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HARMONIC AND HOLOMORPHIC 1-FORMS ON COMPACT BALANCED HERMITIAN MANIFOLDS

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Abstract

On compact balanced Hermitian manifolds we obtain obstructions to the existence of harmonic 1-forms, ∂-harmonic (1,0)-forms and holomorphic (1,0)-forms in terms of the Ricci tensors with respect to the Riemannian curvature and the Hermitian curvature. Vanishing of the first Dolbeault cohomology groups of the twistor space of a compact irreducible hyper Kähler manifold is shown. A necessary and sufficient condition the (1,0)-part of a harmonic 1-form to be holomorphic and vice versa, a real 1-form with a holomorphic (1,0)-part to be harmonic are found.

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1 Introduction

The well-known vanishing theorem of Bochner says that if the Ricci tensor of a compact Riemannian manifold is nonnegative, then every harmonic 1-form is parallel; moreover, if the Ricci tensor is nonnegative and positive at least at one point, then there are no nonzero harmonic 1-forms and the first Betti number $b_1 = 0$.

It is also well known that on a compact Kähler manifold the (1,0)-part of a harmonic 1-form is holomorphic, i.e. it is $\bar{\partial}$-closed; conversely, every holomorphic (1,0)-form is ∂-closed or equivalently, the corresponding real 1-form is harmonic [29]. Certainly, the Bochner theorem is true for compact Kähler manifolds and could be expressed also in terms of holomorphic forms.

However, the (1,0)-part of a harmonic 1-form may not be holomorphic on a compact Hermitian manifold. Nevertheless, there exists a Bochner type theorem for holomorphic (1,0)-forms on a compact Hermitian manifold. This theorem is formulated in terms of the Chern connection and its mean curvature. In fact, on a compact Hermitian manifold with nonnegative mean curvature every holomorphic (1,0)-form is parallel with respect to the Chern connection; if in addition the mean curvature is positive at least at one point, then there are no nonzero holomorphic (1,0)-forms [21, 14, 27, 20] and the Hodge number $h^{1,0} = 0$. We note that this is a part of the general result for the nonexistence of holomorphic sections of a holomorphic vector bundle over a compact Hermitian manifold [21, 14] (see also [27, 20]).

In the present paper we consider questions of existence of harmonic 1-forms, holomorphic (1,0)-forms and find relations between them on compact balanced Hermitian manifolds.

Balanced Hermitian manifolds are Hermitian manifolds with a co-closed fundamental form or equivalently with a zero Lee form. They have been studied intensively in [22, 1, 2, 3]; in [14] they are called semi-Kähler of special type. This class of manifolds includes the class of Kähler manifolds but also many important classes of non-Kähler manifolds, such as: complex solvmanifolds, twistor spaces of oriented Riemannian 4-manifolds, 1-dimensional families of Kähler manifolds (see [22]), some compact Hermitian manifolds with a flat Chern connection (see [16]), twistor spaces of oriented distinguished Weyl structure on compact self-dual 4-manifolds [15], twistor spaces of quaternionic Kähler manifolds [25, 4], manifolds obtained as modification of compact Kähler manifolds [1] and of compact balanced manifolds [2] (see also [3]).

On a balanced Hermitian manifold (M, g, J) there are two Ricci tensors ρ and ρ^* associated with the Levi-Civita connection ∇ of the metric g and two Ricci tensors k and k^* associated with the canonical Chern connection D generated by the metric g and the complex structure J. We note that the $(1,1)$-form corresponding to the tensor k represents the first Chern class of M and the $(1,1)$-form corresponding to the tensor k^* is the mean curvature. All these Ricci tensors coincide on a Kähler manifold.

Let (M, g, J) be a Hermitian manifold. If X is an arbitrary C^∞ vector field on M, we denote by ω_X its corresponding 1-form with respect to the metric g and use the decomposition

$\omega_X = \omega_X^{1,0} + \omega_X^{0,1}$ with respect to the complex structure J. We find obstructions to the existence of harmonic and holomorphic 1-forms in terms of the Ricci tensors of the Levi-Civita and Chern connection. The aim of the paper is to prove the following

Theorem 1.1 *Let (M, g, J) be a compact balanced Hermitian manifold.*

*i) If the *-Ricci tensor ρ^* is nonnegative on M, then:*

a) every holomorphic (1,0)-form $\omega_X^{1,0}$ is ∂-harmonic (ω_X is harmonic);

b) every ∂-harmonic (1,0)-form $\omega_X^{1,0}$ satisfies the conditions

$$\rho^*(X, X) = 0, \qquad \nabla''\omega_X = 0,$$

where $\nabla''\omega_X$ is the (2,0)-part of $\nabla\omega_X$.

ii) If the tensor ρ^* is nonnegative on M and positive at least at one point in M, then there are neither holomorphic (1,0)-forms, nor ∂-harmonic (1,0)-forms other than zero. Consequently, the Hodge numbers $h^{1,0}(M) = h^{0,1}(M) = 0$ and the first Betti number $b_1(M) = 0$.

iii) If the tensor $c\rho + (1-c)\rho^*$ is nonnegative on M for some constant $c \geq 0$, then any harmonic 1-form ω_X is ∇-parallel and satisfies the conditions

$$\rho(X, X) = \rho^*(X, X) = 0.$$

iv) If the tensor $c\rho + (1-c)\rho^*$ is nonnegative on M and positive at least at one point in M, then there are no harmonic 1-forms other than zero and $b_1 = 0$.

Note that these conditions agree with the classical Bochner conditions on Kähler manifolds.

In Example 1 we apply Theorem 1.1 to the complex twistor space $(\mathbf{Z}, J)$ of a compact hyper Kähler manifold which holonomy group is exactly Sp(n) to show the vanishing of the cohomology group $H^1(\mathbf{Z}, \mathcal{O}_{\mathbf{Z}})$ (see Theorem 5.1 in the last section).

On a compact balanced Hermitian manifold we find necessary and sufficient conditions for a ∂-harmonic (1,0)-form to be $\bar{\partial}$-harmonic (holomorphic) in terms of the Ricci tensors of the Levi-Civita and Chern connections and show that it is also necessary and sufficient condition for a $\bar{\partial}$-harmonic (1,0)-form to be ∂-harmonic. Constructing the tensor $H = 2\rho^* - k - k^*$ we prove

Theorem 1.2 *On a compact balanced Hermitian manifold the following conditions are equivalent:*

(i) The (1,0)-part of a harmonic 1-form ω_X is holomorphic;

(ii) A real 1-form ω_X with a holomorphic (1,0)-part is harmonic;

(iii) $\int_M H(X, X)\, dv = 0$.

We note that the tensor H vanishes identically on a Kähler manifold and measures the deviation of a balanced Hermitian manifold from a Kähler one (see section 3 below).

Finally, in Example 2 we show that the third condition of Theorem 1.2 is essential.

2 Preliminaries

Let (M, g, J) be a 2n-dimensional Hermitian manifold with metric g and complex structure J. The algebra of all C^∞ vector fields on M will be denoted by $\mathcal{X}M$. The Kähler form Ω of the Hermitian structure (g, J) is defined by $\Omega(X, Y) = g(JX, Y); \quad X, Y \in \mathcal{X}M$. The associated Lee form θ is given by $\theta = -\delta\Omega \circ J$.

We denote by ∇ and $R = [\nabla, \nabla] - \nabla_{[,]}$ the Levi-Civita connection of the metric g and the Riemannian curvature tensor, respectively. The corresponding curvature tensor of type (0,4) is given by the equality $R(X, Y, Z, V) = g(R(X, Y)Z, V), \quad X, Y, Z, V \in \mathcal{X}M$.

Further ρ and ρ^* will stand for the Ricci tensor and *-Ricci tensor, respectively. We have

$$\rho^*(X, Y) = \sum_{j=1}^{2n} R(e_j, X, JY, Je_j), \quad X, Y \in \mathcal{X}M.$$

Henceforth $\{e_1, ..., e_{2n}\}$ will denote an orthonormal frame.

We denote by D, T and K the canonical Chern (Hermitian) connection of the Hermitian structure, its torsion tensor and its curvature tensor (Hermitian curvature tensor), respectively. We recall that the Chern connection D is the unique linear connection preserving the metric g and the complex structure J, so that the torsion tensor T of D has the property $T(JX, Y) = T(X, JY), \quad X, Y \in \mathcal{X}M$. This implies (e.g. [5]):

$$T(JX, Y) = JT(X, Y), \quad X, Y \in \mathcal{X}M. \tag{2.1}$$

The corresponding torsion tensor of type (0,3) is defined by the equality

$$T(X, Y, Z) = g(T(X, Y), Z), \quad X, Y, Z \in \mathcal{X}M.$$

The curvature tensor K of D has the following properties:

$$K(JX, JY)Z = K(X, Y)Z, \quad K(X, Y)JZ = JK(X, Y)Z, \quad X, Y, Z \in \mathcal{X}M. \tag{2.2}$$

The Ricci identity for the Chern connection is expressed in the following form:

$$D_X D_Y Z - D_Y D_X Z = K(X, Y)Z - D_{T(X,Y)}Z, \qquad X, Y, Z \in \mathcal{X}M. \tag{2.3}$$

The two connections ∇ and D are related by the following identity

$$g(\nabla_X Y, Z) = g(D_X Y, Z) + \frac{1}{2} d\Omega(JX, Y, Z), \quad X, Y, Z \in \mathcal{X}M. \tag{2.4}$$

This equality implies that

$$T(X, Y, Z) = -\frac{1}{2} d\Omega(JX, Y, Z) - \frac{1}{2} d\Omega(X, JY, Z), \quad X, Y, Z \in \mathcal{X}M. \tag{2.5}$$

There are three Ricci-type tensors k, k^* and s associated with the curvature tensor K defined by

$$k(X, Y) = -\frac{1}{2} \sum_{j=1}^{2n} g(K(X, JY)e_j, Je_j); \quad k^*(X, Y) = -\frac{1}{2} \sum_{j=1}^{2n} g(K(e_j, Je_j)X, JY);$$

$$s(X,Y)=\sum_{j=1}^{2n} g(K(e_j,X)Y,e_j), \quad X,Y\in\mathcal{X}M.$$

The corresponding scalar curvatures are defined by $\tau = tr\rho, \tau^* = tr\rho^*, u = trk = trk^*$, $v = trs$.

The $(1,1)$-form κ corresponding to the tensor k represents the first Chern class of $\mathbf{M}$ (further we shall call it the Chern form) and the (1,1)-form κ^* corresponding to the tensor k^* is the mean curvature of the holomorphic tangent bundle $T^{1,0}M$ with the hermitian metric induced by g.

For an arbitrary vector field X in $\mathcal{X}M$ we denote by ω_X its dual 1-form defined by $\omega_X(Y) = g(X,Y), \quad Y\in\mathcal{X}M$. From (2.4) it follows that

$$\delta\omega_X = -\sum_{i=1}^{2n}(D_{e_i}\omega_X)e_i - \theta(X). \tag{2.6}$$

3 Balanced Hermitian manifolds

We recall the definition of a balanced Hermitian manifold and some equivalent conditions given in [14, 22] for completeness:

Definition: A Hermitian manifold (M,g,J) is said to be *balanced* if it satisfies one of the following equivalent conditions:

i) $\delta\Omega = 0 \quad (\theta = 0)$;

ii) $d\Omega^{n-1} = 0$;

iii) $\Delta_\partial f = \Delta_{\bar\partial} f = \frac{1}{2}\Delta_d f$ for every smooth function f on M, where $\Delta_\partial, \Delta_{\bar\partial}$ and Δ_d denote the Laplacians with respect to the operators $\partial, \bar\partial$ and d, respectively.

We shall use local holomorphic coordinates $\{z^\alpha\}, \alpha = 1,...,n$ and the corresponding frame field

$$\{\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^{\bar\alpha}} = \overline{\frac{\partial}{\partial z^\alpha}}\}, \quad \alpha = 1,...,n; \quad \bar\alpha = \bar1,...,\bar n$$

for some calculations. The first Bianchi identity for the Hermitian curvature K with respect to local holomorphic coordinates gives

$$K_{\alpha\bar\beta\gamma\bar\lambda} - K_{\gamma\bar\beta\alpha\bar\lambda} = -D_{\bar\beta}T_{\alpha\gamma\bar\lambda}. \tag{3.7}$$

By the condition $\delta\Omega = 0$ from (3.7) it follows that [7]

$$s(X,Y) = s(Y,X) = k(X,Y), \quad X,Y\in\mathcal{X}M. \tag{3.8}$$

It is immediate from (2.6) that on a balanced Hermitian manifold we have:

$$\delta\omega = -\sum_{i=1}^{2n}(D_{e_i}\omega)e_i. \tag{3.9}$$

Now let a be a tensor of type (0,2) and denote by a^t the tensor of type (0,2) defined by $a^t(X,Y) = a(Y,X), \quad X,Y\in\mathcal{X}M$. The symmetric part and the skew-symmetric part of the tensor a are given by

$$Sym(a) = \frac{1}{2}(a + a^t), \qquad Skew(a) = \frac{1}{2}(a - a^t),$$

respectively. The induced by the metric g scalar product in the vector space of (0,2)-tensors will be denoted by the same letter. For two tensors a, b of type (0,2) we have

$$g(a,b) = \sum_{i,j=1}^{2n} a(e_i,e_j)b(e_i,e_j); \qquad g(a^t,b) = g(a,b^t) = \sum_{i,j=1}^{2n} a(e_i,e_j)b(e_j,e_i).$$

For a fixed vector field X we obtain the following (0,2)-tensors $i_X T$ and $j_X T$ from the torsion tensor T:

$$i_X T(Y,Z) = T(X,Y,Z); \quad j_X T(Y,Z) = T(Y,Z,X), \quad Y,Z \in \mathcal{X}M. \tag{3.10}$$

The equalities (2.1) and (3.10) imply that the tensor $i_X T$ is J-invariant while the tensor $j_X T$ is J-antiinvariant, i.e.

$$(i_X T)(JX,JY) = (i_X T)(Y,Z), \quad (j_X T)(JY,JZ) = -(j_X T)(Y,Z).$$

The next statement, proved in [13], gives relations between the tensors ρ and ρ^*.

Proposition 3.1 *[13] Let (M,g,J) be a balanced Hermitian manifold. Then the Ricci tensors of the Riemannian and Hermitian curvature satisfy the following identities*

$$\rho^*(X,Y) = \rho^*(JX,JY) = \rho^*(Y,X), \quad X,Y \in \mathcal{X}M; \tag{3.11}$$

$$\rho(X,Y) - \rho(JX,JY) = -g(i_X T, (i_Y T)^t), \quad X,Y \in \mathcal{X}M. \tag{3.12}$$

$$k(X,Y) - \rho^*(X,Y) = \frac{1}{4}g(j_X T, j_Y T), \tag{3.13}$$

$$k(X,Y) + k^*(X,Y) - \frac{1}{2}(\rho(X,Y) + \rho(JX,JY)) - \rho^*(X,Y) = \frac{1}{2}g(i_X T, i_Y T), \tag{3.14}$$

$$k(X,X) + k^*(X,X) - \rho(X,X) - \rho^*(X,X) = \|Sym(i_X T)\|^2, \tag{3.15}$$

where $X,Y \in \mathcal{X}M$, and $\|.\|^2$ is the usual tensor norm.

We have

Corollary 3.2 *Let (M,g,J) be a balanced Hermitian manifold. Then*

i) $\tau = \tau^*$;

ii) (M,g,J) is Kähler iff $\tau = u$.

Proof: Taking traces in (3.13) and (3.14) we find

$$u - \tau^* = \frac{1}{4}\|T\|^2, \qquad 2u - \tau - \tau^* = \frac{1}{2}\|T\|^2.$$

Hence $\tau = \tau^*$ and $u - \tau = \frac{1}{4}\|T\|^2$. The last two equalities imply i) and ii). **QED**

Let η be a 1-form. Further we denote by $d'\eta, D'\eta$ and $\nabla'\eta$ the (1,1)-part (with respect to the complex structure J) of the exterior derivative $d\eta$, the covariant derivative $D\eta$ with respect to the Chern connection and the covariant derivative $\nabla\eta$ with respect to the Levi-Civita connection of η, respectively. For the ((2,0) + (0,2))-parts of $d\eta, D\eta$ and $\nabla\eta$ we use the denotations $d''\eta, D''\eta$ and $\nabla''\eta$, respectively. For example,

$$d'\eta(X,Y) = \frac{1}{2}(d\eta(X,Y) + d\eta(JX,JY)); \quad d''\eta(X,Y) = \frac{1}{2}(d\eta(X,Y) - d\eta(JX,JY)).$$

The next integral formulas are essential for the proof of our main results.

Proposition 3.3 *Let (M,g,J) be a compact balanced Hermitian manifold. Then for any vector field $X \in \mathcal{X}M$ we have*

$$\int_M 2\|Skew(D''\omega_X)\|^2\, dv = \int_M \{\|D''\omega_X\|^2 + k(X,X) - \frac{1}{2}(\delta\omega_X)^2 - \frac{1}{2}(\delta\omega_{JX})^2\}\, dv; \tag{3.16}$$

$$\int_M 2\|Skew(D''\omega_X)\|^2\, dv = \tag{3.17}$$

$$\int_M \{\|D'\omega_X\|^2 + k(X,X) - k^*(X,X) - \frac{1}{2}(\delta\omega_X)^2 - \frac{1}{2}(\delta\omega_{JX})^2\}\, dv.$$

Proof. Let $\omega_X = \omega_\alpha dz^\alpha + \omega_{\bar\alpha} dz^{\bar\alpha}$. We consider the following real 1-form

$$\varphi = D_\alpha\omega_\beta X^\alpha dz^\beta + D_{\bar\alpha}\omega_{\bar\beta} X^{\bar\alpha} dz^{\bar\beta}$$

and compute its co-differential $\delta\varphi$. Here and further the summation convention is assumed.

Using (3.9) and taking into account the Ricci identity (2.3) for the Chern connection, (2.1), (2.2) and (3.8), we obtain

$$-\delta\varphi = g(D''\omega_X, (D''\omega_X)^t) + k(X,X) - \frac{1}{2}X\delta\omega_X - \frac{1}{2}JX\delta\omega_{JX}.$$

Integrating this equality over M we find

$$\int_M \{g(D''\omega_X, (D''\omega_X)^t) + k(X,X) - \frac{1}{2}(\delta\omega_X)^2 - \frac{1}{2}(\delta\omega_{JX})^2\}\, dv = 0. \tag{3.18}$$

On the other hand we have

$$2\|Skew(D''\omega_X)\|^2 = -g(D''\omega_X, (D''\omega_X)^t) + \|D"\omega_X\|^2.$$

Then the last equality and (3.18) imply (3.16).

By similar calculations for the real 1-form

$$(D_\alpha\omega_\beta X^\beta - D_\alpha\omega_{\bar\beta} X^{\bar\beta})dz^\alpha + (D_{\bar\alpha}\omega_{\bar\beta} X^{\bar\beta} + D_{\bar\alpha}\omega_\beta X^\beta)dz^{\bar\alpha}$$

we find

$$\int_M \{\|D''\omega_X\|^2 - \|D'\omega_X\|^2 + k^*(X,X)\}\, dv = 0. \tag{3.19}$$

Now (3.16) and (3.19) imply (3.17). **QED**

Proposition 3.4 *Let (M,g,J) be a compact balanced Hermitian manifold. Then for any vector field $X \in \mathcal{X}M$ we have*

$$\int_M 2\|Skew(D'\omega_X)\|^2\, dv = \tag{3.20}$$
$$\int_M \{\|D'\omega_X\|^2 - \frac{1}{2}(\delta\omega_X)^2 + \frac{1}{2}(\delta\omega_{JX})^2 + g(j_X T, (D'\omega_X)^t)\}\, dv;$$

$$\int_M g(i_X T, D'\omega)\, dv = \int_M \{k(X,X) - k^*(X,X) + g(j_X T, D''\omega_X)\}\, dv. \tag{3.21}$$

Proof. Let $\omega_X = \omega_\alpha dz^\alpha + \omega_{\bar\alpha} dz^{\bar\alpha}$. We consider the real 1-form

$$D_{\bar\alpha}\omega_\beta X^{\bar\alpha} dz^\beta + D_\alpha \omega_{\bar\beta} X^\alpha dz^{\bar\beta}$$

and compute its co-differential. Integrating over M the obtained equality we find that

$$\int_M \{g(D'\omega_X, (D'\omega_X)^t) - \frac{1}{2}(\delta\omega_X)^2 + \frac{1}{2}(\delta\omega_{JX})^2 + g(j_X T, (D'\omega_X)^t)\}\, dv = 0. \tag{3.22}$$

On the other hand

$$2\|Skew(D'\omega_X)\|^2 = \|D'\omega_X\|^2 - g(D'\omega_X, (D'\omega_X)^t).$$

By virtue of the last equality and (3.22) we obtain (3.20).

To prove (3.21) we consider the real 1-form

$$T_{\alpha\beta\bar\gamma} X^\beta X^{\bar\gamma} dz^\alpha + T_{\bar\alpha\bar\beta\gamma} X^{\bar\beta} X^\gamma dz^{\bar\alpha}$$

and compute its co-differential. Taking into account (3.7) after an integration over M we get (3.21). **QED**

4 Proof of the theorems

Let X be a real vector field in $\mathcal{X}M$ and $\omega_X = \omega_\alpha dz^\alpha + \omega_{\bar\alpha} dz^{\bar\alpha} = \omega_X^{(1,0)} + \omega_X^{(0,1)}$ be its dual 1-form.

The (1,0)-form $\omega_X^{1,0} = \omega_\alpha dz^\alpha$ is ∂-harmonic iff

$$d\omega_{\alpha\beta} = D_\alpha\omega_\beta - D_\beta\omega_\alpha + T^\sigma_{\alpha\beta}\omega_\sigma = 0, \quad \delta\omega_X = \delta\omega_{JX} = 0. \tag{4.23}$$

The real 1-form ω_X is harmonic iff

$$d\omega_{\alpha\beta} = D_\alpha\omega_\beta - D_\beta\omega_\alpha + T^\sigma_{\alpha\beta}\omega_\sigma = 0, \quad d\omega_{\alpha\bar\beta} = D_\alpha\omega_{\bar\beta} - D_{\bar\beta}\omega_\alpha = 0, \tag{4.24}$$

$$\delta\omega_X = 0. \tag{4.25}$$

The second equality of (4.24) implies that $\delta\omega_{JX} = 0$.

The (1,0)-form $\omega_X^{1,0} = \omega_\alpha dz^\alpha$ is holomorphic iff

$$D_{\bar\alpha}\omega_\beta = 0. \tag{4.26}$$

It is immediate from this equality that $\delta\omega_X = \delta\omega_{JX} = 0$.

4.1 Proof of Theorem 1.1

i) Let $\omega_X^{1,0} = \omega_\alpha dz^\alpha$ be a holomorphic (1,0)-form. Taking into account the condition (4.26) from (3.19) it follows that

$$\int_M \{\|D''\omega_X\|^2 + k^*(X,X)\}\, dv = 0. \tag{4.27}$$

Since

$$\nabla''\omega_X = D''\omega_X + \frac{1}{2} j_X T, \tag{4.28}$$

then

$$\|\nabla''\omega_X\|^2 = \|D''\omega_X\|^2 + g(D''\omega_X, j_X T) + \frac{1}{4}\|j_X T\|^2. \tag{4.29}$$

Under the condition (4.26) the equality (3.21) implies

$$\int_M \{g(j_X T, D''\omega_X) + k(X,X) - k^*(X,X)\}\, dv = 0. \tag{4.30}$$

By virtue of (4.29), (4.27), (4.30) and (3.13) it follows that

$$\int_M \{\|\nabla''\omega_X\|^2 + \rho^*(X,X)\}\, dv = 0. \tag{4.31}$$

This formula proves a).

In order to prove b) we shall show that (4.31) is also true for any ∂-harmonic (1,0)-form. Indeed, let $\omega_X^{1,0} = \omega_\alpha dz^\alpha$ be ∂-harmonic. Then (4.23) implies

$$Skew(D''\omega_X) = -\frac{1}{2} j_X T. \tag{4.32}$$

Using this equality and (3.13) we find

$$\|Skew(D''\omega_X)\|^2 = \frac{1}{4}\|j_X T\|^2 = k(X,X) - \rho^*(X,X).$$

The last equality, (4.23) and (3.16) imply

$$\int_M \|D''\omega_X\|^2\, dv = \int_M (k(X,X) - 2\rho^*(X,X))\, dv. \tag{4.33}$$

Since the tensor $j_X T$ is skew-symmetric, then (4.32) leads to

$$g(D''\omega_X, j_X T) = g(Skew(D''\omega_X), j_X T) = -\frac{1}{2}\|j_X T\|^2.$$

We obtain from (4.29) that

$$\|\nabla''\omega_X\|^2 = \|D''\omega_X\|^2 - \frac{1}{4}\|j_X T\|^2 = \|D''\omega_X\|^2 - k(X,X) + \rho^*(X,X).$$

Integrating the last equality and taking into account (4.33), we obtain (4.31) which proves b).

The statement ii) follows ¿from (4.31) by applying the Dolbeault theory to the $\bar\partial$-operator and the well known inequality (see e.g.[17], Section 3.5)

$$b_1(M) \le h^{1,0}(M) + h^{0,1}(M) \tag{4.34}$$

To prove iii) and iv) let $\omega_X = \omega_\alpha dz^\alpha + \omega_{\bar\alpha} dz^{\bar\alpha}$ be a harmonic 1-form. From (4.24) and (3.13) we have

$$g(i_X T, D''\omega_X) = -\frac{1}{2}\|i_X T\|^2 = 2\rho^*(X,X) - 2k(X,X).$$

Combining this equality with (3.21) we get

$$\int_M g(j_X T, D'\omega)\, dv = \int_M \{2\rho^*(X,X) - k(X,X) - k^*(X,X)\}\, dv \tag{4.35}$$

The last equality, (4.25) and (3.20) imply

$$\int_M \{\|D'\omega_X\|^2 + 2\rho^*(X,X) - k(X,X) - k^*(X,X)\}\, dv = 0. \tag{4.36}$$

On the other hand we have

$$\nabla'\omega_X = D'\omega_X + Sym(i_X T)$$

and

$$\|\nabla'\omega_X\|^2 = \|D'\omega_X\|^2 + 2g(D'\omega_X, Sym(i_X T)) + \|Sym(i_X T)\|^2.$$

Integrating the last equality and taking into account (4.35), (4.36) and (3.15) we find

$$\int_M \{\|\nabla'\omega_X\|^2 + \rho(X,X) - \rho^*(X,X)\}\, dv = 0. \tag{4.37}$$

Let c be a positive constant. Combining (4.37) with (4.31) we obtain

$$\int_M \{c\|\nabla'\omega_X\|^2 + \|\nabla''\omega_X\|^2 + c\rho(X,X) + (1-c)\rho^*(X,X)\}\, dv = 0. \tag{4.38}$$

This formula implies immediately iii). The statement iv) also follows from (4.38) by using the Hodge theory. **QED**

4.2 Proof of Theorem 1.2

We define the tensor H of type (0,2) by the equality

$$H(X,Y) = 2\rho^*(X,Y) - k(X,Y) - k^*(X,Y); \quad X, Y \in \mathcal{X}M.$$

Let $\omega_X = \omega_\alpha dz^\alpha + \omega_{\bar\alpha} dz^{\bar\alpha}$ be a harmonic 1-form. By virtue of (4.35) we have

$$\int_M \Big(\|D'\omega_X\|^2 + H(X,X)\Big)\, dv = 0, \tag{4.39}$$

Now the equivalence i) $\Leftrightarrow$ iii) follows immediately from (4.39).

To prove the equivalence ii) $\Leftrightarrow$ iii) let $\omega_X^{1,0}$ be a holomorphic (1,0)-form. Since $d''\omega_X = 2Skew(D''\omega_X) + j_X T$, then

$$\|d''\omega_X\|^2 = 4\|Skew(D''\omega_X)\|^2 + 4g(j_X T, D''\omega_X) + \|j_X T\|^2. \tag{4.40}$$

Taking into account (3.16) and (3.19) we find

$$\int_M \{2\|Skew(D''\omega_X)\|^2 + k^*(X,X) - k(X,X)\}\, dv = 0. \tag{4.41}$$

By virtue of the equalities (3.21), (4.40), (4.41) and (3.15) we obtain

$$\int_M \{\frac{1}{2}\|d''\omega_X\|^2 + H(X,X)\}\, dv = 0.$$

The last equality implies the equivalence ii) $\Leftrightarrow$ iii) which completes the proof of Theorem 1.2. **QED**

In the next theorem we find obstructions to the existence of holomorphic (1,0)-forms in terms of the Ricci tensors of Chern connection. We have

Theorem 4.1 *Let (M,g,J) be a compact balanced Hermitian manifold.*

i) If the tensor $k+k^*$ is nonnegative on M, then any holomorphic (1,0)-form $\omega_X^{1,0}$ satisfies the conditions

$$k(X,X) + k^*(X,X) = 0, \qquad Sym(D''\omega_X) = 0.$$

ii) If the tensor $k+k^*$ is nonnegative on M and positive at least at one point in M, then there are no holomorphic 1-forms other than zero and $h^{1,0}=0$.

Proof. Let $\omega_X^{1,0}$ be a holomorphic (1,0)-form. The identity

$$\|Sym(D''\omega_X)\|^2 + \|Skew(D''\omega_X)\|^2 = \|D''\omega_X\|^2$$

and the equality (3.16) give

$$\int_M \{2\|Sym(D''\omega_X)\|^2 - \|D''\omega_X\|^2 + k(X,X)\}\, dv = 0.$$

Combining the last formula with (4.27) we obtain

$$\int_M \{2\|Sym(D''\omega_X)\|^2 + k(X,X) + k^*(X,X)\}\, dv = 0. \tag{4.42}$$

Now the statements i) and ii) follow from formula (4.42). **QED**

We obtain as a corollary from the proof of Theorem 4.1 and formulas in Proposition 3.1 the following

Proposition 4.2 *Let (M,g,J) be a compact balanced Hermitian manifold.*

i) If the tensor $\rho+\rho^*$ is nonnegative on M, then any holomorphic (1,0)-form $\omega_X^{1,0}$ satisfies the conditions

$$\rho(X,X) + \rho^*(X,X) = k(X,X) + k^*(X,X) = i_X T = 0$$

and the vector field X is Killing.

ii) If $\rho+\rho^*$ is nonnegative on M and positive at least at one point in M, then there are no holomorphic 1-forms other than zero and the Hodge number $h^{1,0}=0$.

Proof. We recall that a real vector field X is said to be Killing if $L_X g = 0$, where L_X denotes the Lie derivative with respect to X. In terms of the Chern connection the Killing condition is expressed by the equalities

$$Sym(D''\omega_X) = 0, \tag{4.43}$$

$$(4.44) \qquad Sym(D'\omega_X) = -Sym(i_X T).$$

Let $\omega_X^{1,0}$ be a holomorphic (1,0)-form. By virtue of (3.15) we can apply Theorem 4.1, which implies $Sym(D''\omega_X) = 0$ and $k(X,X) + k^*(X,X) = 0$. Taking into account (3.15) we find $\rho(X,X) + \rho^*(X,X) = 0, i_X T = 0$. From (4.43) and (4.44) it follows that X is Killing.

The second statement follows immediately from (3.15) and Theorem 1.2. **QED**

5 Examples

Example 1. Let $(\mathbf{M}^{4n}, g)$ be a compact $4n$-dimensional hyper-Kähler manifold, i.e. there are three anticommuting complex structures which are parallel with respect to the Levi-Civita connection of g; for $n = 1$ $(\mathbf{M}^4, g)$ means a self-dual Ricci flat manifold. It is well known that every hyper-Kähler manifold can be considered as a Ricci-flat quaternionic Kähler manifold. The twistor space of $\mathbf{M}^{4n}$ is a 2-sphere bundle $\mathbf{Z}$ over $\mathbf{M}^{4n}$ whose fibre at any point $p \in \mathbf{M}^{4n}$ consists of all complex structures on the tangent space $T_p\mathbf{M}^{4n}$ at p which are compatible with the given hyper-Kähler structure. There are two natural distributions on $\mathbf{Z}$, namely, the vertical 2-dimensional distribution V consisiting of all vector fields tangent to the fibre and a horizontal $4n$-dimensional distribution H induced by the Levi-Civita connection. The $(4n+2)$-dimensional twistor space $\mathbf{Z}$ admits a complex structure J [6, 26]. There exists a natural 1-parameter family of hermitian metrics $h_c, c > 0$ on $(\mathbf{Z}, J)$ such that the projection $\pi : \mathbf{Z} \to \mathbf{M}^{4n}$ is a Riemannian submersion with totally geodesic fibres [12]. The twistor space $(\mathbf{Z}, J, h_c), c > 0$ is a compact balanced Hermitian manifold [23, 4]. The curvature of $(\mathbf{Z}, h_c)$ for $n = 1$ has been calculated by many authors [9, 10, 11, 12, 19, 28]. The $*$-Ricci tensor ρ_c^* of $(\mathbf{Z}, h_c, J)$ for $n \geq 2$ is given in [4] by formulas (3.12). The latter formulas are also valid when $\mathbf{M}^4$ is an oriented self-dual Ricci-flat Riemannian manifold. Substituting $s = 0$ into (3.12) from [4], we obtain

$$(5.45) \qquad \rho_c^*(X^v, X^v) > 0, \quad \rho_c^*(Y^h, Y^h) = \rho_c^*(Y^h, X^v) = 0, \quad X^v \in V, \quad Y^h \in H.$$

The formula (5.45) shows that the tensor ρ_c^* is non-negative on $\mathbf{Z}$. An application of Theorem 1.1 leads to

Theorem 5.1 *Let $(\mathbf{Z}, J)$ be the twistor space of a compact hyper-Kähler manifold $\mathbf{M}$ endowed with the natural complex structure J. Then we have*

$$(5.46) \qquad h^{0,1}(\mathbf{Z}) = dim H^1(\mathbf{Z}, \mathcal{O}_{\mathbf{Z}}) = b_1(\mathbf{Z}).$$

In particular, if the hyper Kähler manifold $\mathbf{M}$ is irreducible then $H^1(\mathbf{Z}, \mathcal{O}_{\mathbf{Z}}) = 0$

Proof. Let $\omega_X^{1,0}$ be a ∂-harmonic (1,0)-form on $(\mathbf{Z}, J, h_c)$. The condition $\rho_c^*(X, X) = 0$ of Theorem 1.1 together with (5.45) implies that the vector field X has to be horizontal i.e. $X = X^h$. Using the general formula $g(\nabla_{JY} J)Z, U) = g(T(Z, U), Y)$, which is a simple consequence of (2.4) and (2.5), we derive from (2.9) in [4] that $j_{X^h} T = 0, X^h \in H$. Then the condition

$\nabla''\omega_X = 0$ of Theorem 1.1 together with the formula (4.28) implies $D''\omega_X = 0$ which means that X is a (real) holomorphic vector field on $(\mathbf{Z}, J)$. Hence, it generates a non-zero Killing vector field on $(\mathbf{M}, g)$ (see e.g. [18, 26, 24]). The dimension of the space of Killing vector fields on $(\mathbf{M}, g)$ is equal to $b_1(\mathbf{M})$ since $(\mathbf{M}, g)$ is Ricci flat. Applying Dolbeault theory, we obtain

$$h^{0,1}(\mathbf{Z}) \leq b_1(\mathbf{M}). \tag{5.47}$$

It is well known that $h^{1,0}(\mathbf{Z}) = 0$ and $b_1(\mathbf{M}) = b_1(\mathbf{Z})$ [26]. The assertion follows from (4.34), (5.47) and the last two equalities. If $(\mathbf{M}, g)$ is irreducible then it is simply connected [8]. Hence, $b_1(\mathbf{M}) = 0$ and (5.46) implies $H^1(\mathbf{Z}, \mathcal{O}_{\mathbf{Z}}) = 0$ **Q.E.D.**

The next example shows that the third condition in Theorem 1.3 is essential.

Example 2. Consider the complex Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \quad z_1, z_2, z_3 \in \mathbf{C} \right\},$$

with multiplication. The complex Iwasawa manifold is the compact quotient space $M = G/\Gamma$ formed from the right cosets of the discrete group Γ given by the matrices whose entries z_1, z_2, z_3 are Gaussian integers. The 1-forms

$$dz_1, \quad dz_2, \quad dz_3 - z_1 dz_2 \tag{5.48}$$

are left invariant by G and certain by Γ. These 1-forms pass to the quotient M. We denote by $\alpha_1, \alpha_2, \alpha_3$ the corresponding 1-forms on M, respectively. Consider the Hermitian manifold (M, g, J), where J is the natural complex structure on M arising from the complex coordinates z_1, z_2, z_3 on G and the metric g is determined by $g = \sum_{i=1}^{3} \alpha_i \otimes \bar{\alpha}_i$. The Chern connection D is determined by the conditions that the 1-forms $\alpha_1, \alpha_2, \alpha_3$ are parallel. The torsion tensor of D is given by

$$T(\alpha_i^{\#}, \alpha_j^{\#}) = -[\alpha_i^{\#}, \alpha_j^{\#}], \quad i, j = 1, 2, 3,$$

where $\alpha_i^{\#}$ is the vector field corresponding to α_i via g. The nonzero term is only $T(\alpha_1^{\#}, \alpha_2^{\#}) = -\alpha_3^{\#}$ and its complex conjugate. Thus, the space (M, g, J) is a compact balanced Hermitian (non Kähler) manifold with a flat Chern connection.

It is easy to compute that

$$H(\alpha_1^{\#}, \alpha_1^{\#}) = H(\alpha_2^{\#}, \alpha_2^{\#}) = 0, \qquad H(\alpha_3^{\#}, \alpha_3^{\#}) = -2.$$

The conclusions of Theorem 1.3 agree with the fact that the holomorphic (1,0)-forms α_1 and α_2 are closed while the holomorphic $(1,0)$-form α_3 is not closed (indeed, from (5.48) it follows that $d\alpha_3 = -\alpha_1 \wedge \alpha_2$).

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