

# A Note on Interactions of (Non-Commutative) Instantons Via AdS/CFT

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We consider the interaction between instantons and anti-instantons in four-dimensional  $\mathcal{N} = 4$  super-Yang–Mills theory at large  $N$  and large 't Hooft coupling as described by D-instantons via AdS/CFT duality. We give an estimate of the strength of the interaction in various regimes. We discuss also the case of Non-Commutative super-Yang–Mills theory where the interaction between instantons and anti-instantons can be used as a way to probe the locality properties of the theory in the supergravity picture, without explicit reference to the definition of local operators.

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## 1. Introduction and Conclusions

One of the most successful tests of the AdS/CFT correspondence of refs. [1,2] is provided by instanton physics. In supersymmetric theories, many aspects of instanton dynamics (such as the moduli space and the measure) are of BPS-type, i.e. they are protected and thus provide useful tests of the duality.

The AdS/CFT picture of Yang–Mills instantons is in terms of  $D(p-4)$ -brane probes in the background geometry (vacuum) of  $Dp$ -branes [3]. Concentrating in the case of the  $\mathcal{N} = 4$  SYM theory, dual to type IIB string theory on  $AdS_5 \times \mathbf{S}^5$ , we can write the background metric as

$$\frac{ds^2}{R^2} = \frac{1}{\rho^2} (d\rho^2 + d\vec{x}^2) + d\Omega_5^2, \quad (1.1)$$

where the radius  $R$  is determined by the Regge slope and string coupling by  $R^4 = 4\pi \alpha'^2 g_s N = \alpha'^2 \lambda$ . We have also defined the 't Hooft coupling  $\lambda = g_s N = g_{\text{YM}}^2 N$  controlling the large- $N$  expansion of the gauge theory. The coordinates  $\vec{x}$  parametrize the  $\mathbf{R}^4$  boundary giving space-time data for the gauge theory, while  $\rho$  represents the gauge-theory length scale according to the UV/IR relation. For example, a D-instanton probe sitting at  $(\rho, \vec{x})$  is interpreted as a Yang–Mills instanton of size parameter  $\rho$  located at point  $\vec{x}$  in  $\mathbf{R}^4$ .

The calculations of instanton-dominated BPS amplitudes using the AdS/CFT rules match dramatically with the perturbative computations recently done at weak coupling [4]. In this paper instead we are interested in studying non-BPS configurations containing both instantons and anti-instantons (thus breaking all supersymmetries) still using semi-classical physics<sup>2</sup>. Of course in this case we do not expect the perturbative results to match the supergravity ones, which should describe the physics at strong coupling. The results we obtain are then predictions which could help us in understanding the strong coupling regime of large- $N$   $\mathcal{N} = 4$  SYM theory.

Since the dual description is gauge invariant, it is problematic to describe the collective coordinates associated to relative gauge orientations in a multi-instanton configuration. Presumably we should interpret the gravitational description as providing the result of having integrated out the gauge collective coordinates. On the other hand, the D-instanton geometrical moduli corresponding to the location in  $\mathbf{S}^5$  do have gauge-theory interpretation in terms of new collective coordinates parametrizing large  $N$  saddle-point approximations of the instanton measure [4]. Therefore, as long as we have the hierarchy  $1 \ll \lambda \ll N$  we should be able to study instanton dynamics of the SYM theory via dilute D-instantons in the supergravity approximation to IIB strings on  $AdS_5 \times \mathbf{S}^5$ .

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<sup>2</sup> See [5] for other aspects of non-supersymmetric instanton physics.

Since the D-instantons are fully localized in ten dimensions, we expect to find physical quantities in the four-dimensional SYM theory revealing (at least for  $\lambda \gg 1$ ) a ten-dimensional scaling. The interaction of I/A pairs is a convenient probe of the localization properties, in the sense that the interaction will show ten-dimensional features if the proper distance between the I/A pair is much smaller than the curvature radius of  $AdS_5 \times S^5$ . For sufficiently large values of the 't Hooft coupling, this condition is compatible with the dilute character of the D-instanton gas in supergravity. In fact, we find that the dilute D-instanton gas in supergravity corresponds to the dilute gas of SYM instantons *only* in the limit where the I/A distances are much larger than the curvature radius, i.e. when no ten-dimensional features are revealed. On the other hand, the fully localized regime mentioned above is to be interpreted as a transient phase of overlapping YM instantons, i.e. the geometrical notions of “diluteness” are slightly different on both sides of the AdS/CFT correspondence.

The occurrence in four-dimensional gauge theories of regimes with ten-dimensional features via the AdS/CFT duality is not new. For example, a similar phenomenon is found in the study of the dependence of the entropy on the energy scales of the theory, i.e. there are intermediate energy regimes, visible at large  $\lambda$ , which show ten-dimensional scaling of the entropy [6]. One of these regimes is characterized by fully localized configurations in  $AdS_5 \times S^5$  where the density of states is well approximated by that of ten-dimensional Schwarzschild black holes. We could present our results as the euclidean, semiclassical analog of these localized states. The analogy with the entropy estimates is rather close, since we are also looking at non-BPS quantities and the large- $\lambda$  results are to be interpreted as predictions of AdS/CFT, rather than tests to be satisfied.

We shall furthermore compare the results with the case where we add a non-vanishing Neveu–Schwarz (NS)  $B$ -field to the background configuration. Indeed, given the embedding of Non-Commutative Yang–Mills theories (NCYM) in open string dynamics with non-vanishing Neveu–Schwarz (NS)  $B$ -field [7,8,9], the possibility opens up of studying the large  $N$  limit of these systems by means of generalizations of the AdS/CFT correspondence. In the recent works [10] and [11] a gravitational dual of large  $N$  NCYM was proposed as the near-horizon geometry of  $Dp$ -branes with non-trivial profiles of NS  $B$ -fields, under the appropriate scaling of parameters that isolates the low energy regime with NCYM physics.

We will consider the interaction action of an instanton/anti-instanton pair as an intermediate probe into the locality properties of the NCYM theory. Although the dependence of such quantity on the pair’s spatial separation is providing some local information, one can compute it using the AdS/CFT rules without explicitly defining local operators via boundary behaviour, a rather problematic procedure in view of the results of [11]. In particular, one can define this quantity in terms of the *bulk* D-instanton/anti-D-instanton

interaction action in Type IIB superstring theory

We will thus be able to see how the non-locality properties of NCYM are reflected in a different behaviour of the I/A interactions. We find evidence of an effective delocalization of instantons of size smaller than the non-commutativity length scale.

## 2. The UV/IR relation and I/A interactions

In presence of a D-instanton half of the supersymmetries are broken, and there are sixteen supersymmetric zero modes or, equivalently, supersymmetric collective coordinates. To have a non vanishing interaction between the D-instanton and anti-D-instanton, the sixteen zero modes must be saturated. Integrating over the fermionic collective coordinates is equivalent to summing over all components of the supersymmetric multiplet to which the D-instanton belongs. We can give a fully fledged string treatment in the context of the boundary state formalism, for the case of D-instanton interactions in flat space [12,13,14]. In the low-energy approximation, the D-instanton multiplet is replaced by a set of effective operators with up to sixteen fermionic legs. By the usual supersymmetric power-counting, a pair of fermions corresponds to one derivative. Therefore, to leading order in the low-energy expansion, an effective operator with  $N_f$  fermions must have  $8 - N_f/2$  derivatives to saturate the sixteen zero modes.

We should think of these effective operators as the result of having integrated out the massive string modes. Therefore, for consistency, the instanton/anti-instanton gases that we consider must be *dilute* in the sense of the superstring background, i.e. the proper distance between topological defects should be much larger than the string scale. In this way the short-distance stringy singularities of the I/A interaction are also avoided. The restriction to dilute configurations is also technically required for the I/A pair to be well-defined as an approximate non-perturbative configuration. This is important because the I/A pairs are in the same topological sector as the perturbative excitations.

The full set of effective operators in type IIB string theory, contributing up to relative order  $O(\alpha'^3)$  can be summarized in the  $(R_{\mu\nu\rho\sigma})^4$  terms of the type IIB supergravity together with their superpartners. Such operators are explicitly constructed for all channels in refs. [14]. Therefore, the supergravity approximation can be extended to the analysis of I/A interactions in curved backgrounds, such as  $AdS_5 \times S^5$ .

We can thus compute the D-instanton/anti-D-instanton interaction by evaluating the vacuum amplitudes of these effective operators. Among them, the most characteristic is the operator with sixteen dilatinos and no derivatives (this operator determines the instanton measure), for which the I/A interaction is simply given by the single fermionic zero mode

exchange. The corresponding effective operator reads (in the string frame):

$$I_{\lambda^{16}} = \frac{1}{\alpha'} \int d^{10}x \sqrt{-g} e^{-\phi/2} f_{(16)}(\tau, \bar{\tau}) \varepsilon_{[16]} (\lambda^\alpha)^{16}, \quad (2.1)$$

where  $\varepsilon_{[16]}$  denotes the completely antisymmetric tensor and  $f(\tau, \bar{\tau})$  is a modular function of the type IIB complex coupling  $\tau = ie^{-\phi} - \chi$ . To leading order in the weak-coupling expansion, the single-instanton contribution to  $f_{(16)}$  is proportional to  $e^{-12\phi} e^{2\pi i\tau}$ , where we recognize the typical instanton factor  $e^{-2\pi/g_s} = e^{-8\pi^2/g_{\text{YM}}^2}$ .

In the rest of the paper we will concentrate mostly on this contribution to the I/A interaction action, postponing for another publication a more detailed analysis of all channels. The corresponding vacuum diagrams take the form

$$W_{\text{I/A}} = (\text{V.F.}) \prod_{\text{Fermi lines}} \langle I | \not{D}^{-1} | A \rangle. \quad (2.2)$$

with instantons or anti-instantons contributing sixteen oriented fermion lines each. The term (V.F.) represents the contribution of vertex factors coming from (2.1), with the instanton action  $e^{-2\pi/g_s}$  and the coupling constant dependence of the instanton measure, both included in  $f_{(16)}$ , as well as a totally antisymmetric tensor contracting all spinor indices at each vertex. Explicitly, in Poincaré coordinates:

$$(\text{V.F.}) = \prod_{\text{vertices}} \sqrt{N} R^{8 \cdot 9} N^{-16} \varepsilon_{[16]} \int \frac{d\rho d\vec{x}}{\rho^5} d\Omega_5 f_{(16)}, \quad (2.3)$$

where the power of  $R$  comes from the  $\alpha'$  and  $g_s$  dependence in (2.1) and the rescaling  $\lambda^\alpha \rightarrow N^{-1} R^4 \lambda^\alpha$  to have canonically normalized dilatinos.

The chirality selection rules forbid Fermi exchange interaction between I/I or A/A pairs, a fact that we shall see explicitly below. We will be primarily interested in the spatial dependence of the interaction action. Therefore, we shall focus on the structure of the Dirac propagator between an instanton located at  $\vec{x}$  and an anti-instanton at  $\vec{y}$ , “far apart”, but with approximately the same scale size  $\rho_x \sim \rho_y$ , and at the same point in the five-sphere  $\Omega_x = \Omega_y$ . Our main observation is the existence of two dynamical regimes for I/A interactions, that is two main regimes of interest for the evaluation of the propagator.

If the geodesic distance  $d_{xy}$  is small compared to the AdS and sphere radius  $R \sim \sqrt{\alpha'} \lambda^{1/4}$ , but still large compared to the string length scale:  $\ell_s = \sqrt{\alpha'} \ll d_{xy} \ll R$ , then the propagation is locally equivalent to the ten-dimensional fermion propagation in flat space, i.e. we have the following scaling of the propagator:

$$S(x, y) = \left( \frac{1}{\not{D}} \right)_{xy} \simeq -\not{\partial} \left( \frac{1}{-\partial^2} \right)_{xy} \simeq -\not{\partial} \left( \frac{1}{d_{xy}} \right)^8, \quad (2.4)$$

up to corrections of order  $O(d_{xy}/R)$ . On the other hand, the geodesic distance in terms of Poincaré coordinates for  $d_{xy} \ll R$  is proportional to the so-called “cordal distance”:

$$(d_{xy})_{\text{AdS}}^2 \simeq R^2 \frac{|\vec{x} - \vec{y}|^2 + (\rho_x - \rho_y)^2}{\rho_x \rho_y} \quad (2.5)$$

so that, for instantons of the same scale size, and located at the same point in the  $\mathbf{S}^5$  we have, up to corrections again of order  $O(d_{xy}/R)$

$$S(x, y) \simeq -\oint \left( \frac{1}{d_{xy}} \right)^8 \sim R^{-9} \left( \frac{\rho}{|\vec{x} - \vec{y}|} \right)^9 \vec{\Gamma} \cdot \vec{u}_{xy} \quad (2.6)$$

where  $\vec{u}_{xy} \equiv (\vec{x} - \vec{y})/|\vec{x} - \vec{y}|$ . Finally, putting all pieces together, we find

$$(W_{\text{I/A}})^{\text{overlapping}} = (\text{V.F.}) \prod_{\text{Fermi lines}} \left( \frac{\rho}{|\vec{x} - \vec{y}|} \right)^9 \mathcal{P}_x \vec{\Gamma} \mathcal{P}_y \cdot \vec{u}_{xy} + \dots \quad (2.7)$$

The dots standing for corrections proportional to  $\rho_x - \rho_y$  and  $\Omega_{xy}$ , in addition to higher orders in the small  $d_{xy}/R$  expansion. In particular, for a single I/A pair we have a total power of sixteen in the previous expression. We have made explicit the chiral structure by the insertion of the ten-dimensional chiral projectors  $\mathcal{P}_x, \mathcal{P}_y$ , since the instanton vertices are chiral. A Fermi line must connect an I/A pair, rather than a I/I or A/A pair, in order for the action to be non-vanishing, i.e. we have the selection rule:  $\mathcal{P}_x(1 - \mathcal{P}_y) = \mathcal{P}_x$ . We have also suppressed the explicit power of  $R$  coming from eq. (2.6). In fact, it cancels against the explicit  $R$ -dependence of the vertex factors in (2.3), since propagators and vertices are in relation eight to one.

Notice that in this regime  $|\vec{x} - \vec{y}| \ll \rho$ , which can be interpreted as the fact that the size of the instantons is much larger than their distance, and thus they strongly overlap. This is a situation which cannot be studied in perturbation theory, but it is consistent in the supergravity approximation, because the D-instanton system is still dilute in  $AdS_5 \times \mathbf{S}^5$  for  $d_{xy} \gg \ell_s$ . The elementary transition amplitude in string units

$$\ell_s^9 S(x, y) \sim \lambda^{-9/4} \left( \frac{\rho}{|\vec{x} - \vec{y}|} \right)^9, \quad (2.8)$$

is still small in the window  $|\vec{x} - \vec{y}| \ll \rho \ll \lambda^{1/4} |\vec{x} - \vec{y}|$ , which is wide for  $\lambda \gg 1$ . This is entirely analogous to other situations in AdS/CFT for non-BPS quantities. Namely, large renormalizations of physical scale by powers of  $\lambda$  are frequent. Examples include the renormalization of the topological susceptibility in the models of [15], and the different scales of finite size effects in thermal partition functions on the torus, studied in [16].

In the other dynamical regime, the geodesic distance is much larger than the AdS and sphere radius:  $d_{xy} \gg R$ . In this case  $|\vec{x} - \vec{y}| \gg \rho$  and we are in the more standard (in field theory terms) dilute instanton regime. Now the scale of propagation is sensitive to the curvature of the background. In practice, we have just to evaluate the fermionic propagator in the bulk of  $AdS_5$ . Indeed, since higher harmonics of  $\mathbf{S}^5$  have a large Kaluza–Klein mass of order  $M_{\text{KK}} \sim 1/R$ , their contribution is exponentially suppressed for propagation over distances  $d_{xy} \gg R$ . Moreover, the dilatini component along  $\mathbf{S}^5$  is given by a Killing spinor, and the net effect of the sphere is summarized in a dimensional factor of the volume:

$$S(x, y)_{10\text{d}} \simeq R^{-5} S(x, y)_{\text{AdS}} . \quad (2.9)$$

Upon Kaluza–Klein reduction on the  $\mathbf{S}^5$ , the dilatini acquire an effective mass  $m_f = -3/(2R)$ , c.f. [17]. The fermionic propagator in the bulk of  $AdS_5$  has been already discussed in the literature [18,19], but it can be easily obtained as follows. In Poincaré coordinates, it holds <sup>3</sup>

$$(\gamma^\mu D_\mu)^2 = \nabla^2 + 4 + \rho \gamma^5 \vec{\gamma} \cdot \vec{\partial}, \quad [\gamma^\mu D_\mu, \gamma^5] = 2\rho \vec{\gamma} \gamma^5 \vec{\partial}, \quad (2.10)$$

where

$$\gamma^\mu D_\mu = \rho \gamma^5 \partial_\rho + \rho \vec{\gamma} \cdot \vec{\partial} - 2\gamma^5. \quad (2.11)$$

$\gamma^\mu$  are the five-dimensional Dirac matrices,  $\gamma^5$  refers to the  $\rho$  coordinate and  $\nabla^2$  is the five-dimensional scalar laplacian. From these equations one obtains that the Dirac propagator for a fermion with mass  $m_f$  satisfying  $(\gamma^\mu D_\mu - m_f)S = \delta^{(5)}$  is given by

$$S(x, y) = -\sqrt{\frac{\rho_x}{\rho_y}} \left[ \gamma_\mu D_y^\mu + \frac{1}{2} \gamma^5 + m_f \right] \left\langle x \left| \left( \nabla_y^2 - \left[ \left( \frac{1}{2} \gamma^5 + m_f \right)^2 - 4 \right] \right)^{-1} \right| y \right\rangle . \quad (2.12)$$

Thus the Dirac propagator for the component of a fermion, with well-defined eigenvalue of  $\gamma^5$  and mass  $m_f$ , is related to the propagator of a scalar with an effective mass

$$m_{\text{eff}}^2 = \left( \frac{1}{2} \gamma^5 + m_f \right)^2 - 4 . \quad (2.13)$$

The bulk bosonic propagator for a scalar is well known [20,18], and given by

$$G_\Delta^B(u) = \frac{\Gamma(\Delta)\Gamma(\Delta - 3/2)}{(4\pi)^{5/2}\Gamma(2\Delta - 3)} u^\Delta F(\Delta, \Delta - 3/2, 2\Delta - 3, -u), \quad (2.14)$$

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<sup>3</sup> We rescale  $R$  out of all the following equations, setting also  $m_f = -3/2$ .

where  $u = 4\rho_x \rho_y / (|\vec{x} - \vec{y}|^2 + (\rho_x - \rho_y)^2)$  and using eq. (2.13)

$$\Delta = 2 + \left| \frac{1}{2} \gamma^5 + m_f \right|. \quad (2.15)$$

It is then easy to see that for instantons of the same scale size but large separation, i.e. large  $|\vec{x} - \vec{y}|$ , the fermionic propagator behaves as

$$\mathcal{P}_\pm S(x, y) \simeq c_1 \mathcal{P}_\pm \left( m_f + \left( \Delta_\pm - \frac{3}{2} \right) \gamma^5 \right) \cdot \left( \frac{\rho}{|\vec{y} - \vec{x}|} \right)^{2\Delta_\pm} + c_2 \mathcal{P}_\pm \vec{\gamma} \cdot \vec{u}_{xy} \left( \frac{\rho}{|\vec{y} - \vec{x}|} \right)^{2\Delta_\pm + 1} \quad (2.16)$$

where the matrices  $\mathcal{P}_\pm = \frac{1}{2}(1 \pm \gamma^5)$  project onto definite  $\pm 1$  eigenvalues of  $\gamma^5$ , which is interpreted as the chirality eigenvalue in the four-dimensional boundary, i.e. the gauge-theory space-time.  $\Delta_\pm$  is given by  $\Delta$  in eq. (2.15) above with  $\gamma^5 = \pm 1$ . In our case  $m_f = -3/2$  and the terms with  $\Delta_+ = 3$  dominate over those with  $\Delta_- = 4$  in the limit of large  $|\vec{y} - \vec{x}|/\rho$ . Of these, the naively leading one is proportional to  $c_1 \mathcal{P}_+ \mathcal{P}_- = 0$  and thus vanishes. The actual leading term is chiral in four-dimensional terms and leads to a net I/A interaction of the form

$$(W_{I/A})^{\text{dilute}} = (\text{V.F.}) \prod_{\text{Fermi lines}} \left( \frac{\rho}{|\vec{y} - \vec{x}|} \right)^7 \mathcal{P}_+ \vec{\gamma} \cdot \vec{u}_{xy} + \dots \quad (2.17)$$

with the dots representing higher corrections in powers of  $\rho_x - \rho_y$  and integer powers of  $u$ . Thus, we can distinguish the dilute and overlapping regimes in this channel by the overall power of the physical separation in the gauge theory  $|\vec{y} - \vec{x}|$ , provided  $\lambda$  is large enough to justify the various approximations. Notice that for a A/I pair the four-dimensional chirality of the amplitude turns out to be opposite, i.e.  $\mathcal{P}_-$  appears in  $W_{A/I}$ , as one expects from the action of a CP transformation.

### 3. The Non-Commutative case

In the recent works [10] and [11] a gravitational dual of large  $N$  NCYM was proposed as the near-horizon geometry of Dp-branes [21,22,23] with non-trivial profiles of NS  $B$ -fields, under the appropriate scaling of parameters that isolates the low energy regime with NCYM physics. Let us consider the particular case of an euclidean D3-brane system endowed with a constant  $B$ -field background of rank two with skew-eigenvalues  $B_z, B_w$  in the respective planes  $z = x_1 + ix_2$ ,  $w = x_3 + ix_4$ . We have the string-frame geometry:

$$\frac{ds^2}{R^2} = \frac{1}{\rho^2} \left( \hat{f}_z(\rho) |dz|^2 + \hat{f}_w(\rho) |dw|^2 \right) + \frac{d\rho^2}{\rho^2} + d\Omega_5^2 \quad (3.1)$$



in terms of the functions

$$\hat{f}_z(\rho) = \frac{\rho^4}{\rho^4 + \rho_z^4}, \quad \hat{f}_w(\rho) = \frac{\rho^4}{\rho^4 + \rho_w^4}. \quad (3.2)$$

The relevant length scales introduced in the radial profile by the noncommutativity properties are related to the perturbative non-commutative length squared  $[z, \bar{z}] = -2i\theta_z$ ,  $[w, \bar{w}] = -2i\theta_w$  by the formulas

$$\rho_z = (\lambda\theta_z^2)^{1/4}, \quad \rho_w = (\lambda\theta_w^2)^{1/4}. \quad (3.3)$$

In these expressions,  $\lambda = g_{\text{YM}}^2 N$  is the 't Hooft coupling of the Yang–Mills theories normalized in the large- $\rho$  region — that is in terms of the ordinary commutative theory that appears in the infrared. The dilaton and NS  $B$ -field have the profiles

$$e^{2\phi} = \frac{\lambda^2}{16\pi^2 N^2} \hat{f}_z(\rho) \hat{f}_w(\rho), \quad B_z = \frac{s_z}{\theta_z} \frac{\rho_z^4}{\rho^4 + \rho_z^4}, \quad B_w = \frac{s_w}{\theta_w} \frac{\rho_w^4}{\rho^4 + \rho_w^4}. \quad (3.4)$$

Here,  $s_z$  and  $s_w$  are sign factors controlling the sign  $S_{\text{Pf}} = s_z s_w$  of the pfaffian  $\text{Pf}(B) = B_z B_w$  (we can choose, without loss of generality  $\theta_i \geq 0$ ).

Ramond–Ramond (RR) fields coupling to any non-trivial product  $B \wedge B \wedge \cdots \wedge B$  on the  $Dp$ -brane world-volume are also excited. For the case of interest here, the type IIB two- and zero-form RR field strengths are excited. In particular, the axion profile is given by [11]

$$\chi = -\frac{\theta_{\text{YM}}}{2\pi} + i \frac{4\pi N}{\lambda} S_{\text{Pf}} \frac{\rho_z^2 \rho_w^2}{\rho^4}, \quad (3.5)$$

where  $\theta_{\text{YM}}$  is the Yang–Mills vacuum angle, also normalized in the infrared.

Since  $\hat{f}(\rho \rightarrow 0) \rightarrow 0$ , there is a significant distortion of the metric and various field profiles (3.1), (3.4) in the ultraviolet regime. This fact has consequences for the interpretation of holography in these dual large  $N$  descriptions of the NCYM theory. For example, there are problems in defining correlators of local operators [11] related to ambiguities in the ultraviolet renormalization.

A case that exposes these difficulties in a dramatic way is that of euclidean D3-branes with a (anti-) self-dual antisymmetric tensor  $B^+ = 0$  ( $B^- = 0$ ) or  $\theta_z = \theta_w = \theta$ ,  $\rho_z = \rho_w = \rho_\theta$  in the previous formulas. The Einstein frame metric, appropriate to the discussion of massless field propagation, is given by

$$ds_E^2 = \left(\frac{r_\theta}{R}\right)^2 \frac{1}{\sqrt{1 + (r_\theta/r)^4}} d\vec{x}^2 + \left(\frac{R}{r_\theta}\right)^2 \sqrt{1 + (r_\theta/r)^4} (dr^2 + r^2 d\Omega_5^2). \quad (3.6)$$

With the identifications  $r = R^2/\rho$ ,  $r_\theta = R^2/\rho_\theta$  and certain rescalings of the coordinates, this is simply the full D3-brane metric asymptotic to flat space as  $r \rightarrow \infty$  (the other fields

do not correspond however to this solution). Since the resulting manifold is asymptotic to flat  $\mathbf{R}^{10}$  it is not obvious in what way this description can be holographic. The crossover from the “throat” to the flat region is at  $\rho = \rho_\theta$ , i.e. at the onset of non-commutative effects in the gauge theory.

Despite these problems in making sense of local operators in the geometric picture, one can easily compute certain observables, such as the thermodynamic quantities, that can be written as integrals over spacetime of local operators [23,24].

Non-commutative instantons provide an interesting probe into the locality properties of NCYM theories [25,9]. It is natural to try to draw some lesson from the AdS/CFT picture of these in terms of D-instantons in the geometry (3.6). Some general properties expected for instantons can be obtained from the Dirac–Born–Infeld action of a D-instanton probe:

$$S_{\text{DBI}} = 2\pi (e^{-\phi} - i\chi) \ , \quad (3.7)$$

with anti-D-instantons coupling instead to the conjugate combination of type IIB dilaton and axion:  $e^{-\phi} + i\chi$ . Upon substitution of the previous formulas we find

$$S_{\text{DBI}} = \frac{8\pi^2}{g_{\text{YM}}^2} \left[ \sqrt{\left(1 + \frac{\rho_z^4}{\rho^4}\right) \left(1 + \frac{\rho_w^4}{\rho^4}\right)} + S_{\text{Pf}} \frac{\rho_z^2 \rho_w^2}{\rho^4} \right] + i\theta_{\text{YM}} \ . \quad (3.8)$$

We see that, for very large instantons  $\rho \gg \rho_z, \rho_w$  compared to the non-commutativity scales, the action reduces to the usual  $i\theta_{\text{YM}} + 8\pi^2/g_{\text{YM}}^2$ . On the other hand, for  $\rho \sim \rho_z, \rho_w$  and  $\rho \ll \rho_z, \rho_w$  there are significative modifications. In fact, the radial coordinate  $\rho$  is not an exact moduli for D-instantons in the generic case. In other words, the D-instanton probe is to be considered an approximate or constrained instanton. For  $S_{\text{Pf}} = 1$  the large action for small “sizes” could be interpreted as the absence of a small instanton singularity in instanton moduli space. In particular, for  $B^- = 0$  we have  $S_{\text{Pf}} = 1$  and

$$S_{B^-=0} = \frac{8\pi^2}{g_{\text{YM}}^2} + i\theta_{\text{YM}} + \frac{16\pi^2}{g_{\text{YM}}^2} \left(\frac{\rho_\theta}{\rho}\right)^4 \ , \quad (3.9)$$

giving a strong suppression of the “small” instanton regime. On the other hand, for  $B^+ = 0$  one has  $S_{\text{Pf}} = -1$  and

$$S_{B^+=0} = \frac{8\pi^2}{g_{\text{YM}}^2} + i\theta_{\text{YM}} \quad (3.10)$$

exactly as in YM theory. Namely, non-commutative instantons in a self-dual  $B$ -field (conversely anti-instantons in an anti-self-dual  $B$ -field) have a standard commutative moduli space. In particular there is an apparent small instanton singularity. It is very satisfying to see these results emerge in such an elementary way from the AdS/CFT picture in terms of D-instanton probes.

On the other hand, even if the moduli space of instantons only depends on the self-dual part of the non-commutative deformation parameter, in general one expects the precise form of the instanton solutions to depend also on  $B^-$ . Therefore, it is not clear to what extent the region  $\rho \ll \rho_\theta$  really represents small non-commutative instantons, in a physical sense.

A preliminary test of the locality properties of such  $\rho \ll \rho_\theta$  or “small” instantons can be extracted from our previous analysis of the I/A interaction. This is a particularly interesting quantity in the case at hand, since it does not involve directly the specification of boundary behaviour at  $\rho = 0$ , and thus it should be free of renormalization ambiguities.

The D-instanton probe description of non-commutative instantons is generically off-shell for  $\rho \ll \rho_\theta$ . For example, let  $B^+ = 0$  and place a Dirichlet-(I/A) pair at large  $\rho$ . From eq. (3.10) it follows that we can “drag” the D-instanton to  $\rho \ll \rho_\theta$  with no cost in action, but from eq. (3.9) it follows that it is not possible, without a large cost in action, to drag the corresponding anti-D-instanton to  $\rho \ll \rho_\theta$ .

If we insist in maintaining the off-shell Dirichlet-(I/A) pair at “small” size (i.e.  $\rho \ll \rho_\theta$ ), as an approximate or “constrained” configuration, then the fermionic exchange interaction over distances  $d_{xy} \gg \rho_\theta$  must be evaluated in the geometry (3.6), which is asymptotic to flat  $\mathbf{R}^{10}$  in the region of interest ( $\rho \ll \rho_\theta$ ) for both “instantons”. Therefore, the appropriate fermion propagator shows ten-dimensional scaling

$$S(x, y) = \left( \frac{1}{\mathcal{D}} \right)_{xy} \simeq -\not{\partial} \left( \frac{1}{d_{xy}} \right)^8 \sim R^{-9} \left( \frac{\rho_\theta}{|\vec{y} - \vec{x}|} \right)^9 \vec{\Gamma} \cdot \vec{u}_{xy}, \quad (3.11)$$

where we have used the geodesic distance computed in the metric (3.6). We find an effective interaction of the overlapping type, see eq. (2.6), with a characteristic scale independent of the size  $\rho$  and fixed at the non-commutative length  $\rho_\theta \gg \rho$ .

A possible interpretation of this result is that, as the instantons decrease in size to the scale where non-commutative effects start to be relevant, they actually “delocalize”, behaving again as if they were large, but with an effective size dictated by the non-commutative geometry.

Finally, when considering the interaction of a D-instanton and an anti-D-instanton of large size (i.e.  $\rho \gg \rho_\theta$ ), the results of the previous section apply since the metric (3.6) in this region is asymptotically given by  $AdS_5 \times \mathbf{S}^5$ .

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