Pre-Big Bang Cosmology and Quantum Fluctuations

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The quantum fluctuations of a homogeneous, isotropic, open pre-big bang model are discussed. By solving exactly the equations for tensor and scalar perturbations we find that particle production is negligible during the perturbative Pre-Big Bang phase.

1. INTRODUCTION

In the framework of string theory, the Pre-Big Bang scenario [1,2] provides an alternative to the standard inflationary paradigm. In the Pre-Big Bang model [1] the inflationary solutions, driven by the kinetic energy of the dilaton, emerge naturally via the duality symmetries of string theory.

It has, however, been argued [3] that, even though the two classical moduli of the open ($\mathcal{K} = -1$), homogeneous and isotropic solution [4,5] lie deeply inside the perturbative regime, the vacuum quantum fluctuations drastically modify the classical behaviour preventing the occurrence of an appreciable amount of inflation.

Quantum fluctuations in a non-spatially flat Universe are considerably harder to study than in the flat case [6,7]. In Ref. [8] we thoroughly studied the quantum fluctuations around the $\mathcal{K} = -1$ solution [4,5]. In that work we showed that the perturbation equations can be *exactly* integrated in terms of standard hypergeometric functions. We found that particle production (i.e. the amplification of vacuum fluctuations) is strongly suppressed at very early times and remains small through the whole perturbative PBB phase, and hence, does not impede the occurrence of PBB inflation.

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[1]An updated collection of papers on the PBB scenario is available at `http://www.to.infn.it/∼gasperin/`.

2. THE SECOND-ORDER ACTION

The (string-frame) open homogeneous, isotropic PBB-type solution was first found in [4] and then rederived and discussed in [5]. The solution contains two arbitrary moduli, L and ϕ_{in}, reflecting the symmetries of the classical equations under a constant shift of the dilaton and a constant rescaling of the metric. These two parameters are to be chosen appropriately (see Refs. [3,5,9–14]) in order to ensure the occurrence of a sufficient amount of PBB inflation. Such a solution describes a universe which is almost trivial (Milne-like) at $\eta \to -\infty$ and inflating in $-\infty < \eta < \mathcal{O}(1)$, having an initial curvature $\mathcal{O}(L^{-2})$ and coupling $\mathcal{O}(\exp(\phi_{\mathrm{in}}/2))$, until it enters the strong curvature and/or strong coupling regime at $\eta \sim \eta_1$. The critical value η_1 is easily determined in terms of the integration constants L and ϕ_{in} as $(-\eta_1) = \max(e^{\phi_{\mathrm{in}}/\sqrt{3}}, (\ell_s/L)^{1+1/\sqrt{3}})$.

It is well known [1] that studying perturbations is technically simpler in the so-called Einstein-frame which is related to the string-frame by a conformal transformation. The action is

$$S^{(E)} = \frac{1}{2\ell_P^2} \int d^4x \sqrt{-g} \left(R(g) - \frac{1}{2}(\partial\phi)^2 \right) , \quad (1)$$

where ϕ_{today} is the present value of the dilaton, $\ell_P \equiv \sqrt{8\pi G} = \exp(\phi_{\mathrm{today}}/2)\ell_s \sim 0.1\ell_s$ refers to the present value of the Planck-length with $\hbar = 1$. Usually one computes perturbations in the Einstein frame and then transforms the results back

to the string frame for a physical interpretation.

The $\mathcal{K} = -1$ solution is:

$$
\begin{aligned}
a(\eta) &= \ell \left(-\sinh\eta\cosh\eta\right)^{\frac{1}{2}} \\
\phi(\eta) &= -\sqrt{3}\,\ln(-\tanh\eta) + \phi_{\text{in}}\,,\ \ \eta < 0\,,\ \ (2)
\end{aligned}
$$

where the modulus ℓ is given by $\ell^2 = L^2 \exp(\phi_{\text{today}} - \phi_{\text{in}})$. Generic perturbations are defined by

$$
g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}\,,\ \ \phi = \phi^{(0)} + \delta\phi \tag{3}
$$

where superscript (0) refers to the background solution and we shall use isotropic-spatial coordinates.

3. QUANTUM FLUCTUATIONS

3.1. Tensor perturbations

Since the tensor metric perturbations are automatically gauge-invariant and decoupled from the scalar perturbations, they are easier to study. They are defined as

$$
\delta g_{\mu\nu}^{(T)} = \text{diag}(0, a^2 h_{ij})\,, \tag{4}
$$

where the symmetric three-tensor h_{ij} satisfies the transverse-traceless (TT) conditions.

We then find:

$$
\begin{aligned}
\delta^{(2)} S^{(T)} &= \frac{1}{4\ell_P^2} \int d^4x \sqrt{\gamma}\, a^2 \left(h'^{ij} h'_{ij}\right. \\
&\left. - \nabla^l h^{ij} \nabla_l h_{ij} - 2\mathcal{K} h^{ij} h_{ij}\right)\,. \tag{5}
\end{aligned}
$$

By expanding the tensor perturbations in TT tensor-pseudospherical harmonics (as $\mathcal{K} = -1$) [15], we eventually get the simple equation

$$
u_{nlm}'' + \left(n^2 + \frac{1}{12}\phi'^2\right) u_{nlm} = 0\,, \tag{6}
$$

where $u_{nlm} \equiv a h_{nlm}$ is the canonical variable of perturbation. For the background (2) Eq. (6) can be exactly solved in terms of the standard hypergeometric function [16] as

$$
\begin{aligned}
u_N(\eta) &= C_1 \left[\text{csch}^2(2\eta)\right]^{-\frac{in}{4}} \times \\
&F\left[\frac{1-in}{4}, \frac{1-in}{4}, \frac{2-in}{2}, -\text{csch}^2(2\eta)\right] \\
&+ C_2 \text{ c.c.}\,, \tag{7}
\end{aligned}
$$

where N stands for the collection of indices (nlm) and $C_{1,2}$ are (classically arbitrary) integration constants. At early times, $n^2 \gg \phi'^2$, and thus u is a free canonical field. Hence, imposing the standard commutation relations, as $\eta \to -\infty$, we get

$$
u_N(\eta) \to u_N^{-\infty}(\eta) \equiv \frac{2\ell_P}{\sqrt{n}} e^{-in\eta}. \tag{8}
$$

Since $F[a, b, c, 0] = 1$, Eq. (8) fixes the integration constants as $|C_1| = 2\ell_P/\sqrt{n}, C_2 = 0$. The deviation from a trivial plane-wave behaviour can easily be computed from the small argument limit of F. We find

$$
u_N(\eta) = u_N^{-\infty}(\eta) \left(1 + \alpha_n e^{4\eta - i\beta_n}\right)\,, \tag{9}
$$

where α_n, β_n are n-dependent constants fixed from the Taylor expansion of the hypergeometric function. It is worth noting that the correction to the vacuum amplitude dies off as $e^{4\eta}$, i.e. as t^{-4} in terms of cosmic time $t \sim -e^{-\eta}$.

We can also estimate the behaviour of the solution near the singularity, i.e. for $\eta \to 0$. By virtue of the small η behaviour $a \simeq \ell |\eta|^{1/2}$, we find

$$
|h_N| \simeq 2\sqrt{\frac{2}{\pi}} \frac{\ell_P}{\ell} \sqrt{\coth\left(\frac{n\pi}{2}\right)} \ln|\eta|\,. \tag{10}
$$

We shall come back to this result after deriving a similar expression for scalar perturbations.

3.2. Scalar perturbations

Consider now scalar metric-dilaton perturbations (3). The scalar part of metric perturbations is defined by [6]

$$
\delta g_{\mu\nu}^{(S)} \equiv -a^2(\eta) \begin{pmatrix} 2\varphi & \nabla_i B \\ \nabla_i B & 2(\psi\gamma_{ij} + \nabla_i\nabla_j E) \end{pmatrix}. \tag{11}
$$

In the second-order action the variables B, φ are Lagrange multipliers, providing two constraints. We can introduce the gauge-invariant variable Ψ by (see Ref. [7])

$$
\Psi = \frac{4}{\phi'}\left[\psi + \mathcal{H}(B - E')\right]\,, \tag{12}
$$

and, after using the constraints, the action reads

$$
\begin{aligned}
\delta^{(2)} S^{(S)} &= \frac{1}{2\ell_P^2} \int d^4x\, a^2 \sqrt{\gamma} \times \\
&(\nabla^2 + 3\mathcal{K})\Psi \left[\partial_\eta^2 - \nabla^2 + 2(\mathcal{H}' + \mathcal{K})\right]\Psi. \tag{13}
\end{aligned}
$$

One can now make use of the constraints to eliminate the variable $(B - E')$ from the action (13) in terms of φ, ψ and $\delta\phi$. The latter variables are not independent either, being related by a linear combination of the two constraints. After its implementation the action (13) contains only true degrees of freedom.

As was in the case of tensor perturbations, we introduce a canonical field Ψ_c and expand it as

$$\Psi_c \equiv a\Psi = \int dn \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Psi_{nlm}(\eta)Q_{nlm}(\mathbf{x}), \quad (14)$$

where $Q_{nlm}(\mathbf{x})$ are the scalar pseudospherical harmonics [15]. We then get a simple equation for $\bar{\Psi}_N \equiv \sqrt{n^2 + 4}\Psi_N$, namely

$$\bar{\Psi}_N'' + (n^2 - \frac{1}{4}\phi'^2)\bar{\Psi}_N = 0 , \quad (15)$$

where we must impose, as $\eta \to -\infty$,

$$\bar{\Psi}_N(\eta) \to \bar{\Psi}_N^{-\infty}(\eta) \equiv \frac{\ell_P}{\sqrt{n}}e^{-in\eta} . \quad (16)$$

Eq. (15) can again be transformed (for the background (2)) into a hypergeometric equation. We hence find

$$\bar{\Psi}_N(\eta) = \tilde{C}_1 \left[\operatorname{csch}^2(2\eta)\right]^{-\frac{in}{4}} \times$$
$$F\left[\frac{-1-in}{4}, \frac{3-in}{4}, \frac{2-in}{2}, -\operatorname{csch}^2(2\eta)\right]$$
$$+\tilde{C}_2 \text{ c.c. }, \quad (17)$$

where, as before, we have to take $|\tilde{C}_1| = \ell_P/\sqrt{n}, \tilde{C}_2 = 0$. Corrections to the free plane wave can be easily computed and, again, are suppressed by four powers of $1/t$:

$$\bar{\Psi}_N(\eta) = \bar{\Psi}_N^{-\infty}(\eta) \left(1 + \tilde{\alpha}_n e^{4\eta - i\tilde{\beta}_n}\right) , \quad (18)$$

where $\bar{\Psi}_N^{-\infty}$ is given by (16) and $\tilde{\alpha}_n, \tilde{\beta}_n$ are n-dependent constants fixed from the expansion of the hypergeometric function.

Estimating the behaviour of (17) near $\eta \simeq 0$ [16], we obtain:

$$|\bar{\Psi}_N| \simeq \ell_P \sqrt{\frac{n^2+1}{2\pi}} \sqrt{\coth\left(\frac{n\pi}{2}\right)} \times$$
$$\left(-|\eta|^{3/2} \ln|\eta| + \frac{2}{n^2+1}|\eta|^{-1/2}\right) . \quad (19)$$

4. CONCLUSIONS

Let us choose the off-diagonal gauge [17,18], defined by setting $\psi = E = 0$ in (11). By using Eq. (12) one can reconstruct the scalar field fluctuation $\delta\phi$ from Ψ as

$$\delta\phi = \Psi' + \frac{\mathcal{K} - \mathcal{H}'}{\mathcal{H}}\Psi , \quad (20)$$

implying that $\delta\phi$ represents, in this gauge, a gauge-invariant object.

In the presence of spatial curvature, the field $v = a\delta\phi$ plays the role of the canonical field in the far past, when η is large and negative. Eq. (20) tells us that the behaviour of v in the far past follows directly from that of $\bar{\Psi}_N$, given in Eqs. (16) and (18):

$$v^{-\infty}(\eta) \equiv \frac{\ell_P}{\sqrt{n}}\sqrt{\frac{2-in}{2+in}}e^{-in\eta} . \quad (21)$$

Corrections to (21) are again suppressed as t^{-4}

$$v(\eta) = v^{-\infty}(\eta) \left(1 + \hat{\alpha}_n e^{4\eta - i\hat{\beta}_n}\right) , \quad (22)$$

where $\hat{\alpha}_n, \hat{\beta}_n$ are n-dependent constants.

The behaviour of $\delta\phi$ near $\eta \simeq 0$ is :

$$|\delta\phi_N| \simeq \frac{\ell_P}{\ell}\sqrt{\frac{n^2+1}{2\pi}}\sqrt{\frac{\coth(\frac{n\pi}{2})}{n^2+4}} \ln|\eta| . \quad (23)$$

Lastly, let us compare the energy contained in the quantum fluctuations of the dilaton and that in the classical solution near the singularity. Note that the expansion (19) can be trusted only up to some maximum n for which $1 \ll n_{\max} \sim 1/|\eta|$. Consequently, the ratio of the kinetic energy densities near $|\eta| \simeq 0$ (up to constant prefactors of $\mathcal{O}(1)$) becomes

$$\frac{\mathcal{E}_Q}{\mathcal{E}_C} = \frac{\int d^3x\sqrt{\gamma} \, a^2(\delta\phi')^2}{\int d^3x\sqrt{\gamma} \, a^2\phi'^2} \simeq \frac{\ell_P^2}{\ell^2}\int^{n_{\max}} \frac{dn}{n} \, n^3 . \quad (24)$$

We can express the above result in terms of the value of the physical Hubble parameter $H(\eta) \equiv \mathcal{H}/a$ at horizon crossing of the scale n, $H_{\mathrm{HC}}(n)$, i.e.

$$H_{\mathrm{HC}}(n) \sim \frac{1}{\eta a}(\eta \sim 1/n) \sim n^{3/2}/\ell . \quad (25)$$

Thus Eq. (24) takes the suggestive form

$$\frac{\mathcal{E}_{\mathrm{Q}}}{\mathcal{E}_{\mathrm{C}}} = \ell_P^2 \int^{n_{\max}} \frac{dn}{n} H_{\mathrm{HC}}^2(n) \ . \qquad (26)$$

In order to draw physical conclusion we should transform the results back to the string frame. However, in our case, this is hardly necessary. As far as the importance of vacuum fluctuations is concerned, as $\eta \to 0$, the final result (26) expresses the relative importance of quantum and classical fluctuations near the singularity in terms of a frame-independent quantity: the ratio of the effective Planck length to the size of the horizon. Since, by definition of the perturbative dilaton phase, the Hubble radius is always larger than the string scale, the relative importance of quantum fluctuations is always bounded by the ratio ℓ_P/ℓ_s which is always less than one in the perturbative phase.

Let us now come to the more subtle issue of the far-past behaviour of tensor and scalar quantum fluctuations. Computations may be done in either frame, since the dilaton is approximately constant in the far past. Our results, expressed in Eqs. (9) and (22), show that corrections to the trivial quantum fluctuations are of relative order $e^{4\eta} \sim t^{-4}$, i.e. of order t^{-3} relative to the (homogeneous) classical perturbation. This suggests that quantum effects do not modify appreciably classical behaviour in the far past, contrary to the claim of [3]. This result is also supported by the structure of the superstring one-loop effective-action (which is well-defined thanks to the string cutoff). Because of supersymmetry, neither a cosmological term nor a renormalization of Newton's constant are generated at one-loop, but only terms containing at least four derivatives. Thus, quantum corrections to early-time classical behaviour are of relative order t^{-6}, i.e just like our corrections $(\delta\phi'/\phi')^2$. Note also that that generating a cosmological constant by quantum corrections would upset completely the whole PBB scenario.

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