2 Vector Potential

In current-free regions, the magnetic field fulfills $\vec{\nabla} \times \vec{B} = \vec{0}$ and $\vec{\nabla} \cdot \vec{B} = 0$. It can be derived either from a scalar potential ϕ or a vector potential $\vec{A}$, as $\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla}\phi$. If the field does not depend on z, the differential operators act only in the two transverse dimensions. In this case, the general form of the transverse magnet field is the standard multipole expansion:

$$B_y + iB_x = \sum_{n=1}^{\infty} [b_n + ia_n]\ [x + iy]^{n-1}/r_0^{n-1} \tag{1}$$

where r_0 denotes a reference radius. This is the usual situation without fringe fields. The longitudinal field component $B_z = B_{z0}$ is constant and equal to zero, except in a solenoid. The corresponding scalar potential for a normal quadrupole field ($b_2 \neq 0$) is

$$\Phi_2 = \frac{b_2}{r_0} xy, \tag{2}$$

for a normal octupole ($b_4 \neq 0$)

$$\Phi_{4n} = \frac{b_4}{r_0^3}[x^3 y - xy^3], \tag{3}$$

and for a skew octupole ($a_4 \neq 0$)

$$\Phi_{4s} = \frac{a_4}{4r_0^3}[y^4 - 6x^2 y^2]. \tag{4}$$

Now consider a quadrupole of finite length and aperture, whose field depends on the longitudinal position z. In this case, the scalar potential Φ contains z-dependent terms and obeys the three-dimensional Laplace equation. In polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$, the scalar potential can be written as [4, 5]

$$\Phi(r, \theta, z) = G(r, z) r^2 \frac{\sin 2\theta}{2!} = [G_{20}(z) + G_{22}(z) r^2 + \ldots] r^2 \frac{\sin 2\theta}{2} \tag{5}$$

The first term in the square brackets on the right-hand side, $G_{20}(z)$, parametrises the field variation on the magnet axis, via $G_{20}(z) = \partial B_y / \partial x(z)|_{r=0}$. Its derivative gives rise to a longitudinal field component.

The second term in Eq. (5) is related to the second derivative of G_{20}:

$$G_{22}(z) = -\frac{1}{12}\frac{d^2G_{20}(z)}{dz^2} \tag{6}$$

The scalar potential associated with this term is proportional to

$$r^4\frac{\sin 2\theta}{2} = [x^3y + y^3x] \tag{7}$$

Comparison with Eqs. (3) shows that this polynomial differs from that of an ordinary octupole by the relative sign of its two arguments. In addition, as already mentioned, derivatives with respect to z introduce longitudinal field components, which are absent for fields that are independent of z. Thus, for several reasons the fringe field effect cannot be described by the usual multipole expansion [6][1].

The integrated effect on a particle trajectory is conventionally described by a Hamiltonian which contains the vector potential $\vec{A}$ and not the scalar potential Φ. Thus, the polynomial form of the Hamiltonian form is different from that of the scalar potential.

For example, a normal quadrupole ($b_2 \neq 0$) corresponds to the Hamiltonian

$$H_{2n} = \frac{1}{2}K_{2n}[x^2 - y^2] \tag{8}$$

where $K_2 = b_2l_Q/(B\rho)/r_0$, l_Q denotes the length of the magnet, and $(B\rho)$ the magnetic rigidity. Similarly, the Hamiltonians for a normal ($b_4 \neq 0$) or skew octupole ($a_4 \neq 0$) are

$$\begin{aligned} H_{4n} &= \frac{1}{24}K_{4n}[x^4 - 6x^2y^2 + y^4], & (9)\\ H_{4s} &= \frac{1}{6}K_{4s}[y^3x - x^3y], & (10) \end{aligned}$$

with $K_{4n} = 6b_4/(B\rho)/r_0^3$, and $K_{4s} = 6a_4/(B\rho)/r_0^3$. The evolution of a particle trajectory then follows from Hamilton's equations: $dx'/dz = -\partial H/\partial x$, and $dy'/dz = -\partial H/\partial y$.

To represent the fringe field effect by a Hamiltonian, we must find the vector potential $\vec{A}$. For simplicity, we rewrite Eq. (5) as

$$\Phi(r, \theta, z) = \Phi_0(r, z)\sin 2\theta \tag{11}$$

so that only the quadrupolar azimuthal dependence is explicit. We know that $B_r = \partial\Phi/\partial r$, $B_z = \partial\Phi/\partial z$, and $B_\theta = 1/r(\partial\Phi/\partial\theta)$. One choice of vector potential

[1] By placing several families of octupoles at positions with large and small β_x/β_y ratios, respectively, it might still be possible using octupoles to globally compensate the two terms proportional to x^3y and y^3x.

which gives the same magnetic field is [7]

$$
\begin{aligned}
A_r &= \frac{1}{2} r \frac{\partial \Phi_0}{\partial z} \cos 2\theta && (12) \\
A_z &= -\frac{1}{2} r \frac{\partial \Phi_0}{\partial r} \cos 2\theta && (13) \\
A_\theta &= 0 && (14)
\end{aligned}
$$

This can be verified explicitly:

$$
\begin{aligned}
B_r &= \frac{1}{r}\frac{\partial A_z}{\partial \theta} = \frac{\partial \Phi_0}{\partial r} \sin 2\theta = \frac{\partial \Phi}{\partial r} && (15) \\
B_z &= -\frac{1}{r}\frac{\partial A_r}{\partial \theta} = \frac{\partial \Phi_0}{\partial z} \sin 2\theta = \frac{\partial \Phi}{\partial z} && (16) \\
B_\theta &= \frac{\partial A_r}{\partial z} - \frac{\partial A_s}{\partial r} = \frac{1}{r}\frac{\partial}{\partial \theta}\Phi_0 \sin 2\theta && (17)
\end{aligned}
$$

where in the last line we used the fact that the scalar potential $\Phi = \Phi_0 \sin 2\theta$ satisfies the Laplace equation:

$$
\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right)\Phi + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\Phi = 0 \tag{18}
$$

The Taylor expansion of A_r and A_z as a function of the radius r reads

$$
\begin{aligned}
A_r &\approx \frac{r^3}{4}\left[\frac{dG_{20}(z)}{dz} + r^2 \frac{dG_{22}(z)}{dz} + \ldots\right] \cos 2\theta && (19) \\
A_z &\approx -\frac{r^2}{2}\left[G_{20}(z) + 2r^2 G_{22}(z) + \ldots\right] \cos 2\theta. && (20)
\end{aligned}
$$

In the following, we only retain the lowest-order terms.

3 Hamiltonian and Tune Shift

The non-vanishing components of the vector potential, A_z and A_r in Eqs. (19) and (20), can be inserted into the general form of the Lie-algebraic Hamiltonian [7]:

$$
\begin{aligned}
H &= \frac{1}{2}\left(p_r - \frac{qA_r}{p_0(1+\delta)}\right)^2 + \frac{p_\theta^2}{2r^2} - \frac{qA_z}{p_0(1+\delta)} && (21) \\
&\approx H_{\text{lin}} - \frac{q}{p_0} p_r A_r - \frac{q}{p_0} A_z && (22)
\end{aligned}
$$

Here q denotes the charge of the particle, and p_r, p_θ the radial and angular momenta, respectively, and we omit the δ-dependence. Keeping again only the two lowest-order nonlinear terms (up to 4th power in r and p_r) we obtain

$$
H \approx H_{\text{lin}} - \frac{1}{B\rho}\frac{dG_{20}}{dz}\frac{1}{4} r^3 p_r \cos 2\theta + r^4 \frac{1}{B\rho} G_{22}(z) \cos 2\theta \tag{23}
$$

where the linear part, $H_{\rm lin}$, includes the usual kinematic term, $\frac{1}{2}[p_r^2 + p_\theta^2/r^2]$, and also the linear quadrupole focusing, $\frac{1}{2}K_Q r^2 \cos 2\theta$, with $K_Q = G_{20}/(B\rho)$. The nonlinear perturbation, $H_{\rm pert} = H - H_{\rm lin}$, can be expressed in cartesian coordinates, x and y, as

$$H_{\rm pert} \approx -\frac{1}{B\rho}\frac{dG_{20}(z)}{dz}\frac{1}{4}(x^2-y^2)(xp_x+yp_y) - \frac{1}{12}\frac{1}{B\rho}\frac{d^2G_{20}}{dz^2}(x^4-y^4) \tag{24}$$

Next we integrate the Hamiltonian over the incoming or outgoing side of the magnet. We assume that the fringe field extends over a longitudinal distance $\pm\Delta$ around the edge of the magnet. The distance Δ is proportional to the magnet aperture. To evaluate the integral

$$\hat{H} = \int_{-\Delta}^{\Delta} H_{\rm pert}(z)dz \tag{25}$$

we perform a Taylor expansion of the transverse coordinates in terms of z, around the entrance or exit points of the magnet [2] [7]. These two reference points are taken to be the positions where the field gradient is 1/2 of its value at the center of the magnet. We assume that the field fall-off is symmetric about each of these points.

For example, the second argument in Eq. (24) is expanded as

$$(x^4-y^4) = \left\{[x^4-y^4]_0 + z\left[\frac{d}{dz}(x^4-y^4)\right]_0 + \frac{z^2}{2}\left[\frac{d^2}{dz^2}(x^4-y^4)\right]_0 + \ldots\right\}. \tag{26}$$

The subindex 0 refers to the expansion point. Inserting this and the equivalent expansion of $(x^2-y^2)(xp_x+yp_y)$ into Eq. (25), we obtain integrals of the form

$$\frac{1}{B\rho}\int_{-\Delta}^{\Delta} G'_{20}dz = K_Q, \tag{27}$$

$$\frac{1}{B\rho}\int_{-\Delta}^{\Delta} G'_{20}z dz = 0, \tag{28}$$

$$\frac{1}{B\rho}\int_{-\Delta}^{\Delta} G'_{20}z^2 dz \approx \frac{1}{3}\Delta^2 K_Q, \tag{29}$$

$$\frac{1}{B\rho}\int_{-\Delta}^{\Delta} G''_{20}dz = 0, \tag{30}$$

$$\frac{1}{B\rho}\int_{-\Delta}^{\Delta} G''_{20}z dz = -K_Q, \tag{31}$$

$$\frac{1}{B\rho}\int_{-\Delta}^{\Delta} G''_{20}z^2 dz = 0, \tag{32}$$

[2] For a special form of the field fall-off, and considering one dimension only, M. Venturini recently computed the integrated fringe-field Hamiltonian without resorting to a Taylor map expansion [8].

where we used the assumption that the fringe fall-off is symmetric about the entrance (or exit) point. All the results quoted are for the incoming edge. For the outgoing edge, the signs on the right-hand-side are inverted.

Three terms, corresponding to the three non-vanishing integrals above, contribute to the integral Eq. (25), up to second order in Δ. We make this transparent by writing $\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_3$. The first term results from the first term on the right-hand-side of Eq. (24) and the nonzero integral in Eq. (27). Adding the contributions from magnet entrance and exit, it reads

$$\hat{H}_1 = -\frac{1}{4}K_Q\left[(x^2-y^2)_i(xp_x+yp_y)_i-(x^2-y^2)_o(xp_x+yp_y)_o,\right] \tag{33}$$

where the subindices i and o indicate coordinates at the incoming and outgoing sides, respectively, and K_Q is the normalized quadrupole gradient in units of inverse squarea meters, or $K_Q = G_{20}(0)/(B\rho)$.

The second term arises from the second term in Eq. (240 and from Eq. (31):

$$\hat{H}_2 = \frac{1}{3}K_Q\left[(x^3p_x-y^3p_y)_i-(x^3p_x-y^3p_y)_o.\right] \tag{34}$$

Adding the two previous equations, we get

$$\begin{aligned}\hat{H}_{1+2} &= \frac{1}{12}K_Q\left[xp_x(x^2+3y^2)-yp_y(y^2+3x^2)\right]_i \\ &\quad -\frac{1}{12}K_Q\left[xp_x(x^2+3y^2)-yp_y(y^2+3x^2)\right]_o.\end{aligned} \tag{35}$$

This agrees with the effect of an ideal hard-edge fringe field, which was calculated by Lee-Whiting [9] and, more recently and in more general form by E. Forest and J. Milutinovic [10]. This term, which is independent of the fringe field length Δ, will turn out to be the dominant nonlinear effect, in good agreement with Venturini's result for a 1-dimensional fringe field [8].

Finally, the last term, which derives from the integral in Eq. (29) and, again, from the first part of Eq. (24), depends on the fringe length:

$$\begin{aligned}\hat{H}_3 &= -\frac{1}{24}\Delta^2K_Q\Big[\Big(6xp_x^3-6yp_y^3-10K_Qx^3p_x-10K_Qy^3p_y \\ &\quad +6K_Qx^2yp_y+6K_Qy^2xp_x+2xp_xp_y^2-2yp_yp_x^2\Big)_i \\ &\quad -\Big(6xp_x^3-6yp_y^3-10K_Qx^3p_x-10K_Qy^3p_y \\ &\quad +6K_Qx^2yp_y+6K_Qy^2xp_x+2xp_xp_y^2-2yp_yp_x^2\Big)_o\Big]\end{aligned} \tag{36}$$

The coordinates at the outgoing side, 'o', can be expressed by those at the entrance of the magnet using the linear transformation through the quadrupole. We assume that $(\sqrt{K_Q}l_Q)$ is sufficiently small, that we can linearly expand the $\sin(\sqrt{K_Q}l_Q)$ or $\sinh(\sqrt{K_Q}l_Q)$ functions in the elements of the R matrix. In our example below, $\sqrt{K_Q}l_Q$ is about 0.13. We will also assume that the quadrupole is short, and that the beta function at the quadrupole is large, or specifically that

$$K_Q \gg 1/\beta^2 \tag{37}$$

and

$$l_Q \ll \beta. \tag{38}$$

Under these conditions, the transverse coordinates are approximately constant within the magnet

$$x_0 \approx x_i, \tag{39}$$
$$y_0 \approx y_i, \tag{40}$$

while only the values of the trajectory slope change, roughly as

$$p_{xo} \approx p_{xi} - (K_Q l_Q)x_i, \tag{41}$$
$$p_{yo} \approx p_{yi} + (K_Q l_Q)y_i. \tag{42}$$

The Δ-independent part of $\hat{H}$ becomes

$$\hat{H}_{1+2} \approx \frac{1}{12}(K_Q l_Q)K_Q\left[x^4 + 6x^2y^2 + y^4\right], \tag{43}$$

where the coordinates x and y may now be taken to be those at the center of the magnet.

Again using Eqs. (41) and (42) and keeping only the largest components, the next term in the Hamiltonian, Eq. (36), can be approximated as

$$\hat{H}_3 \approx \frac{5}{12}\Delta^2 K_Q^2 (K_Q l_Q)\left[x^4 - y^4\right]. \tag{44}$$

Expressing the transverse positions in terms of action angle coordinates, $x = \sqrt{2I_x\beta_x}\cos\phi_x$ and $y = \sqrt{2I_y\beta_y}\cos\phi_y$, and averaging the Hamiltonian $\hat{H} = (\hat{H}_{1+2} + \hat{H}_3)$ over the betatron phases ϕ_x and ϕ_y using $< \cos^4\phi >= 3/8$ and $< \cos^2\phi >= 1/2$, the nonlinear Hamiltonian representing the effect of the fringe fields reads

$$\begin{aligned} < \hat{H} > \approx\ & \frac{1}{8}\sum_Q (K_Q l_Q)K_Q\left[\beta_{x,Q}^2 I_x^2 + 4\beta_{x,Q}\beta_{y,Q}I_xI_y + \beta_{y,Q}^2 I_y^2\right] \\ & + \frac{5}{8}\sum_Q \Delta^2 K_Q^2 (K_Q l_Q)[\beta_x^2 I_x^2 - \beta_y^2 I_y^2]. \end{aligned} \tag{45}$$

The sum is over all quadrupoles Q, and $K_Q > 0$ for a horizontally focusing quadrupole. The derivatives of $< \hat{H} >$ with respect to $I_{x,y}$ yield the amplitude dependent tune shifts:

$$\begin{aligned}\Delta Q_x &\approx \frac{1}{8\pi}\sum_Q (K_Q l_Q) K_Q \left[\beta_{x,Q}^2 I_x + 2\beta_{x,Q}\beta_{y,Q} I_y\right] \\ &+\frac{5}{8\pi}\sum_Q \Delta^2 l_Q K_Q^3 \beta_{x,Q}^2 I_x, \end{aligned} \tag{46}$$

$$\begin{aligned}\Delta Q_y &\approx \frac{1}{8\pi}\sum_Q (K_Q l_Q) K_Q \left[\beta_{y,Q}^2 I_y + 2\beta_{x,Q}\beta_{y,Q} I_x\right] \\ &-\frac{5}{8\pi}\sum_Q \Delta^2 l_Q K_Q^3 \beta_{y,Q}^2 I_y, \end{aligned} \tag{47}$$

where the sums are over the various quadrupoles. All three tune shifts, $\Delta Q_x/\Delta I_x$, $\Delta Q_x/\Delta I_y = \Delta Q_y/\Delta I_x$, and $\Delta Q_y/\Delta I_y$, are positive and of comparable magnitude.

4 Example

We consider the FNAL muon storage ring, whose optics is shown in Fig. 1. The ring consists of three parts: a neutrino production straight, a return straight, and the (two) arcs. We first evaluate the tune shift from the arcs. A detailed view of the arc optics is shown in Fig. 2. There are a total of 31 arc cells, each comprising two quadrupoles. Using maximum and minimum beta functions of $\beta_{x,y}$ of 16 m and 3 m, respectively, a quadrupole length $l_Q = 1$ m, strength $K_Q = 0.31$ m^{-2}, and zero fringe extent ($\Delta = 0$), we estimate $\Delta Q_x/\Delta I_x \approx \Delta Q_y/\Delta I_y \approx 31$ m^{-1}, and $\Delta Q_x/\Delta I_y = \Delta Q_y/\Delta I_x \approx 23$ m^{-1}. We can compare these estimates with an exact calculation using the program COSY INFINITY [1, 11], which gives $\Delta Q_x/\Delta I_x = 30$ m^{-1}, $\Delta Q_x/\Delta I_y = 28$ m^{-1} and $\Delta Q_y/\Delta I_y = 34$ m^{-1}. The agreement between COSY and our first-order estimate is quite satisfactory.

The same comparison can be made for the neutrino production straight. Here the maximum and minimum beta functions are about 430 m and 300 m, the quadrupole strength $K_Q \approx 0.0019$ m^{-2}, the length $l_Q = 3$ m, and the total number of cells is 5. We then obtain $\Delta Q_x/\Delta I_x \approx \Delta Q_y/\Delta I_y \approx 0.6$ m^{-1}, and $\Delta Q_x/\Delta I_y = \Delta Q_y/\Delta I_x \approx 1.1$ m^{-1}. These values almost perfectly agree with the COSY results of 0.6 m^{-1} and 1.0 m^{-1}, respectively. The product $[\beta^2 K_Q^2 l_Q]$ scales about as $1/\beta$, which explains why the tune shift induced in the arcs is much larger than that from the production straight.

The actual value of the tune shift at 1σ can be estimated by setting $I_{x,y}$ in the above expressions for $\Delta Q_{x,y}$ equal to half the rms geometric emittance $\epsilon_{x,y}/2$.

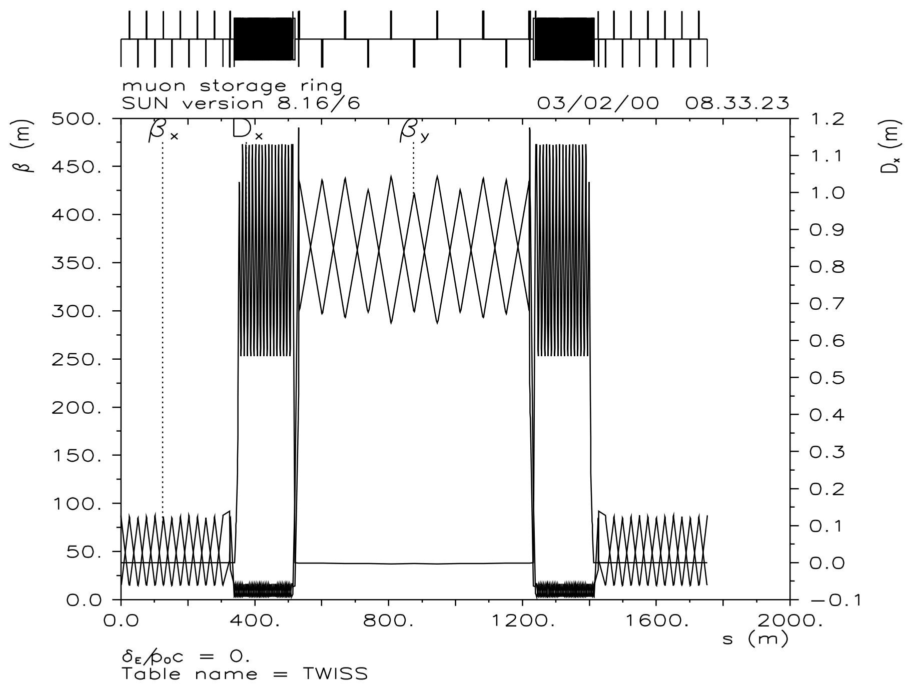

Figure 1: Optics for the FNAL muon storage ring; courtesy of C. Johnstone.

For the nominal rms emittance, $\epsilon_{x,y} \approx 7\ \mu$m, the tune shift due to fringe fields in arcs and straight section is small.

Equations. (46) and (47) indicate that the fringe fields of quadrupoles in the matching section between arcs and production straight are the dominant perturbation, since here the beta functions are comparable to those in the production straight, while the quadrupole strengths are 100 or 1000 times larger. Indeed, for the FNAL muon storage ring, the tune shift induced by the matching quadrupoles is a few orders of magnitude higher than that generated in the rest of the ring. The dynamic aperture can, therefore, be improved by lengthening the matching quadrupoles. This was explicitly demonstrated for the CERN muon storage ring [12], where an analogous fringe effect occurred.

The part of the tune shift quadratic in Δ is suppressed compared to the Δ-independent part by a factor $5\Delta^2 K_Q$. For quadrupoles in the production straight with $K_Q = 0.002$ m^{-2} and $\Delta \approx 0.17$ m (the magnet half aperture), this suppression factor is a few 10^{-4}. We expect that the contributions from higher-order terms are even less important.

5 Conclusion

We have derived an analytical expression of the lowest-order nonlinear Hamiltonian generated by quadrupole fringe fields, partly reproducing earlier results

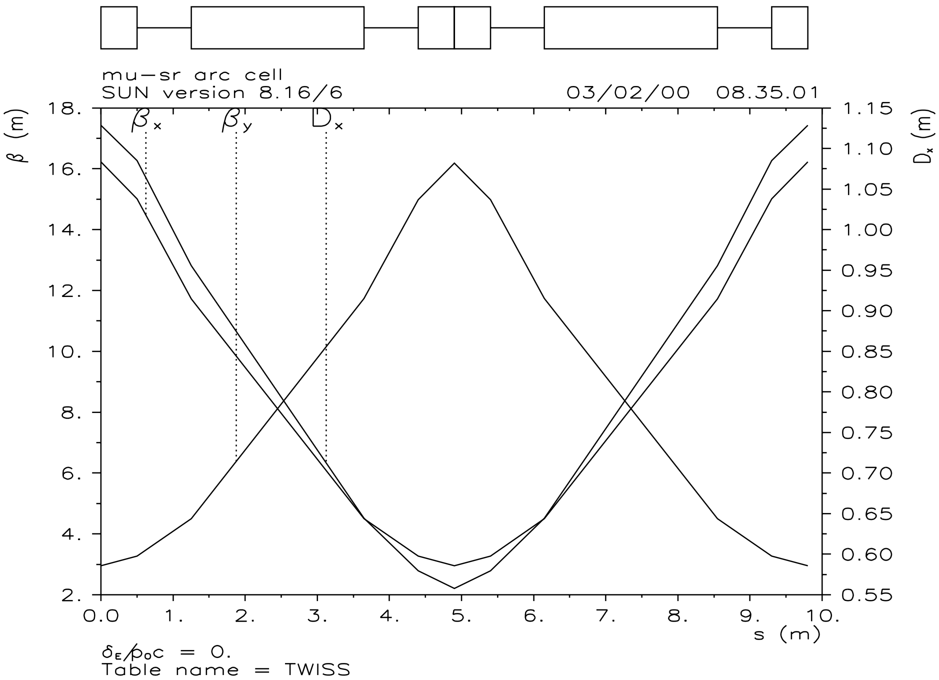

Figure 2: Optics for an arc cell of the FNAL muon storage ring; courtesy of C. Johnstone.

[7, 9, 10]. Using this Hamiltonian, approximate formulae were obtained for the linear tune shifts with amplitude, Eqs. (46) and (47), which are valid in the common situation that $\beta_Q^2 \gg 1/K_Q \gg l_Q^2$, and if the longitudinal extent of the fringe field, Δ, is small compared with $1/\sqrt{K_Q}$. For two typical examples, the analytical formulae are consistent with results from the computer code COSY INFINITY.

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References

[1] M. Berz et al., COSY INFINITY web page. http://cosy.nscl.msu.edu

[2] M. Berz, K. Makino and B. Erdelyi, "Fringe Field Effects in Muon Rings", presented at HEMC'99 Montauk, New York (1999).

[3] F. Zimmermann, C. Johnstone, M. Berz, B. Erdelyi, K. Makino, W. Wan, "Fringe Fields and Dynamic Aperture in Muon Storage Rings", unpublished.

[4] M. Bassetti and C. Biscari, "Effects of Fringing Fields on Single Particle Dynamics", Proc. of 14th Advanced ICFA workshop on e^+e^- factories, Frascati Physics Series Vol. X, p. 247 (1998).

[5] M. Bassetti and C. Biscari, "Analytical Formulae for Magnetic Multipoles", Part. Acc. 52, p. 221 (1996).

[6] C. Iselin, private communication (2000).

[7] J. Irwin, lectures on Lie algebra methods in beam optics, SLAC, unpublished (1994); J. Irwin's fringe field treatment partly followed E. Forest's Ph.D. thesis.

[8] M. Venturini, "Scaling of Third-Order Quadrupole Aberrations with Fringe Field Extensions", ICAP 98, Monterey (1998).

[9] G.E. Lee-Whiting, "Third-Order Aberrations of a Magnetic Quadrupole Lens", Nuclear Instruments and Methods 83, p. 232 (1970).

[10] E. Forest and J. Mulotinovic, "Leading order Hard Edge Fringe Fields Effects Exact in $(1+\delta)$ and consistent with Maxwell's Equations for Rectilinear Magnets", Nuclear Instruments and Methods A269, p. 474 (1988).

[11] K. Makino and M. Berz, "COSY INFINITY version 8", NIM A 427, p. 338 (1999).

[12] Private discussion with E. Keil.