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ANOTHER ALGEBRA FROM THE YOKONUMA-HECKE ALGEBRA

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1. Introduction

Last year, finite dimensional algebras of one parameter (or more) defined by generators and relations that involve the braid relations has been studied rather intensely (e.g. the Hecke algebra, the Temperley-Lieb algebra, the Birman-Wenzl-Murakami algebra, among others). A motivation for this is to take matricial representations for the braid group, polynomial invariants for knots (via Markov trace over the algebra), as applications in 2D lattice in statistical mechanics.

Now, to define such an algebra is not a trivial matter. Usually, these algebras arise as quotients of the groups algebras of the braid group, as centraliser algebras of the tensor powers of the quantised enveloping algebras of classical Lie algebras, or they are defined by generators and relations motived by reasons arising from Physics.

In this note, we introduce and begin to do a systematic study of an algebra of one parameter, whose generators satisfy certain braid relations. On the contrary to the classical definitions, our algebra arises from the Yokonuma-Hecke algebra[6]; that is, from the algebra of endomorphisms of the permutation representation of the group $GL_n(\mathbb{F}_q)$, relative to its maximal upper unipotent subgroup. More precisely, the definition of our algebra is by generators and relations, which are motived from the non-standard generators of the Yokonuma-Hecke algebra introduced by the author in [4].

Here is the outline of the note: In section 2 our algebra is defined, and the word "another" in the title is explained (see subsection 2.1). In section 3 we construct a good system of generators for our algebra (theorem 20). The reason for to be good is that we have experimental reason for to conjecture that is, in fact, a basis for our algebra. In section 4 our algebra is realized as a subalgebra of Yokonuma-Hecke algebra (see theorem 23). In section 5, we prove that the algebra can be Yang-Baxterized. Finally, in section 6 some technical lemmas are proved.

Notations. In this work $\mathcal{P}(n)$ denote the power-set of $\{1,\dots,n-1\}$. We denote by S_n the symmetric group on $\{1,\dots,n\}$, and by s_i the transposition $(i,i+1)$ of S_n.

$\mathbb{C}(u)$ denotes the field of rational functions in an indeterminate u, over the complex number $\mathbb{C}$.

2. The algebra $\mathcal{E}_n(u)$

Definition 1. *Let n be a natural number and u an indeterminate over $\mathbb{C}$. Let $\mathcal{E}_n(u)$ be the associative algebra over $\mathbb{C}(u)$, with generators:*

$$1, T_1, \dots, T_{n-1}, E_1, \dots, E_{n-1}$$

subject to the following relations:

$$E_i^2 = E_i \qquad E_iE_j = E_jE_i, \quad \forall i,j \tag{1}$$

$$E_iT_i = T_iE_i \tag{2}$$

$$[T_i, T_j] = [T_i, E_j] = 0 \qquad \text{if} \quad |i-j| > 1. \tag{3}$$

$$T_i^2 = 1 + (u-1)E_iT_i(1-T_i) \tag{4}$$

$$E_iT_i(1-T_i) = -u^{-1}E_i(1-T_i) \tag{5}$$

And when $|i-j| = 1$, we impose:

$$T_iT_jT_i = T_jT_iT_j \tag{6}$$

$$T_iT_jE_i = E_jT_iT_j \tag{7}$$

$$E_iE_jT_j = E_iT_jE_i = T_jE_iE_j \tag{8}$$

$$T_iE_jT_i - T_jE_iT_j = (1-u^{-1})(E_jT_iE_j - E_iT_jE_i). \tag{9}$$

Notice that relation (5) and (9) are superfluous[2].

It is easy to check that the map $T_i \mapsto (i, i+1)$, $E_i \mapsto 0$ define a homomorphism of $\mathcal{E}_n(u)$ onto the group algebra of the symmetric group. Also we have a morphism of $\mathcal{E}_n(u)$ onto the Iwahori-Hecke algebra $\mathcal{H}_n(u)$ via the map

$$\begin{aligned} T_i &\mapsto (1-u^{-1}) - u^{-1}L_i \\ E_i &\mapsto 1, \end{aligned}$$

where $L_1, \ldots, L_{n-1}$ are the standard generators of $\mathcal{H}_n(u)$, that is, the L_i's satisfies the braid relations and the quadratic relation $L_i^2 = u + (u-1)L_i$.

Now, from (4) we deduce that T_i is invertible,

$$T_i^{-1} = T_i - (u-1)E_i(1-T_i). \tag{10}$$

And we get the following notable relation, cf. section 5,

$$T_i^{-1}T_jT_i^{-1} + uT_jT_i^{-1}T_j = T_j^{-1}T_iT_j^{-1} + uT_iT_j^{-1}T_i. \tag{11}$$

Set $P_i = u/(u+1)E_i(1-T_i)$. It is not difficult to prove the following useful relations:

$$(u+1)^3(P_iP_jP_i - P_jP_iP_j) = u^2E_iE_j(T_j - T_i). \tag{12}$$

$$(u+1)P_iP_jP_i + uT_iP_jP_i = uP_jP_i, \tag{13}$$

(14) $E_iP_i = P_iE_i = (u-1)P_i$
(15) $E_iP_j = P_jE_i, \qquad \forall i, j$
(16) $P_i^2 = P_i$
(17) $T_iP_i = P_iT_i = -u^{-1}P_i$
(18) $T_i^2 = 1 - u^{-1}(T_i - T_i^{-1}) = 1 + (u^{-2}-1)P_i = 1 + (u^{-1}-1)E_i(1-T_i)$
(19) $T_i^3 + u^{-1}T_i^2 - T_i - u^{-1} = (T_i + u^{-1})(T_i+1)(T_i-1) = 0$
(20) $P_iT_jT_i = T_jT_iP_j$, if $|i-j| = 1$ (from (6) and (18)).

2.1. **The algebra $\mathcal{J}_n(u)$.** In [5] we have defined an algebra $\mathcal{J}_n(u)$. We know that this algebra is finite dimensional up to $n = 5$. Now, a standard method to know if an algebra, defined by generators and relations that involved braid relations, is finite dimensional is to solve the so-called *problem of the words*, that is, to write any word (in its defining generators) as a linear combination of words having only at most one element in certain pre-fixed subfamily of generators. We cann't solve this problem for the algebra $\mathcal{J}_n(u)$.

On the other side, motived by the realization of $\mathcal{J}_n(u)$ as a subalgebra of Yokonuma-Hecke algebra, it was natural to consider a certain linear decomposition of a part of the generators of $\mathcal{J}_n(u)$. Thus obtaining another algebra $\mathcal{E}_n(u)$, in which the problem of the words is solved (see proposition 4). Hence in some sense, the algebra $\mathcal{E}_n(u)$ is a certain linear decomposition for the algebra $\mathcal{J}_n(u)$, in the sense that a part of the generators of $\mathcal{J}_n(u)$ is linearly decomposed in $\mathcal{E}_n(u)$, see proposition 2.

Recall that the algebra $\mathcal{J}_n(u)$ is the associative algebra over $\mathbb{C}(u)$ defined by the generators

$$1, \tau_1, \ldots, \tau_{n-1}, \tau_1^{-1}, \ldots, \tau_{n-1}^{-1}, \pi_1, \ldots, \pi_{n-1},$$

and the defining relations:

$$\begin{gathered} \tau_i\tau_i^{-1} = \tau_i^{-1}\tau_i = 1 \\ \tau_i - \tau_i^{-1} = (u - u^{-1})\pi_i \\ \tau_i\pi_i = \pi_i\tau_i = -u^{-1}\pi_i \\ [\tau_i, \tau_j] = [\pi_i, \pi_j] = [\tau_i, \pi_j] = 0 \qquad \text{if} \qquad |i-j| > 1. \end{gathered}$$

And when $|i-j| = 1$,

$$\tau_i\tau_j\tau_i = \tau_j\tau_i\tau_j$$

[2]F. Aircardi pointed out that relation (9) is superfluous.

$$\tau_i^{-1}\tau_j\tau_i^{-1} + u\tau_j\tau_i^{-1}\tau_j = \tau_j^{-1}\tau_i\tau_j^{-1} + u\tau_i\tau_j^{-1}\tau_i$$
$$(u+1)\pi_i\pi_j\pi_i + u\tau_i\pi_j\pi_i = u\pi_j\pi_i.$$

Proposition 2. *The map $\tau_i \mapsto T_i$, $\pi_i \mapsto P_i$ defines a morphism from the algebra $\mathcal{J}_n(u)$ to the algebra $\mathcal{E}_n(u)$.*

Proof. The proof follows from the defining relations for $\mathcal{E}_n(u)$, and relations (11), (13), (17) and (18). ∎

3. The linear part

In order to prove that $\mathcal{E}_n(u)$ is finite dimensional we need, in addition to the above relations, the following lemma.

Lemma 3. *For all i, j such that $|i-j| = 1$, we have:*

(3.1) $(u-1)(T_iP_jT_i - T_jP_iT_j) = -(u-u^{-1})^2(P_iP_jP_i - P_jP_iP_j)$

(3.2) $(u-1)(P_iT_jP_i - P_jT_iP_j) = u(u-u^{-1})(P_iP_jP_i - P_jP_iP_j)$

(3.3) $(u-1)(P_iP_jT_i - T_jP_iP_j) = -(u-u^{-1})(P_iP_jP_i - P_jP_iP_j)$.

Proof. It is a consequence of proposition 2 and lemma 2.10[5]. ∎

Proposition 4. *In the algebra $\mathcal{E}_n(u)$ any word in $1, T_1, \ldots, T_{n-1}$, $E_1, \ldots, E_{n-1}$ is a linear combination of words in T_i's, E_i's having at most one R_{n-1}, where $R_{n-1} \in \{T_{n-1}, E_{n-1}, P_{n-1}\}$. Thus $\mathcal{E}_n(u)$ is finite dimensional.*

Proof. We use the same argument of induction used in lemma 3.1[1]. For $n = 2$ the claim is obvious. Now suppose the lemma for $n-1$. Thus, we must prove that all words below can be written as linear combination of words having at most one $R_n \in \{T_n, E_n, P_n\}$ (we will say the word have a reduction):

(i) $T_nR_{n-1}T_n$
(ii) $T_nR_{n-1}P_n$
(iii) $T_nR_{n-1}E_n$
(iv) $P_nR_{n-1}T_n$
(v) $P_nR_{n-1}P_n$
(vi) $P_nR_{n-1}E_n$
(vii) $E_nR_{n-1}T_n$
(viii) $E_nR_{n-1}P_n$
(ix) $E_nR_{n-1}E_n$,

where $R_{n-1} \in \{1, T_{n-1}, P_{n-1}, E_{n-1}\}$.

Let us see the case whenever $R_{n-1} = 1$: The reduction of (i) is from (18). The reduction of (ii), (iv) is from (17). The reduction of (v) is from (16). The reduction of (vi) and (viii) is from (14). The reduction of (ix) is from (1). For (iii) and (vii) recall

$$T_nE_n = E_nT_n = E_n - (1+u^{-1})P_n.$$

Now, let us consider the following three cases:

Case $R_{n-1} = T_{n-1}$. The reduction of (ii) and (iv) is from (20). The reduction of (v) is from lemma 3 and (12). The reduction of (iii) and (vii) is from (7). The reduction of (ix) is from (8). For (vi) we have

$$\begin{aligned} P_nT_{n-1}E_n &= P_nE_nT_{n-1}E_n && \text{(from (14))} \\ &= P_nE_nE_{n-1}T_{n-1} && \text{(from (8))} \\ &= P_nE_{n-1}T_{n-1} && \text{(from (14)).} \end{aligned}$$

In the same way we get a reduction for (viii).

Case $R_{n-1} = P_{n-1}$. The reduction of (vi) and (viii) is from (15) and (14). The reduction of (ix) is from (15) and (1). The reduction of (i), (ii), (iv) and (v) is from (12) and lemma 3. For (iii) we have

$$\begin{aligned} T_n P_{n-1} E_n &= T_n E_n P_{n-1} \qquad \text{(from (15))} \\ &= (E_n - (1+u^{-1})P_n) P_{n-1} \\ &= E_n P_{n-1} - (1+u^{-1}) P_n P_{n-1}. \end{aligned}$$

In the same way we get a reduction for (vii).

Case $R_{n-1} = E_{n-1}$. The reduction of (i) is from (9) and (8). The reduction of (ii) and (iv) is from (15) and (17). The reduction of (v) is from (15) and (16). The reduction of (vi) and (viii) is from (15) and (14). The reduction of (ix) is from (1).

For (iii) we have

$$\begin{aligned} T_n E_{n-1} E_n &= T_n E_n E_{n-1} \qquad \text{(from (1))} \\ &= (E_n - (1+u^{-1})P_n) E_{n-1} \\ &= E_n E_{n-1} - (1+u^{-1}) P_n E_{n-1}. \end{aligned}$$

In the same way we get a reduction for (vii).

For $n = 2$ the algebra is dimension 4. Now, $\mathcal{E}_n(u) = \sum \mathcal{E}_{n-1}(u) R_n \mathcal{E}_{n-1}(u)$, where the sum is over $R_n \in \{1, T_n, E_n, P_n\}$. Thus, we deduce by induction that the algebra $\mathcal{E}_n(u)$ is finite dimensional. ∎

Remark. From the definition of the P_i's it is obvious that the preceding proposition holds if $R_{n-1} \in \{1, T_{n-1}, E_{n-1}, E_{n-1}T_{n-1}\}$. And from (10) the proposition also holds if $R_{n-1} \in \{1, T_{n-1}, E_{n-1}, T_{n-1}^{-1}\}$.

We are going to construct a good system of generators for $\mathcal{E}_n(u)$. First, we prove that the algebra is linearly generated by certain "standard words" (see proposition 12), and then we will take from them the good system of generators (theorem 20).

As usual, we denote by ℓ the length function on S_n relative to $\{s_1, \ldots, s_{n-1}\}$.

Also, we use the fact that all elements w in S_n admit a writing reduced of the form

$$w = (s_{i_1} s_{i_1 - 1} \cdots)(s_{i_2} s_{i_2 - 1} \cdots) \cdots (s_{i_m} s_{i_m - 1} \cdots), \tag{21}$$

where $i_1 < i_2 < \cdots < i_m$.

Let us consider the following elements, in form reduced, in S_{n+1}:

$$\theta_r := s_r s_{r+1} \cdots s_n \qquad (r \leq n).$$

Lemma 5. *For all $w \in S_n$, we have in S_{n+1}:*

(5.1) $\ell(\theta_r w) = \ell(\theta_r) + \ell(w)$

(5.2) $(s_{m+1} s_m)(s_{m+2} s_{m+1}) \cdots (s_n s_{n-1}) s_n = (s_m s_{m+1} \cdots s_{n-1} s_n)(s_m s_{m+1} \cdots s_{n-1})$.

Proof. Trivial. ∎

In virtue of the braid relation (6) and a well-known theorem of H. Matsumoto, we have that if $w = s_{i_1} \cdots s_{i_m} \in S_n$ is an expression reduced for w, then the element

$$T_w := T_{i_1} \cdots T_{i_m},$$

is well-defined.

Proposition 6. *Let $w \in S_n$, and set $s = s_i$. We have*

$$\begin{aligned} T_w T_i &= T_{ws} \qquad \textit{if} \quad \ell(w s_i) = \ell(w) + 1 \\ T_w T_i &= T_{ws} + (u^{-1} - 1) E_i (T_{ws} - T_w) \qquad \textit{if} \quad \ell(w s_i) = \ell(w) - 1. \end{aligned}$$

Proof. The proof is by the same procedure used for to prove the analogous in the case of Iwahori-Hecke algebra; that is, using induction over length of w and (18). ∎

We will denote by L_i the element T_iE_i.

Lemma 7. *For all i, j such that $|i-j| = 1$, we have:*

(7.1) $E_jT_i = T_iT_jE_iT_j + (u^{-1}-1)(T_iT_jE_iE_j + T_iE_iE_j)$

(7.2) $L_jT_i = T_iT_jL_iT_j + (u^{-1}-1)(T_iT_jL_iE_j + T_jL_iE_j)$.

Proof. From (7) we get

$$\begin{aligned}(T_iT_jE_i)T_j &= E_jT_iT_j^2\\ &= E_jT_i(1+(u^{-1}-1)(E_j - E_jT_j)) \qquad \text{(from (18))}\\ &= E_jT_i + (u^{-1}-1)(T_iE_iE_j - T_iT_jE_iE_j) \qquad \text{(from (8))}.\end{aligned}$$

Thus the assertion (7.1) follows.

Multiplying (7.1) on the left by T_j, and after using (6) we take (7.2). ∎

Now, as in the Iwahori-Hecke algebra we can take a system of linear generates for $\mathcal{E}_n(u)$ (which in our case will be redundant) in the following way: we define $U_1 = \{1, T_1, E_1, L_1\}$, and U_i is defined by

$$U_i := \{1\} \cup T_iU_{i-1} \cup E_iU_{i-1} \cup L_iU_{i-1} \qquad (2 \le i \le n).$$

Using induction and proposition 4 we deduce that $\mathcal{E}_n(u)$ is generated linearly by all the products of the form $u_1u_2\cdots u_{n-1}$, where $u_i \in U_i$. From where we deduce

$$\mathcal{E}_{n+1}(u) = \sum_{1\le i\le n} Y_iY_{i+1}\cdots Y_n\mathcal{E}_n(u) + \mathcal{E}_n(u), \tag{22}$$

where $Y_j \in \{T_j, E_j, L_j\}$.

In order to take a best system of linear generators (let us say in terms of *standard words*) we will need the following three technical lemmas, whose proofs will be done in section 6.

Lemma 8. *Let $m > i$. The word $E_iT_{i+1}\cdots T_{m-1}T_m \in \mathcal{E}_{m+1}(u)$, is a linear combination the words of the type:*

$$(T_{i+1}T_i)(T_{i+2}T_{i+1})\cdots(T_mT_{m-1})E_m\alpha \tag{8.1}$$

$$T_{i+1}T_{i+2}\cdots T_mE_m\beta \tag{8.2}$$

$$T_{i+1}T_{i+2}\cdots T_{m-1}T_mT_jT_{j+1}\cdots T_{m-1}E_m\gamma_j, \qquad (i+1 \le j \le m-1) \tag{8.3}$$

where α, β, $\gamma_j \in \mathcal{E}_m(u)$.

Lemma 9. *Let $m > i$. The word $L_iT_{i+1}\cdots T_{m-1}T_m \in \mathcal{E}_{m+1}(u)$, is a linear combination the words of the type:*

$$(T_{i+1}T_i)(T_{i+2}T_{i+1})\cdots(T_mT_{m-1})L_m\alpha \tag{9.1}$$

$$T_iT_{i+1}T_{i+2}\cdots T_{m-1}L_m\beta \tag{9.2}$$

$$T_{m-j}T_{m-j+1}\cdots T_mT_iT_{i+1}\cdots T_{m-1}L_m\gamma_j, \qquad (0 \le j \le m-2) \tag{9.3}$$

where α, β, $\gamma_j \in \mathcal{E}_m(u)$.

Lemma 10. *Let $Y_l \in \{T_l, E_l, L_l\}$, and $m > i$. The word $Y_iY_{i+1}\cdots Y_{m-1}Y_m \in \mathcal{E}_{m+1}(u)$, can be written as a linear combination of words of the form*

$$E_iE_{i+1}\cdots E_{m-1}E_m, \quad T_iT_{i+1}\cdots T_{m-1}T_m \tag{10.1}$$

$$T_rT_{r+1}\cdots T_{m-1}L_m\alpha \tag{10.2}$$

(10.3) $$T_iT_{i+1}\cdots T_{r-1}(T_{r+1}T_r)(T_{r+2}T_{r+1})\cdots(T_mT_{m-1})F_m\beta \qquad F_m=E_m,L_m$$

(10.4) $$T_rT_{r+1}\cdots T_{m-1}T_mT_jT_{j+1}\cdots T_{m-1}E_m\gamma_j \qquad (r\le j\le m-1)$$

(10.5) $$T_{m-j}T_{m-j+1}\cdots T_mT_iT_{i+1}\cdots T_{m-1}L_m\delta_j, \qquad (0\le j\le m-2)$$

where $i\le r\le m$, α, β, γ_j, $\delta_j\in\mathcal{E}_m(u)$.

Definition 11. (11.1) *For* $I:=\{i_1,\dots,i_m\}\in\mathcal{P}(n)$, *we define* E_I *as the product* $E_{i_1}\cdots E_{i_m}$. *If* I *is the empty set, we assume that* E_I *is equal to* 1.

(11.2) *We define* $W_1:=\{1\}$, $W_2:=\{1,E_2T_1\}$, $W_3=\{1,E_3T_2,E_3T_2T_1\}$, *so on.*

Proposition 12. *The algebra* $\mathcal{E}_n(u)$ *is generated linearly by the words of the form*

$$T_wX_{n-1}\cdots X_2X_1E_I, \qquad (\textit{standard word})$$

where $w\in S_n$, $X_i\in W_i$ *and* $I\in\mathcal{P}(n)$.

Proof. The proof is by induction over n. For $n=2$ the proposition is true, because $\mathcal{E}_2(u)$ is linearly generated by the elements $1,T_1,E_1,T_1E_1$. Set $n>2$. According to (22), it is sufficient to prove that any product of the form $Y_iY_{i+1}\cdots Y_nZ$ ($Z\in\mathcal{E}_n(u)$) is a linear combination of standard words in $\mathcal{E}_{n+1}(u)$.

Now, from lemma 10 we deduce that $Y_iY_{i+1}\cdots Y_nZ$ is a linear combination of elements of the form

(A1) $$E_iE_{i+1}\cdots E_{n-1}E_nZ_0, \quad T_iT_{i+1}\cdots T_{n-1}T_nZ_1$$

(A2) $$T_rT_{r+1}\cdots T_{n-1}T_nE_nZ_2,$$

(A3) $$T_iT_{i+1}\cdots T_{r-1}(T_{r+1}T_r)(T_{r+2}T_{r+1})\cdots(T_nT_{n-1})F_nZ_3, \qquad F_n=E_n,L_n$$

(A4) > $$T_rT_{r+1}\cdots T_{n-1}T_nT_jT_{j+1}\cdots T_{n-1}E_nZ_4 \qquad (r\le j\le n-1)$$

(A5) $$T_{n-j}T_{n-j+1}\cdots T_nT_iT_{i+1}\cdots T_{n-1}L_nZ'_j, \qquad (0\le j\le n-2)$$

where $Z_0,\dots,Z_4$, $Z'_j\in\mathcal{E}_n(u)$.

Consequently, it is enough to prove that the elements in A1 to A5 are a linear combination of standard words in $\mathcal{E}_{n+1}(u)$. This will be done in the cases A1 to A5.

First notice that by the hypothesis of induction the elements $Z_0,\dots,Z_4$, Z'_j are linear combinations of standard words in $\mathcal{E}_n(u)$. Thus, we can put

$$T_wX_{n-1}\cdots X_2E_K, \qquad (w\in S_n, K\in\mathcal{P}(n))$$

to the place of Z_i in Ai ($i=0,4$), and to the place of Z'_j in A5.

Furthermore, we will use the expression reduced (21) for w.

Case A1. (i) From (5.1) and proposition 6, it is clear that

$$T_i\cdots T_nZ_1=(T_i\cdots T_nT_w)X_{n-1}\cdots X_2E_K,$$

is a standard word in $\mathcal{E}_{n+1}(u)$.

(ii) On the other hand,

$$E_iE_{i+1}\cdots E_{n-1}E_nZ_0=E_n(E_iE_{i+1}\cdots E_{n-1}Z_0).$$

Using induction on the elements between parenthesis, we have that $E_iE_{i+1}\cdots E_{n-1}E_nZ_0$ is a linear combination of elements A_1 of the form

$$A_1=E_nT_wX_{n-1}\cdots X_2E_K \qquad (w\in S_n, K\in\mathcal{P}(n)).$$

In the case that w does not contain s_{n-1}, we have that A_1 is the standard word

$$T_wX_{n-1}\cdots X_2E_nE_K.$$

If w contains s_{n-1}, we put $w = w'(s_{n-1}s_{n-2}\cdots)$, where $w' \in S_n$. Then A_1 is the standard word

$$A_1 = T_{w'}X_nX_{n-1}\cdots X_2E_K \qquad (X_n = E_nT_{n-1}T_{n-2}\cdots).$$

Case A2. If in (21) w does not contain s_{n-1}. Then A2 becomes

$$T_rT_{r+1}\cdots T_nT_wX_{n-1}\cdots X_2E_nE_K,$$

which is a standard word in $\mathcal{E}_{n+1}(u)$, because according to (5.1) and proposition 6 the element $T_rT_{r+1}\cdots T_nT_w$ is of the form $T_{w'}$.

Suppose that w contains s_{n-1}. Put $w = w's_{n-1}s_{n-2}\cdots$, where $w' \in S_{n-1}$. We get

$$T_rT_{r+1}\cdots T_{n-1}E_nT_wX_{n-1}\cdots X_2E_K = (T_rT_{r+1}\cdots T_nT_{w'})X_nX_{n-1}\cdots X_2E_K.$$

where $X_n = E_nT_{n-1}T_{n-2}\cdots$. Using (5.1) and proposition 6 on the word between parenthesis, we have that the element in A2 becomes a standard word in $\mathcal{E}_{n+1}(u)$.

Case A3. First let us note that

$$(23)\qquad \begin{cases} (T_{m+1}T_m)(T_{m+2}T_{m+1})\cdots(T_nT_{n-1}) = (T_{m+1}\cdots T_n)(T_m\cdots T_{n-1}) \\ (T_{m+1}T_m)(T_{m+2}T_{m+1})\cdots(T_nT_{n-1})T_n = (T_m\cdots T_n)(T_m\cdots T_{n-1}) \end{cases} \qquad (m < n).$$

Set $T_\xi = T_iT_{i+1}\cdots T_{r-1}$, where $\xi = s_is_{i+1}\cdots s_{r-1}$. Now, we distinguish between the cases: $F_n = E_n$ or $F_n = L_n$.

Case $F_n = E_n$ in A3. In this case the element in A3 case take the form

$$\begin{aligned} A_3 &= T_\xi(T_{r+1}T_r)(T_{r+2}T_{r+1})\cdots(T_nT_{n-1})E_nT_wX_{n-1}\cdots X_2E_K \\ &= T_\xi T_uT_vE_nT_wX_{n-1}\cdots X_2E_K \qquad \text{(from (23))} \\ &= T_uT_\xi T_vE_nT_wX_{n-1}\cdots X_2E_K, \\ &= T_uT_{\xi v}E_nT_wX_{n-1}\cdots X_2E_K \qquad \text{(from (5.1) and proposition 6)} \end{aligned}$$

where $u = s_{r+1}s_{r+2}\cdots s_{n-2}s_n$, and $v = s_rs_{r+1}\cdots s_{n-1}$.

We are going to distinguish the cases: w contains or not s_{n-1}.

† If w does not contain s_{n-1}, then we can write A_3 as

$$A_3 = T_u(T_{\xi v}T_wX_{n-1}\cdots X_2E_K)E_n.$$

Using the hypothesis of induction over the element between parenthesis we obtain that A_3 can be re-written as

$$A_3 = T_uT_wX_{n-1}\cdots X_2E_K,$$

where $w \in S_n$, $K \in \mathcal{P}(n+1)$.

Now from (5.1) the product T_uT_w is of the form T_θ, with $\theta \in S_{n+1}$. So, A_3 is a standard word in $\mathcal{E}_{n+1}(u)$.

† In the case that w contains s_{n-1}, we put $w = w's_{n-1}s_{n-2}\cdots$, where $w' \in S_{n-1}$. Thus A_3 is

$$\begin{aligned} A_3 &= T_uT_{\xi v}E_nT_wX_{n-1}\cdots X_2E_K \\ &= T_uT_{\xi v}T_{w'}X_nX_{n-1}\cdots X_2E_K \qquad (X_n = E_nT_{n-1}T_{n-2}\cdots). \end{aligned}$$

Using proposition 6 and (5.1) over $T_{\xi v}T_{w'}$, and after on $T_u(T_{\xi v}T_{w'})$, we deduce that A_3 is a standard word in $\mathcal{E}_{n+1}(u)$.

Case $F_n = L_n$ in A3. In this case according to (23) we have that the element in A3 takes the form

$$A_3 = T_\xi T_uT_vE_nT_wX_{n-1}\cdots X_2E_K,$$

where $u = s_rs_{r+1}\cdots s_{n-1}s_n$, and $v = s_rs_{r+1}\cdots s_{n-1}$.

According to (5.1) and proposition 6, $T_\xi T_u = T_{\xi u}$. Then

$$A_3 = T_{\xi u}T_vE_nT_wX_{n-1}\cdots X_2E_K.$$

Again we distinguish the cases: w contains or not s_{n-1}.

† If w does not contain s_{n-1}, then the word in A_3 take the form

$$A_3 = T_{\xi u}(T_vT_wX_{n-1}\cdots X_2E_K)E_n.$$

Using the hypothesis of induction over the elements between parenthesis, A_3 can be re-written as

$$A_3 = T_{\xi u} T_w X_{n-1} \cdots X_2 E_K,$$

where $w \in S_n$, and $K \in \mathcal{P}(n+1)$.

Applying (5.1) and proposition 6 over $T_{\xi u} T_w$, we deduce that A_3 is a standard word in $\mathcal{E}_{n+1}(u)$.

† In the case that w contains s_{n-1}, we put $w = w' s_{n-1} s_{n-2} \cdots$, where $w' \in S_{n-1}$. Then we get that A_3 is the following standard word in $\mathcal{E}_{n+1}(u)$,

$$(T_{\xi u} T_v T_{w'}) X_n X_{n-1} \cdots X_2 E_K, \qquad X_n = E_n T_{n-1} T_{n-2} \cdots$$

(notice that $T_{\xi u} T_v T_{w'}$ is of the form T_θ, $\theta \in S_{n+1}$).

Case A4. From (5.1) and proposition 6 we have

$$T_r T_{r+1} \cdots T_{n-1} T_n T_j T_{j+1} \cdots T_{n-1} = T_u T_v$$

where $u = s_r s_{r+1} \cdots s_{n-1} s_n$, and $v = s_j s_{j+1} \cdots s_{n-1}$.

† In the case that w does not contain s_{n-1}, we have that in A4 one write

$$A_4 = T_u (T_v T_w X_{n-1} X_{n-2} \cdots X_2 E_K) E_n.$$

Now, applying the hypothesis of induction on the elements between parenthesis, we obtain that A_4 is re-written as

$$A_4 = T_u T_w X_{n-1} X_{n-2} \cdots X_2 E_K.$$

where now $w \in S_n$ and $K \in \mathcal{P}(n+1)$.

Using (5.1) and proposition 6 on $T_u T_w$, we deduce that A_4 is a standard word in $\mathcal{E}_{n+1}(u)$.

† In the case that w contains s_{n-1}, we put $w = w' s_{n-1} s_{n-2} \cdots$. And then A_4 is

$$A_4 = T_u T_v T_{w'} X_n X_{n-1} \cdots X_2 E_K, \qquad (X_n = E_n T_{n-1} \cdots)$$

where $K \in \mathcal{P}(n)$.

The fact that A_4 is a standard word follows applying first (5.1) and proposition 6 on $T_v T_{w'}$ and after on $T_u(T_v T_{w'})$.

Case A5. First we note that

$$\begin{aligned} T_{n-j} \cdots T_n T_i T_{i+1} \cdots T_{n-1} T_n &= T_i \cdots T_{n-j-2} (T_{n-j} T_{n-j-1}) \cdots (T_{n-1} T_{n-2})(T_n T_{n-1}) T_n \\ &= T_i \cdots T_{n-j-2} (T_{n-j-1} \cdots T_n)(T_{n-j-1} \cdots T_{n-1}) \\ &= T_u T_v, \end{aligned}$$

where $u = s_i \cdots s_{n-j-2} s_{n-j-1} \cdots s_n$, $v = s_{n-j-1} \cdots s_{n-1}$.

Thus, the element in A5 takes the form

$$A_5 = T_u T_v E_n T_w X_{n-1} \cdots E_K \qquad (w \in S_n, K \in \mathcal{P}(n)).$$

Again we distinguish between the cases: w contains or not s_{n-1}.

† In the case w does not contain s_{n-1}, we have $A_5 = T_u T_v T_w X_{n-1} \cdots E_K E_n$. Applying (5.1) and proposition 6 on $T_v T_w$ and after on $T_u(T_v T_w)$ we obtain that A_5 is a standard word in $\mathcal{E}_{n+1}(u)$.

† In the case that w contains s_{n-1}, one can write $w = w'(s_{n-1} s_{n-2} \cdots)$. Then

$$A_5 = T_u T_v T_{w'} X_n X_{n-1} \cdots X_2 E_K, \qquad (X_n = E_n T_{n-1} T_{n-2} \cdots)$$

from where A_5 is a standard word in $\mathcal{E}_{n+1}(u)$, because in virtue of (5.1) and proposition 6, $T_u T_v T_{w'}$ is of the form T_ξ, with $\xi \in S_{n+1}$. ∎

Lemma 13. *Any $E \in \{E_i, E_{i-1} \cdots E_j, E_{i+1} E_i \cdots E_j\}$ commute with:*
(13.1) $T_i T_{i-1} \cdots T_j$
(13.2) *the elements of W_r, for all $r \leq i$.*

Proof. (i) Set $E = E_i \cdots E_j$. We can write

$$T_i T_{i-1} \cdots T_j E = T_i T_{i-1} E_i T_{i-2} E_{i-1} T_{i-3} \cdots T_{j+1} E_{j+2} T_j E_{j+1} E_j.$$

Using repetitively the relation (8) from the right to left, we deduce the claim. In the same way we obtain the proof for the case $E = E_{i+1}E_i \cdots E_j$.

(ii) Let E be $E_iE_{i-1}\cdots E_j$, and put $X_r = E_rT_{r-1}\cdots E_s \in W_r$, with $r \le i$. We have

$$X_rE = E_J(E_rT_{r-1}E_{r-1}T_{r-3}\cdots T_sE_{s-1}),$$

where $\{i, i-1, \ldots, j\} = J \cup \{r, r-1, \ldots, s\}$.

The result follows, using repetitively the relation (8) from the right to left in the expression between parenthesis. In similar way we take (13.2) for $E_{i+1}E_i\cdots E_j$. ∎

Corollary 14. *If I contains $\{i, \ldots, j\}$, then E_I commutes with $T_iT_{i-1}\cdots T_j$, and with all the elements of W_r $(r \le i)$. In particular, we have that $E_{\{1,\ldots,n\}}$ is in the center of $\mathcal{E}_n(u)$.*

Definition 15. *Set $X = T_{w_1}X_i\cdots X_2E_I$ and $Y = T_{w_2}Y_j\cdots Y_2E_J$ with $Y_j \neq 1$, two standard words in $\mathcal{E}_n(u)$. We say X reduces to Y if the product YX, is a linear combination of standard words of the form*

$$T_wZ_l\cdots Z_2E_K \qquad (l < j),$$

where $w \in S_n$, $Z_i \in W_i$, $K \in \mathcal{P}(n)$.

Lemma 16. *We have that $E_{i-1}T_{i-2}$ reduce to E_iT_{i-1}.*

Proof. This follows directly from (8):

$$(E_iT_{i-1}E_{i-1})T_{i-2} = T_{i-1}E_iE_{i-1}T_{i-2} = T_{i-1}(E_{i-1}T_{i-2})E_i. \quad ∎$$

Lemma 17. *E reduces to $E_iT_{i-1}\cdots T_j \in W_i$, for all $E \in \{E_i, E_{i-1}\cdots E_j\}$.*

Proof. (i) We have

$$\begin{aligned}(E_iT_{i-1}\cdots T_j)E_i &= (E_iT_{i-1}E_i)T_{i-2}\cdots T_j\\ &= T_{i-1}E_{i-1}E_iT_{i-2}\cdots T_j \qquad \text{(from (8))}\\ &= T_{i-1}(E_{i-1}T_{i-2}\cdots T_j)E_i.\end{aligned}$$

The words between parenthesis belong to W_{i-1}, thus the lemma holds for $E = E_i$.

(ii) In the case $E = E_{i-1}\cdots E_j$, we use lemma 13:

$$(E_iT_{i-1}\cdots T_j)E = E_iET_{i-1}T_{i-2}\cdots T_j = T_{i-1}T_{i-2}\cdots T_jE_iE. \quad ∎$$

Remark. It is obvious that any E_I reduces to $E_iT_{i-1}\cdots T_j \in W_i$, if I contains $\{i\}$ or $\{i-1, \ldots, j\}$.

Proposition 18. *Let $X_i \in W_i$, $(2 \le i \le m)$, let us put $X_m = E_mT_{m-1}\cdots T_j$, and let E be in $\{E_m, E_{m-1}\cdots E_j\}$. We have that $X_mX_{m-1}\cdots X_2E$ is a linear combination of standard words of the form*

$$T_wV_lV_{l-1}\cdots V_2E_K \in \mathcal{E}_{m+1}(u) \qquad (l < m),$$

where $w \in S_m$, $V_i \in W_i$, $K \in \mathcal{P}(m+1)$.

Proof. (i) Suppose $E = E_m$. In the case $X_m = 1$ the assertion is trivial. Set $X_m = E_mT_{m-1}T_{m-2}\cdots T_j$; we have

$$\begin{aligned}X_mX_{m-1}\cdots X_2E &= X_mEX_{m-1}\cdots X_2\\ &= (E_mT_{m-1}ET_{m-2}\cdots T_j)X_{m-1}\cdots X_2\\ &= T_{m-1}E_{m-1}ET_{m-2}\cdots T_jX_{m-1}\cdots X_2\\ &= \{(T_{m-1}E_{m-1}T_{m-2}\cdots T_j)X_{m-1}\cdots X_2\}E.\end{aligned}$$

Using proposition 12 on the word between curly brackets, the assertion follows.

(ii) Set $E = E_{m-1}\cdots E_j$. In the same way as in the proof of lemma 13 part (ii) and using (13.2), we get

$$\begin{aligned} X_m X_{m-1}\cdots X_2 E &= X_m E X_{m-1}\cdots X_2 \\ &= E_m E T_{m-1}\cdots T_j X_{m-1}\cdots X_2 \\ &= T_{m-1}\cdots T_j E_m E X_{m-1}\cdots X_2 \\ &= (T_{m-1}\cdots T_j E X_{m-1}\cdots X_2)E_m. \end{aligned}$$

As the expression in the parenthesis belongs to $\mathcal{E}_m(u)$, we deduce the claim from proposition 12. ∎

Corollary 19. *The above proposition holds for $E = E_K$, for all K that contains to $\{i\}$ or $\{i-1,\dots,j\}$.*

Set $X_{m,j_m} := E_m T_{m-1}\cdots T_{m-j_m} \in W_m$, where $1 \leq j_m \leq m-1$. And we set $X_{m,0} = 1$. With these notations, the above corollary, lemma 13 and proposition 12, we deduce the following theorem

Theorem 20. *The algebra $\mathcal{E}_n(u)$ is generated linearly by the standard words of the form*

$$T_w X_{n-1,j_{n-1}} X_{n-2,j_{n-2}} \cdots X_{2,j_2} E_K, \qquad (w \in S_n, K \in \mathcal{P}(n))$$

where if $j_m \neq 0$, then K does not contain $\{m\}$ nor $\{m-1,\dots,j_{m-1}\}$; and if $j_m = 1$, then $j_{m-1} \neq 1$.

We conjecture that the family described in the above theorem is a basis for $\mathcal{E}_n(u)$. This conjecture is supported by the case $n = 2, 3$ and 4. For instance, the family of generators $\{1, T_1, E_1, T_1E_1\}$ is a basis for $\mathcal{E}_2(u)$. And we take from the theorem a system of generators for $\mathcal{E}_3(u)$ formed by

$$T_w E_I, T_w E_2 T_1 \qquad (w \in S_3,\ I \in \mathcal{P}(3)).$$

In the next section we will prove that this family is a basis for the algebra $\mathcal{E}_3(u)$ realized as a subagebra of the Yokonuma-Hecke algebra.

4. Our algebra as subalgebra of the Yokonuma-Hecke algebra

In the following we denote by k the finite field with q elements $\mathbb{F}_q$.

Let $G = GL_n(k)$. Let B be the upper triangular subgroup of G, and let U be the unitriangular subgroup of B. Let us recall that B has a decomposition as a semidirect product $B = D \rtimes U$, where D denotes the diagonal subgroup of G. We denote by M the normalizer of D in G, which consists of all monomial matrices in G. Let us recall that the Weyl group M/D of G is isomorphic to the symmetric group S_n. Thus we can think the transposition $s_i = (i, i+1)$ as the elementary matrix

$$\begin{array}{c} \\ \\ i \\ i+1 \\ \\ \\ \end{array} \begin{pmatrix} 1 & & \cdots & & 0 \\ & \ddots & & & \\ \vdots & & 0 \quad 1 & & \vdots \\ & & 1 \quad 0 & & \\ & & & \ddots & \\ 0 & & \cdots & & 1 \end{pmatrix}.$$

We have $M \simeq D \rtimes S_n \simeq k^\times \wr S_n$.

Set $r \in k^\times$, and $1 \leq i \leq n-1$, we define the element $h_i(r)$, as the diagonal matrix with r in the position (i,i), r^{-1} in the position $(i+1, i+1)$, and 1 without. We have that the product

$h_{i,j}(r) := s_j h_i(r) s_j$ is the matrix

$$h_{i,j}(r) = \begin{pmatrix} 1 & & & \cdots & & & 0 \\ & \ddots & & & & & \\ & & r & & & & \\ \vdots & & & 1 & & & \vdots \\ & & & & r^{-1} & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & 1 \end{pmatrix}, \qquad (|i-j|=1)$$

where r is in the position (j,j) if $j < i$, and is in the position (i,i) if $j > i$. We have

$$s_i h_i(r) s_i = h_i(r^{-1}).$$

Let us consider the algebra of endomorphims $\mathcal{Y}_n(q)$ of the permutation representation of G with respect to U; which we shall call the Yokonuma-Hecke algebra. From the Bruhat decomposition for G we have that the standard basis of $\mathcal{Y}_n(q)$ is parametrized by the elements of M. Hence the dimension of $\mathcal{Y}_n(q)$ is $(q-1)^n n!$.

We shall call the operator of homothety to the elements H_t in $\mathcal{Y}_n(q)$ corresponding to $t \in D$.

Set $H_i(r)$ the homothety in $\mathcal{Y}_n(q)$ corresponding to $h_i(r)$. And set $H_{i,j}(r)$ the homothethy corresponding to $h_{i,j}(r)$. We define F_i and $F_{i,j}$ as

$$F_i := \frac{1}{q-1} \sum_{r \in k^\times} H_i(r),$$

$$F_{i,j} := \frac{1}{q-1} \sum_{r \in k^\times} H_{i,j}(r) \qquad (|i-j|=1).$$

Now, let us consider the operators $J_1 \dots, J_{n-1}$ in $\mathcal{Y}_n(q)$ defined in [4]. These operators joined with the operators of homotheties give a full description for $\mathcal{Y}_n(q)$; namely

Theorem 21 (see [4]). *The algebra $\mathcal{Y}_n(q)$ is generated by $J_1 \dots, J_{n-1}$, and H_t, $(t \in D)$. And these generators with the below relations give a presentation for $\mathcal{Y}_n(q)$.*

$$\begin{array}{rcl} J_i^2 & = & 1 + (q^{-1}-1) F_i (1 - J_i) \\ J_i J_j & = & J_j J_i \quad \text{if} \quad |i-j| > 1 \\ J_i J_j J_i & = & J_j J_i J_j \quad \text{if} \quad |i-j| = 1 \\ J_i H_t & = & H_{t'} J_i \quad \text{where} \quad t' = s_i t s_i \\ H_r H_s & = & H_{rs} \quad (r, s \in D). \end{array}$$

Corollary 22. *For all i, j, such that $|i-j| = 1$. We have:*

(22.1) $F_{i,j} = F_{j,i}$

(22.2) $F_{i,j} F_i = F_i F_{i,j} = F_i F_j$

(22.3) $J_i F_j = F_{i,j} J_i$, *and* $F_j J_i = J_i F_{i,j}$.

Now, from the braid relations between $J_1 \dots, J_{n-1}$, and the theorem of H. Matsumoto, we can define $J_w := J_{i_1} \cdots J_{i_m}$, where $w \in S_n$ take a reduced expression of the form $w = s_{i_1} \cdots s_{i_m}$.

Proposition 23 (see [4]). *A basis for $\mathcal{Y}_n(q)$ is $\{J_w H_t \,;\, w \in S_n,\, t \in D\}$.*

Theorem 24. *The operators J_i's and F_i of the Yokonuma-Hecke algebra satisfy the relations (1) to (9), when we put J_i in the place of T_i, F_i in the place of E_i, and q in the place of u.*

Proof. One deduces the relations (1), (2), (3), (4) and (6) immediately from theorem 16. The other relations follows with little effort using corollary 22; for example, we shall check the relation (9):

$$\begin{aligned} J_iF_jJ_i - J_jF_iJ_j &= F_{i,j}J_i^2 - F_{j,i}J_j^2 \qquad \text{(from 22.3)} \\ &= (q^{-1}-1)F_{i,j}(F_i(1-J_i) - F_j(1-J_j)) \\ &= (1-q^{-1})(F_{i,j}J_i - F_{i,j}J_j) \qquad \text{(from 22.2)} \\ &= (1-q^{-1})(F_jJ_iF_j - F_iJ_jF_i). \quad \blacksquare \end{aligned}$$

Proposition 25. *The family of generators in theorem 20 is a basis for $\mathcal{E}_3(q)$.*

Proof. We must prove that if

$$\sum_{\substack{w\in S_3\\ I\in\mathcal{P}(3)}} \alpha_{w,I}T_wF_I + \sum_{w\in S_3}\beta_wJ_wF_2J_1 = 0,$$

then $\alpha_{w,I} = \beta_w = 0$, for all $w \in S_3$, $I \in \mathcal{P}(3)$.

Now, from (22.3) we have $J_wF_2J_1 = J_wJ_1F_3$, where $F_3 := F_{1,2}$. Using (4) on J_1, and corollary 21, we get

$$\begin{aligned} J_1J_1F_3 &= F_3 + \lambda F - \lambda J_1F \\ J_2J_1J_1F_3 &= J_2F_3 + \lambda J_2F - \lambda J_2J_1F \\ J_1J_2J_1J_1F_3 &= J_1J_2F_3 + \lambda J_1J_2F - \lambda J_1J_2J_1F, \end{aligned}$$

where $F := F_1F_3 = F_2F_3$, and $\lambda = q^{-1} - 1$.

Thus the equation in question can be written as

$$\sum_{\substack{w\in S_3\\ \{1,2\}\neq I\in\mathcal{P}(3)}} \alpha_{w,I}J_wF_I + \sum_{w\in S_3}\beta_wJ_wF_3 + \sum_{w'}(\alpha_{w'}+\lambda\beta_{w's_1})J_{w'}F + \sum_{w''}(\alpha_{w''}-\lambda\beta_{w''})J_{w''}F = 0,$$

where $w' \in \{1, s_2, s_1s_2\}$, $w'' \in \{s_1, s_2s_1, s_1s_2s_1\}$.

The assertion follows using proposition 23, and the elementary argument of the linear algebra on substitution in a basis. $\blacksquare$

5. Yang-Baxterization

Our algebra has no BW-structure, but admits the procedure of the Yang-Baxterization of [2].

Proposition 26. *The algebra $\mathcal{E}_n(u)$ can be Yang-Baxterized.*

Proof. We will use the notations of §3.[2]. First, from (19) the element T_i is a solution of the cubic equation $x^3 + u^{-1}x^2 - x - u^{-1} = 0$. Let us put $\lambda_1 = -u^{-1}$, $\lambda_2 = -1$ and $\lambda_3 = 1$. We get

$$f_3^+ = -\frac{1}{u}, \qquad f_3^- = -\frac{1}{u^2}, \qquad f_2 = f_1^+ = f_1^- = 0.$$

According to theorem §3.[2], the algebra can be Yang-Baxterized if the following equation holds $f_3^+\theta_3^+ + f_3^-\theta_3^- + f_2\theta_2 + f_1^+\theta_1^+ + f_1^-\theta_1^- = 0$. Now, this equation takes the form

$$u\theta_3^+ + \theta_3^- = 0,$$

where $\theta_3^+ = T_iT_{i+1}^{-1}T_i - T_{i+1}T_i^{-1}T_{i+1}$ and $\theta_3^- = T_i^{-1}T_{i+1}T_i^{-1} - T_{i+1}^{-1}T_iT_{i+1}^{-1}$. The proposition follows from (11). $\blacksquare$

6. Proof of lemmas 8,9 and 10

In this section m is a natural number greater than i.

6.1. **Proof of lemma 8.** The proof is by induction over m. For $m = i+1$ the lemma is the part 7.1 of lemma 7. Suppose the lemma is true for any natural number less than m. Let us take the word $V = (E_i T_{i+1} \cdots T_{m-1}) T_m \in \mathcal{E}_{m+1}(u)$. Using the hypothesis of induction on the word between parenthesis we have that V is a linear combination of elements of the type:

$$\begin{aligned} X &= (T_{i+1}T_i)\cdots(T_{m-1}T_{m-2})E_{m-1}T_m A \\ Y &= T_{i+1}T_{i+2}\cdots T_{m-1}E_{m-1}T_m B \\ Z &= T_{i+1}T_{i+2}\cdots T_{m-1}T_j\cdots T_{m-2}E_{m-1}T_m C_j, \qquad (i+1 \leq j \leq m-2) \end{aligned}$$

where A, B, $C_j \in \mathcal{E}_{m-1}(u)$.

Now from lemma 7, we get

$$E_{m-1}T_m = T_m T_{m-1} E_m x + T_m E_m y \qquad (x, y \in \mathcal{E}_m(u)). \tag{8A}$$

Using this relation we will prove that X, Y and Z are linear combinations of elements of type (8.1), (8.2) and (8.3).

• From 8A we have that X is a linear combination of elements of the type:

$$\begin{aligned} X_1 &= (T_{i+1}T_i)\cdots(T_{m-1}T_{m-2})T_m T_{m-1}E_m x A, \\ X_2 &= (T_{i+1}T_i)\cdots(T_{m-1}T_{m-2})T_m E_m y A. \end{aligned}$$

Now, for X_2 we have

$$\begin{aligned} X_2 &= (T_{i+1}T_i)\cdots(T_{m-2}T_{m-3})(T_{m-1})T_m E_m T_{m-2} y A \\ &\vdots \\ X_2 &= (T_{i+1}\cdots T_m E_m)(T_i T_{i+1}\cdots T_{m-3}T_{m-2} y A). \end{aligned}$$

(notice that the elements in the parenthesis belong to $\mathcal{E}_m(u)$). We have that X_1 is of the form (8.1), and X_2 is of the form (8.2). Therefore X is a linear combination of the desired elements .

• From 8A, Y is a linear combination of elements of the type

$$\begin{aligned} Y_1 &= T_{i+1}T_{i+2}\cdots T_{m-1}T_m T_{m-1}E_m x B, \\ Y_2 &= T_{i+1}T_{i+2}\cdots T_{m-1}T_m E_m y B. \end{aligned}$$

Thus Y_1 is the form (8.3), and Y_2 is the form (8.2). Consequently Y is a linear combination of the desired elements.

• From 8A, Z is a linear combination of elements of the type

$$\begin{aligned} Z_1 &= T_{i+1}T_{i+2}\cdots T_{m-1}T_j\cdots T_{m-2}T_m T_{m-1}E_m x C_j, \\ Z_2 &= T_{i+1}T_{i+2}\cdots T_{m-1}T_j\cdots T_{m-2}T_m E_m y C_j. \end{aligned}$$

In Z_1 moving T_m to the left, we get

$$Z_1 = T_{i+1}T_{i+2}\cdots T_{m-1}T_m T_j\cdots T_{m-2}T_{m-1}E_m x C_j,$$

which is an element of the form (8.3).

In Z_2 we can move $T_m E_m$ to the left, then we get

$$Z_2 = T_{i+1}T_{i+2}\cdots T_{m-1}T_m E_m (T_j\cdots T_{m-2} y C_j).$$

As the element in the parenthesis is in $\mathcal{E}_m(u)$, we obtain that Z_2 is of the form (8.2). ∎

6.2. **Proof of lemma 9.** Again the proof is by induction over m. For $m = i+1$ the lemma is the part 7.2 of lemma 7. Suppose the lemma is true for any natural number less than m. Let $V = (L_i T_{i+1}\cdots T_{m-1})T_m \in \mathcal{E}_{m+1}(u)$. By hypothesis of induction on the word between parenthesis we have that V is a linear combination of elements of the type:

$$\begin{aligned} X &= (T_{i+1}T_i)\cdots(T_{m-1}T_{m-2})L_{m-1}T_m A \\ Y &= T_i T_{i+1}T_{i+2}\cdots T_{m-2}L_{m-1}T_m B \\ Z &= T_{m-1-j}T_{m-j}\cdots T_{m-1}T_i T_{i+1}\cdots T_{m-2}L_{m-1}T_m C_j, \qquad (0 \leq j \leq m-3) \end{aligned}$$

where A, B, $C_j \in \mathcal{E}_{m-1}(u)$.

From part 7.2 of lemma 7, we get

$$L_{m-1}T_m = T_mT_{m-1}L_mx + T_{m-1}L_my \qquad (x, y \in \mathcal{E}_m(u)). \tag{9A}$$

We are going to prove that X, Y and Z are linear combinations of word of the type (9.1), (9.2) and (9.3).

• From 9A, X is a linear combination of

$$\begin{aligned} X_1 &= (T_{i+1}T_i)\cdots(T_{m-1}T_{m-2})T_mT_{m-1}L_mxA, \\ X_2 &= (T_{i+1}T_i)\cdots(T_{m-1}T_{m-2})T_{m-1}L_myA. \end{aligned}$$

It is obvious that X_1 is the form (9.1). In X_2 we have

$$\begin{aligned} X_2 &= \{(T_{i+1}T_i)\cdots(T_{m-2}T_{m-3})(T_{m-1}T_{m-2})T_{m-1}\}L_myA \\ &= (T_i\cdots T_{m-1})(T_i\cdots T_{m-2})L_myA \quad \text{(from (23))} \\ &= T_i\cdots T_{m-1}L_m(T_i\cdots T_{m-2}yA). \end{aligned}$$

As the element in parenthesis is in $\mathcal{E}_m(u)$, we have that X_2 is of the form (9.2).

• From 9A, Y is a linear combination of:

$$\begin{aligned} Y_1 &= T_iT_{i+1}T_{i+2}\cdots T_{m-2}T_mT_{m-1}L_mxB, \\ Y_2 &= T_iT_{i+1}T_{i+2}\cdots T_{m-2}T_{m-1}L_myB. \end{aligned}$$

In Y_1 moving T_m to the left, we get $Y_1 = T_mT_iT_{i+1}T_{i+2}\cdots T_{m-2}T_{m-1}L_mxA$, which is of the form (9.3).

It is obvious that Y_2 is of the form (9.2).

• Using 9A in Z, we get that Z is a linear combination of elements Z_1 and Z_2:

$$\begin{aligned} Z_1 &= T_{m-1-j}T_{m-j}\cdots T_{m-1}T_iT_{i+1}\cdots T_{m-2}T_mT_{m-1}L_mxC_j \\ Z_2 &= T_{m-1-j}T_{m-j}\cdots T_{m-1}T_iT_{i+1}\cdots T_{m-2}T_{m-1}L_myC_j. > \end{aligned}$$

Moving T_m to the left in Z_1, we get

$$Z_1 = T_{m-1-j}T_{m-j}\cdots T_{m-1}T_mT_iT_{i+1}\cdots T_{m-2}T_{m-1}L_mxC_j,$$

which is the form (9.3).

In Z_2 moving T_{m-1} to the right and using the braid relation on $T_{m-1}T_{m-2}T_{m-1}$, we have

$$\begin{aligned} Z_2 &= T_{m-1-j}T_{m-j}\cdots T_{m-2}T_iT_{i+1}\cdots T_{m-3}T_{m-1}T_{m-2}T_{m-1}L_myC_j \\ &= T_{m-1-j}T_{m-j}\cdots T_{m-2}T_iT_{i+1}\cdots T_{m-3}T_{m-2}T_{m-1}T_{m-2}L_myC_j \\ &= T_{m-1-j}T_{m-j}\cdots T_{m-2}T_iT_{i+1}\cdots T_{m-3}T_{m-2}T_{m-1}L_mT_{m-2}yC_j \\ &\vdots \\ Z_2 &= T_iT_{i+1}\cdots T_{m-2}T_{m-1}L_m(T_{m-2-j}\cdots T_{m-2}yC_j). \end{aligned}$$

As the element in parenthesis is in $\mathcal{E}_m(u)$, we have that Z_2 is of the form (9.2). ∎

6.3. **Proof of lemma 10.** The lemma is obvious, if the product $\Upsilon_i := Y_i\cdots Y_m \in \mathcal{E}_{m+1}(u)$ is a product that contains only $T's$, or only E's. In the case when Υ_i contains only one $F_l \in \{E_l, L_l\}$, the lemma follows from lemmas 8 and 9.

Suppose that Υ_l contains d ($d > 1$) elements F's. Let r be the first position, from left to right, where appear one F in Υ_i. We are going to see that in Υ_i the number of F's "can be reduced". More precisely, we will see that Υ_i is a linear combination of elements of the form

$$\Upsilon_k\alpha \quad \text{and} \quad \Upsilon'_i\alpha, \tag{10A}$$

where $k > r$, $\alpha \in \mathcal{E}_m(u)$, and the first F in Υ'_i appears in the position l, $l > r$.

Then the lemma will follow by an inductive argument.

We have

$$\Upsilon_i = T_iT_{i+1}\cdots T_{r-1}F_rY_{r+1}\cdots Y_m \in \mathcal{E}_{m+1}(u). \qquad (i \le r)$$

Now, we distinguish between two cases: the successive F is in the position $r+1$ (case 1) or not (case 2).

Case 1. Here $\Upsilon_i = T_iT_{i+1}\cdots T_{r-1}F_rF_{r+1}Y_{r+2}\cdots Y_{m-1}Y_m$.

(i) If $F_r = E_r$, we have

$$\begin{aligned}\Upsilon_i &= T_iT_{i+1}\cdots T_{r-1}E_rF_{r+1}Y_{r+2}\cdots Y_m \\ &= T_iT_{i+1}\cdots T_{r-1}F_{r+1}E_rY_{r+2}\cdots Y_m \qquad \text{(from (8))} \\ &= F_{r+1}Y_{r+2}\cdots Y_m(T_iT_{i+1}\cdots T_{r-1}E_r).\end{aligned}$$

So Υ_i is of the form (10A).

(ii) Set $F_r = L_r = E_rT_r$. In the case $F_{r+1} = E_{r+1}$, we have

$$\begin{aligned}\Upsilon_i &= T_iT_{i+1}\cdots T_{r-1}L_rE_{r+1}Y_{r+2}\cdots Y_m \\ &= T_iT_{i+1}\cdots T_rE_{r+1}E_rY_{r+2}\cdots Y_m \qquad \text{(from (8))} \\ &= T_iT_{i+1}\cdots T_rE_{r+1}Y_{r+2}\cdots Y_mE_r.\end{aligned}$$

Thus Υ_i is of the form 10A.

If $F_{r+1} = L_{r+1}$, using (8) we get $L_rL_{r+1} = T_rL_{r+1}E_r$. Then,

$$\Upsilon_i = T_iT_{i+1}\cdots T_rL_{r+1}Y_{r+2}\cdots Y_mE_r,$$

which is of the form (10A).

Case 2. In this case Υ_i one write

$$\text{(10B)} \qquad \Upsilon_i = T_iT_{i+1}\cdots T_{r-1}F_rT_{r+1}\cdots T_lF_{l+1}Y_{l+2}\cdots Y_m \qquad (l > r).$$

We will prove that (10B) is a linear combination of elements of the form (10A). For this we distinguish four possibilities, according to $F_r \in \{E_r, L_r\}$, and $F_{l+1} \in \{E_{l+1}, L_{l+1}\}$.

Case $F_r = E_r$ and $F_{l+1} = E_{l+1}$. In this case Υ_i becomes

$$\Upsilon_i = T_iT_{i+1}\cdots T_{r-1}(E_rT_{r+1}\cdots T_l)E_{l+1}Y_{l+2}\cdots Y_m.$$

Using lemma 8 on the word in parenthesis, we have that (10B) is a linear combination of the following words:

$$\begin{aligned}X_1 &= T_iT_{i+1}\cdots T_{r-1}((T_{r+1}T_r)\cdots(T_lT_{l-1})E_l)E_{l+1}Y_{l+2}\cdots Y_mA \\ X_2 &= T_iT_{i+1}\cdots T_{r-1}(T_{r+1}T_{r+2}\cdots T_lE_l)E_{l+1}Y_{l+2}\cdots Y_mB \\ X_3 &= T_iT_{i+1}\cdots T_{r-1}(T_{r+1}T_{r+2}\cdots T_{l-1}T_lT_j\cdots T_{l-1}E_l)E_{l+1}Y_{l+2}\cdots Y_mC_j,\end{aligned}$$

where A, B and $C_j \in \mathcal{E}_m(u)$.

Moving $T_{l-1}E_l$ to the right, we get

$$\begin{aligned}X_1 &= T_iT_{i+1}\cdots T_{r-1}(T_{r+1}T_r)\cdots(T_{l-1}T_{l-2})T_lE_{l+1}Y_{l+2}\cdots Y_mT_{l-1}E_lA \\ &= T_iT_{i+1}\cdots T_{r-1}(T_{r+1}\cdots T_{l-1})(T_r\cdots T_{l-2})T_lE_{l+1}Y_{l+2}\cdots Y_mT_{l-1}E_lA \\ &= (T_{r+1}\cdots T_{l-1})T_iT_{i+1}\cdots T_{r-1}(T_r\cdots T_{l-2})T_lE_{l+1}Y_{l+2}\cdots Y_mT_{l-1}E_lA \\ &= (T_{r+1}\cdots T_{l-1})T_lE_{l+1}Y_{l+2}\cdots Y_m(T_iT_{i+1}\cdots T_{r-1}T_r\cdots T_{l-2})T_{l-1}E_lA.\end{aligned}$$

Then X_1 is of the form (10A).

In X_2, we have

$$\begin{aligned}X_2 &= T_iT_{i+1}\cdots T_{r-1}(T_{r+1}T_{r+2}\cdots T_l)E_{l+1}Y_{l+2}\cdots Y_mE_lB \\ &= T_{r+1}T_{r+2}\cdots T_lE_{l+1}Y_{l+2}\cdots Y_m(T_iT_{i+1}\cdots T_{r-1})E_lB,\end{aligned}$$

which is an element of the form (10A).

In X_3, we have

$$\begin{aligned}X_3 &= T_iT_{i+1}\cdots T_{r-1}(T_{r+1}T_{r+2}\cdots T_{l-1}T_lT_j\cdots T_{l-1})E_{l+1}Y_{l+2}\cdots Y_mE_lC_j \\ &= T_iT_{i+1}\cdots T_{r-1}(T_{r+1}T_{r+2}\cdots T_{l-1})T_lE_{l+1}Y_{l+2}\cdots Y_mT_j\cdots T_{l-1}E_lC_j \\ &= T_{r+1}T_{r+2}\cdots T_{l-1}T_lE_{l+1}Y_{l+2}\cdots Y_m(T_iT_{i+1}\cdots T_{r-1}T_j\cdots T_{l-1}E_l)C_j,\end{aligned}$$

which is of the form (10A).

Thus as X_1, X_2 and X_3 are of the form (10A) it follows that in this case (10B) is a linear combination of the elements of the form (10A).

Case $F_r = E_r$ and $F_{l+1} = L_{l+1}$. In this case for Υ_i we have

$$\Upsilon_i = T_i T_{i+1} \cdots T_{r-1}(E_r T_{r+1} \cdots T_l) L_{l+1} Y_{l+2} \cdots Y_m.$$

As for the preceding case, we have that Υ_i, is a linear combination of words of the type

$$\begin{aligned}
Y_1 &= T_i T_{i+1} \cdots T_{r-1}((T_{r+1}T_r) \cdots (T_l T_{l-1}) E_l) L_{l+1} Y_{l+2} \cdots Y_m A \\
Y_2 &= T_i T_{i+1} \cdots T_{r-1}(T_{r+1} T_{r+2} \cdots T_l E_l) L_{l+1} Y_{l+2} \cdots Y_m B \\
Y_3 &= T_i T_{i+1} \cdots T_{r-1}(T_{r+1} T_{r+2} \cdots T_{l-1} T_l T_j \cdots T_{l-1} E_l) L_{l+1} Y_{l+2} \cdots Y_m C_j,
\end{aligned}$$

where A, B and $C_j \in \mathcal{E}_m(u)$.

From case 1(i) we have

$$\begin{aligned}
Y_1 &= T_i T_{i+1} \cdots T_{r-1}((T_{r+1}T_r) \cdots (T_l T_{l-1})) L_{l+1} Y_{l+2} \cdots Y_m E_l A \\
&= T_i T_{i+1} \cdots T_{r-1}(T_{r+1} \cdots T_l)(T_r \cdots T_{l-1}) L_{l+1} Y_{l+2} \cdots Y_m E_l \\
&= (T_{r+1} \cdots T_l)(T_i T_{i+1} \cdots T_{r-1})(T_r \cdots T_{l-1}) L_{l+1} Y_{l+2} \cdots Y_m E_l A \\
&= (T_{r+1} \cdots T_l) L_{l+1} Y_{l+2} \cdots Y_m (T_i T_{i+1} \cdots T_{r-1})(T_r \cdots T_{l-1}) E_l A
\end{aligned}$$

which is of the form (10A).

$$\begin{aligned}
Y_2 &= T_i T_{i+1} \cdots T_{r-1}(T_{r+1} T_{r+2} \cdots T_l) L_{l+1} Y_{l+2} \cdots Y_m E_l B \\
&= T_{r+1} T_{r+2} \cdots T_l L_{l+1} Y_{l+2} \cdots Y_m (T_i T_{i+1} \cdots T_{r-1} E_l B),
\end{aligned}$$

then Y_2 is of the form (10A).

$$\begin{aligned}
Y_3 &= T_i T_{i+1} \cdots T_{r-1}(T_{r+1} T_{r+2} \cdots T_{l-1} T_l)(T_j \cdots T_{l-1}) L_{l+1} Y_{l+2} \cdots Y_m E_l C_j \\
&= T_i T_{i+1} \cdots T_{r-1}(T_{r+1} T_{r+2} \cdots T_{l-1} T_l) L_{l+1} Y_{l+2} \cdots Y_m (T_j \cdots T_{l-1}) E_l C_j \\
&= T_{r+1} T_{r+2} \cdots T_{l-1} T_l L_{l+1} Y_{l+2} \cdots Y_m (T_i T_{i+1} \cdots T_{r-1})(T_j \cdots T_{l-1}) E_l C_j,
\end{aligned}$$

which is of the form (10A).

Therefore, in this case, (10B) is a linear combination of the elements of the type (10A).

Case $F_r = L_r$ and $F_{l+1} = L_{l+1}$. In this case

$$\Upsilon_i = T_i T_{i+1} \cdots T_{r-1}(L_r T_{r+1} \cdots T_l) E_{l+1} Y_{l+2} \cdots Y_m.$$

Using lemma 9, we have that Υ_i is a linear combination of the following elements

$$\begin{aligned}
Z_1 &= T_i T_{i+1} \cdots T_{r-1}((T_{r+1}T_r) \cdots (T_l T_{l-1}) L_l) E_{l+1} Y_{l+2} \cdots Y_m A \\
Z_2 &= T_i T_{i+1} \cdots T_{r-1}(T_r T_{r+1} T_{r+2} \cdots T_{l-1} L_l) E_{l+1} Y_{l+2} \cdots Y_m B \\
Z_3 &= T_i T_{i+1} \cdots T_{r-1}(T_{l-j} T_{l-j+1} \cdots T_{l-1} T_l T_r T_{r+1} \cdots T_{l-1} L_l) E_{l+1} Y_{l+2} \cdots Y_m C_j,
\end{aligned}$$

where A, B and $C_j \in \mathcal{E}_m(u)$.

Using case 1, we get for Z_1, Z_2, and Z_3:

$$\begin{aligned}
Z_1 &= T_i T_{i+1} \cdots T_{r-1}\{(T_{r+1}T_r) \cdots (T_l T_{l-1}) T_l\} E_{l+1} Y_{l+2} \cdots Y_m E_l A \\
&= (T_i T_{i+1} \cdots T_{r-1})(T_r \cdots T_l)(T_r \cdots T_{l-1}) E_l Y_{l+1} \cdots Y_m E_l A \qquad \text{(from (23))} \\
&= (T_i T_{i+1} \cdots T_{r-1})(T_r \cdots T_l) E_l Y_{l+1} \cdots Y_m (T_r \cdots T_{l-1}) E_l A \\
&= T_i T_{i+1} \cdots T_l E_{l+1} Y_{l+2} \cdots Y_m (T_r \cdots T_{l-1} E_l A).
\end{aligned}$$

$$Z_2 = T_i T_{i+1} \cdots T_{r-1}(T_r T_{r+1} T_{r+2} \cdots T_{l-1} T_l) E_{l+1} Y_{l+2} \cdots Y_m E_l B.$$

$$
\begin{aligned}
Z_3 &= T_i T_{i+1} \cdots T_{r-1}(T_{l-j} T_{l-j+1} \cdots T_{l-1} T_l T_r T_{r+1} \cdots T_{l-1} T_l) E_{l+1} Y_{l+2} \cdots Y_m E_l C_j \\
&= T_i T_{i+1} \cdots T_{r-1}(T_{l-j} T_{l-j+1} \cdots T_{l-1} T_r T_{r+1} \cdots T_l T_{l-1} T_l) E_{l+1} Y_{l+2} \cdots Y_m E_l C_j \\
&= T_i T_{i+1} \cdots T_{r-1}(T_{l-j} T_{l-j+1} \cdots T_{l-1} T_r T_{r+1} \cdots T_{l-1} T_l T_{l-1}) E_{l+1} Y_{l+2} \cdots Y_m E_l C_j \\
&= T_i T_{i+1} \cdots T_{r-1}(T_{l-j} T_{l-j+1} \cdots T_{l-1} T_r T_{r+1} \cdots T_{l-1} T_l) E_{l+1} Y_{l+2} \cdots Y_m T_{l-1} E_l C_j \\
&\ \vdots \\
&= T_i T_{i+1} \cdots T_{l-1} T_l E_{l+1} Y_{l+2} \cdots Y_m T_{l-j-1} T_{l-j} \cdots T_{l-1} E_l C_j .
\end{aligned}
$$

Thus Z_1, Z_2, and Z_3 are of the form (10A). Then in this case (10B) is a linear combination of elements of the type (10A).

Case $F_r = L_r$ and $F_{l+1} = L_{l+1}$. In this case Υ_i takes the form

$$\Upsilon_i = T_i T_{i+1} \cdots T_{r-1}(L_r T_{r+1} \cdots T_l) L_{l+1} Y_{l+2} \cdots Y_m .$$

From lemma 9, we get

$$
\begin{aligned}
V_1 &= T_i T_{i+1} \cdots T_{r-1}((T_{r+1} T_r) \cdots (T_l T_{l-1}) L_l) L_{l+1} Y_{l+2} \cdots Y_m A \\
V_2 &= T_i T_{i+1} \cdots T_{r-1}(T_r T_{r+1} T_{r+2} \cdots T_{l-1} L_l) L_{l+1} Y_{l+2} \cdots Y_m B \\
V_3 &= T_i T_{i+1} \cdots T_{r-1}(T_{l-j} T_{l-j+1} \cdots T_{l-1} T_l T_r T_{r+1} \cdots T_{l-1} L_l) L_{l+1} Y_{l+2} \cdots Y_m C_j ,
\end{aligned}
$$

where A, B and $C_j \in \mathcal{E}_m(u)$.

To finish this proof, it is enough to see that V_1, V_2, and V_3 are of the form (10A). We do this below.

$$
\begin{aligned}
V_1 &= T_i T_{i+1} \cdots T_{r-1}((T_{r+1} T_r) \cdots (T_l T_{l-1}) T_l) L_{l+1} Y_{l+2} \cdots Y_m E_l A \\
&= T_i T_{i+1} \cdots T_{r-1}(T_r \cdots T_l)(T_r \cdots T_{l-1}) L_{l+1} Y_{l+2} \cdots Y_m E_l A \qquad \text{(from (23))} \\
&= T_i T_{i+1} \cdots T_{r-1}(T_r \cdots T_l) L_{l+1} Y_{l+2} \cdots Y_m (T_r \cdots T_{l-1}) E_l A \\
&= T_i T_{i+1} \cdots T_l L_{l+1} Y_{l+2} \cdots Y_m (T_r \cdots T_{l-1} E_l A).
\end{aligned}
$$

$$
\begin{aligned}
V_2 &= T_i T_{i+1} \cdots T_{r-1}(T_r T_{r+1} T_{r+2} \cdots T_{l-1} L_l) L_{l+1} Y_{l+2} \cdots Y_m B \\
&= T_i T_{i+1} \cdots T_{r-1}(T_r T_{r+1} T_{r+2} \cdots T_{l-1} T_l) L_{l+1} Y_{l+2} \cdots Y_m E_l B.
\end{aligned}
$$

$$
\begin{aligned}
V_3 &= T_i T_{i+1} \cdots T_{r-1}(T_{l-j} T_{l-j+1} \cdots T_{l-1} T_l T_r T_{r+1} \cdots T_{l-1} T_l) L_{l+1} Y_{l+2} \cdots Y_m E_l C_j \\
&= T_i T_{i+1} \cdots T_{r-1}(T_{l-j} T_{l-j+1} \cdots T_{l-1} T_r T_{r+1} \cdots T_l T_{l-1} T_l) L_{l+1} Y_{l+2} \cdots Y_m E_l C_j \\
&= T_i T_{i+1} \cdots T_{r-1}(T_{l-j} T_{l-j+1} \cdots T_{l-1} T_r T_{r+1} \cdots T_{l-1} T_l T_{l-1}) L_{l+1} Y_{l+2} \cdots Y_m E_l C_j \\
&= T_i T_{i+1} \cdots T_{r-1}(T_{l-j} T_{l-j+1} \cdots T_{l-1} T_r T_{r+1} \cdots T_{l-1} T_l) L_{l+1} Y_{l+2} \cdots Y_m T_{l-1} E_l C_j \\
&\ \vdots \\
&= T_i T_{i+1} \cdots T_{l-1} T_l L_{l+1} Y_{l+2} \cdots Y_m T_{l-j-1} \cdots T_{l-1} E_l C_j . \quad \blacksquare
\end{aligned}
$$

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