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The Pressure in 2, 2+1 and 3 Flavour QCD

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ABSTRACT

We calculate the pressure in QCD with two and three light quarks on a lattice of size $16^3 \times 4$ using tree level improved gauge and fermion actions. We argue that for temperatures $T \gtrsim 2T_c$ systematic effects due to the finite lattice cut-off and non-vanishing quark masses are below 15% in this calculation and give an estimate for the continuum extrapolated pressure in QCD with massless quarks. We find that the flavour dependence of the pressure is dominated by that of the Stefan-Boltzmann constant. Furthermore we perform a calculation of the pressure using 2 light ($m_{u,d}/T = 0.4$) and one heavier quark ($m_s/T = 1$). In this case the pressure is reduced relative to that of three flavour QCD. This effect is stronger than expected from the mass dependence of an ideal Fermi gas.

1 Introduction

One of the central goals in studies of the QCD thermodynamics on the lattice is the calculation of the equation of state for QCD with a realistic mass spectrum. Controlling the influence of a heavier strange quark on the QCD phase transition as well as understanding its contribution to bulk thermodynamic observables, e.g. the pressure or energy density, is of fundamental importance for the analysis and interpretation of heavy ion experiments which look for signals from the quark-gluon plasma phase of QCD. While the former problem is closely related to the chiral structure of QCD and requires numerical computations with light quarks close to the chiral limit [1] the latter question can be addressed in computations with moderate quark masses already.

An increase in the relative abundance of strange particles at high temperature and density is being discussed as a signature for the formation of a quark-gluon plasma [2]. A first analysis of the contribution of the strange quark sector to the energy density has been performed already some time ago in a lattice calculation [3]. This calculation suggested that even at temperatures a few times the critical temperature the strange quark contribution to the overall energy density is strongly suppressed. The energy density in the strange quark sector was found to reach only half the value of a non-interacting Fermi gas. This conclusion, however, has been drawn by separately analyzing operators which in a non-interacting gas would describe the contribution of light and heavy quarks, respectively, to the overall energy density. Such a separation of different contributions is questionable at temperatures a few times T_c. In fact, the experience gained from lattice calculations of the equation of state in the pure gauge sector [4], the failure of a perturbative description of its high temperature behaviour [5] as well as the success of hard thermal loop resummed perturbative calculations [6, 7] suggest that the high temperature phase of QCD remains non-perturbative even at temperatures several times T_c. This prohibits an isolated analysis of the contribution of different parton sectors to bulk thermodynamic observables, e.g. the free energy density $f(T)$ which in the thermodynamic limit yields the pressure, $p(T) = -f(T)$. One rather has to analyze the variation of $f(T)$ with the number of flavours and its quark mass dependence to deduce the effect of a non-vanishing strange quark mass on the thermodynamics.

The pioneering calculation of the strange quark contribution to the QCD equation of state [3] uses the standard Wilson gauge and staggered fermion actions. It still is strongly influenced by cut-off effects and moreover, makes use of partly perturbative relations in the calculation of thermodynamic quantities. Both problems can be handled now much better. The use of the *integral method* [8] allows an entirely non-perturbative calculation of thermodynamic observables. Cut-off effects can be strongly reduced in calculations with improved gauge [9, 10, 11, 12] and fermion [13, 14] actions.

We will present here results from a calculation with improved gauge and improved staggered fermion actions for two and three quarks of equal mass as well as two light and a heavier strange quark. Based on an analysis of the remaining cut-off effects and the quark mass dependence we will give an estimate for the free energy density ($\sim$ pressure) of QCD with massless quarks at temperatures $T \gtrsim 2T_c$. In addition we discuss the contribution of the heavier strange quark to the free energy density in this temperature interval.

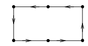
This paper is organized as follows. In the next Section we will describe the lattice action used for our calculations and discuss the cut-off dependence of thermodynamic observables in the infinite temperature, ideal gas limit. In Section 3 we present details of our numerical calculation and give the basic numerical data entering our analysis of the pressure which is performed in Section 4. In Section 5 we discuss the extrapolation of our results to the continuum limit for which we give a first estimate. Our conclusions are given in Section 6.

2 Improved action, Cut-off dependence

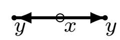
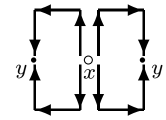
The equation of state for QCD with two light quarks has been analyzed recently on lattices with temporal extent $N_\tau = 4$ [15] and 6 [16]. In these calculations, which have been performed with the standard Wilson gauge and staggered fermion actions, a sizeable cut-off dependence has been observed. The general pattern, a strong overshooting of the continuum ideal gas limit, is in accordance with the known cut-off effects for an ideal quark-gluon gas calculated with these actions on lattices with finite temporal extent. Also the first thermodynamic studies performed for four flavour QCD with an improved fermion action [13] indicate that the qualitative features of the cut-off dependence closely follow the pattern seen for an ideal gas. Analyzing cut-off effects in the ideal gas limit thus provides useful guidance for selecting an improved action for thermodynamic calculations.

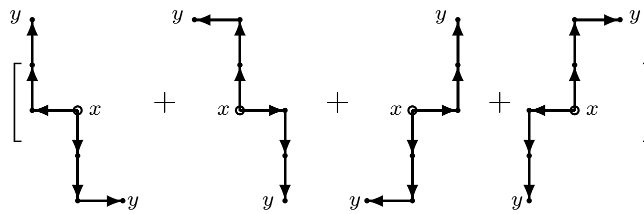
In our calculations we use a tree level, Symanzik improved gauge action, which in addition to the standard Wilson plaquette term also includes the planar 6-link Wilson loop, and a staggered fermion action with 1-link and bended 3-link terms,

$$
\begin{aligned}
Z(T,V) &= \int \prod_{x,\mu} \mathrm{d}U_{x,\mu} \mathrm{e}^{-\beta S_G} \prod_f \Big(\int \prod_x \mathrm{d}\bar{\chi}_x \mathrm{d}\chi_x \ \mathrm{e}^{-S_F(m_{f,L})} \Big)^{1/4} \\
S_G &= c_4 \ S_{plaquette} + c_6 \ S_{planar} \\
&\equiv \sum_{x,\nu>\mu} \frac{5}{3} \Big(1 - \frac{1}{3} \mathrm{Re\ Tr} \quad \Box
\end{aligned}
$$

$$-\frac{1}{6}\left(1-\frac{1}{6}\text{Re Tr}\left(\right.\right.$$ 

$$S_F(m_{f,L}) = c_1^F S_{1-link,fat}(\omega) + c_3^F S_{3-link} + m_{f,L}\sum_x \bar{\chi}_x^f \chi_x^f$$

$$\equiv \sum_x \bar{\chi}_x^f \sum_\mu \eta_\mu(x)\left(\frac{3}{8}\left[\right.\right.$$  $+\ \omega \sum_{\nu\neq\mu}$  $\Big]$

$$+\frac{1}{96}\sum_{\nu\neq\mu}\Big[$$  $\Big]\Big)\chi_y^f$

$$+m_{f,L}\sum_x \bar{\chi}_x^f \chi_x^f \quad . \tag{2.3}$$

Here we have made explicit the dependence of the action on different quark flavours, f, and the corresponding bare quark masses $m_{f,L}$ and give an intuitive graphical representation of the action. With $\eta_\mu(x)$ the staggered fermion phase factors are denoted. Further details on the definition of the action are given in [14]. The tree level coefficients c_1^F and c_3^F appearing in S_F have been fixed by demanding rotational invariance of the free quark propagator at $\mathcal{O}(p^4)$ ("p4-action"). Moreover, the 1-link term of this action has been modified by introducing "fat" links [17] with a weight $\omega = 0.2$. In the infinite temperature limit the fat link term does not contribute to thermodynamic observables nor to their cut-off dependence. Also at $\mathcal{O}(g^2)$ its effect has been found to be small [14]. It thus is expected to be of little importance for our current analysis, which is focused on the high temperature behaviour of the pressure. The fat links are, however, known to improve the flavour symmetry of the staggered fermion action [17]. Their contribution thus should become of significance in thermodynamic calculations with light quarks close to T_c.

Even with our improved action cut-off effects are still quadratic in the lattice spacing a. In thermodynamic calculations this translates into a quadratic dependence on the finite temporal extent of the lattice, $(aT)^2 = 1/N_\tau^2$. Compared to the standard gauge and staggered fermion actions these contributions are, however, drastically reduced in magnitude. For $\beta \to \infty$ the p4-action yields a much more rapid approach to the continuum ideal gas limit and deviates little from it already on lattices with rather small temporal extent N_τ. The deviations are smaller than 5% already on lattices with temporal extent $N_\tau = 6$. Even with an improved gauge sector this accuracy is reached with the standard staggered fermion action only for $N_\tau \geq 16$. The cut-off distortion at some small values of N_τ is given in Table 1.

N_τ	$p_G(N_\tau)/p_{\mathrm{SB,G}}$	$p_F(N_\tau)/p_{\mathrm{SB,F}}$		$(\epsilon_F(N_\tau)-3p_F(N_\tau))/T^4$	
	gauge action	p4-action	Naik-action	p4-action	Naik-action
4	0.9284	0.5932	0.8070	1.3393	1.5991
6	0.9925	0.9378	0.7718	-0.0867	0.8079
8	0.9983	0.9778	0.9303	0.0236	0.1156
10	0.9994	0.9889	0.9809	0.0155	0.0202

Table 1: Cut-off dependence of the pressure on lattices with temporal extent N_τ. We separately give the gluonic contribution and that for massless fermions normalized to the corresponding continuum ideal gas values. The second column gives the result for the tree level improved gauge action defined in Eq. (2.2) and the third column is for the p4-action defined in Eq. (2.3). In the third column we give for comparison the results for the $\mathcal{O}(a^2)$ improved Naik action [18]. The last two columns show the violation of a basic thermodynamic identity for the ideal Fermi gas (one flavour) due to finite cut-off effects.

In the last two columns of Table 1 we also show $(\epsilon_F - 3p_F)/T^4$ calculated by using the lattice versions of standard thermodynamic relations, i.e. $p/T = V^{-1}\ln Z$ and $\epsilon = -V^{-1}\partial \ln Z/\partial(1/T)$. The resulting integrals which have been evaluated numerically are given for a free massless fermion gas,

$$\begin{aligned} \frac{p_F(N_\tau)}{T^4} &= \\ & \frac{3}{8} n_f N_\tau^4 \, \frac{1}{(2\pi)^3} \int_0^{2\pi} \mathrm{d}^3\vec{p}\Big[N_\tau^{-1} \sum_{n_0=0}^{N_\tau-1} \ln\Big(\omega^2(\vec{p}) + 4f^2((2n_0+1)\pi/N_\tau)\Big) \\ & -\frac{1}{(2\pi)} \int_0^{2\pi} \mathrm{d}p_0 \, \ln\Big(\omega^2(\vec{p}) + 4f^2(p_0)\Big)\Big] \quad , \end{aligned} \tag{2.4}$$

$$\begin{aligned} \frac{\epsilon_F(N_\tau)}{T^4} &= \\ & 3 n_f N_\tau^4 \, \frac{1}{(2\pi)^3} \int_0^{2\pi} \mathrm{d}^3\vec{p}\Big[N_\tau^{-1} \sum_{n_0=0}^{N_\tau-1} \frac{f^2((2n_0+1)\pi/N_\tau)}{\omega^2(\vec{p}) + 4f^2((2n_0+1)\pi/N_\tau)} \\ & -\frac{1}{(2\pi)} \int_0^{2\pi} \mathrm{d}p_0 \, \frac{f^2(p_0)}{\omega^2(\vec{p}) + 4f^2(p_0)}\Big] \quad , \end{aligned} \tag{2.5}$$

where the zero temperature contributions to p_F/T^4 and ϵ_F/T^4 have been subtracted. Here the function $\omega^2(\vec{p}) \equiv 4\sum_{\mu=1}^{3} f^2(p_\mu)$ is introduced where in the case of the Naik and p4 action $f(p_\mu)$ is given by

$$f(p_\mu) \quad = \quad \frac{9}{16}\sin(p_\mu) - \frac{1}{48}\sin(3p_\mu) \quad (\mathrm{Naik-action}) \tag{2.6}$$

$$f(p_\mu) \quad = \quad \frac{3}{8}\sin(p_\mu) + \frac{1}{48}\ 2\ \sin(p_\mu)\sum_{\nu\neq\mu}\cos(2p_\nu) \quad (\mathrm{p4-action}). \tag{2.7}$$

In the temporal direction only the discrete Matsubara modes $p_0 = (2n_0+1)\pi/N_\tau$ with $n_0 = 0,\ 1, ...,\ (N_\tau - 1)$ contribute. The deviations from zero in $(\epsilon_F - 3p_F)/T^4$ are a direct measure of the violation of basic thermodynamic identities, valid for an ideal gas, due to finite cut-off effects. This also supports our preference for using the p4-action in thermodynamic calculations rather than the Naik-action.

3 The numerical calculation

Our calculations for two and three flavour QCD have been performed with quarks of mass $m_{u,d;L} = 0.1$ on lattices of size $16^3 \times 4$, i.e. $m_{u,d}/T = 0.4$. In addition we perform calculations with two light quarks of mass $m_{u,d;L} = 0.1$ and a heavier quark of mass $m_{s;L} = 0.25$, i.e. $m_s/T = 1.0$. To normalize the pressure and also in order to extract a physical temperature scale additional *zero temperature* calculations have been performed on symmetric 16^4 lattices. We have used the standard Hybrid R algorithm [19] with a step size $\Delta\tau \lesssim m_{u,d;L}/2$ and a trajectory length $\tau = 0.8$ to update the gauge and fermion fields. On the $16^3 \times 4$ lattice we collected 2000 to 3000 trajectories for values of the gauge coupling near the critical point and about 1000 trajectories for values away from it. In the zero temperature simulations up to 800 trajectories were generated to obtain a statistical error comparable to the finite temperature calculations.

In a first step we determine the transition region for our choice of parameters. A pseudo-critical temperature has been determined on the $16^3\times 4$ lattice by locating the peak position of the susceptibility of the Polyakov-loop and the chiral condensate, respectively. The location of both peaks has been found to coincide within errors. The signal in the chiral susceptibilities is, however, far more pronounced than in the Polyakov loop susceptibility. The resulting pseudo-critical couplings are given in Table 2.

In order to determine a physical scale for our finite temperature calculations we have calculated the heavy quark potential and also performed spectrum calculations at $T = 0$, i.e. on the 16^4 lattice. The heavy quark potential has been determined in the usual way from smeared Wilson loops (for details see [11]). From its long distance behaviour we extract the string tension which then yields $T_c/\sqrt{\sigma}$ and also defines a temperature scale for our calculations of the pressure. The resulting critical parameters are also given in Table 2.

We note that $T_c/\sqrt{\sigma}$ shows little dependence on the number of flavours, although the present analysis certainly is not yet indicative for the behaviour in the

flavour content	β_c	σa^2	$T_c/\sqrt{\sigma}$	$m_{\rm ps}a$	$m_{\rm v}a$
2	3.646 (4)	0.271 (10)	0.480 (10)	0.958 (2)	1.377 (25)
2+1	3.543 (2)	0.271 (11)	0.480 (10)	0.962 (3)	1.343 (20)
3	3.475 (2)	0.283 (11)	0.470 (9)	0.967 (1)	1.415 (15)

Table 2: Pseudo-critical couplings, string tensions calculated at these couplings and the resulting pseudo-critical temperatures for $n_f = 2$ and 3 as well as QCD with two light and a heavier strange quark. In addition the masses for the light pseudo-scalar and vector mesons are given.

chiral limit where $T_c/\sqrt{\sigma}$ seems to be about 20% smaller than our current result [20]. Furthermore we have performed a spectrum calculation on a 16^4 lattice at the pseudo-critical couplings. For the ratio of pseudo-scalar and vector meson masses we find in all three cases, $m_{\rm ps}/m_{\rm v} \simeq 0.7$ (see Table 2). This also indicates that the quark masses used in our current analysis are certainly too large to investigate in more detail the temperature interval close to T_c. In the high temperature phase, however, the dependence on the bare quark masses is strongly reduced [13, 16] and a reliable estimate of the pressure in the chiral limit becomes possible.

4 The pressure

The free energy density, $f = -TV^{-1} \ln Z$, which in the thermodynamic limit directly yields the pressure, $p = -f$, can be calculated using the integral method [8]. As the logarithm of the partition function is not directly accessible in a Monte Carlo calculation one first differentiates the partition function, Eq. (2.1), with respect to the coupling $\beta \equiv 6/g^2$ and subsequently integrates the resulting expectation values of the gauge action,

$$\frac{f}{T^4}\Big|_{\beta_0}^{\beta} = -\left(\frac{N_\tau}{N_\sigma}\right)^3 \int_{\beta_0}^{\beta} d\beta' \left(\langle S_G \rangle_0 - \langle S_G \rangle_T\right) \quad . \tag{4.1}$$

Here N_σ and N_τ denote the spatial and temporal extent of the finite temperature lattice, respectively. In the limit $N_\sigma \to \infty$, Eq. (4.1) gives the pressure, $p(T) = -f(T)$. Experience from earlier calculations shows that aside from a small temperature interval around T_c this limit is well approximated by $N_\sigma \geq 4N_\tau$. Moreover, $\langle S_G \rangle_T$ is the expectation value of the gluonic part of the action at finite temperature. The zero temperature contribution $\langle S_G \rangle_0$ calculated on the 16^4 lattice is subtracted to normalize the pressure to zero at $T = 0$. Strictly speaking Eq. (4.1) gives the difference between the ratios $f(T)/T^4$ calculated at two different temperatures, $T = T(\beta)$

and $T_0 = T(\beta_0)$. We choose the latter such that $f(T_0)/T_0^4 \simeq 0$. Within our numerical accuracy this is the case for $T_0 \simeq 0.6T_c$. Calculations of the differences of the action expectation values have then been performed at about 15 different values of the coupling which cover the temperature range $0.6\,T_c \leq T \leq 4\,T_c$.

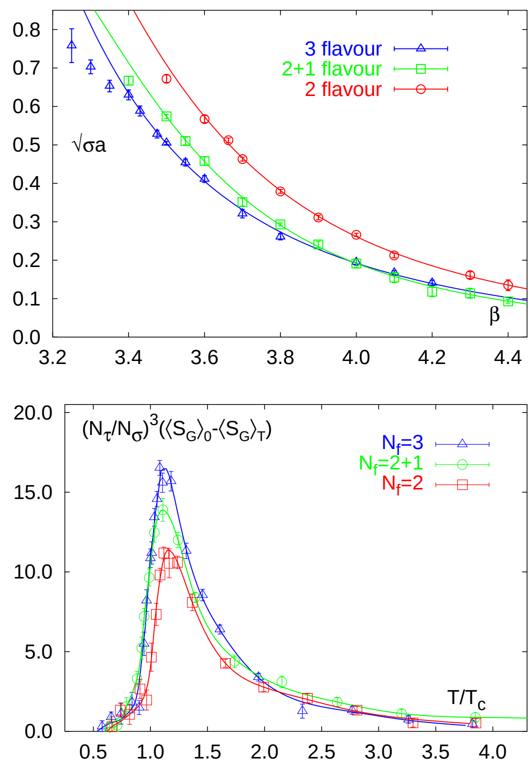


Figure 1: String tension calculated on 16^4 lattices (a) and action differences from 16^4 and $16^3 \times 4$ lattices (b) for $n_f = 2$, 3 and (2+1).

The basic input for our analysis of the QCD thermodynamics thus consists of a zero temperature calculation of the string tension, which is used to fix the temperature scale, and a calculation of action differences. The latter can then be integrated analytically to give the pressure according to Eq. (4.1). These data are shown in Figure 1.

For the interpolation of the string tension data we use a renormalization group inspired ansatz [21],

$$\sqrt{\sigma a^2}(\beta) = R(\beta)(1 + c_2\hat{a}^2(\beta) + c_4\hat{a}^4(\beta))/c_0 \qquad (4.2)$$

with $\hat{a} \equiv R(\beta)/R(\bar{\beta})$. The interpolation parameters are given in Table 3. Although this ansatz is, in principle, suitable for an extrapolation to the continuum limit we stress that it is used here only as an interpolation for the string tension data and as such is valid only in the interval indicated in Table 3.

flavour content	$[\beta_{min}, \beta_{max}]$	$\bar{\beta}$	c_0	c_2	c_4
2	[3.6,4.4]	3.70	0.0570 (35)	0.669 (208)	-0.0822 (1088)
2+1	[3.5,4.4]	3.60	0.0526 (32)	1.026 (224)	-0.1964 (1065)
3	[3.4,4.2]	3.50	0.0448 (15)	0.507 (115)	-0.0071 (677)

Table 3: Fit parameters used for the interpolation of string tension data.

The systematic increase in the action differences with increasing number of flavours visible in Figure 1 leads to the increase of the pressure with increasing number of the degrees of freedom, which is apparent from Figure 2a, where we show the pressure for $n_f = 2$ and 3 as well as the (2+1)-flavour case. In fact, in the case of the simulations with two and three light quarks, respectively, we observe that this flavour dependence can almost completely be attributed to that of an ideal quark-gluon gas,

$$\frac{p_{\rm SB}}{T^4} = \left(16 + \frac{21}{2} g_f\right) \frac{\pi^2}{90} \quad . \tag{4.3}$$

Here g_f counts the effective number of degrees of freedom of a massive Fermi gas. For a massless gas we have, of course, $g_f = n_f$. In general we define

$$g_f = \sum_{f=u,d,..} g(m_f/T) \quad , \tag{4.4}$$

with

$$g(m/T) = \frac{360}{7\pi^4} \int_{m/T}^{\infty} {\rm d}x x \sqrt{x^2 - (m/T)^2} \ln(1 + {\rm e}^{-x}) \quad . \tag{4.5}$$

For the quark mass values used here one gets $g(0.4) = 0.9672$ and $g(1) = 0.8275$, respectively. The correspondingly normalized curves are given in Figure 2b. This indicates that in the presence of a heavier quark the deviations of the pressure from the ideal gas value is larger than in the massless limit. This is in qualitative agreement with the observations made in [3].

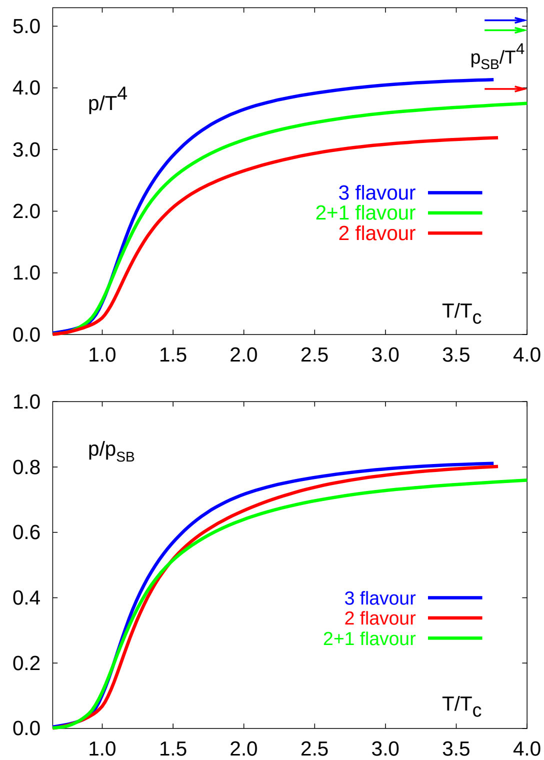


Figure 2: The pressure for $n_f = 2$, 2+1 and 3 calculated with the p4-action (a) and the normalized values $p/p_{\rm SB}$ (b).The arrows indicate the continuum ideal gas limits for two and three flavour QCD with quarks of mass $m/T = 0.4$ as well as the case of two flavour QCD with $m/T = 0.4$ and an additional heavier quark of mass $m_s/T = 1$.

5 Continuum Limit: An Estimate

Our current analysis is restricted to a single temporal lattice size, i.e. $N_\tau = 4$. We are thus not yet in the position of performing a complete extrapolation to the continuum limit. With our improved action finite cut-off effects are, however, strongly reduced at high temperature. We thus may attempt to give an estimate for the continuum equation of state for massless QCD with two or three quark flavours at temperatures not too close to T_c, e.g. $T \geq 2T_c$.

The influence of a non-zero, yet small to moderately large quark mass is small at high temperatures [13, 15, 16] and, moreover, seems to be well described by that of an ideal Fermi gas, Figure 2b. The ratio of Stefan-Boltzmann factors for QCD with two and three light quarks of mass $m/T = 0.4$ and massless QCD is 0.981 and 0.978, respectively. Thus, this seems to be a minor source for systematic deviations from the massless continuum limit. The main source for systematic errors clearly still are finite cut-off effects.

The analyzes of finite cut-off effects in the pure gauge theory have shown that at temperatures $T \simeq \ (2-4)\ T_c$ the ideal gas calculations correctly describe qualitative features of the cut-off dependent terms. However, they overestimate their influence by roughly a factor 2. If this carries over to calculations with light quarks, which similar to thermal gluons also acquire a thermal mass of $\mathcal{O}(g(T)T)$ we may expect that the finite cut-off distortion in our numerical calculations is also reduced by a similar factor. From Table 1 we find that in the ideal gas limit our improved action leads to results which are 26% and 29% below the continuum value for $n_f = 2$ and 3, respectively. Combined with the small systematic errors resulting from the use of non-zero quark masses we thus expect that the continuum equation of state for massless QCD at temperatures $T \gtrsim 2T_c$ is about 15% above the values currently obtained in our analysis. This estimate for the continuum limit is shown for two-flavour QCD in Figure 3 where we also show results from a calculation with the standard Wilson gauge and staggered fermion action on lattices with temporal extent $N_\tau = 4$ and 6 [16]. These latter data lie substantially higher which is in accordance with the larger cut-off effects for the unimproved actions.

The lattice studies of the QCD free energy density, or equivalently the pressure, indicate that deviations from the infinite temperature, ideal gas limit are about (15-20)% in the temperature interval $2T_c \ < \ T \ < \ 4T_c$. This is quite similar to what has been found in the pure gauge sector. We note that also the HTL resummed perturbative calculations [22, 23] suggest similar deviations from ideal gas behaviour. This does form a basis for more phenomenological quasi-particle models for the QCD equation of state [24, 25].

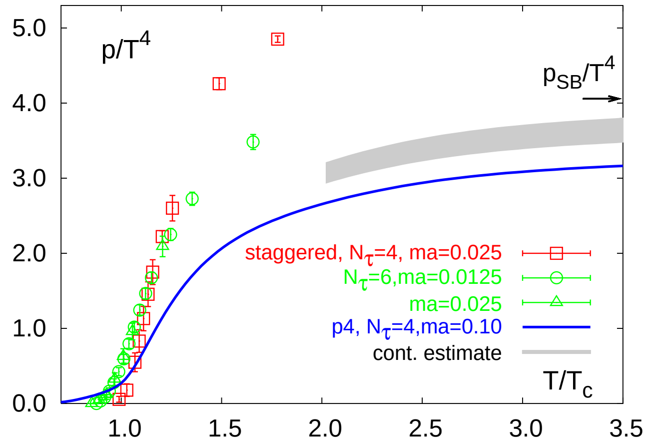


Figure 3: The pressure for $n_f = 2$. Shown are results obtained with the p4-action on lattices with temporal extent $N_\tau = 4$ (line) as well as with the standard staggered fermion action on $N_\tau = 4$ (squares) and 6 (circles, triangles) lattices. Also shown is an estimate of the continuum equation of state for massless QCD (dashed band), based on the assumption that the systematic error of the current analysis is (15±5)%.

6 Conclusions

We have calculated the pressure in the high temperature phase of QCD using improved gauge and fermion actions. Our analysis focuses on the flavour dependence of the pressure and its dependence on a heavier (strange) quark mass in the high temperature plasma phase. We find that the quark mass dependence closely follows the pattern expected from the analysis of an ideal Fermi gas. We observe, however, a significant reduction of the contribution of strange quarks relative to that in an ideal gas. Interactions, which at temperatures a few times T_c lead to a strong reduction of the pressure in QCD relative to that of an ideal gas thus also show a significant quark mass dependence. We note, however, that the current analysis has been performed with a fixed ratio m_s/T rather than a fixed heavy quark mass. The latter will be needed to arrive at quantitative predictions for the plasma phase of QCD.

So far we only can give an estimate for the continuum extrapolated pressure in the high temperature phase of QCD. To complete this analysis and in particular in order to analyze the transition region and the nature of the transition in (2+1) flavour QCD further calculations on lattices with larger temporal extent and smaller quark masses are needed. As the p4-action strongly reduces the cut-off dependence it provides a suitable framework for such more detailed studies with staggered fermions. Eventually one would like to confirm the studies of the equation of state also within another discretization scheme, in general with Wilson fermions. Some work in this direction is underway [26]. However, also here one will have to look for suitable improvements of the action in order to reduce the cut-off dependence of thermodynamic observables.

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References

[1] for recent discussions of the flavour dependence of the QCD phase diagram and further references see:
S. Aoki et al. (JLQCD collaboration), Nucl. Phys. B [Proc.Suppl.] 73 (1999) 456;
A. Peikert, F. Karsch, E. Laermann and B. Sturm, Nucl. Phys. B [Proc.Suppl.] 73 (1999) 468.

[2] for a recent review see: J. Rafelski and J. Letessier, *Diagnosis of QGP with Strange Hadrons*, hep-ph/9910300.

[3] J.B. Kogut and D.K. Sinclair, Phys. Rev. Lett. 60 (1988) 1250.

[4] G. Boyd, J. Engels, F. Karsch, E. Laermann, C. Legeland, M. Lütgemeier and B. Petersson, Phys. Rev. Let. 75 (1995) 4169 and Nucl. Phys. B469 (1996) 419.

[5] E. Braaten and A. Nieto, Phys. Rev. Lett. 76 (1996) 1417 and Phys. Rev. D53 (1996) 3421.

[6] J.O. Andersen, E. Braaten and M. Strickland, Phys. Rev. Lett. 83 (1999) 2139.

[7] J.P. Blaizot, E. Iancu and A. Rebhan, Phys. Rev. Lett. 83 (1999) 2906.

[8] J. Engels, J. Fingberg, F. Karsch, D. Miller and M. Weber, Phys. Lett. B252 (1990) 625.

[9] B. Beinlich, F. Karsch and E. Laermann, Nucl. Phys. B462 (1996) 415.

[10] A. Papa, Nucl. Phys. B478 (1996) 335.

[11] B. Beinlich, F. Karsch, E. Laermann and A. Peikert, Eur. Phys. J. C6 (1999) 133.

[12] M. Okamoto et al. (CP-PACS collaboration), Phys. Rev. D60 (1999) 094510.

[13] J. Engels, R. Joswig, F. Karsch, E. Laermann, M. Lütgemeier and B. Petersson, Phys. Lett. B396 (1997) 210.

[14] U.M. Heller, F. Karsch and B. Sturm, Phys. Rev. D60 (1999) 114502.

[15] T. Blum, L. Kärkkäinen, D. Toussaint and S. Gottlieb, Phys. Rev. D51 (1995) 5153.

[16] C. Bernard,T. Blum, C. De Tar, S. Gottlieb, K. Rummukainen, U.M. Heller, J.E. Hetrick, D. Toussaint, L. Kärkkäinen, R.L. Sugar and M. Wingate, Phys. Rev. D55 (1997) 6861.

[17] T. Blum, C. De Tar, S. Gottlieb, K. Rummukainen, U.H. Heller, J.E. Hetrick, D. Toussaint, R.L. Sugar and M. Wingate, Phys. Rev. D55 (1997) 1133

[18] S. Naik, Nucl. Phys. B316 (1989) 238.

[19] S. Gottlieb, W. Liu, D. Toussaint, R.L. Renken and R.L. Sugar, Phys. Rev. D35 (1987) 2531.

[20] for a recent review see: F. Karsch, *Lattice QCD at Finite Temperature and Density*, hep-lat/9908046.

[21] C. Allton, Nucl. Phys. B [Proc. Suppl.] 53 (1997) 867 and hep-lat/9610016.

[22] J.O. Andersen, E. Braaten and M. Strickland, *Hard Thermal Loop Resummation of the Free Energy of a Hot Quark Gluon Plasma*, hep-ph/9908323.

[23] J.P. Blaizot, E. Iancu, A. Rebhan, *Self-Consistent Hard-Thermal-Loop Thermodynamics for the Quark-Gluon Plasma*, hep-ph/9910309.

[24] A. Peshier, B. Kämpfer, O.P. Pavlenko and G. Soff, Phys. Rev. D54 (1996) 2399.

[25] P. Lévai and U. Heinz, Phys. Rev. C57 (1998) 1879.

[26] A. Ali Khan et al. (CP-PACS Collaboration), *Equation of State in Finite Temperature QCD with Improved Wilson Quarks*, hep-lat/9909075.