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Renormalization-Group-Improved Effective Potential for the MSSM Higgs Sector with Explicit CP Violation

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ABSTRACT

We perform a systematic study of the one-loop renormalization-group-improved effective potential of the minimal supersymmetric extension of the Standard Model (MSSM), including CP violation induced radiatively by soft trilinear interactions related to squarks of the third generation. We calculate the charged and neutral Higgs-boson masses and couplings, including the two-loop logarithmic corrections that arise from QCD effects, as well as those associated with the top- and bottom-quark Yukawa couplings. We also include the potentially large two-loop non-logarithmic corrections induced by one-loop threshold effects on the top- and bottom-quark Yukawa couplings, due to the decoupling of the third-generation squarks. Within this minimal CP-violating framework, the charged and neutral Higgs sectors become intimately related to one another and therefore require a unified treatment. In the limit of a large charged Higgs-boson mass, $M_{H^\pm} \gg M_Z$, the lightest neutral Higgs boson resembles that in the Standard Model (SM), and CP violation occurs only in the heavy Higgs sector. Our analysis shows that sizeable radiative effects of CP violation in the Higgs sector of the MSSM may lead to significant modifications of previous studies for Higgs-boson searches at LEP 2, the Tevatron and the LHC. In particular, CP violation could enable a relatively light Higgs boson to escape detection at LEP 2.

1 Introduction

The violation of CP was first observed in the neutral kaon system [1], where a deviation from the superweak theory has recently been confirmed [2], and CP violation in B -meson decays [3] is strongly suggested by recent experiments [4]. In addition to its interest for particle physics, CP non-conservation provides a key ingredient for cosmological baryogenesis, namely for explaining the underlying mechanism which caused matter to dominate over anti-matter in our observable universe [5]. Although a fundamental understanding of the origin of CP violation is still lacking, most of the scenarios proposed in the existing literature indicate that Higgs interactions play a key role in mediating CP violation. For instance, CP violation is broken explicitly in the Standard Model (SM) by complex Yukawa couplings of the Higgs boson to quarks. Another appealing scheme of CP violation occurs in models with an extended Higgs sector, in which the CP symmetry of the theory is broken spontaneously by the ground state of the Higgs potential [6]. Supersymmetric (SUSY) theories, including the minimal supersymmetric extension of the Standard Model (MSSM), predict an extended Higgs sector, and therefore may realize either or both the above two schemes of explicit and spontaneous CP violation. However, the Higgs potential of the MSSM is invariant under CP at the tree level, and any explicit or spontaneous breakdown of CP symmetry can arise only via radiative corrections. The case of purely spontaneous CP violation in the MSSM [7] leads to an unacceptably light CP-odd scalar [8], as a result of the Georgi-Pais theorem [9], and hence such a scenario is ruled out experimentally.

It has recently been shown [10] that the tree-level CP invariance of the MSSM Higgs potential may be violated sizeably by loop effects involving soft CP-violating trilinear interactions of the Higgs bosons to top and bottom squarks. A detailed study [11] has shown that significant CP-violating effects of level crossing in the Higgs sector can take place in such a minimal SUSY scenario of explicit CP violation, which may lead to drastic modifications of the tree-level couplings of the Higgs particles to fermions [11,12] and to the W^\pm and Z bosons [11]. The latter can have important phenomenological consequences on the production rates of the lightest Higgs particle, even though the upper bound on its mass was found [11] to be very similar to that found previously in the CP-conserving case [13–18,20,21]. The MSSM predicts an upper bound on the lightest Higgs boson mass of approximately 110 (130) GeV for small (large) values of the ratio of Higgs vacuum expectation values $\tan\beta \approx 2$ (> 15). On the other hand, experiments at LEP 2, running at center-of-mass energies $\sqrt{s} = 196\text{--}202$ GeV, have placed a severe lower bound of approximately 108 GeV on the mass of the SM Higgs boson [22]. LEP 2 is expected to run at center-of-mass energies up to $\sqrt{s} = 206$ GeV during the year 2000. Consequently, for

stop masses smaller than 1 TeV, a significant portion of the parameter space spanned by $\tan\beta$ and the CP-odd scalar mass M_A can be tested for the CP-conserving case in this next round of experiments at LEP 2 [11]. However, the explorable region of parameters is smaller for larger amounts of stop mixing and/or larger CP-violating phases. A decisive test of such a scenario can only be provided by the upgraded Tevatron collider and the LHC.

The earlier study of the renormalization-group (RG) improved effective potential of the MSSM with explicit CP violation in [11] was based on an expansion of the Higgs quartic couplings in inverse powers of the arithmetic average of stop and sbottom masses, under the assumption that the mass splittings of the left- and right-handed stop and sbottom masses are small. The mass expansion of the one-loop effective potential was truncated up to renormalizable operators of dimension four. Although the above approach captures the basic qualitative features of the underlying dynamics under study, it is known [18,11] that such a mass expansion has limitations when the third-generation squark mixing is large. Since the dominant CP-violating loop contributions to the effective Higgs-boson masses and mixing angles occur for large values of the third-generation squark-mixing parameters, it is necessary to provide a more complete one-loop computation of the effective MSSM Higgs potential with explicit radiative breaking of CP invariance, including non-renormalizable operators below the heavy scalar-quark scale and without resorting to any other kinematic approximations.*

To this end, we consider here the two-loop leading logarithms induced by top- and bottom-quark Yukawa couplings as well as those associated with QCD corrections, by means of RG methods. In the calculation of the RG-improved effective potential, we also include the leading logarithms generated by one-loop gaugino and higgsino quantum effects [14]. Finally, we implement the potentially large two-loop corrections that are induced by the one-loop stop and sbottom thresholds in the top- and bottom-quark Yukawa couplings, which may become particularly relevant in the large- $\tan\beta$ regime. On the basis of the RG-improved effective potential, we present predictions for the Higgs-boson mass spectrum and the effective Higgs couplings to fermions and gauge bosons. The results of the analysis are compared with those obtained in the CP-conserving case [18,19] and also with the results obtained by truncating the one-loop RG-improved effective potential up to renormalizable operators [11].

In this analysis, it is important to consider the constraints on the low-energy CP-

*We recall that the diagrammatic computation of scalar-pseudoscalar transitions is the focus of [10], whilst sbottom contributions and relevant D -term effects on the effective potential are not considered in [12].

violating parameters of the MSSM that originate from experimental upper limits on the electron [23] and neutron [24] electric dipole moments (EDMs) [25–35]. Most of the EDM constraints affect the CP-violating couplings of the first two generations [25,26]. Thus, making the first two generations of squarks rather heavy, much above the TeV scale [27,28], is a possibility that can drastically reduce one-loop contributions to the neutron EDM, without suppressing the CP-violating phases of the theory. Another interesting possibility for avoiding any possible CP crisis in the MSSM is to arrange for cancellations among the different EDM terms, either at the level of short-distance diagrams [29] or (for the neutron EDM) at the level of the strong-interaction matrix elements for operators with s , u and d quark flavours [30,31]. Alternatively, one might require a specific form of non-universality in the soft trilinear Yukawa couplings [32,34].

However, third-generation squarks can also give rise by themselves to observable effects on the electron and neutron EDMs through the three-gluon operator [33], through the effective coupling of the ‘CP-odd’ Higgs boson to the gauge bosons [34], and through two-loop gaugino/higgsino-mediated EDM graphs [35]. These different EDM contributions of the third generation can also have different signs and add destructively to the electron and neutron EDMs. In our phenomenological discussion, we take into account the relevant EDM constraints related to the CP-violating parameters of the stop and sbottom sectors.

This paper is organized as follows: in Section 2 we calculate the complete one-loop CP-violating effective potential, and derive the analytic expressions for the effective charged and neutral Higgs-boson mass matrices. Technical details are given in Appendices A and B. Section 3 describes our approach to determining the RG-improved Higgs-boson mass matrices, after including the leading two-loop logarithms associated with Yukawa and QCD corrections. Section 4 is devoted to the calculation of the effective top- and bottom-quark Yukawa couplings, in which one-loop threshold effects of the third-generation squarks are implemented. In Section 5, we discuss the phenomenological implications of representative CP-violating scenarios compatible with EDM constraints for direct Higgs searches at LEP 2 and the upgraded Tevatron collider. We also compare the results of our analysis with those obtained using the mass-expansion method [11]. Finally, in Section 6 we summarize our conclusions.

2 CP-Violating One-Loop Effective Potential

In this Section, we first describe the basic low-energy structure of the MSSM that contains explicit CP-violating sources, such as soft CP-violating trilinear interactions. Then we calculate the general one-loop CP-violating effective potential. Finally, after implementing the

minimization tadpole conditions related to the Higgs ground state, we derive the effective charged and neutral Higgs-boson mass matrices.

CP violation is introduced into the MSSM through the Higgs superpotential and the soft SUSY-breaking Lagrangian:

$$W = h_l \widehat{H}_1^T i\tau_2 \widehat{L} \widehat{E} + h_d \widehat{H}_1^T i\tau_2 \widehat{Q} \widehat{D} + h_u \widehat{Q}^T i\tau_2 \widehat{H}_2 \widehat{U} - \mu \widehat{H}_1^T i\tau_2 \widehat{H}_2, \quad (2.1)$$

$$\begin{aligned} -\mathcal{L}_{\text{soft}} = & -\frac{1}{2} \left(m_{\tilde{g}} \lambda_{\tilde{g}}^a \lambda_{\tilde{g}}^a + m_{\tilde{W}} \lambda_{\tilde{W}}^i \lambda_{\tilde{W}}^i + m_{\tilde{B}} \lambda_{\tilde{B}} \lambda_{\tilde{B}} + \text{h.c.} \right) + \widetilde{M}_L^2 \widetilde{L}^\dagger \widetilde{L} + \widetilde{M}_Q^2 \widetilde{Q}^\dagger \widetilde{Q} \\ & + \widetilde{M}_U^2 \widetilde{U}^* \widetilde{U} + \widetilde{M}_D^2 \widetilde{D}^* \widetilde{D} + \widetilde{M}_E^2 \widetilde{E}^* \widetilde{E} + m_1^2 \widetilde{\Phi}_1^\dagger \widetilde{\Phi}_1 + m_2^2 \Phi_2^\dagger \Phi_2 - \left(B\mu \widetilde{\Phi}_1^T i\tau_2 \Phi_2 \right. \\ & \left. + \text{h.c.} \right) + \left(h_l A_l \Phi_1^\dagger \widetilde{L} \widetilde{E} + h_d A_d \Phi_1^\dagger \widetilde{Q} \widetilde{D} - h_u A_u \Phi_2^T i\tau_2 \widetilde{Q} \widetilde{U} + \text{h.c.} \right), \quad (2.2) \end{aligned}$$

where $\widetilde{\Phi}_1 = i\tau_2 \Phi_1^*$ is the scalar component of the Higgs chiral superfield \widehat{H}_1 and τ_2 is the usual Pauli matrix. The conventions followed throughout this paper, including the quantum-number assignments of the fields under the SM gauge group, are displayed in Table 1.

As can be seen from (2.1) and (2.2), the MSSM includes additional complex parameters with new CP-odd phases that are absent in the SM. These new CP-odd phases may reside in the following parameters: (i) the mass parameter μ describing the bilinear mixing

Superfields	Bosons	Fermions	$\text{SU}(3)_c \otimes \text{SU}(2)_L \otimes \text{U}(1)_Y$
<u>Gauge multiplets</u>			
\widehat{G}^a	$G_\mu^a \frac{1}{2} \lambda^a$	\tilde{g}^a	$(8, 1, 0)$
\widehat{W}	$W_\mu^i \frac{1}{2} \tau_i$	\widetilde{W}^i	$(1, 3, 0)$
\widehat{B}	B_μ	\tilde{B}	$(1, 1, 1)$
<u>Matter multiplets</u>			
\widehat{L}	$\widetilde{L}^T = (\tilde{\nu}_l, \tilde{l})_L$	$L^T = (\nu_l, l)_L$	$(1, 2, -1)$
\widehat{E}	$\widetilde{E} = \tilde{l}_R^*$	$(e_R)^C = (e^C)_L$	$(1, 1, 2)$
\widehat{Q}	$\widetilde{Q}^T = (\tilde{u}, \tilde{d})_L$	$Q^T = (u, d)_L$	$(3, 2, \frac{1}{3})$
\widehat{U}	$\widetilde{U} = \tilde{u}_R^*$	$(u_R)^C = (u^C)_L$	$(3, 1, -\frac{4}{3})$
\widehat{D}	$\widetilde{D} = \tilde{d}_R^*$	$(d_R)^C = (d^C)_L$	$(3, 1, \frac{2}{3})$
\widehat{H}_1	$\widetilde{\Phi}_1^T = (\phi_1^{0*}, -\phi_1^-)$	$(\bar{\psi}_{H_1}^0, \psi_{H_1}^-)$	$(1, 2, -1)$
\widehat{H}_2	$\Phi_2^T = (\phi_2^+, \phi_2^0)$	$(\psi_{H_2}^+, \psi_{H_2}^0)$	$(1, 2, 1)$

Table 1: *The field content of the MSSM.*

of the two Higgs chiral superfields in the superpotential; (ii) the soft SUSY-breaking gaugino masses $m_{\tilde{g}}$, $m_{\tilde{W}}$ and $m_{\tilde{B}}$ of the gauge groups $SU(3)_c$, $SU(2)_L$ and $U(1)_Y$, respectively; (iii) the soft bilinear Higgs-mixing mass $B\mu$; and (iv) the soft SUSY-breaking trilinear couplings A_f of the Higgs bosons to sfermions. In addition, there may exist other large CP-odd phases, associated with flavor off-diagonal soft SUSY-breaking masses of squarks and sleptons. We assume that these off-diagonal masses are small and therefore do not give sizeable contributions to the effective Higgs potential. The number of independent CP-odd phases may be reduced if one prescribes a universality condition for all gaugino masses at the unification scale M_X ; the gaugino masses will then have a common phase. Correspondingly, the different trilinear couplings A_f may be considered to be all equal at M_X , *i.e.*, $A_f \equiv A$. In this case, however, because of the different RG running of the phases of the trilinear couplings, their values at low energies will be different.[†] Two CP-odd phases may further be eliminated by employing the following two global symmetries that govern the dimension-four operators in the MSSM Lagrangian:

- (i) The $U(1)_Q$ symmetry specified by $Q(\widehat{H}_1) = 1$, $Q(\widehat{H}_2) = -2$, $Q(\widehat{Q}) = Q(\widehat{L}) = 0$, $Q(\widehat{U}) = 2$ and $Q(\widehat{D}) = Q(\widehat{E}) = -1$. This $U(1)_Q$ symmetry is broken by the μ and m_{12}^2 parameters.
- (ii) The $U(1)_R$ symmetry acting on the Grassmann-valued coordinates, *i.e.*, the θ coordinate of superspace carries charge 1. Under the R transformation, the matter superfields and gaugino fields carry charge 1, whilst the Higgs superfields are R -neutral. The R symmetry is violated by the gaugino masses, the trilinear couplings A_f and the parameter μ .

We concentrate on the parameters which may have a dominant CP-violating effect on the MSSM Higgs potential, under the assumption of a common phase for the gauginos; the latter is made less important by the fact that the one-loop gaugino corrections are subdominant compared to the ones induced by the third-generation squarks. As has been mentioned above, two CP-odd phases of the complex parameters $\{\mu, m_{12}^2, m_\lambda, A\}$ may be removed by employing the global symmetries (i) and (ii). Specifically, one of the Higgs doublets and the common phase of the gaugino fields can be rephased in a way such that the gaugino masses and $B\mu$ become real numbers. As a consequence, $\arg(\mu)$ and $\arg(A_{t,b})$ are the only physical CP-violating phases in the MSSM which affect the Higgs sector in a relevant way.

It is obvious from (2.1) and (2.2) that the Yukawa interactions of the third-generation quarks, $Q^T = (t_L, b_L)$ and t_R, b_R , as well as their SUSY bosonic counterparts, $\tilde{Q}^T = (\tilde{t}_L, \tilde{b}_L)$

[†]For a discussion of the RG effects, see [36].

and \tilde{t}_R , \tilde{b}_R , play the most significant role in radiative corrections to the Higgs sector. Therefore, it is useful to give the interaction Lagrangians related to the F and D terms of the third generation:

$$\begin{aligned}
-\mathcal{L}_F &= h_b^2 |\Phi_1^\dagger \tilde{Q}|^2 + h_t^2 |\Phi_2^T i\tau_2 \tilde{Q}|^2 \\
&\quad - \left(\mu h_b \tilde{Q}^\dagger \Phi_2 \tilde{b}_R + \mu h_t \tilde{Q}^\dagger i\tau_2 \Phi_1^* \tilde{t}_R + \text{h.c.} \right) \\
&\quad - \left(h_b \tilde{b}_R \Phi_1^T i\tau_2 + h_t \tilde{t}_R \Phi_2^\dagger \right) \left(h_b i\tau_2 \Phi_1^* \tilde{b}_R^* - h_t \Phi_2 \tilde{t}_R^* \right), \tag{2.3}
\end{aligned}$$

$$\begin{aligned}
-\mathcal{L}_D &= \frac{g_w^2}{4} \left[2|\Phi_1^T i\tau_2 \tilde{Q}|^2 + 2|\Phi_2^\dagger \tilde{Q}|^2 - \tilde{Q}^\dagger \tilde{Q} (\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2) \right] \\
&\quad + \frac{g'^2}{4} (\Phi_2^\dagger \Phi_2 - \Phi_1^\dagger \Phi_1) \left[\frac{1}{3} (\tilde{Q}^\dagger \tilde{Q}) - \frac{4}{3} (\tilde{t}_R \tilde{t}_R^*) + \frac{2}{3} (\tilde{b}_R \tilde{b}_R^*) \right]. \tag{2.4}
\end{aligned}$$

where g_w and g' are the usual $SU(2)_L$ and $U(1)_Y$ gauge couplings. Further, the interaction Lagrangian of the Higgs bosons to the top and bottom quarks is given by

$$-\mathcal{L}_{\text{fermions}} = h_b (\bar{b}_R Q^T \Phi_1^* + \text{h.c.}) + h_t (\bar{t}_R Q^T i\tau_2 \Phi_2 + \text{h.c.}), \tag{2.5}$$

where h_t and h_b are the top and bottom Yukawa couplings, respectively.

With the help of the Lagrangians (2.3)–(2.5), we now proceed with the calculation of the one-loop effective potential. More explicitly, in the $\overline{\text{MS}}$ scheme, the one-loop CP-violating effective potential is determined by

$$-\mathcal{L}_V = -\mathcal{L}_V^0 + \frac{3}{32\pi^2} \sum_{q=t,b} \left[\sum_{i=1,2} \tilde{m}_{q_i}^4 \left(\ln \frac{\tilde{m}_{q_i}^2}{Q^2} - \frac{3}{2} \right) - 2\tilde{m}_q^4 \left(\ln \frac{\tilde{m}_q^2}{Q^2} - \frac{3}{2} \right) \right]. \tag{2.6}$$

In (2.6), \mathcal{L}_V^0 is the tree-level Lagrangian of the MSSM Higgs potential

$$\begin{aligned}
\mathcal{L}_V^0 &= \mu_1^2 (\Phi_1^\dagger \Phi_1) + \mu_2^2 (\Phi_2^\dagger \Phi_2) + m_{12}^2 (\Phi_1^\dagger \Phi_2) + m_{12}^{*2} (\Phi_2^\dagger \Phi_1) + \lambda_1 (\Phi_1^\dagger \Phi_1)^2 \\
&\quad + \lambda_2 (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1) (\Phi_2^\dagger \Phi_2) + \lambda_4 (\Phi_1^\dagger \Phi_2) (\Phi_2^\dagger \Phi_1), \tag{2.7}
\end{aligned}$$

with

$$\begin{aligned}
\mu_1^2 &= -m_1^2 - |\mu|^2, & \mu_2^2 &= -m_2^2 - |\mu|^2, & m_{12}^2 &= B\mu, \\
\lambda_1 &= \lambda_2 = -\frac{1}{8} (g_w^2 + g'^2), & \lambda_3 &= -\frac{1}{4} (g_w^2 - g'^2), & \lambda_4 &= \frac{1}{2} g_w^2. \tag{2.8}
\end{aligned}$$

Further, in (2.6), \tilde{m}_i^2 (with $i = t, b$) and $\tilde{m}_{q_k}^2$ (with $q_k = t_1, b_1, t_2, b_2$) denote the eigenvalues of the quark and squark mass matrices $\mathcal{M}^\dagger \mathcal{M}$ and $\tilde{\mathcal{M}}^2$, respectively, which depend on the Higgs background fields. Specifically, $\mathcal{M}^\dagger \mathcal{M}$ reads

$$\mathcal{M}^\dagger \mathcal{M} = \begin{pmatrix} h_t^2 |\Phi_2|^2 & h_t h_b \Phi_1^T i\tau_2 \Phi_2 \\ h_t h_b \Phi_1^\dagger i\tau_2 \Phi_2^* & h_b^2 |\Phi_1|^2 \end{pmatrix}, \tag{2.9}$$

with eigenvalues

$$\bar{m}_{t(b)}^2 = \frac{1}{2} \left[h_b^2 |\Phi_1|^2 + h_t^2 |\Phi_2|^2 + (-) \sqrt{(h_b^2 |\Phi_1|^2 + h_t^2 |\Phi_2|^2)^2 - 4h_t^2 h_b^2 |\Phi_1^\dagger \Phi_2|^2} \right]. \quad (2.10)$$

It is easy to see that, for $\phi_{1,2}^\pm = 0$, (2.10) simplifies to the known expressions: $\bar{m}_t^2 = h_t^2 |\phi_2^0|^2$ and $\bar{m}_b^2 = h_b^2 |\phi_1^0|^2$.

The (4×4) squark mass matrix $\widetilde{\mathcal{M}}^2$ is more complicated. In the weak basis $\{\widetilde{Q}^T = (\tilde{t}_L, \tilde{b}_L), \widetilde{U} = \tilde{t}_R^*, \widetilde{D} = \tilde{b}_R^*\}$, $\widetilde{\mathcal{M}}^2$ may be cast in the form:

$$\widetilde{\mathcal{M}}^2 = \begin{pmatrix} (\widetilde{\mathcal{M}}^2)_{\widetilde{Q}^\dagger \widetilde{Q}} & (\widetilde{\mathcal{M}}^2)_{\widetilde{Q}^\dagger \widetilde{U}^*} & (\widetilde{\mathcal{M}}^2)_{\widetilde{Q}^\dagger \widetilde{D}^*} \\ (\widetilde{\mathcal{M}}^2)_{\widetilde{U} \widetilde{Q}} & (\widetilde{\mathcal{M}}^2)_{\widetilde{U} \widetilde{U}^*} & (\widetilde{\mathcal{M}}^2)_{\widetilde{U} \widetilde{D}^*} \\ (\widetilde{\mathcal{M}}^2)_{\widetilde{D} \widetilde{Q}} & (\widetilde{\mathcal{M}}^2)_{\widetilde{D} \widetilde{U}^*} & (\widetilde{\mathcal{M}}^2)_{\widetilde{D} \widetilde{D}^*} \end{pmatrix}, \quad (2.11)$$

with

$$\begin{aligned} (\widetilde{\mathcal{M}}^2)_{\widetilde{Q}^\dagger \widetilde{Q}} &= \widetilde{M}_Q^2 \mathbf{1}_2 + h_b^2 \Phi_1 \Phi_1^\dagger + h_t^2 (\Phi_2^\dagger \Phi_2 \mathbf{1}_2 - \Phi_2 \Phi_2^\dagger) - \frac{1}{2} g_w^2 (\Phi_1 \Phi_1^\dagger - \Phi_2 \Phi_2^\dagger) \\ &\quad + \left(\frac{1}{4} g_w^2 - \frac{1}{12} g'^2 \right) (\Phi_1^\dagger \Phi_1 - \Phi_2^\dagger \Phi_2) \mathbf{1}_2, \\ (\widetilde{\mathcal{M}}^2)_{\widetilde{U} \widetilde{Q}} &= (\widetilde{\mathcal{M}}^2)_{\widetilde{Q}^\dagger \widetilde{U}^*}^\dagger = -h_t A_t \Phi_2^T i\tau_2 + h_t \mu^* \Phi_1^T i\tau_2, \\ (\widetilde{\mathcal{M}}^2)_{\widetilde{D} \widetilde{Q}} &= (\widetilde{\mathcal{M}}^2)_{\widetilde{Q}^\dagger \widetilde{D}^*}^\dagger = h_b A_b \Phi_1^\dagger - h_b \mu^* \Phi_2^\dagger, \\ (\widetilde{\mathcal{M}}^2)_{\widetilde{U} \widetilde{U}^*} &= \widetilde{M}_t^2 + h_t^2 \Phi_2^\dagger \Phi_2 + \frac{1}{3} g'^2 (\Phi_1^\dagger \Phi_1 - \Phi_2^\dagger \Phi_2), \\ (\widetilde{\mathcal{M}}^2)_{\widetilde{D} \widetilde{D}^*} &= \widetilde{M}_b^2 + h_b^2 \Phi_1^\dagger \Phi_1 - \frac{1}{6} g'^2 (\Phi_1^\dagger \Phi_1 - \Phi_2^\dagger \Phi_2), \\ (\widetilde{\mathcal{M}}^2)_{\widetilde{U} \widetilde{D}^*} &= (\widetilde{\mathcal{M}}^2)_{\widetilde{D} \widetilde{U}^*}^* = h_t h_b \Phi_1^T i\tau_2 \Phi_2. \end{aligned} \quad (2.12)$$

It is rather difficult to express the four eigenvalues $\widetilde{m}_{q_k}^2$ ($q_k = t_1, b_1, t_2, b_2$) of $\widetilde{\mathcal{M}}^2$ in a simple form. However, as we detail in Appendix A.3, it is not necessary to know the analytic form of $\widetilde{m}_{q_k}^2$ in order to evaluate the Higgs-boson masses and mixing angles [37]. Of course, for $\phi_{1,2}^\pm = 0$, the field-dependent squark eigenvalues simplify to

$$\begin{aligned} \widetilde{m}_{t_1(t_2)}^2 &= \frac{1}{2} \left[\widetilde{M}_Q^2 + \widetilde{M}_t^2 + 2h_t^2 |\phi_2^0|^2 + \frac{g_w^2 + g'^2}{4} (|\phi_1^0|^2 - |\phi_2^0|^2) \right. \\ &\quad \left. + (-) \sqrt{[\widetilde{M}_Q^2 - \widetilde{M}_t^2 + x_t (|\phi_1^0|^2 - |\phi_2^0|^2)]^2 + 4h_t^2 |A_t \phi_2^0 - \mu^* \phi_1^0|^2} \right], \\ \widetilde{m}_{b_1(b_2)}^2 &= \frac{1}{2} \left[\widetilde{M}_Q^2 + \widetilde{M}_b^2 + 2h_b^2 |\phi_1^0|^2 - \frac{g_w^2 + g'^2}{4} (|\phi_1^0|^2 - |\phi_2^0|^2) \right. \\ &\quad \left. + (-) \sqrt{[\widetilde{M}_Q^2 - \widetilde{M}_b^2 - x_b (|\phi_1^0|^2 - |\phi_2^0|^2)]^2 + 4h_b^2 |A_b^* \phi_1^0 - \mu \phi_2^0|^2} \right], \end{aligned} \quad (2.13)$$

where $x_t = \frac{1}{4}(g_w^2 - \frac{5}{3}g'^2)$ and $x_b = \frac{1}{4}(g_w^2 - \frac{1}{3}g'^2)$.

After having set the stage, we now derive the minimization conditions governing the ground state of the MSSM one-loop effective potential and then determine the Higgs-boson mass matrices. As usual, we consider the following linear expansion of the Higgs doublets Φ_1 and Φ_2 around the ground state:

$$\Phi_1 = \begin{pmatrix} \phi_1^+ \\ \frac{1}{\sqrt{2}}(v_1 + \phi_1 + ia_1) \end{pmatrix}, \quad \Phi_2 = e^{i\xi} \begin{pmatrix} \phi_2^+ \\ \frac{1}{\sqrt{2}}(v_2 + \phi_2 + ia_2) \end{pmatrix}, \quad (2.14)$$

where v_1 and v_2 are the moduli of the vacuum expectation values (VEVs) of the Higgs doublets and ξ is their relative phase. Following [10,11], we require the vanishing of the total tadpole contributions

$$\begin{aligned} T_{\phi_1(\phi_2)} &\equiv \left\langle \frac{\partial \mathcal{L}_V}{\partial \phi_{1(2)}} \right\rangle = v_{1(2)} \left[\mu_{1(2)}^2 + \frac{v_1 v_2}{v_{1(2)}^2} \text{Re}(m_{12}^2 e^{i\xi}) + \lambda_{1(2)} v_{1(2)}^2 + \frac{1}{2} (\lambda_3 + \lambda_4) v_{2(1)}^2 \right] \\ &\quad - \frac{3}{16\pi^2} \sum_{q=t,b} \left[\sum_{i=1,2} \left\langle \frac{\partial \widetilde{m}_{q_i}^2}{\partial \phi_{1(2)}} \right\rangle m_{q_i}^2 \left(\ln \frac{m_{q_i}^2}{Q^2} - 1 \right) - 2 \left\langle \frac{\partial \widetilde{m}_q^2}{\partial \phi_{1(2)}} \right\rangle m_q^2 \right. \\ &\quad \left. \times \left(\ln \frac{m_q^2}{Q^2} - 1 \right) \right], \end{aligned} \quad (2.15)$$

$$\begin{aligned} T_{a_1(a_2)} &\equiv \left\langle \frac{\partial \mathcal{L}_V}{\partial a_{1(2)}} \right\rangle = +(-) v_{2(1)} \text{Im}(m_{12}^2 e^{i\xi}) - \frac{3}{16\pi^2} \sum_{q=t,b} \sum_{i=1,2} \left\langle \frac{\partial \widetilde{m}_{q_i}^2}{\partial a_{1(2)}} \right\rangle m_{q_i}^2 \\ &\quad \times \left(\ln \frac{m_{q_i}^2}{Q^2} - 1 \right), \end{aligned} \quad (2.16)$$

where $\langle \widetilde{m}_{q_i}^2 \rangle = m_{q_i}^2$, and the tadpole derivatives $\langle \partial \widetilde{m}_q^2 / \partial \phi_{1(2)} \rangle$, $\langle \partial \widetilde{m}_{q_k}^2 / \partial \phi_{1(2)} \rangle$ and $\langle \partial \widetilde{m}_{q_k}^2 / \partial a_{1(2)} \rangle$ are given in (A.1) and (A.3). Moreover, from (A.3), we readily see that $T_{a_1} = -\tan \beta T_{a_2}$, with $\tan \beta = v_2/v_1$. This last fact [10,11] allows us to perform an orthogonal rotation in the space spanned by the ‘CP-odd’ scalars a_1 and a_2 ,

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} G^0 \\ a \end{pmatrix}. \quad (2.17)$$

The Higgs potential then has a flat direction with respect to the G^0 field, *i.e.*, $\langle \partial \mathcal{L}_V / \partial G^0 \rangle = 0$, and the G^0 field becomes the true would-be Goldstone boson eaten by the longitudinal component of the Z boson.

We observe from (2.16) that a relative phase ξ between the two Higgs vacuum expectation values is induced radiatively in the $\overline{\text{MS}}$ scheme [10,11]. However, we should stress that the phase ξ is renormalization-scheme dependent. For example, one may adopt a renormalization scheme, slightly different from the $\overline{\text{MS}}$ one, in which ξ is set to zero order-by-order in perturbation theory [10]. This can be achieved by requiring the bilinear Higgs-mixing mass m_{12}^2 to be real at the tree level, but to receive an imaginary counter-term

(CT), $\text{Im } m_{12}^2$, at higher orders, which is determined by the vanishing of the CP-odd tadpole parameters T_{a_1} and T_{a_2} for $\xi = 0$. As we detail below, the scheme of renormalization of $\text{Im}(m_{12}^2 e^{i\xi})$ does not directly affect the renormalization scheme of other physical kinematic parameters of the theory to one loop, such as Higgs-boson masses and $\tan \beta$. In fact, it has been explicitly demonstrated in [10] that physical CP-violating transition amplitudes, such as scalar-pseudoscalar transitions, are independent of the renormalization subtraction point Q^2 and the choice of phase ξ . In the following, we adopt the $\xi = 0$ scheme of renormalization, as irrelevant ξ -dependent phases in the effective chargino and neutralino mass matrices can thereby be completely avoided.

In the remaining part of this section, we evaluate the one-loop effective Higgs-boson mass matrices. Employing the tadpole conditions $T_{\phi_1} = T_{\phi_2} = 0$ and $T_{a_1} = T_{a_2} = 0$ allows one to substitute the mass parameters μ_1^2 and μ_2^2 , and $\text{Im } m_{12}^2$ into the effective potential (2.6). After performing the above substitutions, we can express the charged-Higgs-boson mass matrix as follows:

$$\begin{aligned}
(\mathcal{M}_{\pm}^2)_{ij} = & (-1)^{i+j} \frac{v_1 v_2}{v_i v_j} \left(\text{Re } m_{12}^2 + \frac{1}{4} g_w^2 v_1 v_2 \right) + \frac{3}{16\pi^2} \sum_{q=t,b} \left[\sum_{k=1,2} \left(\left\langle \frac{\partial^2 \widetilde{m}_{qk}^2}{\partial \phi_i^+ \partial \phi_j^-} \right\rangle \right. \right. \\
& - \frac{\delta_{ij}}{v_i} \left\langle \frac{\partial \widetilde{m}_{qk}^2}{\partial \phi_j} \right\rangle - \frac{i(1 - \delta_{ij})}{v_i} \left\langle \frac{\partial \widetilde{m}_{qk}^2}{\partial a_j} \right\rangle \left. \right) m_{qk}^2 \left(\ln \frac{m_{qk}^2}{Q^2} - 1 \right) \\
& - 2 \left(\left\langle \frac{\partial^2 \widetilde{m}_q^2}{\partial \phi_i^+ \partial \phi_j^-} \right\rangle - \frac{\delta_{ij}}{v_i} \left\langle \frac{\partial \widetilde{m}_q^2}{\partial \phi_j} \right\rangle \right) m_q^2 \left(\ln \frac{m_q^2}{Q^2} - 1 \right) \left. \right]. \quad (2.18)
\end{aligned}$$

Since $\det \mathcal{M}_{\pm}^2 = 0$ in (2.18), the square of the charged Higgs-boson mass $M_{H^\pm}^2$ may be determined by the matrix element $(\mathcal{M}_{\pm}^2)_{12}$:

$$\begin{aligned}
M_{H^\pm}^2 = & \frac{v^2}{v_1 v_2} \left\{ \left(\text{Re } m_{12}^2 + \frac{1}{4} g_w^2 v_1 v_2 \right) - \frac{3}{16\pi^2} \sum_{q=t,b} \left[\sum_{k=1,2} \left(\left\langle \frac{\partial^2 \widetilde{m}_{qk}^2}{\partial \phi_1^+ \partial \phi_2^-} \right\rangle - \frac{i}{v_1} \left\langle \frac{\partial \widetilde{m}_{qk}^2}{\partial a_2} \right\rangle \right) \right. \right. \\
& \times m_{qk}^2 \left(\ln \frac{m_{qk}^2}{Q^2} - 1 \right) - 2 \left\langle \frac{\partial^2 \widetilde{m}_q^2}{\partial \phi_1^+ \partial \phi_2^-} \right\rangle m_q^2 \left(\ln \frac{m_q^2}{Q^2} - 1 \right) \left. \right] \left. \right\}. \quad (2.19)
\end{aligned}$$

The self-energy derivatives appearing in (2.18) and (2.19): $\langle \partial^2 \widetilde{m}_q^2 / \partial \phi_1^+ \partial \phi_2^- \rangle$ and $\langle \partial^2 \widetilde{m}_{qk}^2 / \partial \phi_1^+ \partial \phi_2^- \rangle$, as well as the tadpole terms $\langle \partial \widetilde{m}_{qk}^2 / \partial \phi_j \rangle$, $\langle \partial \widetilde{m}_{qk}^2 / \partial \phi_j \rangle$ and $\langle \partial \widetilde{m}_{qk}^2 / \partial a_j \rangle$, are exhibited in Appendix A.

By analogy, in the weak basis $\{\phi_1, \phi_2, a_1, a_2\}$, the neutral-Higgs-boson mass matrix takes on the form[‡]

$$\mathcal{M}_0^2 = \begin{pmatrix} \mathcal{M}_S^2 & \mathcal{M}_{SP}^2 \\ (\mathcal{M}_{SP}^2)^T & \mathcal{M}_P^2 \end{pmatrix}, \quad (2.20)$$

[‡]Notice that our convention differs from that given in [11], as the neutral-Higgs-boson mass matrix \mathcal{M}_0^2 in that work is expressed in the weak basis $\{a_1, a_2, \phi_1, \phi_2\}$.

where \mathcal{M}_S^2 , \mathcal{M}_P^2 and \mathcal{M}_{SP}^2 denote the two-by-two matrices of the scalar, pseudoscalar and scalar-pseudoscalar squared mass terms of the neutral Higgs bosons. Observe that the presence of CP-violating self-energy terms leads to mass eigenstates with no well-defined CP quantum numbers. Therefore the CP-odd Higgs-boson mass M_A cannot be identified with any of the neutral Higgs-boson masses. The individual matrix elements of \mathcal{M}_0^2 are given by

$$\begin{aligned}
(\mathcal{M}_S^2)_{ij} &= (-1)^{i+j} \frac{v_1 v_2}{v_i v_j} \text{Re } m_{12}^2 + \frac{1}{4} (g_w^2 + g'^2) v_i v_j + \frac{3}{16\pi^2} \sum_{q=t,b} \left\{ \sum_{k=1,2} \left[\left(\left\langle \frac{\partial^2 \tilde{m}_{qk}^2}{\partial \phi_i \partial \phi_j} \right\rangle \right. \right. \right. \\
&\quad \left. \left. - \frac{\delta_{ij}}{v_i} \left\langle \frac{\partial \tilde{m}_{qk}^2}{\partial \phi_j} \right\rangle \right) m_{\tilde{q}_k}^2 \left(\ln \frac{m_{\tilde{q}_k}^2}{Q^2} - 1 \right) + \left\langle \frac{\partial \tilde{m}_{qk}^2}{\partial \phi_i} \right\rangle \left\langle \frac{\partial \tilde{m}_{qk}^2}{\partial \phi_j} \right\rangle \ln \frac{m_{\tilde{q}_k}^2}{Q^2} \right] \\
&\quad - 2 \left[\left(\left\langle \frac{\partial^2 \tilde{m}_q^2}{\partial \phi_i \partial \phi_j} \right\rangle - \frac{\delta_{ij}}{v_i} \left\langle \frac{\partial \tilde{m}_q^2}{\partial \phi_j} \right\rangle \right) m_q^2 \left(\ln \frac{m_q^2}{Q^2} - 1 \right) \right. \\
&\quad \left. + \left\langle \frac{\partial \tilde{m}_q^2}{\partial \phi_i} \right\rangle \left\langle \frac{\partial \tilde{m}_q^2}{\partial \phi_j} \right\rangle \ln \frac{m_q^2}{Q^2} \right] \right\}, \tag{2.21}
\end{aligned}$$

$$\begin{aligned}
(\mathcal{M}_{SP}^2)_{ij} &= \frac{3}{16\pi^2} \sum_{q=t,b} \sum_{k=1,2} \left[\left(\left\langle \frac{\partial^2 \tilde{m}_{qk}^2}{\partial \phi_i \partial a_j} \right\rangle - \frac{(1 - \delta_{ij})}{v_i} \left\langle \frac{\partial \tilde{m}_{qk}^2}{\partial a_j} \right\rangle \right) m_{\tilde{q}_k}^2 \left(\ln \frac{m_{\tilde{q}_k}^2}{Q^2} - 1 \right) \right. \\
&\quad \left. + \left\langle \frac{\partial \tilde{m}_{qk}^2}{\partial \phi_i} \right\rangle \left\langle \frac{\partial \tilde{m}_{qk}^2}{\partial a_j} \right\rangle \ln \frac{m_{\tilde{q}_k}^2}{Q^2} \right], \tag{2.22}
\end{aligned}$$

$$\begin{aligned}
(\mathcal{M}_P^2)_{ij} &= (-1)^{i+j} \frac{v_1 v_2}{v_i v_j} \text{Re } m_{12}^2 + \frac{3}{16\pi^2} \sum_{q=t,b} \left\{ \sum_{k=1,2} \left[\left(\left\langle \frac{\partial^2 \tilde{m}_{qk}^2}{\partial a_i \partial a_j} \right\rangle - \frac{\delta_{ij}}{v_i} \left\langle \frac{\partial \tilde{m}_{qk}^2}{\partial \phi_j} \right\rangle \right) \right. \right. \\
&\quad \times m_{\tilde{q}_k}^2 \left(\ln \frac{m_{\tilde{q}_k}^2}{Q^2} - 1 \right) + \left\langle \frac{\partial \tilde{m}_{qk}^2}{\partial a_i} \right\rangle \left\langle \frac{\partial \tilde{m}_{qk}^2}{\partial a_j} \right\rangle \ln \frac{m_{\tilde{q}_k}^2}{Q^2} \right] \\
&\quad \left. - 2 \left(\left\langle \frac{\partial^2 \tilde{m}_q^2}{\partial a_i \partial a_j} \right\rangle - \frac{\delta_{ij}}{v_i} \left\langle \frac{\partial \tilde{m}_q^2}{\partial \phi_j} \right\rangle \right) m_q^2 \left(\ln \frac{m_q^2}{Q^2} - 1 \right) \right\}. \tag{2.23}
\end{aligned}$$

Again, the analytic expressions for the self-energy and tadpole derivatives with respect to the background Higgs fields are given in Appendix A.

Since G^0 does not mix with the other neutral fields, the (4×4) matrix \mathcal{M}_0^2 reduces to a (3×3) matrix, which we denote by \mathcal{M}_N^2 . In the weak basis $\{\phi_1, \phi_2, a\}$, the reduced neutral mass-squared matrix \mathcal{M}_N^2 may be expressed by

$$\mathcal{M}_N^2 = \begin{pmatrix} (\mathcal{M}_S^2)_{11} & (\mathcal{M}_S^2)_{12} & \frac{1}{\cos \beta} (\mathcal{M}_{SP}^2)_{12} \\ (\mathcal{M}_S^2)_{21} & (\mathcal{M}_S^2)_{22} & -\frac{1}{\sin \beta} (\mathcal{M}_{SP}^2)_{21} \\ \frac{1}{\cos \beta} (\mathcal{M}_{SP}^2)_{12} & -\frac{1}{\sin \beta} (\mathcal{M}_{SP}^2)_{21} & -\frac{1}{\sin \beta \cos \beta} (\mathcal{M}_P^2)_{12} \end{pmatrix}. \tag{2.24}$$

In writing \mathcal{M}_N^2 in (2.24), we have used the properties of the matrix elements of \mathcal{M}_{SP}^2 : $(\mathcal{M}_{SP}^2)_{11} = -\tan \beta (\mathcal{M}_{SP}^2)_{12}$ and $(\mathcal{M}_{SP}^2)_{22} = -\cot \beta (\mathcal{M}_{SP}^2)_{21}$, and likewise for \mathcal{M}_P^2 .

Using the expressions (2.19) and (2.24), we determine the analytic forms of the RG-improved charged and neutral Higgs-boson masses in the next Section.

3 RG-Improved Higgs-Boson Mass Matrices

In this Section, we perform a one-loop RG improvement of the squared charged Higgs-boson mass $M_{H^+}^2$ and of the squared neutral Higgs-boson mass matrix \mathcal{M}_N^2 . The RG improvement incorporates all leading two-loop logarithmic corrections to the Higgs-boson mass-matrix elements, which were already found in the CP-conserving case to give rise to significant contributions to the Higgs-boson masses and couplings. In particular, the upper bound on the lightest CP-even Higgs mass was found to be strongly affected by the two-loop logarithmic corrections [15–18]. In carrying out the RG improvement, we follow the procedure outlined in [18], in which the improvement of the Higgs-boson mass-matrix elements was performed by carefully applying the process of decoupling of the third-generation squarks.

Within the framework of the RG approach, the dominant contributions to the Higgs-boson mass matrix \mathcal{M}^2 may be written conceptually as a sum of two terms:

$$\mathcal{M}^2(m_t) = \overline{\mathcal{M}}^2(m_t) + \mathcal{M}^{2,\text{th}}(m_t). \quad (3.1)$$

The first term, $\overline{\mathcal{M}}^2(m_t)$, contains the genuine logarithmic contributions which determine the whole scale dependence of the one-loop effective potential. These contributions would be present, even if the left-right mixing of the stop and sbottom states were absent. The second term, $\mathcal{M}^{2,\text{th}}(m_t)$, describes the threshold effect of the decoupling of the heavier stop and sbottom squarks and their respective mixing with the lighter states. At the one-loop level, the second term is manifestly scale independent. In (3.1), we are interested in evaluating the effective potential at m_t , since it has been shown [16] that this is the scale at which two-loop corrections are minimized. As we explain below, the renormalization of the above two contributions must proceed in different ways.

Let us denote by Q_{tb} the scale of the heaviest third-generation squark, which we assume to be higher than the electroweak scale. In the language of the RG approach, we have first to consider that the aforementioned threshold contribution is ‘frozen’ at the scale Q_{tb} , $\mathcal{M}^{2,\text{th}}(Q_{tb}) \equiv \Delta\mathcal{M}^2(Q_{tb})$, with all the involved kinematic parameters defined at this particular scale. Then, we have to rescale the threshold contribution with the anomalous dimension factors of the relevant Higgs fields:

$$\mathcal{M}_{ij}^{2,\text{th}}(m_t) = \Delta\mathcal{M}_{ij}^2(Q_{tb}) \xi_i^{-1}(m_t) \xi_j^{-1}(m_t), \quad (3.2)$$

where $\xi_i(m_t)$ is the anomalous dimension factor of the H_i state to be determined below. The one-loop matrix elements $\Delta\mathcal{M}_{ij}^2(Q_{tb})$ depend on the running quark masses at the scale Q_{tb} , which have to be conveniently re-expressed as functions of the corresponding running masses at m_t . Thus, the anomalous-dimension factors in combination with the one-loop relation between the quark masses at scales Q_{tb} and m_t yield sizeable two-loop corrections to the mass-matrix elements originating from the one-loop threshold effects.

As was already mentioned, the contribution $\overline{\mathcal{M}}^2(m_t)$ of the third-generation squarks to the effective potential describes the genuine one- and two-loop leading-logarithmic running of the Higgs quartic couplings. In this context, there are two important technical details that should be mentioned. First, we notice that, in the MSSM, the tree-level Higgs quartic couplings λ_i , with $i = 1, 2, 3, 4$, are all proportional to the squared gauge couplings g_w^2 and g'^2 (cf. (2.8)). However, the one-loop β functions of λ_i can generally have appreciable values, as they are proportional to the fourth power of the top- and bottom-quark Yukawa couplings. As a result, the low-energy values of λ_i differ significantly from their tree-level ones. The RG-improved approach followed here is crucial for implementing properly the potentially large logarithmic corrections to the Higgs quartic couplings.

The second technical remark pertains to the RG evolution of the Higgs quartic couplings λ_i , with $i = 5, 6, 7$ (for the notation, see [14,11]), which are absent in the Born approximation to the MSSM Higgs potential. On field-theoretic grounds, these quartic couplings must have vanishing one-loop β functions, and cannot be generated by RG running. However, these quartic couplings are radiatively induced by threshold effects, and have already been taken into account in $\Delta\mathcal{M}^2(Q_{tb})$, given by (3.2).

Following the above discussion, we now proceed with the RG improvement of the Higgs-boson mass-matrix elements. To this end, we first need to compute the one-loop values of the quartic couplings λ_i , with $i = 1, 2, 3, 4$, where the decoupling of the stop and sbottom contributions at their appropriate thresholds is properly taken into account. The best way to calculate the latter effects is to consider the logarithmic part of the effective potential (2.6) in the limit where the squark mixing parameters vanish, *i.e.*, $\mu = A_t = A_b = 0$. The pertinent one-loop running quartic couplings, denoted by $\lambda_i^{(1)}$, may then be obtained by

$$\begin{aligned} \lambda_1^{(1)} &= -\frac{3}{32\pi^2} \left[\left(\frac{g_w^2}{4} - \frac{g'^2}{12} \right)^2 \ln \left(\frac{\widetilde{M}_Q^2 + m_t^2}{Q^2} \right) + \left(h_b^2 - \frac{g_w^2}{4} - \frac{g'^2}{12} \right)^2 \ln \left(\frac{\widetilde{M}_Q^2 + m_b^2}{Q^2} \right) \right. \\ &\quad \left. + \frac{g'^4}{9} \ln \left(\frac{\widetilde{M}_t^2 + m_t^2}{Q^2} \right) + \left(h_b^2 - \frac{g'^2}{6} \right)^2 \ln \left(\frac{\widetilde{M}_b^2 + m_b^2}{Q^2} \right) \right], \quad (3.3) \\ \lambda_2^{(1)} &= -\frac{3}{32\pi^2} \left[\left(h_t^2 - \frac{g_w^2}{4} + \frac{g'^2}{12} \right)^2 \ln \left(\frac{\widetilde{M}_Q^2 + m_t^2}{Q^2} \right) + \left(\frac{g_w^2}{4} + \frac{g'^2}{12} \right)^2 \ln \left(\frac{\widetilde{M}_Q^2 + m_b^2}{Q^2} \right) \right] \end{aligned}$$

$$+ \left(h_t^2 - \frac{g'^2}{3} \right)^2 \ln \left(\frac{\widetilde{M}_t^2 + m_t^2}{Q^2} \right) + \frac{g'^4}{36} \ln \left(\frac{\widetilde{M}_b^2 + m_b^2}{Q^2} \right) \Big], \quad (3.4)$$

$$\begin{aligned} \lambda_3^{(1)} = & -\frac{3}{16\pi^2} \left\{ h_t^2 h_b^2 \left[\ln \left(\frac{\widetilde{M}_Q^2 + m_t^2}{Q^2} \right) + \ln \left(\frac{\max(\widetilde{M}_t^2 + m_t^2, \widetilde{M}_b^2 + m_b^2)}{Q^2} \right) \right] \right. \\ & - \left[\left(\frac{g_w^2}{4} + \frac{g'^2}{12} \right) \left(h_t^2 - \frac{g_w^2}{4} \right) - \frac{g'^2}{12} \left(\frac{g_w^2}{4} - \frac{g'^2}{12} \right) \right] \ln \left(\frac{\widetilde{M}_Q^2 + m_t^2}{Q^2} \right) \\ & - \left[\left(\frac{g_w^2}{4} - \frac{g'^2}{12} \right) \left(h_b^2 - \frac{g_w^2}{4} \right) + \frac{g'^2}{12} \left(\frac{g_w^2}{4} + \frac{g'^2}{12} \right) \right] \ln \left(\frac{\widetilde{M}_Q^2 + m_b^2}{Q^2} \right) \\ & \left. + \frac{g'^2}{3} \left(h_t^2 - \frac{g'^2}{3} \right) \ln \left(\frac{\widetilde{M}_t^2 + m_t^2}{Q^2} \right) + \frac{g'^2}{6} \left(h_b^2 - \frac{g'^2}{6} \right) \ln \left(\frac{\widetilde{M}_b^2 + m_b^2}{Q^2} \right) \right\}, \quad (3.5) \end{aligned}$$

$$\begin{aligned} \lambda_4^{(1)} = & \frac{3}{16\pi^2} \left\{ h_t^2 h_b^2 \left[\ln \left(\frac{\widetilde{M}_Q^2 + m_t^2}{Q^2} \right) + \ln \left(\frac{\max(\widetilde{M}_t^2 + m_t^2, \widetilde{M}_b^2 + m_b^2)}{Q^2} \right) \right] \right. \\ & \left. - \frac{g_w^2}{2} \left(h_t^2 - \frac{g_w^2}{4} \right) \ln \left(\frac{\widetilde{M}_Q^2 + m_t^2}{Q^2} \right) - \frac{g_w^2}{2} \left(h_b^2 - \frac{g_w^2}{4} \right) \ln \left(\frac{\widetilde{M}_Q^2 + m_b^2}{Q^2} \right) \right\}. \quad (3.6) \end{aligned}$$

Moreover, we need to know the one-loop running of the soft SUSY-breaking parameter $\text{Re } m_{12}^2$. Gathering the relevant logarithmic terms present in the effective potential (2.6), we find

$$\begin{aligned} \text{Re } m_{12}^{2(1)} = & \frac{3}{16\pi^2} \left[h_t^2 \text{Re}(\mu A_t) \ln \left(\frac{\max(\widetilde{M}_Q^2 + m_t^2, \widetilde{M}_t^2 + m_t^2)}{Q^2} \right) \right. \\ & \left. + h_b^2 \text{Re}(\mu A_b) \ln \left(\frac{\max(\widetilde{M}_Q^2 + m_b^2, \widetilde{M}_b^2 + m_b^2)}{Q^2} \right) \right]. \quad (3.7) \end{aligned}$$

The analytic form of $\Delta \mathcal{M}^2(Q_{tb})$ in (3.2) can now be obtained by subtracting the one-loop Born-improved mass matrix $\mathcal{M}^{2(0)}$ from its total one-loop contribution $\mathcal{M}^{2(1)}$. Here, we have in mind the charged and neutral Higgs-boson mass matrices $\mathcal{M}_\pm^{2(1)}$ and $\mathcal{M}_N^{2(1)}$ calculated in Section 2. More explicitly, $\Delta \mathcal{M}^2(Q_{tb})$ is given by

$$\Delta \mathcal{M}^2(Q_{tb}) = \mathcal{M}^{2(1)}(Q_{tb}) - \mathcal{M}^{2(0)} \left[\text{Re } m_{12}^{2(1)}(Q_{tb}), \lambda_i^{(1)}(Q_{tb}) \right], \quad (3.8)$$

where $\mathcal{M}^{2(0)}$ represents the tree-level functional form of \mathcal{M}^2 , expressed in terms of $\lambda_i^{(1)}$ and $\text{Re } m_{12}^{2(1)}$. Furthermore, it is essential to stress again that the kinematic parameters involved in (3.8), such as masses and couplings, are evaluated at the scale Q_{tb} .

Another important ingredient in the RG improvement of the Higgs-boson mass matrices is the analytic two-loop result for the Higgs quartic couplings $\lambda_1, \dots, \lambda_4$. As has been done in (3.3)–(3.6), we have to include two-loop leading logarithms, by appropriately considering the stop and sbottom thresholds. These two-loop leading logarithmic contributions to the Higgs quartic couplings, which we denote by $\lambda_i^{(2)}$, can be determined by solving

iteratively the RG equations [18]. In this way, we obtain

$$\lambda_1^{(2)} = -\frac{6h_b^4}{(32\pi^2)^2} \left(\frac{3}{2}h_b^2 + \frac{1}{2}h_t^2 - 8g_s^2 \right) \left[\ln^2 \left(\frac{\widetilde{M}_Q^2 + m_b^2}{Q^2} \right) + \ln^2 \left(\frac{\widetilde{M}_b^2 + m_b^2}{Q^2} \right) \right], \quad (3.9)$$

$$\lambda_2^{(2)} = -\frac{6h_t^4}{(32\pi^2)^2} \left(\frac{3}{2}h_t^2 + \frac{1}{2}h_b^2 - 8g_s^2 \right) \left[\ln^2 \left(\frac{\widetilde{M}_Q^2 + m_t^2}{Q^2} \right) + \ln^2 \left(\frac{\widetilde{M}_t^2 + m_t^2}{Q^2} \right) \right], \quad (3.10)$$

$$\lambda_3^{(2)} = -\frac{3h_t^2 h_b^2}{(16\pi^2)^2} \left(h_t^2 + h_b^2 - 8g_s^2 \right) \times \left[\ln^2 \left(\frac{\widetilde{M}_Q^2 + m_t^2}{Q^2} \right) + \ln^2 \left(\frac{\max(\widetilde{M}_t^2 + m_t^2, \widetilde{M}_b^2 + m_b^2)}{Q^2} \right) \right], \quad (3.11)$$

$$\lambda_4^{(2)} = -\lambda_3^{(2)}. \quad (3.12)$$

For later convenience, we define collectively the sum of the tree, one-loop and two-loop quartic couplings as follows:

$$\bar{\lambda}_i = \lambda_i + \lambda_i^{(1)} + \lambda_i^{(2)}, \quad (3.13)$$

with $i = 1, 2, 3, 4$. Similarly, the sum of the tree, one-loop and two-loop contributions to the soft-bilinear Higgs mixing may be defined as

$$\text{Re } \bar{m}_{12}^2 = \text{Re } m_{12}^2 + \text{Re } m_{12}^{2(1)} + \text{Re } m_{12}^{2(2)}. \quad (3.14)$$

As we see below, knowledge of the two-loop contribution $\text{Re } m_{12}^{2(2)}$ is not required in the one-loop RG improvement of the MSSM Higgs potential.

Given the above definitions of the quartic couplings and the soft Higgs-mixing parameter in (3.13) and (3.14), we can express the one- and two-loop leading logarithmic contributions $\overline{\mathcal{M}}^2(m_t)$ to $\mathcal{M}^2(m_t)$ by means of the two-loop Born-improved mass matrix:

$$\overline{\mathcal{M}}^2(m_t) = \mathcal{M}^{2(0)} \left[\text{Re } \bar{m}_{12}^2(m_t), \bar{\lambda}_i(m_t) \right]. \quad (3.15)$$

Note that $\overline{\mathcal{M}}^2(m_t)$ also includes the tree-level terms. As has also been stated explicitly in (3.15), $\overline{\mathcal{M}}^2(m_t)$ is expressed in terms of mass and coupling parameters evaluated at the top-quark-mass scale.

The last ingredient for completing the programme of RG improvement of the Higgs-boson mass matrices is knowledge of the analytic expressions for the anomalous dimension factors that occur in (3.2). These analytic expressions are given below for the charged and neutral Higgs-boson cases separately.

Adopting the framework outlined above [18], it is not difficult to compute the RG-improved charged Higgs-boson mass $M_{H^+}^2(m_t)$ at the top-mass scale through the relation:

$$M_{H^+}^2(m_t) = \overline{M}_{H^+}^2(m_t) + \left[\xi_1^+(m_t) \xi_2^-(m_t) \right]^{-1} (\Delta M_{H^+}^2)^{\tilde{t}\tilde{b}}(Q_{tb}) + (M_{H^+}^{2(1)})^{tb}(m_t), \quad (3.16)$$

where $Q_{tb}^2 = \max(\widetilde{M}_Q^2 + m_t^2, \widetilde{M}_t^2 + m_t^2, \widetilde{M}_b^2 + m_b^2)$, and $\xi_1^+(m_t)$ and $\xi_2^-(m_t)$ are the anomalous dimension factors of the charged Higgs fields ϕ_1^+ and ϕ_2^- , respectively:

$$\xi_1^+(m_t) = 1 + \frac{3h_b^2}{32\pi^2} \ln \frac{Q_{tb}^2}{m_t^2}, \quad \xi_2^-(m_t) = 1 + \frac{3h_t^2}{32\pi^2} \ln \frac{Q_{tb}^2}{m_t^2}. \quad (3.17)$$

Further, $\overline{M}_{H^+}^2(m_t)$ is the squared two-loop Born-improved charged Higgs-boson mass given by

$$\overline{M}_{H^+}^2(m_t) = \frac{\text{Re } \bar{m}_{12}^2(m_t)}{\sin \beta(m_t) \cos \beta(m_t)} + \frac{1}{2} \bar{\lambda}_4(m_t) v^2(m_t) \quad (3.18)$$

and $(\Delta M_{H^+}^2)^{\tilde{t}\tilde{b}}$ is the one-loop scale-invariant part that contains the stop and sbottom contributions:

$$\begin{aligned} (\Delta M_{H^+}^2)^{\tilde{t}\tilde{b}}(Q_{tb}) &= -\frac{1}{\sin \beta(m_t) \cos \beta(m_t)} \left[(\mathcal{M}_{\pm}^{2(1)})_{12}^{\tilde{t}\tilde{b}}(Q_{tb}) + \text{Re } m_{12}^{2(1)}(Q_{tb}) \right. \\ &\quad \left. + \frac{1}{2} \lambda_4^{(1)}(Q_{tb}) v_1(Q_{tb}) v_2(Q_{tb}) \right], \end{aligned} \quad (3.19)$$

where the scale at which the kinematic parameters are to be evaluated has been indicated explicitly. In (3.19), the term between brackets is the threshold contribution to the off-diagonal matrix element of the charged-Higgs-boson mass matrix, where $(\mathcal{M}_{\pm}^{2(1)})_{12}^{\tilde{t}\tilde{b}}$ denotes the one-loop contribution of the third-generation squarks to $(\mathcal{M}_{\pm}^2)_{12}$. Finally, $(M_{H^+}^{2(1)})^{\tilde{t}\tilde{b}}$ describes the one-loop quark contribution to $M_{H^+}^2$ (see (3.16)).

We remark that the charged-Higgs-boson mass matrix receives the common anomalous dimension factor $\xi_1^+(m_t)\xi_2^-(m_t)$, even though different matrix elements of \mathcal{M}_{\pm}^2 are involved. This is because \mathcal{M}_{\pm}^2 must possess a vanishing determinant at any RG scale Q^2 , as one of its mass eigenstates must correspond to the massless would-be Goldstone boson G^+ that forms the longitudinal component of the W^+ boson. As a consequence, the following relations among the matrix elements of the RG-frozen part $\Delta\mathcal{M}_{\pm}^2$ of the charged-Higgs-boson mass matrix are obtained:

$$\begin{aligned} (\Delta\mathcal{M}_{\pm}^2)_{11}(Q_{tb}) &= -\tan \beta(Q_{tb}) (\Delta\mathcal{M}_{\pm}^2)_{12}(Q_{tb}), \\ (\Delta\mathcal{M}_{\pm}^2)_{22}(Q_{tb}) &= -\cot \beta(Q_{tb}) (\Delta\mathcal{M}_{\pm}^2)_{21}(Q_{tb}). \end{aligned} \quad (3.20)$$

After including the RG running due to the Higgs-boson anomalous dimensions, we find

$$\begin{aligned} \frac{(\Delta\mathcal{M}_{\pm}^2)_{11}(Q_{tb})}{[\xi_1^+(m_t)]^2} &= -\tan \beta(m_t) \frac{(\Delta\mathcal{M}_{\pm}^2)_{12}(Q_{tb})}{\xi_1^+(m_t) \xi_2^-(m_t)}, \\ \frac{(\Delta\mathcal{M}_{\pm}^2)_{22}(Q_{tb})}{[\xi_2^-(m_t)]^2} &= -\cot \beta(m_t) \frac{(\Delta\mathcal{M}_{\pm}^2)_{12}(Q_{tb})}{\xi_1^+(m_t) \xi_2^-(m_t)}, \end{aligned} \quad (3.21)$$

where we have used the RG relation:

$$\tan \beta(Q_{tb}) = \frac{\xi_1^+(m_t)}{\xi_2^-(m_t)} \tan \beta(m_t). \quad (3.22)$$

As a consequence of this last relation, it is evident that the RG running of the different matrix elements of $\Delta\mathcal{M}_\pm^2$ may be expressed in terms of the running of $(\Delta\mathcal{M}_\pm^2)_{12}$ and the value of $\tan \beta$ at the scale m_t .

Correspondingly, the RG-improved neutral-Higgs-boson mass matrix \mathcal{M}_N^2 may be computed by

$$\begin{aligned} (\mathcal{M}_N^2)_{ij}(m_t) &= (\widetilde{\mathcal{M}}_N^2)_{ij}(m_t) + [\xi_{ij}^{\tilde{t}}(m_t)]^{-1} (\Delta\mathcal{M}_N^2)_{ij}^{\tilde{t}}(Q_t) + [\xi_{ij}^{\tilde{b}}(m_t)]^{-1} (\Delta\mathcal{M}_N^2)_{ij}^{\tilde{b}}(Q_b) \\ &\quad + (\mathcal{M}_N^{2(1)})_{ij}^{tb}(m_t), \end{aligned} \quad (3.23)$$

with $i, j = 1, 2, 3$, $Q_t^2 = \max(\widetilde{M}_Q^2 + m_t^2, \widetilde{M}_t^2 + m_t^2)$ and $Q_b^2 = \max(\widetilde{M}_Q^2 + m_b^2, \widetilde{M}_b^2 + m_b^2)$. Notice that, unlike the charged-Higgs-boson case, one has to introduce here two decoupling scales Q_t and Q_b , as the stop/top and sbottom/bottom loop effects occur separately in the threshold contributions. The parameters $\xi_{ij}^{\tilde{q}}$ (with $q = t, b$) are the anomalous-dimension factors related to the neutral Higgs-boson fields

$$\xi_{ij}^{\tilde{q}}(m_t) = \xi_{ji}^{\tilde{q}}(m_t) = \begin{cases} \xi_i^{\tilde{q}}(m_t) \xi_j^{\tilde{q}}(m_t), & \text{for } i, j = 1, 2 \\ \xi_1^{\tilde{q}}(m_t) \xi_2^{\tilde{q}}(m_t), & \text{for } i = 1, 2, 3 \text{ and } j = 3 \end{cases}, \quad (3.24)$$

with

$$\begin{aligned} \xi_1^{\tilde{t}}(m_t) &= 1 + \frac{3h_b^2}{32\pi^2} \ln \frac{Q_t^2}{m_t^2}, & \xi_2^{\tilde{t}}(m_t) &= 1 + \frac{3h_t^2}{32\pi^2} \ln \frac{Q_t^2}{m_t^2}, \\ \xi_1^{\tilde{b}}(m_t) &= 1 + \frac{3h_b^2}{32\pi^2} \ln \frac{Q_b^2}{m_t^2}, & \xi_2^{\tilde{b}}(m_t) &= 1 + \frac{3h_t^2}{32\pi^2} \ln \frac{Q_b^2}{m_t^2}. \end{aligned} \quad (3.25)$$

We should observe that, for the very same reasons as in the charged-Higgs-boson case, the vanishing of the determinants of \mathcal{M}_P^2 and \mathcal{M}_{SP}^2 at any Q^2 scale leads to the common anomalous dimension factor $\xi_{i3}(m_t) = \xi_{3i}(m_t) = \xi_1(m_t) \xi_2(m_t)$ in the calculation of $\mathcal{M}_N^2(m_t)$ in (3.23). Since this last fact involves the matrix elements $(\mathcal{M}_P^2)_{12}$, $(\mathcal{M}_{SP}^2)_{12}$ and $(\mathcal{M}_{SP}^2)_{21}$, the corresponding matrix elements of $(\Delta\mathcal{M}_N^2)_{i3}^{\tilde{q}}$ in (3.23) are given by

$$\begin{aligned} (\Delta\mathcal{M}_N^2)_{13}^{\tilde{q}}(Q_q) &= \frac{(\mathcal{M}_{SP}^2)_{12}^{\tilde{q}}(Q_q)}{\cos \beta(m_t)}, & (\Delta\mathcal{M}_N^2)_{23}^{\tilde{q}}(Q_q) &= -\frac{(\mathcal{M}_{SP}^2)_{21}^{\tilde{q}}(Q_q)}{\sin \beta(m_t)}, \\ (\Delta\mathcal{M}_N^2)_{33}^{\tilde{q}}(Q_q) &= -\frac{(\Delta\mathcal{M}_P^2)_{12}^{\tilde{q}}(Q_q)}{\sin \beta(m_t) \cos \beta(m_t)}, \end{aligned} \quad (3.26)$$

where the RG-scale dependence of the involved quantities has been displayed explicitly.

In (3.23), the (3×3) matrices $\overline{\mathcal{M}}_N^2$ and $(\Delta\mathcal{M}_N^2)^{\tilde{q}}$ (with $q = t, b$) describe respectively the two-loop Born-improved effects and the one-loop threshold contributions associated with the decoupling of the heavy squark states:

$$\overline{\mathcal{M}}_N^2(m_t) = \mathcal{M}_N^{2(0)} \left[\text{Re } \bar{m}_{12}^2(m_t), \bar{\lambda}_1(m_t), \bar{\lambda}_2(m_t), \bar{\lambda}_{34}(m_t) \right], \quad (3.27)$$

$$\begin{aligned} (\Delta\mathcal{M}_N^2)^{\tilde{q}}(Q_q) &= \\ & (\mathcal{M}_N^{2(1)})^{\tilde{q}}(Q_q) - \mathcal{M}_N^{2(0)} \left[\text{Re } m_{12}^{2(1),\tilde{q}}(Q_q), \lambda_1^{(1),\tilde{q}}(Q_q), \lambda_2^{(1),\tilde{q}}(Q_q), \lambda_{34}^{(1),\tilde{q}}(Q_q) \right], \end{aligned} \quad (3.28)$$

where $\lambda_{34}^{(1)} = \lambda_3^{(1)} + \lambda_4^{(1)}$ (likewise $\bar{\lambda}_{34} = \bar{\lambda}_3 + \bar{\lambda}_4$) and $\mathcal{M}_N^{2(0)}$ is the tree-level functional form of \mathcal{M}_N^2 :

$$\begin{aligned} \mathcal{M}_N^{2(0)} &= \quad (3.29) \\ \frac{\text{Re } m_{12}^2}{\sin \beta \cos \beta} & \begin{pmatrix} \sin^2 \beta & -\sin \beta \cos \beta & 0 \\ -\sin \beta \cos \beta & \cos^2 \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} - v^2 \begin{pmatrix} 2\lambda_1 \cos^2 \beta & \lambda_{34} \sin \beta \cos \beta & 0 \\ \lambda_{34} \sin \beta \cos \beta & 2\lambda_2 \sin^2 \beta & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

with $\lambda_{34} = \lambda_3 + \lambda_4$. As in the charged-Higgs-boson case, we write $(\mathcal{M}_N^{2(1)})^{\tilde{q}}$ to denote the one-loop part of \mathcal{M}_N^2 containing the contributions of the third-generation squarks, and $(\mathcal{M}_N^{2(1)})^{tb}$ to denote its fermionic one-loop counterpart.

The resulting RG-improved Higgs-boson mass matrix $\mathcal{M}_N^2(m_t)$ is a symmetric, positive-definite (3×3) matrix, and can therefore be diagonalized by an orthogonal transformation as follows:

$$O^T \mathcal{M}_N^2(m_t) O = \text{diag} \left[M_{H_1}^2(m_t), M_{H_2}^2(m_t), M_{H_3}^2(m_t) \right], \quad (3.30)$$

where we have defined the Higgs fields such that their RG-improved masses satisfy the inequality:

$$M_{H_1}(m_t) \leq M_{H_2}(m_t) \leq M_{H_3}(m_t). \quad (3.31)$$

Notice that our convention in (3.30) differs from that chosen in [11], as we assign the Higgs fields in the reversed order. Analytic expressions for $M_{H_i}(m_t)$ and O are presented in Appendix B.

Before closing this Section, two important remarks are in order. First, we observe that the free kinematic parameters of the MSSM Higgs sector are

$$\begin{aligned} M_{H^+}(m_t), \quad \tan \beta(m_t), \quad \mu(Q_{tb}), \quad A_t(Q_{tb}), \quad A_b(Q_{tb}), \\ \widetilde{M}_Q^2(Q_{tb}), \quad \widetilde{M}_t^2(Q_{tb}), \quad \widetilde{M}_b^2(Q_{tb}). \end{aligned} \quad (3.32)$$

In fact, the soft Higgs-mixing parameter $\text{Re } \bar{m}_{12}^2(m_t)$ may be substituted by the squared RG-improved mass $M_{H^+}^2(m_t)$ of the charged Higgs boson (cf. (3.16) and (3.18)) in the neutral Higgs-boson mass matrix $\mathcal{M}_N^2(m_t)$ in (3.23).

Secondly, we reiterate the fact that $\text{Im } m_{12}^2$ can be renormalized independently, without affecting the renormalization of the physical parameters of the theory [10]. As was stressed in Section 3, the $\xi = 0$ scheme of renormalization gives rise to a considerable simplification, since we can get rid of the radiatively-induced phase ξ between the two Higgs vacuum expectation values in the analytic expressions of the Higgs-boson masses and mixing angles. For example, within the above $\xi = 0$ scheme, the mass renormalization of H^+ may be entirely reabsorbed by a corresponding renormalization of $\text{Re } m_{12}^2$ and λ_4 . In other words, it can be shown that M_{H^+} is Q^2 -independent, after including the RG running of $\text{Re } m_{12}^2$ and λ_4 , denoted as $\gamma_{\text{Re } m_{12}^2}$ and β_{λ_4} . For simplicity, we assume that only the third generation of squarks contributes to $\gamma_{\text{Re } m_{12}^2}$, since fermions do not contribute to the RG running of $\text{Im } m_{12}^2$. The analytic forms of $\gamma_{\text{Re } m_{12}^2}$ and β_{λ_4} are given by

$$\gamma_{\text{Re } m_{12}^2} = -\frac{3}{16\pi^2} \left[h_t^2 \text{Re}(\mu A_t) + h_b^2 \text{Re}(\mu A_b) \right], \quad (3.33)$$

$$\beta_{\lambda_4} \equiv \frac{d\lambda_4^{(1)}}{d \ln Q^2} = -\frac{3}{16\pi^2} \left[2h_t^2 h_b^2 - \frac{g_w^2}{2} (h_t^2 + h_b^2) + \frac{g_w^4}{4} \right]. \quad (3.34)$$

Obviously, the RG running of $\text{Re } m_{12}^2$ due to \tilde{t} and \tilde{b} is only relevant for non-zero values of μA_t and μA_b . Employing (2.19) and examining only the $\ln Q^2$ -dependent part, one can verify that

$$\frac{dM_{H^+}^2}{d \ln Q^2} \propto -\gamma_{\text{Re } m_{12}^2} - \frac{1}{2} \beta_{\lambda_4} v_1 v_2 + \frac{3}{32\pi^2} \left(\left\langle \frac{\partial^2 \text{Tr } \widetilde{\mathcal{M}}^4}{\partial \phi_1^+ \partial \phi_2^-} \right\rangle - \frac{i}{v_1} \left\langle \frac{\partial \text{Tr } \widetilde{\mathcal{M}}^4}{\partial a_2} \right\rangle \right) = 0, \quad (3.35)$$

as it should be. As can also be seen from (3.35), an important role in this proof is played by the necessary CP-odd tadpole term $\langle \partial \text{Tr } \widetilde{\mathcal{M}}^4 / \partial a_2 \rangle$ [10].

4 Effective Top and Bottom Yukawa Couplings

In addition to the RG improvement of the Higgs-boson mass matrices discussed in the previous Section, we consider here a further improvement related to the non-logarithmic threshold corrections to the top- and bottom-quark Yukawa couplings. Specifically, apart from the usual RG running, the effective top- and bottom-quark Yukawa couplings obtain additional non-logarithmic threshold contributions, which are induced by the decoupling of the heavy SUSY states at a high scale, *e.g.*, Q_{tb} . For the bottom-quark Yukawa case, the one-loop RG relation between the bottom mass and the bottom-quark Yukawa coupling at the scale Q_{tb} receives quantum corrections that also include terms proportional to $\tan \beta$ [38–40]. Since these last terms can be significant for large values of $\tan \beta$,[§] we must resum them

[§]For the τ -lepton Yukawa coupling, the corresponding enhanced $\tan \beta$ terms are much smaller, because they are proportional to the weak gauge couplings [39,19].

within the RG approach, so that the actual size of the radiative corrections to the Higgs-boson masses and couplings can properly be extracted [19,41]. For the top-quark case, instead, although one-loop suppressed, the respective corrections can still give rise to an enhancement of up to 4 GeV in the prediction for the lightest Higgs-boson mass [42,43], and therefore should be included in the computation.

There may also be important CP-violating one-loop corrections to the bottom- and top-quark Yukawa couplings, in addition to the CP-violating effects induced by the radiative mixing of the Higgs states, which were considered in Sections 2 and 3 in detail. In the leptonic sector, these CP-violating vertex corrections are generally small [44]. However, the CP-violating radiative corrections to the couplings of the Higgs bosons to b quarks are significant [11], because of the large Yukawa and colour-enhanced QCD interactions [38]. In particular, the radiative effects of the Higgs-boson couplings to the bottom quarks can be further enhanced, if the respective Higgs-mass eigenstate couples predominantly to the Higgs doublet Φ_2 [39,40], as the tree-level b -quark Yukawa coupling is suppressed in this case. For a general discussion of the form and the origin of these finite Yukawa corrections to the third-generation quark masses, the reader is referred to the original literature [39,40]. In the following, we give a brief discussion of the non-logarithmic corrections to the top and bottom Yukawa couplings, and pay special attention to the CP-violating vertex effects.

We start our discussion by considering the effective Lagrangian of the b -quark Yukawa coupling [19,11]:

$$-\mathcal{L}_{\phi^0\bar{b}b} = (h_b + \delta h_b) \phi_1^0 \bar{b}_L b_R + \Delta h_b \phi_2^0 \bar{b}_L b_R + \text{h.c.}, \quad (4.1)$$

with

$$\frac{\delta h_b}{h_b} = -\frac{2\alpha_s}{3\pi} m_{\bar{g}} A_b^* I(m_{\bar{b}_1}^2, m_{\bar{b}_2}^2, |m_{\bar{g}}|^2) - \frac{h_t^2}{16\pi^2} |\mu|^2 I(m_{\bar{t}_1}^2, m_{\bar{t}_2}^2, |\mu|^2), \quad (4.2)$$

$$\frac{\Delta h_b}{h_b} = \frac{2\alpha_s}{3\pi} m_{\bar{g}} \mu I(m_{\bar{b}_1}^2, m_{\bar{b}_2}^2, |m_{\bar{g}}|^2) + \frac{h_t^2}{16\pi^2} A_t \mu I(m_{\bar{t}_1}^2, m_{\bar{t}_2}^2, |\mu|^2), \quad (4.3)$$

where $\alpha_s = g_s^2/(4\pi)$ is the $SU(3)_c$ coupling strength, and $I(a, b, c)$ is the one-loop function

$$I(a, b, c) = \frac{ab \ln(a/b) + bc \ln(b/c) + ac \ln(c/a)}{(a-b)(b-c)(a-c)}. \quad (4.4)$$

The b -quark Yukawa coupling $h_b(Q_{tb})$ is then related to the running b -quark mass $m_b(Q_{tb})$ by

$$h_b = \frac{g_w m_b}{\sqrt{2} M_W \cos \beta [1 + \delta h_b/h_b + (\Delta h_b/h_b) \tan \beta]}, \quad (4.5)$$

where $\delta h_b/h_b$ and $\Delta h_b/h_b$ are given in (4.2) and (4.3), respectively. The running b -quark mass $m_b(Q_{tb})$ is obtained by means of the RG running of the b -quark mass from the scale

m_b . In (4.5), we have redefined the right-handed b -quark superfield, so that the physical b -quark mass is positive. Under such a field redefinition, only the Yukawa coupling h_b becomes complex, while the phases of $\delta h_b/h_b$ and $\Delta h_b/h_b$ as well as those of the SUSY-breaking parameters do not change. Moreover, since only the moduli of the Yukawa couplings h_b and h_t enter the field-dependent quark and squark masses, the Higgs-boson mass matrices remain unaffected by the above field redefinition. At this point, it is interesting to observe that the sum $\delta h_b + \Delta h_b \tan \beta$ in (4.5) receives two sorts of quantum corrections, one originating from QCD effects and another from a chargino-mediated graph. The QCD correction is proportional to the hermitean conjugate of the sbottom-mixing parameter $X_b = A_b - \mu^* \tan \beta$, whilst the chargino-induced diagram [39] depends linearly on the stop-mixing parameter $X_t = A_t - \mu^* \cot \beta$.

The effective Lagrangian describing the t -quark Yukawa coupling is given by

$$-\mathcal{L}_{\phi^0 \bar{t} t} = \Delta h_t \phi_1^{0*} \bar{t}_L t_R + (h_t + \delta h_t) \phi_2^{0*} \bar{t}_L t_R + \text{h.c.} \quad (4.6)$$

The corresponding relation for h_t as a function of m_t may easily be determined analogously by the effective Lagrangian (4.6), and reads

$$h_t = \frac{g_w m_t}{\sqrt{2} M_W \sin \beta [1 + \delta h_t/h_t + (\Delta h_t/h_t) \cot \beta]}, \quad (4.7)$$

with

$$\frac{\Delta h_t}{h_t} = \frac{2\alpha_s}{3\pi} m_{\bar{g}} \mu I(m_{\tilde{t}_1}^2, m_{\tilde{t}_2}^2, |m_{\bar{g}}|^2) + \frac{h_b^2}{16\pi^2} A_b \mu I(m_{\tilde{b}_1}^2, m_{\tilde{b}_2}^2, |\mu|^2), \quad (4.8)$$

$$\frac{\delta h_t}{h_t} = -\frac{2\alpha_s}{3\pi} m_{\bar{g}} A_t^* I(m_{\tilde{t}_1}^2, m_{\tilde{t}_2}^2, |m_{\bar{g}}|^2) - \frac{h_b^2}{16\pi^2} |\mu|^2 I(m_{\tilde{b}_1}^2, m_{\tilde{b}_2}^2, |\mu|^2). \quad (4.9)$$

As in the case of the b -quark Yukawa coupling, we have to make a judicious phase rotation of the right-handed t -quark superfield, such that the physical top-quark mass becomes positive. Again, one can show that such a field redefinition does not change the analytic results of the RG analysis.

At this stage, it is important to remark that, within the RG-resummation approach described in Section 3, the non-logarithmic corrections must be treated as threshold effects and hence they should only contribute to the RG-frozen part of the Higgs-boson mass matrices, generically denoted as $\Delta \mathcal{M}^2(Q_{tb})$. Therefore, the decoupling procedure for the heavy squark states requires that the effective b - and t -quark Yukawa couplings given by (4.5) and (4.7) are evaluated at the scale Q_{tb} . As we discuss in Section 5, these additional Yukawa corrections can lead to observable effects in Higgs-boson searches.

It is now straightforward to obtain the interaction Lagrangians of the Higgs-boson mass eigenstates H_i to the up- and down-type quarks, collectively denoted as u and d .

Taking into account both CP-violating self-energy and vertex effects, we find

$$\mathcal{L}_{H\bar{f}f} = -\sum_{i=1}^3 H_i \left[\frac{g_w m_d}{2M_W} \bar{d} \left(g_{H_i dd}^S + i g_{H_i dd}^P \gamma_5 \right) d + \frac{g_w m_u}{2M_W} \bar{u} \left(g_{H_i uu}^S + i g_{H_i uu}^P \gamma_5 \right) u \right], \quad (4.10)$$

with

$$g_{H_i dd}^S = \frac{1}{h_d + \delta h_d + \Delta h_d \tan \beta} \left\{ \text{Re}(h_d + \delta h_d) \frac{O_{1i}}{\cos \beta} + \text{Re}(\Delta h_d) \frac{O_{2i}}{\cos \beta} + \left[\text{Im}(h_d + \delta h_d) \tan \beta - \text{Im}(\Delta h_d) \right] O_{3i} \right\}, \quad (4.11)$$

$$g_{H_i dd}^P = \frac{1}{h_d + \delta h_d + \Delta h_d \tan \beta} \left\{ \left[\text{Re}(\Delta h_d) - \text{Re}(h_d + \delta h_d) \tan \beta \right] O_{3i} + \text{Im}(h_d + \delta h_d) \frac{O_{1i}}{\cos \beta} + \text{Im}(\Delta h_d) \frac{O_{2i}}{\cos \beta} \right\}, \quad (4.12)$$

$$g_{H_i uu}^S = \frac{1}{h_u + \delta h_u + \Delta h_u \cot \beta} \left\{ \text{Re}(h_u + \delta h_u) \frac{O_{2i}}{\sin \beta} + \text{Re}(\Delta h_u) \frac{O_{1i}}{\sin \beta} + \left[\text{Im}(h_u + \delta h_u) \cot \beta - \text{Im}(\Delta h_u) \right] O_{3i} \right\}, \quad (4.13)$$

$$g_{H_i uu}^P = \frac{1}{h_u + \delta h_u + \Delta h_u \cot \beta} \left\{ \left[\text{Re}(\Delta h_u) - \text{Re}(h_u + \delta h_u) \cot \beta \right] O_{3i} + \text{Im}(h_u + \delta h_u) \frac{O_{2i}}{\sin \beta} + \text{Im}(\Delta h_u) \frac{O_{1i}}{\sin \beta} \right\}, \quad (4.14)$$

where the Higgs scalar and pseudoscalar couplings are normalized with respect to their SM values.

Finally, it is interesting to investigate the behaviour of self-energy- and vertex-type CP violation in the decoupling limit of a heavy charged Higgs boson in the MSSM. Thus, for values of the charged Higgs mass $M_{H^\pm} \gg M_Z$, one has $O_{31} \rightarrow 0$, while $O_{11} \rightarrow \cos \beta$ and $O_{21} \rightarrow \sin \beta$. In this limit, the scalar components of the $H_1 dd$ and $H_1 uu$ couplings acquire the known SM form, given by $g_w m_d / (2M_W)$ and $g_w m_u / (2M_W)$, respectively, where

$$\begin{aligned} m_d &= \frac{1}{\sqrt{2}} \left(h_d + \delta h_d + \Delta h_d \tan \beta \right) v_1, \\ m_u &= \frac{1}{\sqrt{2}} \left(h_u + \delta h_u + \Delta h_u \cot \beta \right) v_2 \end{aligned} \quad (4.15)$$

have already been defined to be positive in (4.5) and (4.7). For similar reasons, the pseudoscalar parts of the $H_1 dd$ and $H_1 uu$ couplings vanish, as they are proportional to O_{31} and $\text{Im} m_{d,u} = 0$. On the other hand, in the same large M_{H^\pm} limit, the scalar and pseudoscalar couplings of both the two heaviest Higgs bosons H_2 and H_3 to the up and down fermions do not vanish. We can therefore conclude that CP-violating self-energy and vertex effects do not decouple in the heavy neutral Higgs sector. In the next section, we demonstrate explicitly the aforementioned (non-)decoupling features of CP violation by analyzing specific phenomenological examples.

5 Phenomenological Discussion

In this Section we discuss the phenomenological implications of radiative Higgs-sector CP violation in the MSSM for Higgs-boson searches at high-energy colliders. We focus our attention on the physics potential for discovering Higgs bosons with mixed CP parities at LEP 2 and the upgraded Tevatron collider, and also comment on the enhanced search capabilities offered by the LHC.

At the LEP 2 and Tevatron colliders, neutral Higgs bosons are predominantly produced via the Higgs-strahlung processes in e^+e^- and $q\bar{q}$ collisions, such as $e^+e^- \rightarrow Z^* \rightarrow ZH_i$ [45], $q\bar{q} \rightarrow Z^* \rightarrow ZH_i$ and $q\bar{q} \rightarrow W^{\pm*} \rightarrow W^\pm H_i$ [46], with $i = 1, 2, 3$. If the next-to-lightest Higgs boson is not too heavy, Higgs bosons can also be produced copiously in pairs through the reactions: $e^+e^- \rightarrow Z^* \rightarrow H_i H_j$ and $q\bar{q} \rightarrow Z^* \rightarrow H_i H_j$. In addition to the Higgs-boson masses, the Higgs-boson couplings to the gauge fields play an essential role in our forthcoming discussion. The effective Lagrangians governing the interactions of the Higgs bosons with the W^\pm and Z bosons are given by [11]

$$\mathcal{L}_{HVV} = g_w M_W \sum_{i=1}^3 g_{H_i VV} \left(H_i W_\mu^+ W^{-,\mu} + \frac{1}{2 \cos^2 \theta_w} H_i Z_\mu Z^\mu \right), \quad (5.1)$$

$$\mathcal{L}_{HH^\mp W^\pm} = \frac{g_w}{2} \sum_{i=1}^3 g_{H_i H^\mp W^\pm} (H_i i \overleftrightarrow{\partial}_\mu H^\mp) W^{+,\mu} + \text{h.c.}, \quad (5.2)$$

$$\mathcal{L}_{HHZ} = \frac{g_w}{2 \cos \theta_w} \sum_{j>i=1}^3 g_{H_i H_j Z} (H_i \overleftrightarrow{\partial}_\mu H_j) Z^\mu, \quad (5.3)$$

where $\cos \theta_w = M_W/M_Z$, $\overleftrightarrow{\partial}_\mu \equiv \overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu$, and

$$g_{H_i VV} = \cos \beta O_{1i} + \sin \beta O_{2i}, \quad (5.4)$$

$$g_{H_i H_j Z} = O_{3i} (\cos \beta O_{2j} - \sin \beta O_{1j}) - O_{3j} (\cos \beta O_{2i} - \sin \beta O_{1i}), \quad (5.5)$$

$$g_{H_i H^\mp W^\pm} = \cos \beta O_{2i} - \sin \beta O_{1i} + i O_{3i}. \quad (5.6)$$

For completeness, we have included in (5.2) the interactions of the charged Higgs bosons H^\pm with the neutral Higgs and W^\mp bosons. Note that the couplings $H_i Z Z$ and $H_i W^+ W^-$ are related to the $H_i H_j Z$ couplings through

$$g_{H_k VV} = \varepsilon_{ijk} g_{H_i H_j Z}. \quad (5.7)$$

Moreover, unitarity provides the constraint

$$\sum_{i=1}^3 g_{H_i VV}^2 = 1. \quad (5.8)$$

Evidently, if two Higgs-boson couplings to gauge bosons are known, this is sufficient to determine the complete set of the couplings $g_{H_i V V}$ and $g_{H_i H_j Z}$ [47].

In the above calculation of the effective Higgs-gauge-boson couplings, we have assumed that the dominant contributions arise from Higgs-mixing effects. As opposed to the b -quark Yukawa case discussed in Section 4, proper vertex corrections to the $H_i Z Z$ and $H_i H_j Z$ couplings do not contain strong-coupling- or $\tan\beta$ -enhanced diagrams. Therefore, naive dimensional analysis suggests that these corrections are suppressed relative to their tree-level values by loop factors of the kind: $(3\alpha_w/4\pi)(|\mu A_t|/m_{\tilde{t}_1}^2)$, $(3\alpha_w/4\pi)(|\mu A_t|^2 v^2/m_{\tilde{t}_1}^6) \lesssim 10^{-2}$, where \tilde{t}_1 is the heaviest stop squark (see (2.13)). In the following, we neglect proper vertex corrections to the $H_i Z Z$ and $H_i H_j Z$ couplings.

For our phenomenological discussion of CP violation, we consider the following two representative values for $\tan\beta$: (i) $\tan\beta = 4$ and (ii) $\tan\beta = 20$. For definiteness, unless stated otherwise, the soft SUSY-breaking and μ parameters are set to the values

$$\begin{aligned} M_{\text{SUSY}} = \widetilde{M}_Q^2 = \widetilde{M}_t^2 = \widetilde{M}_b^2 = 0.5 \text{ TeV}, \quad \mu = 2 \text{ TeV}, \quad |A_t| = |A_b| = 1 \text{ TeV}, \\ |m_{\tilde{B}}| = |m_{\tilde{W}}| = 0.3 \text{ TeV}, \quad |m_{\tilde{g}}| = 1 \text{ TeV}, \end{aligned} \quad (5.9)$$

As can be seen in (5.9), we have chosen relatively large values for the stop and sbottom mixing parameters A_t , A_b and μ , as well as a common left- and right-handed squark mass M_{SUSY} , which leads to enhanced CP-violating effects of the CP-odd phases $\arg(\mu A_{t,b})$ on the Higgs sector.

As was mentioned in the Introduction, large CP-odd phases may lead to rather large EDM contributions, thereby violating the known upper bounds on the electron and neutron EDMs d_e and d_n : $d_e/e < 0.5 \cdot 10^{-26}$ cm [23] and $d_n/e < 0.6 \cdot 10^{-25}$ cm [24] at the $2\text{-}\sigma$ level. One phenomenologically interesting possibility for avoiding the possible CP crisis is to make the first two generations of squarks rather heavy with masses much above the TeV scale [27,28], keeping the third generation relatively light with masses of order 0.5 TeV. In such a scenario, CP violation may only reside in the third generation. For our illustrations, we shall take the μ parameter to be real and assume that the only CP-odd phases in the theory are $\arg(A_t) = \arg(A_b)$ and $\arg(m_{\tilde{g}})$. Again, as one way to avoid the one-loop EDM constraints [29], we have taken a gluino mass of 1 TeV in (5.9). However, in such a scheme, one has to worry about the fact that Higgs-boson two-loop contributions to the EDMs [34] might still become sizeable. For the low- $\tan\beta$ scenario in (5.9), the two-loop EDM contributions are of the order of the experimental EDM upper bounds mentioned above. Since these two-loop EDM effects depend almost linearly on $\tan\beta$, one might then need to arrange for cancellations among the different one- and two-loop EDM terms [29] at the

level of 10% for the scenario with $\tan\beta = 20$. We believe that this can be achieved without excessive fine-tuning of the CP-violating parameters of the theory.

As was already noticed in [14,18,11], the radiative corrections to the lightest Higgs boson H_1 depend crucially on the stop mixing parameter $|X_t| = |A_t - \mu^* \cot\beta|$. Specifically, the radiatively-corrected H_1 -boson mass increases as $|X_t|$ increases, reaching a maximum when $|X_t|/M_{\text{SUSY}} \approx 2.45$. Then, as $|X_t|$ further increases, the radiative corrections to H_1 -boson mass decrease and may even become negative, driving the latter to very small, experimentally excluded values. A distinctive feature of the CP-violating SUSY scenario compared to the CP-conserving one is that $|X_t|$ can be increased by varying only the phase $\arg(A_t)$ from zero to higher values, but holding fixed $|A_t|$ and $|\mu|$. For similar reasons, high values of $|X_t|$ induced by large values of $\arg(A_t)$ can make the mass of the lightest stop squark \tilde{t}_2 very low, so as to violate present experimental constraints, *i.e.*, $m_{\tilde{t}_2} \gtrsim 100$ GeV. Furthermore, light stop quarks, with $m_{\tilde{t}_2} \lesssim 300$ GeV and large $|X_t|$ values, can give rise to observably large contributions to the electroweak precision parameter $\Delta\rho$ [48]. For the scenarios under discussion, $m_{\tilde{t}_2}$ is always larger than about 300 GeV, for all the parameter space for which the H_1 -boson mass acquires acceptable values. Therefore, apart from the bounds derived by EDM constraints, the requirement that the lightest Higgs-boson mass is positive can be used naively to set an upper bound on the phase of A_t .

In Fig. 1 we give numerical predictions for the two lightest Higgs-boson masses M_{H_1} and M_{H_2} , and for the three relevant $H_i ZZ$ couplings squared as a function of $\arg(A_t)$, for two different values of $\arg(m_{\tilde{g}})$: $\arg(m_{\tilde{g}}) = 0$ (solid lines) and $\arg(m_{\tilde{g}}) = \pi/2$ (dashed lines). We first discuss the scenario with $\tan\beta = 4$, for which the values of the remaining soft SUSY-breaking parameters and μ are given in (5.9). Since our interest is to analyze dominant CP-violating effects for a light Higgs sector, we present predictions for a relatively small charged-Higgs-boson mass, $M_{H^\pm} = 150$ GeV. In the CP-conserving limit of the theory ($\arg(A_t) = 0$), the mass M_{H_1} of the lightest neutral Higgs boson is close to 85 GeV, while the square of the $H_1 ZZ$ coupling, $g_{H_1 ZZ}^2$, is approximately equal to 0.8. These values of masses and couplings are now excluded by Higgs-boson searches at LEP 2 [22]. However, this situation changes crucially once CP-violating phases become relevant. As the phase of A_t increases, two important effects take place. First, as was mentioned above, the stop mixing parameter $|X_t|$ becomes larger, giving rise to larger H_1 -boson masses. Second, the mass-matrix terms describing the scalar-pseudoscalar mixing are enhanced, thereby effectively leading to large modifications in the couplings of the Higgs bosons to gauge bosons. This second effect can be attributed entirely to CP violation. In fact, as can be seen from Fig. 1(b), for $\arg(A_t) \approx 80$ degrees, $g_{H_1 ZZ}^2$ gets very suppressed, implying that LEP 2 cannot detect the Higgs boson H_1 via $e^+e^- \rightarrow Z^* \rightarrow ZH_1$. On the other hand,

for the same range of values of $\arg(A_t)$, *i.e.*, $\arg(A_t) = 80^\circ\text{--}95^\circ$, the H_2 and H_3 bosons have significant couplings to the Z bosons. Although the H_3 boson is too heavy to be detected at LEP 2 in this case, the H_2 boson has a mass of 105–110 GeV and $g_{H_2ZZ}^2 \approx 0.8\text{--}0.6$, which may be probed at LEP 2 in this year's run. For larger values of the phase of A_t , $95^\circ < \arg(A_t) < 110^\circ$, the discovery of a Higgs boson at LEP 2 is more challenging. The lightest Higgs boson H_1 acquires a mass below 90 GeV, but the H_1ZZ coupling is too small to allow experimental detection through the reaction $e^+e^- \rightarrow Z^* \rightarrow ZH_1$. In addition, the H_2 boson becomes too heavy to allow for discovery via $e^+e^- \rightarrow Z^* \rightarrow ZH_2$, with $g_{H_2ZZ}^2 \lesssim 0.7$.

As was discussed in [11], the H_1 and H_2 bosons may also be searched for in the channel $e^+e^- \rightarrow Z^* \rightarrow H_1H_2$. Since the squared coupling $g_{H_1H_2Z}^2 = g_{H_3ZZ}^2 \approx 0.2$ almost independently of $\arg(A_t)$, a careful experimental analysis of the parameter region of interest will be necessary to determine whether the two lightest Higgs bosons can be observed for such large mass differences ($M_{H_1} - M_{H_2} \geq 40$ GeV) and such small $g_{H_1H_2Z}$ couplings.

In Fig. 1 we also present predictions for a gluino phase of 90 degrees (dashed lines). Since the vertex corrections are generally small for low or moderate values of $\tan\beta$, they are expected to induce only small corrections to the Higgs-boson masses and mixings. This last fact is reflected in Fig. 1, even though the coupling $g_{H_1ZZ}^2$ ($g_{H_2ZZ}^2$) gets slightly smaller (larger) for larger values of the phase of A_t .

Fig. 2 shows the changes in the predictions for the same choice of parameters as in Fig. 1, but with M_{SUSY} , μ and $|A_t| = |A_b|$ rescaled by a factor of 2. This rescaling leads to a slight increase (decrease) of the Higgs-boson mass M_{H_1} (M_{H_2}), while $g_{H_1ZZ}^2$ exhibits a slightly different quantitative dependence on the phase of A_t , especially for the region in which $g_{H_1ZZ}^2$ is very small. Thus, although M_{H_1} becomes small for $\arg(A_t) > 115$ degrees, the H_1ZZ coupling gets sizeable again, well within the capabilities of LEP 2 to test.

We now investigate more quantitatively the predictions of a large- $\tan\beta$ scenario for the Higgs-boson masses and couplings. We adopt the scenario given in (5.9), with $\tan\beta = 20$ and $M_{H^+} = 150$ GeV. In this large- $\tan\beta$ scenario, one has $|\mu| \cot\beta \ll |A_t|$, and the effective stop mixing parameter $|X_t| \approx |A_t|$ is almost independent of $\arg(A_t)$. Therefore, as is seen in Fig. 3(a), the Higgs-boson masses M_{H_1} and M_{H_2} do not exhibit any significant variation as a function of $\arg(A_t)$. In contrast to the Higgs-boson masses, Fig. 3(b) shows that there is a non-trivial dependence of the squared couplings $g_{H_iZZ}^2$ on $\arg(A_t)$. Furthermore, the next-to-lightest Higgs boson H_2 is heavy enough to render its search through the $e^+e^- \rightarrow Z^* \rightarrow ZH_2$ reaction kinematically inaccessible at LEP 2. For similar reasons, we find that, for all values of $\arg(A_t)$, the H_1 and H_2 bosons are rather too heavy to be produced via the $e^+e^- \rightarrow Z^* \rightarrow H_1H_2$ channel at LEP 2. As a re-

sult, Higgs-boson searches at LEP 2 tend to be more efficient for small values of the A_t phases, for which the lightest neutral Higgs-boson mass is close to 100 GeV and $g_{H_1ZZ}^2$ is non-negligible ($g_{H_1ZZ}^2 \approx 0.3$). In addition, for large A_t phases, $\arg(A_t) \gtrsim 80^\circ$, the H_1VV coupling (with $V = Z, W$) is rather suppressed, so that the H_1 Higgs boson, although it becomes lighter with a mass in the range 90–95 GeV, will be elusive at LEP 2, and may also escape detection via the corresponding channel at the upgraded Tevatron. However, the next-to-lightest Higgs boson H_2 has couplings of order unity to the Z and W bosons. Present simulations show that a neutral Higgs boson, such as H_2 , with $M_{H_2} \approx 180$ GeV and a SM-like coupling strength to vector gauge bosons can be tested at the Tevatron collider with a total integrated luminosity of 10 fb^{-1} . However, discovery of such a Higgs boson at the $5\text{-}\sigma$ level would demand a total integrated luminosity of 30 fb^{-1} , and would have a reach up to $M_{H_2} \approx 120$ GeV [49]. Finally, even though the H_1H_3Z coupling is close to unity ($g_{H_1H_3Z}^2 = g_{H_2ZZ}^2$) for $\arg(A_t) > 100^\circ$, the Higgs-pair production of H_1 and H_3 is not kinematically allowed at LEP 2, since $M_{H_3} \approx M_{H^+}$. Further studies will be necessary to investigate the potential of this production mechanism at the Tevatron.

It is interesting to present predictions for the neutral Higgs-boson masses and their couplings to gauge bosons for lower values of the charged Higgs-boson mass M_{H^+} in the above large- $\tan\beta$ scenario. In Fig. 4, we plot numerical estimates for the same kinematic parameters as in Fig. 3, but with $M_{H^+} = 135$ GeV. In this case, the H_1 -boson mass varies approximately between 80 and 65 GeV, and the H_1ZZ coupling rapidly decreases as the phase of A_t increases. The two heaviest neutral Higgs bosons H_2 and H_3 have masses in the range between 120 and 130 GeV. Hence, these two Higgs bosons cannot be produced via $e^+e^- \rightarrow Z^* \rightarrow ZH_2$ or $e^+e^- \rightarrow Z^* \rightarrow ZH_3$ at LEP 2. However, the H_2 and H_3 bosons may still be accessed via $e^+e^- \rightarrow Z^* \rightarrow H_1H_2$ or H_1H_3 . Interestingly, the squared couplings $g_{H_1H_2Z}^2 = g_{H_3ZZ}^2$ and $g_{H_1H_3Z}^2 = g_{H_2ZZ}^2$ exhibit a cross-over as a function of $\arg(A_t)$. The crossing point of the two squared couplings is when $\arg(A_t) \approx 90^\circ$. For $\arg(A_t) = 180^\circ$, one of the squared couplings goes to 0 and the other to 1, depending on the phase of the gluino mass. In this case, the two heaviest neutral Higgs bosons become almost degenerate in mass. For the whole range of values of $\arg(A_t)$, either the H_2 or H_3 Higgs boson can be tested at the upgraded Tevatron collider provided a total integrated luminosity of 10 fb^{-1} per detector is available [49]

Figs. 5(a) and (b) show the degree of mass splitting between the two heaviest neutral Higgs bosons H_2 and H_3 , for the same choice of parameters as in Figs. 1 and 2, but for $M_{H^\pm} = 200, 300, 400, 500$ GeV. As was already observed in [10,11], even though the H_2 and H_3 bosons are almost degenerate in the CP-conserving limit of the theory, they can have a degree of splitting up to 30% for a maximal CP-violating phase $\arg(A_t) \approx 90^\circ$. The

comparison of the Fig. 5(a) with (b) reveals that this last result is almost independent of the common scale factor of M_{SUSY} , μ and $|A_t|$. Also, the degree of mass splitting is not much affected by the value of the gluino phase $\arg(m_{\tilde{g}})$: the results for $\arg(m_{\tilde{g}}) = 90^\circ$ are slightly higher than those of $\arg(m_{\tilde{g}}) = 0$. In this vein, it is interesting to mention that large CP-violating scalar-pseudoscalar mixings can lead to observable phenomena of resonant CP violation at high-energy colliders [50,51].

In Figs. 6 and 7, we examine the behaviours of the scalar and pseudoscalar parts of the $H_1 bb$ coupling as functions of the CP-odd phase $\arg(A_t)$, for two different charged Higgs-boson masses, $M_{H^+} = 150$ and 300 GeV. As was done in [11], we find that the best way of analyzing such a behaviour is in terms of the CP-even and CP-odd quantities: $[(g_{H_1 bb}^S)^2 + (g_{H_1 bb}^P)^2]$ and $2g_{H_1 bb}^S g_{H_1 bb}^P / [(g_{H_1 bb}^S)^2 + (g_{H_1 bb}^P)^2]$. For example, Higgs-boson branching ratios are proportional to the first quantity, while the second one will only occur in CP-violating observables. In other words, $2g_{H_1 bb}^S g_{H_1 bb}^P / [(g_{H_1 bb}^S)^2 + (g_{H_1 bb}^P)^2]$ gives a measure of the CP-violating component in the $H_1 bb$ coupling. If we compare the predictions for $M_{H^+} = 150$ GeV with those for $M_{H^+} = 300$ GeV in Figs. 6 and 7, we find that the CP-violating component of the $H_1 bb$ coupling reduces in magnitude, for large values of the charged Higgs-boson mass. Such a decoupling behaviour of the CP-violating $H_1 bb$ component is in agreement with our observation, which we already made at the end of Section 4. From Figs. 6(b) and 7(b), we see that the impact of the gluino phase on the CP-violating component of the $H_1 bb$ coupling is more important for the large- $\tan\beta$ scenario. This may be attributed to the fact that the radiatively-induced term $\Delta h_b \tan\beta$, which crucially depends on the gluino phase and $\tan\beta$, has a dominant contribution to the $H_1 bb$ coupling.

There can be a cancellation or a strong suppression of the coupling of the lightest Higgs boson H_1 to the bottom quarks, depending on the magnitude of the CP-violating phases and of the products $A_t \mu$, $A_b \mu$ and $m_{\tilde{g}} \mu$. This cancellation usually takes place for moderate values of the charged Higgs mass and large values of $\tan\beta$. Such an effect is also present in the CP-conserving case, for specific signs and magnitudes of the above products involving the trilinear terms $A_{t,b}$ and the gluino mass, and has been discussed in detail in [52]. Figure 8(b) illustrates such a cancellation for the CP-violating SUSY model under discussion. For example, we observe that for the set of SUSY parameters considered in Fig. 8, the $H_1 bb$ coupling can be strongly suppressed for $\arg(A_t) = \arg(A_b) \approx 15^\circ$. Moreover, we have checked that for this same set of parameters the $H_1 ZZ$ coupling is almost SM-like. In addition, Fig. 8(a) shows that the mass of the lightest Higgs boson is of order 105 GeV, practically independent of the CP-violating phase. This is therefore an extremely interesting example, since the lightest Higgs mass is in the mass range that may be within the reach of LEP 2, and its production cross section will be SM-like. However, the main

decay channel, $H_1 \rightarrow b\bar{b}$, can be strongly suppressed if the CP-violating phases $\arg(A_t)$ and $\arg(m_{\tilde{g}})$ lie in a specific range. Therefore, in such a scenario, the detection of the H_1 boson may in principle be impossible, even in the final run of LEP 2. To make a conclusive statement on this possibility, one should study in detail the capability of LEP 2 to detect such a light H_1 boson via its decays into τ pairs, or into other hadronic modes. Most intriguingly, the set of parameters considered in this example also allow for a light right-handed stop squark and moderate mixing parameter, $|X_t|/\widetilde{M}_Q$, of the type necessary to allow the possibility of electroweak baryogenesis [53]. Hence, if such a Higgs boson cannot be discovered at LEP 2 via other decay channels, the final phase of LEP 2 will leave an open window for electroweak baryogenesis. A careful study of the CP-violating phases required for electroweak baryogenesis and the detection capabilities of LEP 2 for alternative decay modes becomes essential for testing this exciting scenario.

We conclude this section by briefly commenting on the enhanced LHC capabilities for Higgs-boson searches [54]. The LHC has a considerably higher reach than LEP 2 and the upgraded Tevatron collider in the search for heavier Higgs bosons, and hence has more chances to unravel the complete Higgs-boson spectrum of the MSSM with explicit CP violation. At the LHC, Higgs bosons may be copiously produced via a wide variety of processes which depend in many different ways on the couplings of the neutral and charged Higgs bosons both to gauge bosons and fermions [54]. In the case of the CP-violating version of the MSSM under study, we have shown that mixing between states with different CP parities can dramatically modify those couplings and, hence, importantly affect the associated production and decay mechanisms. Studies including CP-violating effects on the gluon-fusion production of Higgs bosons at the LHC have already been considered in the literature [55]. Our work provide the basic tool to improve further those studies, and to perform a complete analysis of the CP-violating effects on the many other Higgs-boson search mechanisms available at LHC. The LHC, together with the information gathered from experiments at LEP 2 and the upgraded Tevatron collider, will be capable of providing a thorough test of the MSSM Higgs sector and shed light on the possibility of explicit radiative breaking of CP invariance in supersymmetry.

A Fortran code that computes the Higgs-boson masses and couplings, including CP-violating effects as presented in this work, may be found in [56].

6 Conclusions

We have performed a complete one-loop RG improvement of the effective Higgs potential in the MSSM, in which CP violation is induced radiatively by soft CP-violating trilinear

interactions that involve the Higgs fields and the stop and sbottom squarks. Earlier studies [10–12] of the neutral Higgs-boson mass spectrum were based on a number of particular assumptions and/or kinematic approximations. The present work goes well beyond those studies, and extends the most detailed analysis [11], in which the one-loop RG-improved effective potential was expanded up to renormalizable operators of dimension 4, assuming a moderate mass splitting among the stop squarks. This assumption seemed to impose a serious limitation, given that CP-violating effects exhibit an enhanced behaviour for large values of the stop-mixing parameter X_t . The results obtained using the present RG approach confirm, however, the qualitative phenomenological features found in [11], for the Higgs-boson masses and their couplings to fermions and to the W^\pm and Z bosons. It offers very accurate predictions, at the same level as the most accurate calculations in the CP-conserving case (implying an uncertainty of order 3 GeV) [20,42,43], for the Higgs-boson masses and for the whole of the MSSM parameter space in the presence of non-trivial CP-violating phases. More specifically, the present study also includes two-loop leading logarithms associated with QCD effects and t - and b -quark Yukawa couplings. It also contains all dominant two-loop non-logarithmic contributions to the one-loop effective potential, which are induced by one-loop threshold effects on the t - and b -quark Yukawa couplings due to the decoupling of the third-generation squarks (as considered in [42,43] in the CP-conserving limit). These one-loop threshold terms, $\delta h_{u,d}$ and $\Delta h_{u,d}$, strongly depend on the phase of the gluino mass and so introduce new CP violation into the MSSM effective potential at the two-loop level.

Large radiative effects of CP violation in the Higgs sector of the MSSM can have important phenomenological consequences on Higgs-boson searches at LEP 2, the Tevatron and the LHC. We have explicitly demonstrated that the radiatively-induced CP violation in the MSSM Higgs potential can lead to important effects of mass and coupling level crossing among the three neutral Higgs particles. These CP-violating effects of level crossing in the Higgs sector modify drastically the Higgs-boson couplings to the up- and down-type quarks, and to the W^\pm and Z bosons. In particular, CP violation in the lightest Higgs sector becomes relevant for a relatively light charged Higgs boson, with $M_{H^\pm} \lesssim 160$ GeV. For instance, for $M_{H^\pm} = 150$ GeV and $\tan\beta = 4$, even a neutral Higgs boson as light as 60 GeV may escape detection at LEP 2. However, the upgraded Tevatron may have the physics potential to explore such CP-violating scenarios at low $\tan\beta$ values, which may remain at the edge of accessibility even during the final LEP 2 run.

We have also studied the effects induced by a non-trivial CP-odd gluino phase, which enters the effective potential at the two-loop level. The presence of a gluino phase gives rise to small but non-negligible changes in the Higgs-boson mass spectrum and in the couplings

of the Higgs fields to the W^\pm and Z bosons. In this context, we have also found that the product of the scalar times the pseudoscalar coupling of the lightest Higgs boson H_1 to the bottom quarks has a very strong dependence on the gluino phase. This product of couplings gives a measure of CP violation in the $H_1 bb$ coupling, which can even be of order unity for relatively small charged Higgs-boson masses.

Finally, it is worth stressing that CP violation decouples from the lightest Higgs sector in the large-mass limit of a heavy charged Higgs boson. This decoupling property, which was known to hold for the CP-violating self-energy effect [10,11], has now been shown to be valid for the CP-violating vertex effects as well. As a result, the predictions for the lightest Higgs-boson mass and its couplings to gauge bosons in the above decoupling regime of the theory will be practically identical to the corresponding predictions in the CP-conserving case. However, unlike the lightest Higgs sector, CP violation does not decouple from the heaviest Higgs sector in the MSSM, opening up new possibilities for studying enhanced effects in CP-violating Higgs scalar-pseudoscalar transitions at the LHC or future muon colliders [50,51], where the heavy MSSM Higgs bosons can be resonantly produced.

In conclusion: the present analysis has shown that the MSSM with explicit radiative breaking of CP invariance constitutes a very rich theoretical framework, introducing new challenges in the search for fundamental Higgs scalars at LEP 2, the Tevatron and the LHC.

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Note Added

After completion of the work described here, we saw [57], which also computes the one-loop effective potential of the MSSM with explicit CP violation. Here we further *i)* perform the RG improvement of the effective potential, *ii)* develop a self-consistent treatment of the whole MSSM Higgs sector, taking into account the crucial one-loop relation between the charged Higgs-boson and neutral Higgs-boson mass matrices, and *iii)* include the dominant two-loop non-logarithmic corrections to the effective potential, which may also have an important impact on the $H_1 bb$ coupling. In the limits where a comparison between the results was possible, we find reasonable agreement with [57].

A Derivatives of Background-Field-Dependent Masses

In this Appendix, we list analytic expressions pertaining to derivatives of quark and squark masses with respect to their background Higgs fields. These derivative expressions are very useful, as they constitute the building blocks of the general one-loop effective potential presented in Section 2. We have divided the Appendix into three subsections. In the first subsection, we list the derivatives of quark masses with respect to Higgs fields, while the next two subsections contain the corresponding expressions for derivatives of squark masses with respect to neutral and charged Higgs fields, respectively.

A.1 Quark Derivatives

First, we give the derivatives related to non-vanishing tadpole contributions:

$$\frac{1}{v_2} \left\langle \frac{\partial \bar{m}_t^2}{\partial \phi_2} \right\rangle = h_t^2, \quad \frac{1}{v_1} \left\langle \frac{\partial \bar{m}_b^2}{\partial \phi_1} \right\rangle = h_b^2, \quad (\text{A.1})$$

where the operation $\langle \dots \rangle$ denotes that the above expressions should be evaluated in the ground state of the Higgs potential.

Then, the self-energy-type derivatives involving neutral and charged Higgs bosons may be listed as follows:

$$\begin{aligned} \left\langle \frac{\partial^2 \bar{m}_t^2}{\partial \phi_2^2} \right\rangle &= \left\langle \frac{\partial^2 \bar{m}_t^2}{\partial a_2^2} \right\rangle = h_t^2, & \left\langle \frac{\partial^2 \bar{m}_b^2}{\partial \phi_1^2} \right\rangle &= \left\langle \frac{\partial^2 \bar{m}_b^2}{\partial a_1^2} \right\rangle = h_b^2, \\ \left\langle \frac{\partial^2 \bar{m}_t^2}{\partial \phi_1^+ \partial \phi_1^-} \right\rangle &= \frac{h_b^2 m_t^2}{m_t^2 - m_b^2}, & \left\langle \frac{\partial^2 \bar{m}_b^2}{\partial \phi_1^+ \partial \phi_1^-} \right\rangle &= \frac{h_b^2 m_b^2}{m_b^2 - m_t^2}, \\ \left\langle \frac{\partial^2 \bar{m}_t^2}{\partial \phi_2^+ \partial \phi_2^-} \right\rangle &= \frac{h_t^2 m_t^2}{m_t^2 - m_b^2}, & \left\langle \frac{\partial^2 \bar{m}_b^2}{\partial \phi_2^+ \partial \phi_2^-} \right\rangle &= \frac{h_t^2 m_b^2}{m_b^2 - m_t^2}, \\ \left\langle \frac{\partial^2 \bar{m}_t^2}{\partial \phi_1^+ \partial \phi_2^-} \right\rangle &= \left\langle \frac{\partial^2 \bar{m}_t^2}{\partial \phi_2^+ \partial \phi_1^-} \right\rangle = - \left\langle \frac{\partial^2 \bar{m}_b^2}{\partial \phi_1^+ \partial \phi_2^-} \right\rangle = - \left\langle \frac{\partial^2 \bar{m}_b^2}{\partial \phi_2^+ \partial \phi_1^-} \right\rangle = \frac{h_t h_b m_t m_b}{m_b^2 - m_t^2}. \end{aligned} \quad (\text{A.2})$$

Note that we have not listed derivatives that vanish.

A.2 Derivatives of Squark Masses with Respect to Neutral Higgs Fields

This section contains the derivatives of the field-dependent squark masses $\widetilde{m}_{t_{1,2}}^2$ and $\widetilde{m}_{b_{1,2}}^2$ with respect to neutral Higgs fields $\phi_{1,2}$ and $a_{1,2}$. We first give the tadpole terms:

$$\begin{aligned}
\frac{1}{v_1} \left\langle \frac{\partial \widetilde{m}_{t_1(t_2)}^2}{\partial \phi_1} \right\rangle &= \frac{g_w^2 + g'^2}{8} + (-) \frac{1}{2(m_{t_1}^2 - m_{t_2}^2)} \left[x_t (\widetilde{M}_Q^2 - \widetilde{M}_t^2 + \frac{1}{2} x_t v^2 \cos 2\beta) \right. \\
&\quad \left. - 2h_t^2 (\text{Re}(\mu A_t) \tan \beta - |\mu|^2) \right], \\
\frac{1}{v_1} \left\langle \frac{\partial \widetilde{m}_{b_1(b_2)}^2}{\partial \phi_1} \right\rangle &= h_b^2 - \frac{g_w^2 + g'^2}{8} - (+) \frac{1}{2(m_{b_1}^2 - m_{b_2}^2)} \left[x_b (\widetilde{M}_Q^2 - \widetilde{M}_b^2 - \frac{1}{2} x_b v^2 \cos 2\beta) \right. \\
&\quad \left. + 2h_b^2 (\text{Re}(\mu A_b) \tan \beta - |A_b|^2) \right], \\
\frac{1}{v_2} \left\langle \frac{\partial \widetilde{m}_{t_1(t_2)}^2}{\partial \phi_2} \right\rangle &= h_t^2 - \frac{g_w^2 + g'^2}{8} - (+) \frac{1}{2(m_{t_1}^2 - m_{t_2}^2)} \left[x_t (\widetilde{M}_Q^2 - \widetilde{M}_t^2 + \frac{1}{2} x_t v^2 \cos 2\beta) \right. \\
&\quad \left. + 2h_t^2 (\text{Re}(\mu A_t) \cot \beta - |A_t|^2) \right], \\
\frac{1}{v_2} \left\langle \frac{\partial \widetilde{m}_{b_1(b_2)}^2}{\partial \phi_2} \right\rangle &= \frac{g_w^2 + g'^2}{8} + (-) \frac{1}{2(m_{b_1}^2 - m_{b_2}^2)} \left[x_b (\widetilde{M}_Q^2 - \widetilde{M}_b^2 - \frac{1}{2} x_b v^2 \cos 2\beta) \right. \\
&\quad \left. - 2h_b^2 (\text{Re}(\mu A_b) \cot \beta - |\mu|^2) \right], \\
\frac{1}{v_2} \left\langle \frac{\partial \widetilde{m}_{t_1(t_2)}^2}{\partial a_1} \right\rangle &= -\frac{1}{v_1} \left\langle \frac{\partial \widetilde{m}_{t_1(t_2)}^2}{\partial a_2} \right\rangle = -(+) \frac{h_t^2 \text{Im}(\mu A_t)}{m_{t_1}^2 - m_{t_2}^2}, \\
\frac{1}{v_2} \left\langle \frac{\partial \widetilde{m}_{b_1(b_2)}^2}{\partial a_1} \right\rangle &= -\frac{1}{v_1} \left\langle \frac{\partial \widetilde{m}_{b_1(b_2)}^2}{\partial a_2} \right\rangle = -(+) \frac{h_b^2 \text{Im}(\mu A_b)}{m_{b_1}^2 - m_{b_2}^2}, \tag{A.3}
\end{aligned}$$

where the coupling parameters x_t and x_b are defined after (2.13). Then, the self-energy-type terms $\langle \partial^2 \widetilde{m}_{q_k}^2 / \partial a_i \partial a_j \rangle$, with $q_k = t_1, b_1, t_2, b_2$ and $i, j = 1, 2$, are found to be

$$\begin{aligned}
\left\langle \frac{\partial^2 \widetilde{m}_{t_1(t_2)}^2}{\partial a_1^2} \right\rangle &= \frac{g_w^2 + g'^2}{8} + (-) \frac{1}{2(m_{t_1}^2 - m_{t_2}^2)} \left[x_t (\widetilde{M}_Q^2 - \widetilde{M}_t^2 + \frac{1}{2} x_t v^2 \cos 2\beta) + 2h_t^2 |\mu|^2 \right], \\
\left\langle \frac{\partial^2 \widetilde{m}_{b_1(b_2)}^2}{\partial a_1^2} \right\rangle &= h_b^2 - \frac{g_w^2 + g'^2}{8} - (+) \frac{1}{2(m_{b_1}^2 - m_{b_2}^2)} \left[x_b (\widetilde{M}_Q^2 - \widetilde{M}_b^2 - \frac{1}{2} x_b v^2 \cos 2\beta) \right. \\
&\quad \left. - 2h_b^2 |A_b|^2 \right], \\
\left\langle \frac{\partial^2 \widetilde{m}_{t_1(t_2)}^2}{\partial a_2^2} \right\rangle &= h_t^2 - \frac{g_w^2 + g'^2}{8} - (+) \frac{1}{2(m_{t_1}^2 - m_{t_2}^2)} \left[x_t (\widetilde{M}_Q^2 - \widetilde{M}_t^2 + \frac{1}{2} x_t v^2 \cos 2\beta) \right. \\
&\quad \left. - 2h_t^2 |A_t|^2 \right], \\
\left\langle \frac{\partial^2 \widetilde{m}_{b_1(b_2)}^2}{\partial a_2^2} \right\rangle &= \frac{g_w^2 + g'^2}{8} + (-) \frac{1}{2(m_{b_1}^2 - m_{b_2}^2)} \left[x_b (\widetilde{M}_Q^2 - \widetilde{M}_b^2 - \frac{1}{2} x_b v^2 \cos 2\beta) + 2h_b^2 |\mu|^2 \right],
\end{aligned}$$

$$\begin{aligned}
\left\langle \frac{\partial^2 \widetilde{m}_{t_1(t_2)}^2}{\partial a_1 \partial a_2} \right\rangle &= -(+) \frac{h_t^2 \operatorname{Re}(\mu A_t)}{m_{t_1}^2 - m_{t_2}^2} + (-) \frac{2v_1 v_2 h_t^4 \operatorname{Im}^2(\mu A_t)}{(m_{t_1}^2 - m_{t_2}^2)^3}, \\
\left\langle \frac{\partial^2 \widetilde{m}_{b_1(b_2)}^2}{\partial a_1 \partial a_2} \right\rangle &= -(+) \frac{h_b^2 \operatorname{Re}(\mu A_b)}{m_{b_1}^2 - m_{b_2}^2} + (-) \frac{2v_1 v_2 h_b^4 \operatorname{Im}^2(\mu A_b)}{(m_{b_1}^2 - m_{b_2}^2)^3}.
\end{aligned} \tag{A.4}$$

In addition, the non-vanishing CP-violating self-energy terms $\langle \partial^2 \widetilde{m}_{q_k}^2 / \partial \phi_i \partial a_j \rangle$ are given by

$$\begin{aligned}
\left\langle \frac{\partial^2 \widetilde{m}_{t_1(t_2)}^2}{\partial \phi_1 \partial a_2} \right\rangle &= +(-) \frac{h_t^2 \operatorname{Im}(\mu A_t)}{m_{t_1}^2 - m_{t_2}^2} \left\{ 1 - \frac{v_1^2}{(m_{t_1}^2 - m_{t_2}^2)^2} \left[x_t (\widetilde{M}_Q^2 - \widetilde{M}_t^2 + \frac{1}{2} x_t v^2 \cos 2\beta) \right. \right. \\
&\quad \left. \left. - 2h_t^2 (\operatorname{Re}(\mu A_t) \tan \beta - |\mu|^2) \right] \right\}, \\
\left\langle \frac{\partial^2 \widetilde{m}_{b_1(b_2)}^2}{\partial \phi_1 \partial a_2} \right\rangle &= +(-) \frac{h_b^2 \operatorname{Im}(\mu A_b)}{m_{b_1}^2 - m_{b_2}^2} \left\{ 1 + \frac{v_1^2}{(m_{b_1}^2 - m_{b_2}^2)^2} \left[x_b (\widetilde{M}_Q^2 - \widetilde{M}_b^2 - \frac{1}{2} x_b v^2 \cos 2\beta) \right. \right. \\
&\quad \left. \left. + 2h_b^2 (\operatorname{Re}(\mu A_b) \tan \beta - |A_b|^2) \right] \right\}, \\
\left\langle \frac{\partial^2 \widetilde{m}_{t_1(t_2)}^2}{\partial \phi_2 \partial a_1} \right\rangle &= -(+) \frac{h_t^2 \operatorname{Im}(\mu A_t)}{m_{t_1}^2 - m_{t_2}^2} \left\{ 1 + \frac{v_2^2}{(m_{t_1}^2 - m_{t_2}^2)^2} \left[x_t (\widetilde{M}_Q^2 - \widetilde{M}_t^2 + \frac{1}{2} x_t v^2 \cos 2\beta) \right. \right. \\
&\quad \left. \left. + 2h_t^2 (\operatorname{Re}(\mu A_t) \cot \beta - |A_t|^2) \right] \right\}, \\
\left\langle \frac{\partial^2 \widetilde{m}_{b_1(b_2)}^2}{\partial \phi_2 \partial a_1} \right\rangle &= -(+) \frac{h_b^2 \operatorname{Im}(\mu A_b)}{m_{b_1}^2 - m_{b_2}^2} \left\{ 1 - \frac{v_2^2}{(m_{b_1}^2 - m_{b_2}^2)^2} \left[x_b (\widetilde{M}_Q^2 - \widetilde{M}_b^2 - \frac{1}{2} x_b v^2 \cos 2\beta) \right. \right. \\
&\quad \left. \left. - 2h_b^2 (\operatorname{Re}(\mu A_b) \cot \beta - |\mu|^2) \right] \right\}.
\end{aligned} \tag{A.5}$$

Finally, the CP-conserving self-energy-type derivatives $\langle \partial^2 \widetilde{m}_{q_k}^2 / \partial \phi_i \partial \phi_j \rangle$ have been calculated to be

$$\begin{aligned}
\left\langle \frac{\partial^2 \widetilde{m}_{t_1(t_2)}^2}{\partial \phi_1^2} \right\rangle &= \frac{g_w^2 + g'^2}{8} + (-) \frac{1}{2(m_{t_1}^2 - m_{t_2}^2)} \left[x_t (\widetilde{M}_Q^2 - \widetilde{M}_t^2 + \frac{1}{2} x_t v^2 (1 + 2 \cos 2\beta)) \right. \\
&\quad \left. + 2h_t^2 |\mu|^2 \right] - (+) \frac{v_1^2}{2(m_{t_1}^2 - m_{t_2}^2)^3} \left[x_t (\widetilde{M}_Q^2 - \widetilde{M}_t^2 + \frac{1}{2} x_t v^2 \cos 2\beta) \right. \\
&\quad \left. - 2h_t^2 (\operatorname{Re}(\mu A_t) \tan \beta - |\mu|^2) \right]^2, \\
\left\langle \frac{\partial^2 \widetilde{m}_{b_1(b_2)}^2}{\partial \phi_1^2} \right\rangle &= h_b^2 - \frac{g_w^2 + g'^2}{8} - (+) \frac{1}{2(m_{b_1}^2 - m_{b_2}^2)} \left[x_b (\widetilde{M}_Q^2 - \widetilde{M}_b^2 - \frac{1}{2} x_b v^2 (1 + 2 \cos 2\beta)) \right. \\
&\quad \left. - 2h_b^2 |A_b|^2 \right] - (+) \frac{v_1^2}{2(m_{b_1}^2 - m_{b_2}^2)^3} \left[x_b (\widetilde{M}_Q^2 - \widetilde{M}_b^2 - \frac{1}{2} x_b v^2 \cos 2\beta) \right. \\
&\quad \left. + 2h_b^2 (\operatorname{Re}(\mu A_b) \tan \beta - |A_b|^2) \right]^2, \\
\left\langle \frac{\partial^2 \widetilde{m}_{t_1(t_2)}^2}{\partial \phi_2^2} \right\rangle &= h_t^2 - \frac{g_w^2 + g'^2}{8} - (+) \frac{1}{2(m_{t_1}^2 - m_{t_2}^2)} \left[x_t (\widetilde{M}_Q^2 - \widetilde{M}_t^2 + \frac{1}{2} x_t v^2 (2 \cos 2\beta - 1)) \right.
\end{aligned}$$

$$\begin{aligned}
& -2h_t^2|A_t|^2] - (+) \frac{v_2^2}{2(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2)^3} \left[x_t (\tilde{M}_Q^2 - \tilde{M}_t^2 + \frac{1}{2}x_tv^2 \cos 2\beta) \right. \\
& \left. + 2h_t^2 (\text{Re}(\mu A_t) \cot \beta - |A_t|^2) \right]^2, \\
\left\langle \frac{\partial^2 \tilde{m}_{b_1(b_2)}^2}{\partial \phi_2^2} \right\rangle &= \frac{g_w^2 + g'^2}{8} + (-) \frac{1}{2(m_{\tilde{b}_1}^2 - m_{\tilde{b}_2}^2)} \left[x_b (\tilde{M}_Q^2 - \tilde{M}_b^2 - \frac{1}{2}x_bv^2(2 \cos 2\beta - 1)) \right. \\
& \left. + 2h_b^2|\mu|^2 \right] - (+) \frac{v_2^2}{2(m_{\tilde{b}_1}^2 - m_{\tilde{b}_2}^2)^3} \left[x_b (\tilde{M}_Q^2 - \tilde{M}_b^2 - \frac{1}{2}x_bv^2 \cos 2\beta) \right. \\
& \left. - 2h_b^2 (\text{Re}(\mu A_b) \cot \beta - |\mu|^2) \right]^2, \\
\left\langle \frac{\partial^2 \tilde{m}_{t_1(t_2)}^2}{\partial \phi_1 \partial \phi_2} \right\rangle &= -(+) \frac{1}{2(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2)} \left[\frac{1}{2}x_t^2v^2 \sin 2\beta + 2h_t^2 \text{Re}(\mu A_t) \right] + (-) \frac{v_1v_2}{2(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2)^3} \\
& \times \left[x_t (\tilde{M}_Q^2 - \tilde{M}_t^2 + \frac{1}{2}x_tv^2 \cos 2\beta) - 2h_t^2 (\text{Re}(\mu A_t) \tan \beta - |\mu|^2) \right] \\
& \times \left[x_t (\tilde{M}_Q^2 - \tilde{M}_t^2 + \frac{1}{2}x_tv^2 \cos 2\beta) + 2h_t^2 (\text{Re}(\mu A_t) \cot \beta - |A_t|^2) \right], \\
\left\langle \frac{\partial^2 \tilde{m}_{b_1(b_2)}^2}{\partial \phi_1 \partial \phi_2} \right\rangle &= -(+) \frac{1}{2(m_{\tilde{b}_1}^2 - m_{\tilde{b}_2}^2)} \left[\frac{1}{2}x_b^2v^2 \sin 2\beta + 2h_b^2 \text{Re}(\mu A_b) \right] + (-) \frac{v_1v_2}{2(m_{\tilde{b}_1}^2 - m_{\tilde{b}_2}^2)^3} \\
& \times \left[x_b (\tilde{M}_Q^2 - \tilde{M}_b^2 - \frac{1}{2}x_bv^2 \cos 2\beta) + 2h_b^2 (\text{Re}(\mu A_b) \tan \beta - |A_b|^2) \right] \\
& \times \left[x_b (\tilde{M}_Q^2 - \tilde{M}_b^2 - \frac{1}{2}x_bv^2 \cos 2\beta) - 2h_b^2 (\text{Re}(\mu A_b) \cot \beta - |\mu|^2) \right]. \quad (\text{A.6})
\end{aligned}$$

A.3 Derivatives of Squark Masses with Respect to Charged Higgs Fields

Here we evaluate the derivatives of the field-dependent squark masses with respect to charged Higgs fields. To calculate the expressions $\langle \partial^2 \tilde{m}_{q_k}^2 / \partial \phi_i^+ \partial \phi_j^- \rangle$ directly turns out to be a formidable task. The reason is that $\tilde{m}_{q_k}^2$ are the eigenvalues of a non-trivial (4×4) squark mass matrix $\tilde{\mathcal{M}}^2$ (cf. (2.11)), and their analytic form is very complicated. Therefore, we proceed differently, using a mathematical trick which was first applied in [37].

First, we notice that $\langle \partial \tilde{m}_{q_k}^2 / \partial \phi_i^\pm \rangle = 0$, as a consequence of the fact that the true ground state of the effective potential should conserve charge. Then, one may make use of the eigenvalue equation:

$$\det(\tilde{\mathcal{M}}^2 - \tilde{m}_{q_k}^2 \mathbf{1}_4) = \tilde{m}_{q_k}^8 + A\tilde{m}_{q_k}^6 + B\tilde{m}_{q_k}^4 + C\tilde{m}_{q_k}^2 + D = 0, \quad (\text{A.7})$$

with

$$\begin{aligned}
A &= -\text{Tr} \tilde{\mathcal{M}}^2, \\
B &= \frac{1}{2} (\text{Tr}^2 \tilde{\mathcal{M}}^2 - \text{Tr} \tilde{\mathcal{M}}^4),
\end{aligned}$$

$$\begin{aligned}
C &= \frac{1}{3} \left(\text{Tr}^3 \widetilde{\mathcal{M}}^2 - \text{Tr} \widetilde{\mathcal{M}}^6 \right) - \frac{1}{2} \text{Tr} \widetilde{\mathcal{M}}^2 \left(\text{Tr}^2 \widetilde{\mathcal{M}}^2 - \text{Tr} \widetilde{\mathcal{M}}^4 \right), \\
D &= \det \widetilde{\mathcal{M}}^2 = -\frac{1}{4} \left(\text{Tr} \widetilde{\mathcal{M}}^8 + A \text{Tr} \widetilde{\mathcal{M}}^6 + B \text{Tr} \widetilde{\mathcal{M}}^4 + C \text{Tr} \widetilde{\mathcal{M}}^2 \right), \quad (\text{A.8})
\end{aligned}$$

to obtain

$$\left\langle \frac{\partial^2 \widetilde{m}_{q_k}^2}{\partial \phi_i^+ \partial \phi_j^-} \right\rangle = - \left\langle \frac{A_{ij} \widetilde{m}_{q_k}^6 + B_{ij} \widetilde{m}_{q_k}^4 + C_{ij} \widetilde{m}_{q_k}^2 + D_{ij}}{\prod_{q_l \neq q_k} (\widetilde{m}_{q_k}^2 - \widetilde{m}_{q_l}^2)} \right\rangle, \quad (\text{A.9})$$

with $q_l, q_k = t_1, b_1, t_2, b_2$, and $A_{ij} = \partial^2 A / \partial \phi_i^+ \partial \phi_j^-$, $B_{ij} = \partial^2 B / \partial \phi_i^+ \partial \phi_j^-$, etc.. For our purposes, it is sufficient to calculate the derivatives with respect to ϕ_1^+ and ϕ_2^- . In particular, it proves convenient to use a representation in which the columns and rows ‘2’ and ‘3’ of the (4×4) matrix $\widetilde{\mathcal{M}}^2$ have been interchanged. With such a reordering, we find

$$\langle \widetilde{\mathcal{M}}^2 \rangle \rightarrow \begin{pmatrix} \widetilde{\mathcal{M}}_t^2 & 0 \\ 0 & \widetilde{\mathcal{M}}_b^2 \end{pmatrix}, \quad \left\langle \frac{\partial \widetilde{\mathcal{M}}^2}{\partial \phi_1^+} \right\rangle \rightarrow \begin{pmatrix} 0 & \widetilde{\mathcal{M}}_+ \\ 0 & 0 \end{pmatrix}, \quad \left\langle \frac{\partial \widetilde{\mathcal{M}}^2}{\partial \phi_2^-} \right\rangle \rightarrow \begin{pmatrix} 0 & 0 \\ \widetilde{\mathcal{M}}_- & 0 \end{pmatrix}, \quad (\text{A.10})$$

where $\widetilde{\mathcal{M}}_t^2$ and $\widetilde{\mathcal{M}}_b^2$ are the usual \tilde{t} - and \tilde{b} - (2×2)-mass matrices, respectively, and

$$\widetilde{\mathcal{M}}_+ = \begin{pmatrix} \frac{1}{\sqrt{2}} (h_b^2 - \frac{1}{2} g_w^2) v_1 & h_b A_b^* \\ h_t \mu^* & \frac{1}{\sqrt{2}} h_t h_b v_2 \end{pmatrix}, \quad \widetilde{\mathcal{M}}_- = - \begin{pmatrix} \frac{1}{\sqrt{2}} (h_t^2 - \frac{1}{2} g_w^2) v_2 & h_t A_t^* \\ h_b \mu^* & \frac{1}{\sqrt{2}} h_t h_b v_1 \end{pmatrix}. \quad (\text{A.11})$$

Then, the relevant coefficients A_{12} , B_{12} , C_{12} and D_{12} may be expressed in a compact form as follows:

$$\begin{aligned}
\langle A_{12} \rangle &= 0, \\
\langle B_{12} \rangle &= -\text{Tr} (\widetilde{\mathcal{M}}_+ \widetilde{\mathcal{M}}_-), \\
\langle C_{12} \rangle &= \left(\text{Tr} \widetilde{\mathcal{M}}_t^2 + \text{Tr} \widetilde{\mathcal{M}}_b^2 \right) \text{Tr} (\widetilde{\mathcal{M}}_+ \widetilde{\mathcal{M}}_-) - \text{Tr} (\widetilde{\mathcal{M}}_t^2 \widetilde{\mathcal{M}}_+ \widetilde{\mathcal{M}}_-) - \text{Tr} (\widetilde{\mathcal{M}}_b^2 \widetilde{\mathcal{M}}_- \widetilde{\mathcal{M}}_+), \\
\langle D_{12} \rangle &= -\text{Tr} (\widetilde{\mathcal{M}}_t^2 \widetilde{\mathcal{M}}_+ \widetilde{\mathcal{M}}_b^2 \widetilde{\mathcal{M}}_-) - \text{Tr} (\widetilde{\mathcal{M}}_t^4 \widetilde{\mathcal{M}}_+ \widetilde{\mathcal{M}}_-) - \text{Tr} (\widetilde{\mathcal{M}}_b^4 \widetilde{\mathcal{M}}_- \widetilde{\mathcal{M}}_+) \\
&\quad - \left(\text{Tr} \widetilde{\mathcal{M}}_t^2 + \text{Tr} \widetilde{\mathcal{M}}_b^2 \right) \left[\text{Tr} (\widetilde{\mathcal{M}}_t^2 \widetilde{\mathcal{M}}_+ \widetilde{\mathcal{M}}_-) + \text{Tr} (\widetilde{\mathcal{M}}_b^2 \widetilde{\mathcal{M}}_- \widetilde{\mathcal{M}}_+) \right] \\
&\quad + \frac{1}{2} \left[\text{Tr} \widetilde{\mathcal{M}}_t^4 + \text{Tr} \widetilde{\mathcal{M}}_b^4 - \left(\text{Tr} \widetilde{\mathcal{M}}_t^2 + \text{Tr} \widetilde{\mathcal{M}}_b^2 \right)^2 \right] \text{Tr} (\widetilde{\mathcal{M}}_+ \widetilde{\mathcal{M}}_-). \quad (\text{A.12})
\end{aligned}$$

With the help of (A.12), it is straightforward to obtain the derivatives $\partial^2 \widetilde{m}_{q_k}^2 / \partial \phi_1^+ \partial \phi_2^-$. More explicitly, we have

$$\begin{aligned}
\left\langle \frac{\partial^2 \widetilde{m}_{t_1}^2}{\partial \phi_1^+ \partial \phi_2^-} \right\rangle &= - \frac{\langle B_{12} \rangle m_{t_1}^4 + \langle C_{12} \rangle m_{t_1}^2 + \langle D_{12} \rangle}{(m_{t_1}^2 - m_{b_1}^2) (m_{t_1}^2 - m_{t_2}^2) (m_{t_1}^2 - m_{b_2}^2)}, \\
\left\langle \frac{\partial^2 \widetilde{m}_{t_2}^2}{\partial \phi_1^+ \partial \phi_2^-} \right\rangle &= - \frac{\langle B_{12} \rangle m_{t_2}^4 + \langle C_{12} \rangle m_{t_2}^2 + \langle D_{12} \rangle}{(m_{t_2}^2 - m_{b_1}^2) (m_{t_2}^2 - m_{t_1}^2) (m_{t_2}^2 - m_{b_2}^2)}, \\
\left\langle \frac{\partial^2 \widetilde{m}_{b_1}^2}{\partial \phi_1^+ \partial \phi_2^-} \right\rangle &= - \frac{\langle B_{12} \rangle m_{b_1}^4 + \langle C_{12} \rangle m_{b_1}^2 + \langle D_{12} \rangle}{(m_{b_1}^2 - m_{t_1}^2) (m_{b_1}^2 - m_{t_2}^2) (m_{b_1}^2 - m_{b_2}^2)},
\end{aligned}$$

$$\left\langle \frac{\partial^2 \widetilde{m}_{b_2}^2}{\partial \phi_1^+ \partial \phi_2^-} \right\rangle = - \frac{\langle B_{12} \rangle m_{b_2}^4 + \langle C_{12} \rangle m_{b_2}^2 + \langle D_{12} \rangle}{(m_{b_2}^2 - m_{b_1}^2)(m_{b_2}^2 - m_{t_2}^2)(m_{b_2}^2 - m_{t_1}^2)}. \quad (\text{A.13})$$

B Higgs-Boson Masses and Mixing Angles

Here we present analytic expressions for the Higgs-boson masses $M_{H_i}(m_t)$ ($i = 1, 2, 3$) and the corresponding (3×3) orthogonal matrix O , after diagonalizing the RG-improved Higgs-boson mass matrix $(\mathcal{M}_N^2)(m_t)$.

For notational simplicity, we do not display explicitly the functional dependence of \mathcal{M}_N^2 on m_t . The mass eigenvalues of the (3×3) matrix \mathcal{M}_N^2 are then obtained by solving the characteristic equation of cubic order:

$$x^3 + rx^2 + sx + t = 0, \quad (\text{B.1})$$

with

$$\begin{aligned} r &= -\text{Tr}(\mathcal{M}_N^2), \\ s &= \frac{1}{2} \left[\text{Tr}^2(\mathcal{M}_N^2) - \text{Tr}(\mathcal{M}_N^4) \right], \\ t &= -\det(\mathcal{M}_N^2). \end{aligned} \quad (\text{B.2})$$

To this end, it proves useful to define the following auxiliary parameters:

$$\begin{aligned} p &= \frac{3s - r^2}{3}, \\ q &= \frac{2r^3}{27} - \frac{rs}{3} + t, \\ D &= \frac{p^3}{27} + \frac{q^2}{4}. \end{aligned} \quad (\text{B.3})$$

To ensure that the three eigenvalues are positive, it is necessary and sufficient to require that

$$D < 0, \quad r < 0, \quad s > 0, \quad t < 0. \quad (\text{B.4})$$

Imposing these inequalities on the kinematic parameters of the theory, we may express the three mass eigenvalues of \mathcal{M}_N^2 as

$$\begin{aligned} M_{H_1}^2 &= -\frac{1}{3}r + 2\sqrt{-p/3} \cos\left(\frac{\varphi}{3} + \frac{2\pi}{3}\right), \\ M_{H_2}^2 &= -\frac{1}{3}r + 2\sqrt{-p/3} \cos\left(\frac{\varphi}{3} - \frac{2\pi}{3}\right), \\ M_{H_3}^2 &= -\frac{1}{3}r + 2\sqrt{-p/3} \cos\left(\frac{\varphi}{3}\right), \end{aligned} \quad (\text{B.5})$$

with

$$\varphi = \arccos\left(-\frac{q}{2\sqrt{-p^3/27}}\right) \quad \text{and} \quad 0 \leq \varphi \leq \pi. \quad (\text{B.6})$$

Since the Higgs-boson mass matrix \mathcal{M}_N^2 is symmetric, we can diagonalize it by means of an orthogonal rotation O as stated in (3.30). Furthermore, one can show [58] that the Higgs-boson mass eigenvalues in (B.5) satisfy the desired mass hierarchy in accordance with the inequality of (3.31).

If M_{ij}^2 , with $i, j = 1, 2, 3$, denote the matrix elements of \mathcal{M}_N^2 , the elements O_{ij} can then be obtained by appropriately solving the underdetermined coupled system of equations, $\sum_k M_{ik}^2 O_{kj} = M_{H_j}^2 O_{ij}$:

$$\begin{aligned} (M_{11}^2 - M_{H_1}^2)O_{1i} + M_{12}^2 O_{2i} + M_{13}^2 O_{3i} &= 0, \\ M_{21}^2 O_{1i} + (M_{22}^2 - M_{H_2}^2)O_{2i} + M_{23}^2 O_{3i} &= 0, \\ M_{31}^2 O_{1i} + M_{32}^2 O_{2i} + (M_{33}^2 - M_{H_3}^2)O_{3i} &= 0. \end{aligned} \quad (\text{B.7})$$

More explicitly, we have

$$O = \begin{pmatrix} |x_1|/\Delta_1 & x_2/\Delta_2 & x_3/\Delta_3 \\ y_1/\Delta_1 & |y_2|/\Delta_2 & y_3/\Delta_3 \\ z_1/\Delta_1 & z_2/\Delta_2 & |z_3|/\Delta_3 \end{pmatrix}, \quad (\text{B.8})$$

where

$$\Delta_i = \sqrt{x_i^2 + y_i^2 + z_i^2} \quad (\text{B.9})$$

and

$$\begin{aligned} |x_1| &= \left\| \begin{array}{cc} M_{22}^2 - M_{H_1}^2 & M_{23}^2 \\ M_{32}^2 & M_{33}^2 - M_{H_1}^2 \end{array} \right\|, \quad y_1 = s_{x_1} \left| \begin{array}{cc} M_{23}^2 & M_{21}^2 \\ M_{33}^2 - M_{H_1}^2 & M_{31}^2 \end{array} \right|, \quad z_1 = s_{x_1} \left| \begin{array}{cc} M_{21}^2 & M_{22}^2 - M_{H_1}^2 \\ M_{31}^2 & M_{32}^2 \end{array} \right|, \\ x_2 &= s_{y_2} \left| \begin{array}{cc} M_{13}^2 & M_{12}^2 \\ M_{33}^2 - M_{H_2}^2 & M_{32}^2 \end{array} \right|, \quad |y_2| = \left\| \begin{array}{cc} M_{11}^2 - M_{H_2}^2 & M_{13}^2 \\ M_{31}^2 & M_{33}^2 - M_{H_2}^2 \end{array} \right\|, \quad z_2 = s_{y_2} \left| \begin{array}{cc} M_{12}^2 & M_{11}^2 - M_{H_2}^2 \\ M_{32}^2 & M_{31}^2 \end{array} \right|, \\ x_3 &= s_{z_3} \left| \begin{array}{cc} M_{12}^2 & M_{13}^2 \\ M_{22}^2 - M_{H_3}^2 & M_{23}^2 \end{array} \right|, \quad y_3 = s_{z_3} \left| \begin{array}{cc} M_{13}^2 & M_{11}^2 - M_{H_3}^2 \\ M_{23}^2 & M_{21}^2 \end{array} \right|, \quad |z_3| = \left\| \begin{array}{cc} M_{11}^2 - M_{H_3}^2 & M_{12}^2 \\ M_{21}^2 & M_{22}^2 - M_{H_3}^2 \end{array} \right\|. \end{aligned} \quad (\text{B.10})$$

In (B.10), the abbreviation $s_x \equiv \text{sign}(x)$ is an operation that simply gives the sign of a real expression x .

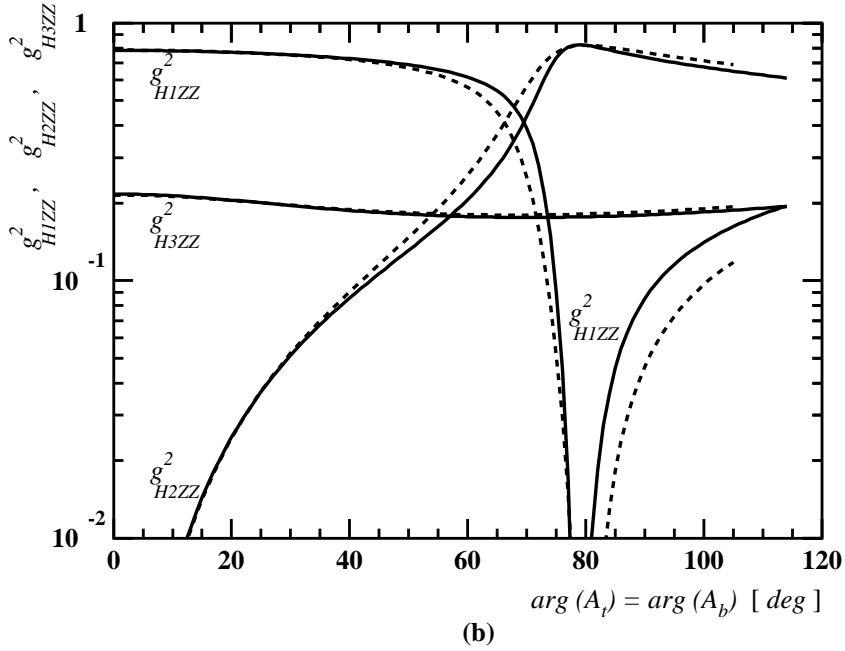
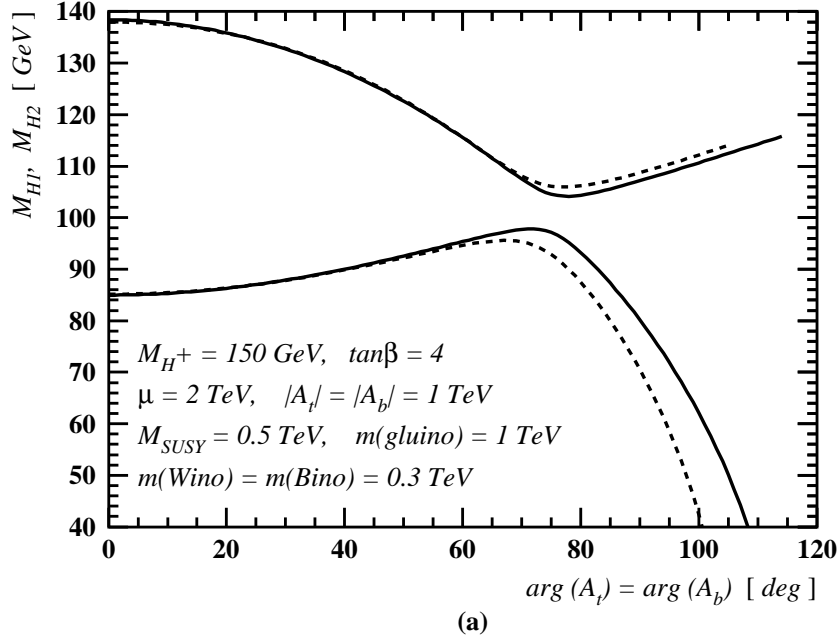
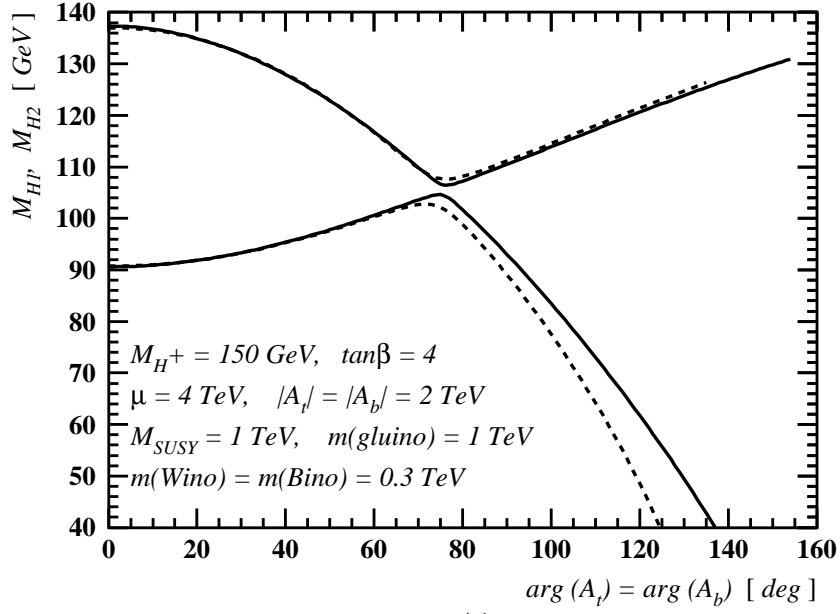
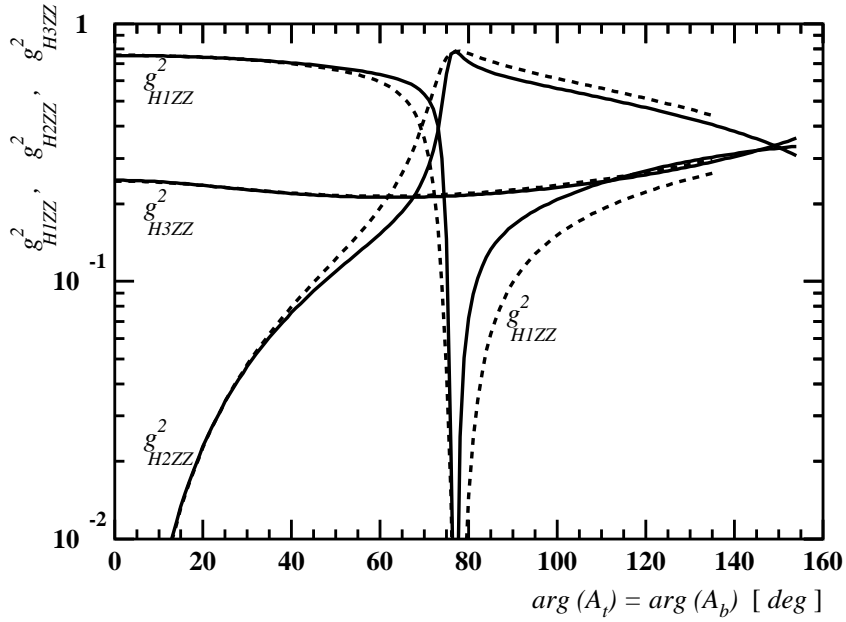


Figure 1: Numerical estimates of (a) M_{H_1} and M_{H_2} and (b) $g_{H_iZZ}^2$ as functions of $\arg(A_t)$, for the indicated choices of MSSM parameters. Solid lines correspond to $\arg(m_{\tilde{g}}) = 0$, dashed lines to $\arg(m_{\tilde{g}}) = 90^\circ$.



(a)



(b)

Figure 2: As Fig. 1, but with $\mu = 4 \text{ TeV}$, $|A_t| = |A_b| = 2 \text{ TeV}$ and $M_{SUSY} = 1 \text{ TeV}$.

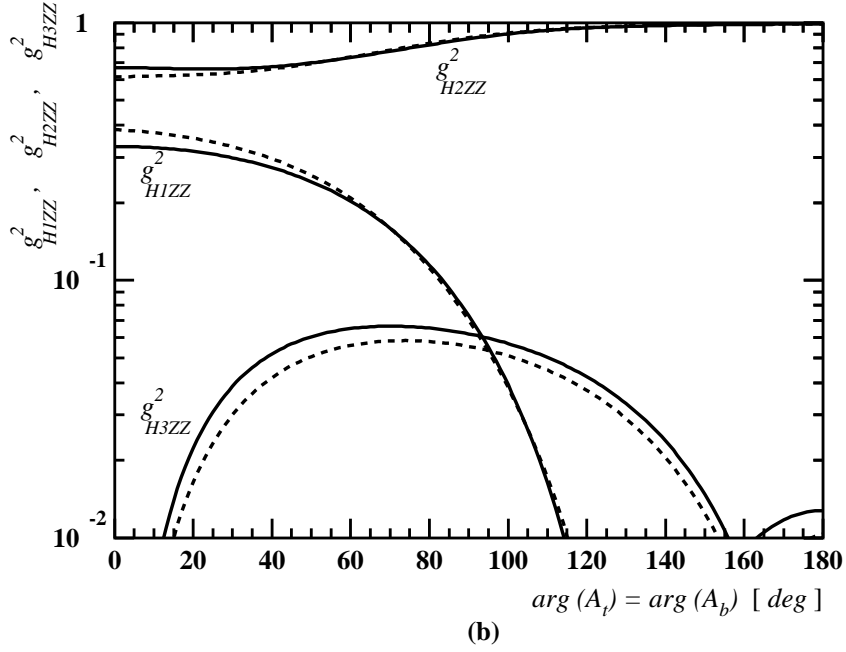
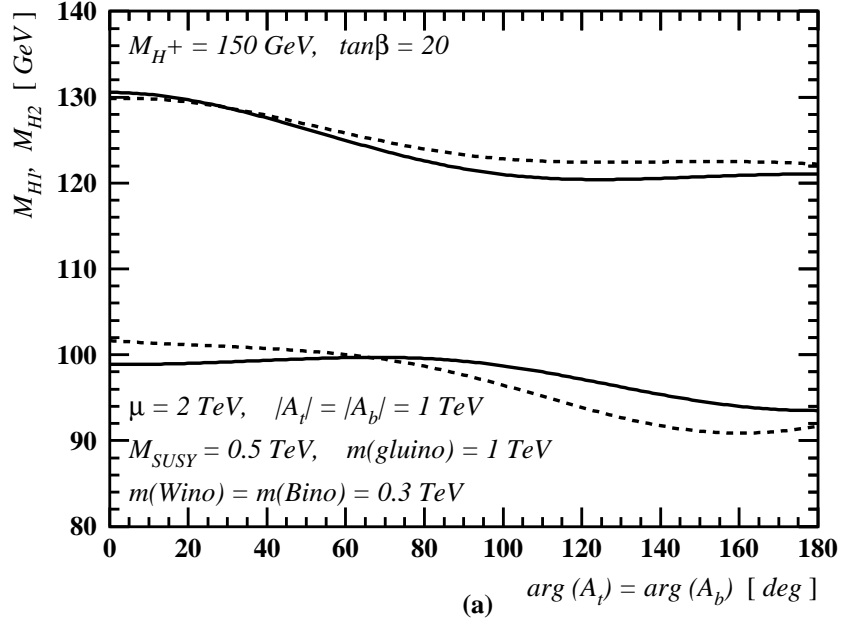


Figure 3: As in Fig. 1, but with $\tan\beta = 20$.

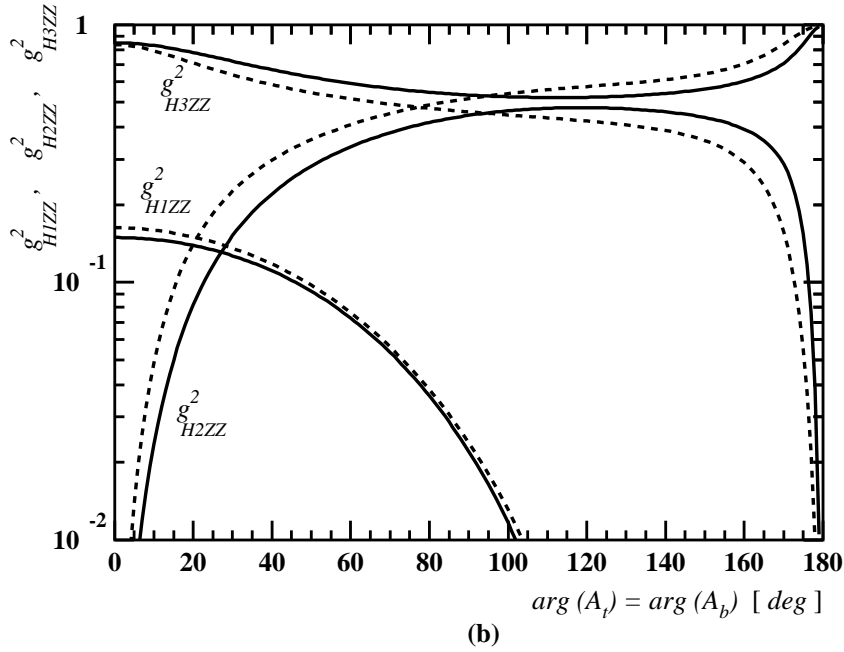
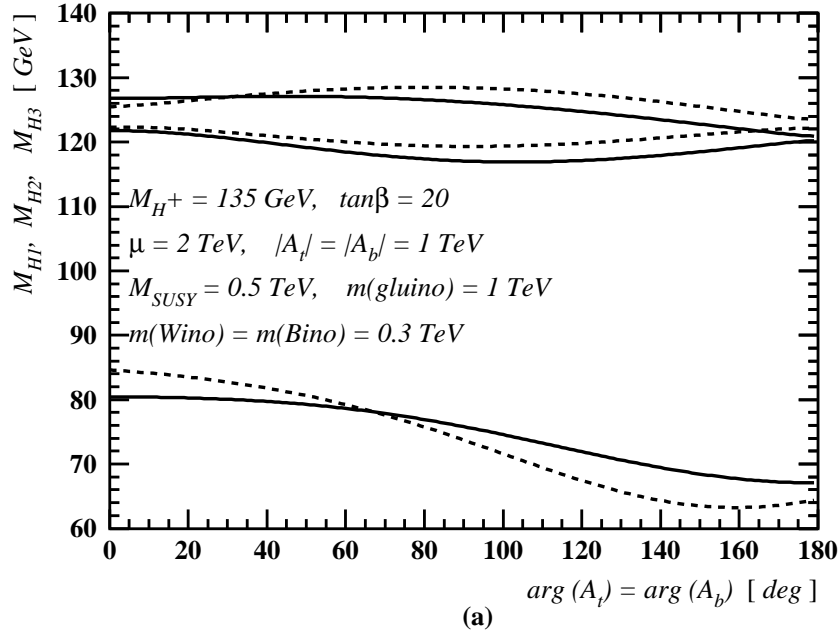


Figure 4: As in Fig. 3, but with $M_{H^+} = 135 \text{ GeV}$.

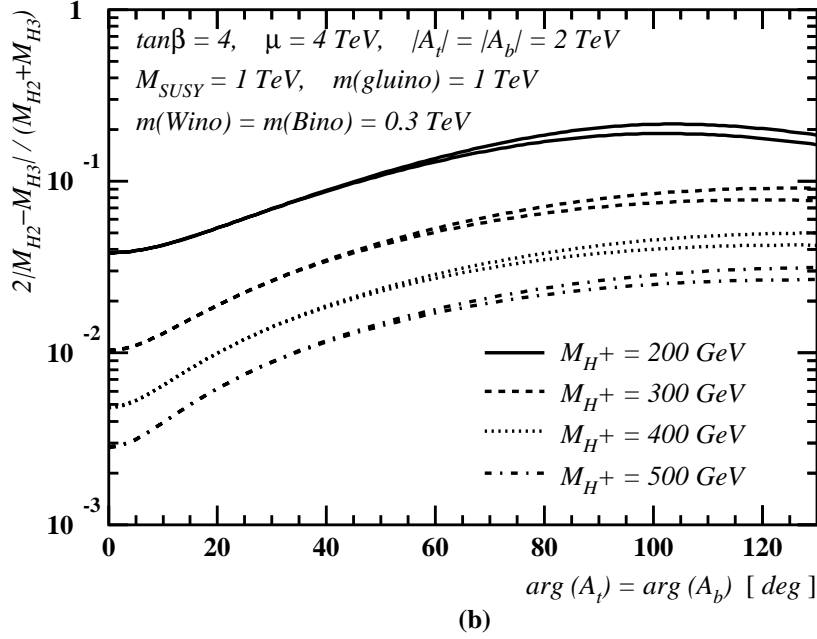
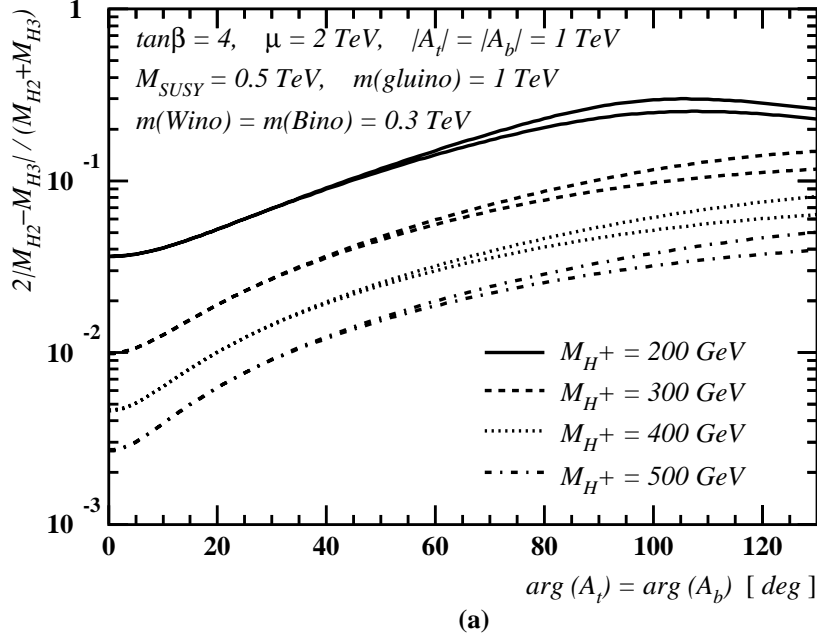


Figure 5: Numerical estimates of (a) M_{H_1} and (b) $2|M_{H_2} - M_{H_3}| / (M_{H_2} + M_{H_3})$ as functions of the CP-violating phase $\arg(A_t)$. Lower values of the same line type correspond to $\arg(m_{\tilde{g}}) = 0$, the higher ones to $\arg(m_{\tilde{g}}) = 90^\circ$.

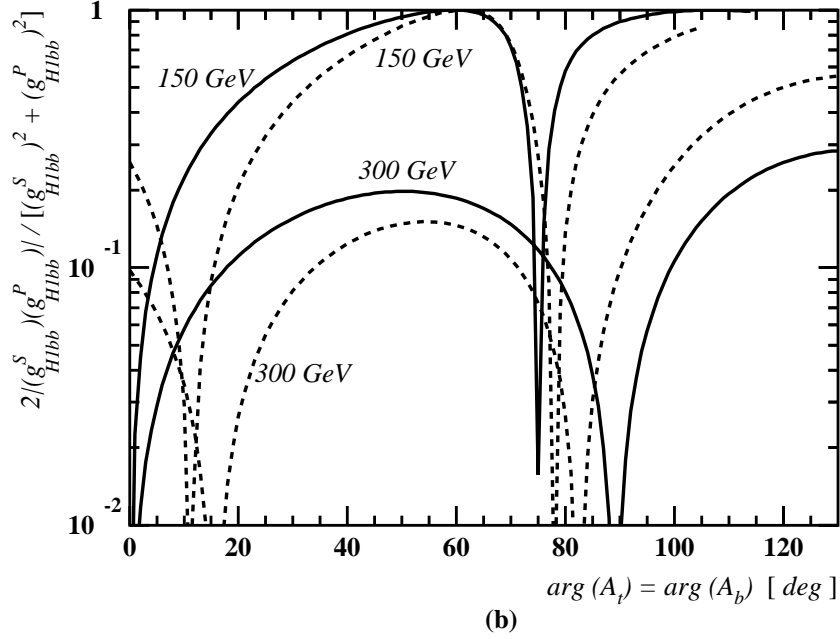
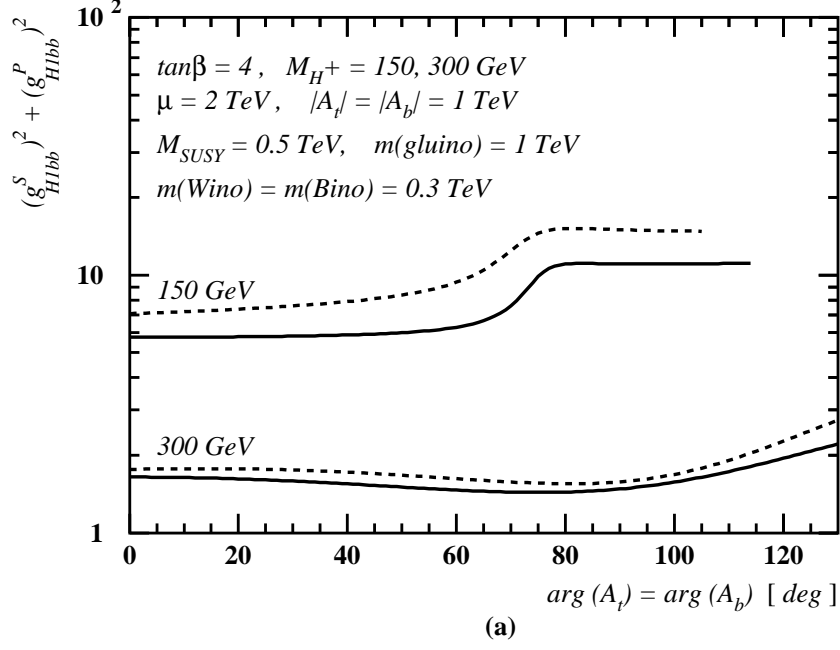


Figure 6: Numerical estimates of (a) $(g_{H1bb}^S)^2 + (g_{H1bb}^P)^2$ and (b) $2|(g_{H1bb}^S)(g_{H1bb}^P)| / [(g_{H1bb}^S)^2 + (g_{H1bb}^P)^2]$ versus $\arg(A_t)$. Solid lines correspond to $\arg(m_{\tilde{g}}) = 0$, dashed ones to $\arg(m_{\tilde{g}}) = 90^\circ$.

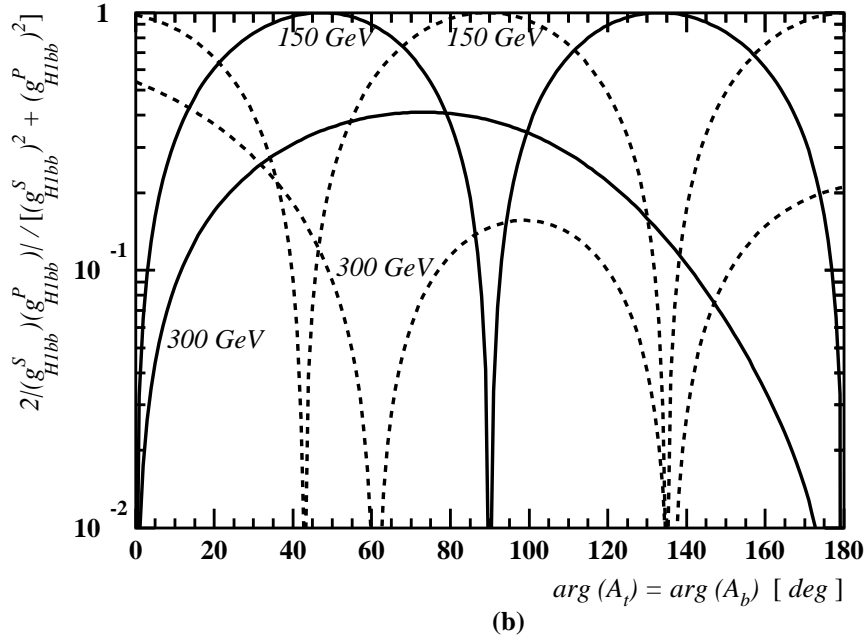
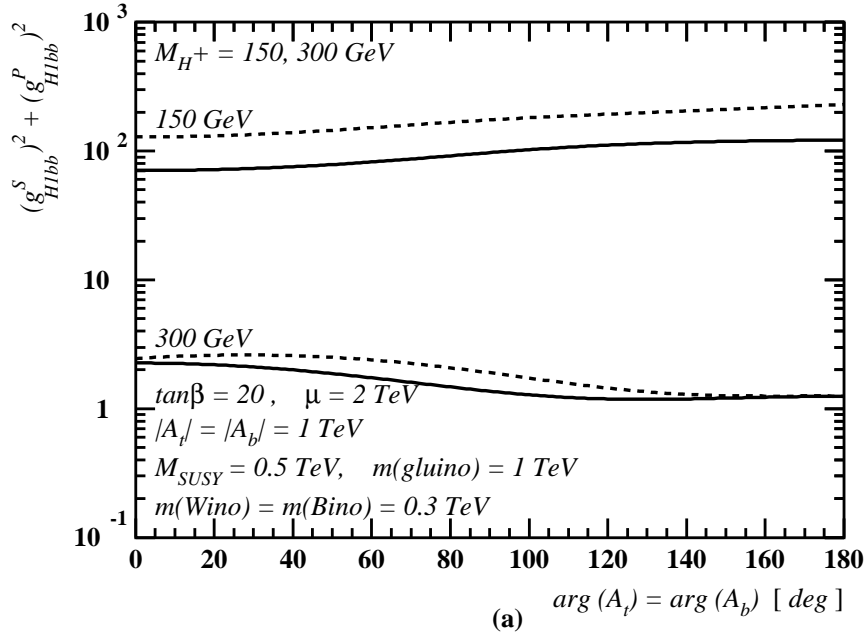


Figure 7: As in Fig. 6, but with $\tan \beta = 20$.

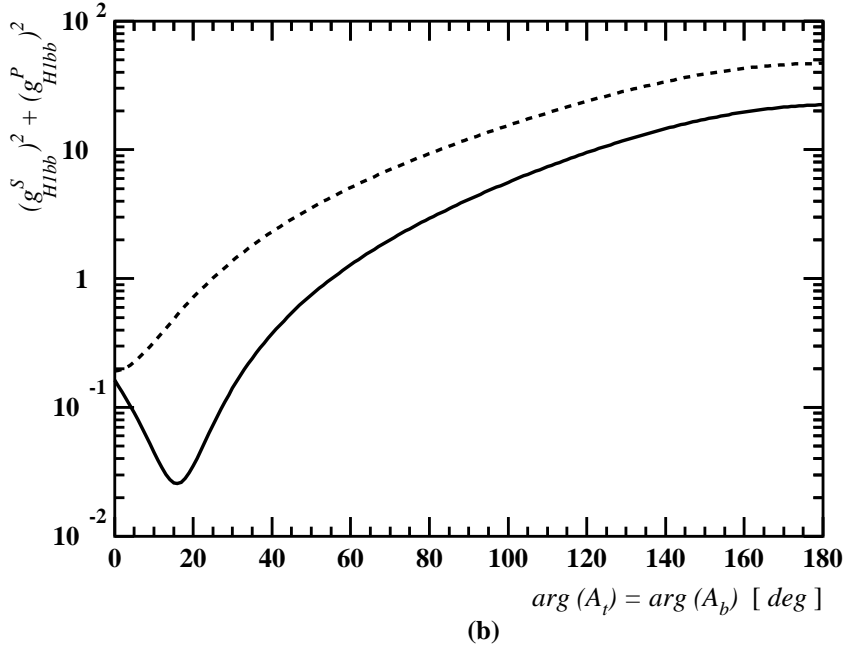
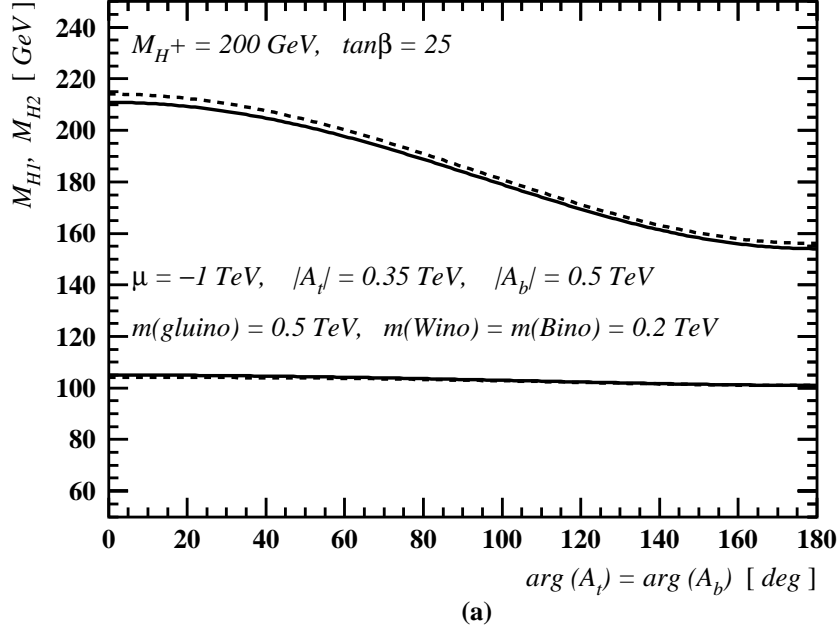


Figure 8: Numerical estimates of (a) M_{H_1} and M_{H_2} and (b) $[(g_{H_1bb}^S)^2 + (g_{H_1bb}^P)^2]$ versus $\arg(A_t)$, with soft squark masses: $\tilde{M}_Q^2 = \tilde{M}_b^2 = 0.6 \text{ TeV}$ and $\tilde{M}_t^2 = 0$. Solid lines are for $\arg(m_{\tilde{g}}) = -90^\circ$ and dashed lines for $\arg(m_{\tilde{g}}) = 0^\circ$.

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