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THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ON THE HOMOLOGY OF FREE 2-STEP NILPOTENT LIE ALGEBRAS

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1 Introduction

The free 2-step nilpotent Lie algebra of rank r is $\mathcal{N}(r) = V \oplus \Lambda^2 V$, where V is an r dimensional vector space over $\mathbb{C}$. The only non-zero Lie brackets are for $v, w \in V$, in which case $[v, w] = v \wedge w \in \Lambda^2 V$. The centre of $\mathcal{N}(r)$ is $\Lambda^2 V$.

The Lie algebra homology $H_*(\mathcal{N}(r))$ has been described explicitly for all r by Sigg [7]. In fact, it turns out that this description can be derived from an earlier description of the Lie algebra cohomology by Kostant [5]. We shall give Sigg's description here and give the derivation from Kostant's in the Appendix. Sigg showed that

$$H_*(\mathcal{N}(r)) = \bigoplus_{I \subset \langle r \rangle} H_I(\mathcal{N}(r)), \tag{1.1}$$

where $\langle r \rangle$ denotes the set of integers from 1 to r and the summand $H_I(\mathcal{N}(r))$ is isomorphic as a representation of $GL(r) = GL(V)$ to the irreducible tensor representation $R_\lambda(V)$ corresponding to the self-conjugate partition $\lambda = (I; I)$ in Frobenius notation (explained in Section 2). Furthermore, the homology grading of $H_I\big(\mathcal{N}(r)\big)$ is $\Sigma(I) = \sum_{i \in I} i$. Hence, one may write a formula for the Poincaré polynomial

$$P(\mathcal{N}(r); t) = \sum_{n \in \mathbb{N}} \dim H_n(\mathcal{N}(r)) t^n = \sum_{I \subset \langle r \rangle} \dim R_{(I;I)}(V) t^{\Sigma(I)} \tag{1.2}$$

and, specializing, for the 'total homology' $T(r) = P(\mathcal{N}(r); 1)$. In principle, these sums may be evaluated using one of the standard formulae for the dimension of an irreducible representation of $GL(r)$. For example, the first nine values of $T(r)$ are as follows.

r	$T(r)$
1	2
2	6
3	36
4	420
5	9800
6	452760
7	41835024
8	7691667984
9	2828336198688

However, the length of the computation (as well as the size of the answer) grows exponentially, because it involves a sum of 2^r positive terms.

A well-known lower bound for the total homology of any 2-step nilpotent Lie algebra is 2^z, where z is the dimension of the centre, since the so-called Toral Rank Conjecture is true in this case. Recently, in [8], the bound has been improved to 2^{z+s}, where $s = [(n+1)/2]$ and n is the codimension of the centre. For the free 2-step nilpotent Lie algebra, we have $z = r(r-1)/2$ and $n = r$, so that $s \geq r/2$ and a lower bound for $T(r)$ is $2^{r^2/2}$.

In this paper we find the following explicit formula for $T(r)$.

Theorem 1.1. *For $n \geq 0$*

$$T(2n+1) = 2^{n+1} \beta(n)^2 \tag{1.3}$$

$$T(2n+2) = 2^{n+1} \beta(n) \beta(n+1) \tag{1.4}$$

where

$$\beta(n) = \prod_{1\le i\le j\le n} \frac{2(i+j)-1}{2i-1} \tag{1.5}$$

$$= \prod_{1\le k\le n} \frac{(4k)!k!^2}{(2k)!^3} \tag{1.6}$$

and, by natural convention, $\beta(0) = 1$.

This result is proved in Section 2 by using Giambelli's determinant formula for the representation dimensions and observing that several simplifications can be made for self-conjugate partitions leading to an expression for $T(r)$ as a single determinant, which can in turn be simpilfied by elementary row and column operations.

In fact, Giambelli's formula is really a formula for the $GL(r)$ character of a representation and the initial simplification gives a single determinant formula (Proposition 2.2) for the $GL(r)$-character of $H_*(\mathcal{N}(r))$. The second simplification only works at the level of $SO(r)$-characters but gives Proposition 2.3 as a character version of Theorem 1.1. In particular, we observe in the process that $\beta(n)$ is actually the dimension of an irreducible $SO(2n+1)$ representation.

While the nature of $H_*(\mathcal{N}(r))$ as an $SO(r)$ representation appears a little mysterious, we are able to explain the multiplicities 2^{n+1} in (1.3) and (1.4) by showing (Proposition 2.5) how to decompose the sum (1.1) into 2^{n+1} isomorphic $SO(r)$-representations where $r = 2n+1$ or $2n+2$.

One important consequence of Theorem 1.1 is that it enables us to show that $2^{r^2/2}$ is the dominant term in the asymptotic behaviour of $T(r)$. More precisely, in Section 3, we analyse the asymptotic behaviour of $\beta(n)$ using (1.6) and deduce the following.

Theorem 1.2. *There is a constant* $\kappa_T \simeq 1.381431394$ *such that*

$$T(r) \sim 2^{r^2/2} r^{1/8} \kappa_T \tag{1.7}$$

We apply this asymptotic analysis to further improve the lower bound on the total homology of any 2-step nilpotent Lie algebra in Section 4.

Now, it is clear from (1.5) that $\beta(n)$ is always odd. For example, the first five values of β are as follows.

n	0	1	2	3	4
$\beta(n)$	1	3	35	1617	297297

Hence the power of 2 that divides $T(r)$ is precisely the one given in (1.3) and (1.4). We may also use (1.6) to find the powers of odd primes dividing $\beta(n)$ and hence find the complete prime factorization of $T(r)$. We do this in Section 5. We also mention two unproven observations about the dimensions of $GL(r)$-representations corresponding to self-conjugate partitions, which we noticed in the course of the MAPLE experiments that lead to the discovery of the results proved in this paper.

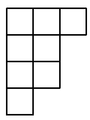
2 Frobenius' notation and Giambelli's formula

A partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$ is often represented by its Young diagram $Y(\lambda)$, a graphical arrangement of λ_i boxes in the i-th row starting in the first column. The conjugate partition λ' of λ has Young diagram $Y(\lambda')$ obtained by reflecting $Y(\lambda)$ in the diagonal.

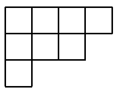
Another way to denote a partition λ is due to Frobenius. Let $d = d_\lambda$ be the number of diagonal boxes of $Y(\lambda)$. For $i = 1, \ldots, d$, let α_i to be the number of boxes in the i-th row to the right of and including the diagonal. Let β_i to be the number of boxes in the i-th column below and including the diagonal. Then one writes $\lambda = (I; J)$ where $I = \{\alpha_1, \ldots, \alpha_d\}$ and $J = \{\beta_1, \ldots, \beta_d\}$. Observe that $\lambda' = (J; I)$.

Note that $\alpha_1 > \cdots > \alpha_d \geq 1$ and $\beta_1 > \cdots > \beta_d \geq 1$, so the sets I and J do determine the sequences α_i and β_i. To compute the Frobenius notation directly from the standard notation $\lambda = (\lambda_1, \ldots, \lambda_k)$, set $d = \#\{i : \lambda_i \geq i\}$, $\alpha_i = \lambda_i - i + 1$ and $\beta_i = \lambda'_i - i + 1$. An example is given below, showing a partition λ and its conjugate λ' in standard notation and Frobenius notation, together with their Young diagrams.

$$\lambda = (3,2,2,1) = (\{3,1\};\{4,2\}) \qquad Y(\lambda) =$$



$$\lambda' = (4,3,1) = (\{4,2\};\{3,1\}) \qquad Y(\lambda') =$$



Note that there are different conventions on the precise form of Frobenius notation and, in particular, [7] uses a slightly different one.

Recall (e.g. [3] §15.5) that the irreducible representations $R_\lambda(r)$ of $GL(r)$ correspond to partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$, where $m \leq r = \dim V$. The 'second Giambelli formula' ([3](24.11)) gives the character of $R_\lambda(V)$ as a determinant:

$$\mathrm{Char}_{GL(r)}\, R_\lambda(V) = \det G_1(\lambda, r), \tag{2.1}$$

where the Giambelli matrix $G_1(\lambda, r)$ is the $n \times n$ matrix with entries

$$G_1(\lambda, r)_{ij} = E\tbinom{r}{\lambda'_i + j - i} \tag{2.2}$$

Here n is the length of the conjugate partition λ' and $E\binom{r}{k}$ is the character of $\Lambda^k V$. The dimension of $R_\lambda(V)$ is obtained by 'dropping the E', that is, substituting the binomial coefficient $\binom{r}{k}$ for the character $E\binom{r}{k}$.

For a self-conjugate partition $\lambda = \lambda'$, we have $n = m \leq r$. If we extend λ' by 0, that is, let $\lambda'_i = 0$ for $n < i \leq r$, then we may use (2.2) to define an $r \times r$ matrix $\widehat{G}_1(\lambda, r)$ with the same determinant as $G_1(\lambda, r)$. The rows of this extended Giambelli matrix $\widehat{G}_1(\lambda, r)$ can be reordered to obtain a matrix with a particularly simple description in terms of the Frobenius notation for λ.

Proposition 2.1. *Let $\lambda = (I; I)$ be a self-conjugate partition in Frobenius notation. Then*

$$\det G_1(\lambda, r) = (-1)^{\# I_0} \det G_2(I, r)$$

where $\#I_0$ is the number of even elements of I and $G_2(I,r)$ is the $r \times r$ matrix with entries

$$G_2(I,r)_{ij} = \begin{cases} E\binom{r}{j+i-1}, & \text{if } i \in I; \\ E\binom{r}{j-i}, & \text{if } i \notin I. \end{cases} \tag{2.3}$$

Proof. We show that the rows of $G_2(I,r)$ are a reordering of the rows of $\widehat{G}_1(\lambda,r)$, as described above. The set $\{\lambda_i - i : i = 1 \ldots r\}$ is the disjoint union of $\{\lambda_i - i \geq 0\}$ and $\{\lambda_i - i < 0\}$. Since $\alpha_i = \lambda_i - i + 1$, it follows that the first set is $\{i-1 : i \in I\}$. On the other hand, if $\lambda_i - i < 0$ then $i - \lambda_i$ is the distance, in the full $r \times r$ square, from λ_i to the main diagonal, that is the height of the i-th column from the main diagonal to $Y(\lambda)$. By looking at the upper triangular part of $Y(\lambda)$ it is clear that these heights and the numbers $\lambda_i - i$, for $i = 1, \ldots, d$ are all different. Hence the second set is $\{-i : i \notin I\}$.

Finally, the reordering of the rows is given by the permutation σ, that from $\langle r \rangle$ moves the set I to the front reversing the order, leaving the complement I^c at the back. It follows by induction on the size $\#I$ of I that $\mathrm{sg}(\sigma) = (-1)^{\#I_0}$, where I_0 is the set of even elements in I. □

From this we immediately obtain the following.

Proposition 2.2.

$$\mathrm{Char}_{GL(r)} H_*(\mathcal{N}(r)) = \det G_3(r)$$

where $G_3(r)$ is the $r \times r$ matrix with entries

$$G_3(r)_{ij} = E\binom{r}{j-i} - (-1)^i E\binom{r}{j+i-1},$$

Proof. Combine (1.1), (2.1) and Proposition 2.1 and use the linearity of det in rows. □

It turns out that the determinant of $G_3(r)$ is quite amenable to simplification by elementary row and column operations and this will yield a proof of our main result Theorem 1.1. However to make the simplification, we make frequent use of the identity $E\binom{r}{k} = E\binom{r}{r-k}$. Therefore, the main step is only valid at the level of characters for $SO(r)$ and not for $GL(r)$. What we may prove is the following.

Proposition 2.3.

$$\mathrm{Char}_{SO(2n+1)} H_*(\mathcal{N}(2n+1)) = 2^{n+1} \det B(n) \det C(n) \tag{2.4}$$

$$\mathrm{Char}_{SO(2n+2)} H_*(\mathcal{N}(2n+2)) = 2^{n+1} \det D(n) \det A(n+1) \tag{2.5}$$

where $A(n)$, $B(n)$, $C(n)$ and $D(n)$ are $n \times n$ matrices with coefficients

$$\begin{aligned} A(n)_{ij} &= E\binom{2n}{n+1+j-2i} + E\binom{2n}{n-2+j+2i} \\ B(n)_{ij} &= \begin{cases} E\binom{2n+1}{n+2-2i} & j = 1 \\ E\binom{2n+1}{n+1+j-2i} + E\binom{2n+1}{n-2+j+2i} & j > 1 \end{cases} \\ C(n)_{ij} &= E\binom{2n+1}{n+1+j-2i} - E\binom{2n+1}{n+j+2i} \\ D(n)_{ij} &= E\binom{2n+2}{n+1+j-2i} - E\binom{2n+2}{n+j+2i} \end{aligned}$$

Furthermore, $\det B(n) = \det C(n)$ *and this is the character of the irreducible $SO(2n+1)$ representation W_a with highest weight $a = (1, \ldots, 1, 2)$.*

Proof. Let $r = 2n+1$ or $r = 2n+2$, depending on which case we are considering. Note that the number of odd numbers in $\langle r \rangle$ is $n+1$, while the number of even numbers is $r-n-1$.

First we apply to $G_3(r)$ the column operations

$$\mathrm{Col}_j \mapsto \mathrm{Col}_j + \mathrm{Col}_{r+1-j} \qquad j = 1, \ldots, r-n-1.$$

After this the first $n+1$ columns in the even rows are 0, while the same columns in the odd rows are all divisible by 2. Note that, when r is odd, the middle column is not changed by the above operations, but this already has the required property. Hence, we next apply

$$\mathrm{Col}_j \mapsto \tfrac{1}{2}\,\mathrm{Col}_j \qquad j = 1, \ldots, n+1.$$

to take the required factor of 2^{n+1} out of the determinant. The non-zero entries of the first $n+1$ columns are now equal to $E\binom{r}{j-i} + E\binom{r}{j+i-1}$. We may then apply

$$\mathrm{Col}_j \mapsto \mathrm{Col}_j - \mathrm{Col}_{r+1-j} \qquad j = n+2, \ldots, r$$

which sets to 0 the last $r-n-1$ columns of the odd rows, without changing the even rows.

Now, we may permute the rows to collect the odd rows at the top and the even rows at the bottom. If we further reverse the order of the first $n+1$ columns, then the combined effect of these permutations does not change the sign of the determinant and we are left with a block diagonal matrix. When i is odd, $G_3(r)_{ij} = G_3(r)_{i(r-j+1)}$ so the first block contains the last $n+1$ columns of the odd rows of $G_3(r)$, except that the factor of 2 has gone from the middle column, when r is odd. The second block contains the last $r-n-1$ columns of the even rows of $G_3(r)$.

One of the two block matrices—the one coming from the odd rows, when r is odd, and from the even rows, when r is even—has last row equal to $(0 \cdots 0\, 1)$. Removing the last row and column of this block, we obtain the main statement of the proposition.

To see that $\det B(n) = \det C(n)$, apply to $B(n)$ successively the operations

$$\begin{aligned} \mathrm{Row}_i &\mapsto \mathrm{Row}_i - \mathrm{Row}_{i+1} \qquad i = 1, \ldots, n-1 \\ \mathrm{Col}_j &\mapsto \mathrm{Col}_j + \mathrm{Col}_{j-2} \qquad j = 3, \ldots, n \end{aligned}$$

Finally we observe (with some surprise) that $\det B(n)$ is one of the Giambelli-type determinant formulae ([3] Corollary 24.35) for the character of the irreducible $SO(2n+1)$ representation W_a with highest weight $a = (1, \ldots, 1, 2)$ in the basis of fundamental weights, where the 2 is associated to the short simple root. □

Proof of Theorem 1.1. We now drop the E's and write simply the binomial coefficients to prove that $\det A = \det B$ and $\det C = \det D$. For this, apply to A or C the column operations

$$\mathrm{Col}_j \mapsto \mathrm{Col}_j + \mathrm{Col}_{j-1} \qquad j = n, \ldots, 2$$

and use the fact that $\binom{r}{k} + \binom{r}{k-1} = \binom{r+1}{k}$. Notice that the first columns of A and B are already equal, as are the first columns of C and D.

Finally, the Weyl dimension formula ([3] Corollary 24.6 & Exercise 24.30) gives

$$\dim W_a = \frac{\displaystyle\prod_{1 \le i < j \le n} 2(j-i)(2(i+j)-1) \prod_{1 \le j \le n} (4j-1)}{\displaystyle\prod_{1 \le j \le n} (2j-1)!}$$

This expression may be simplified to (1.5), which is also a simplification of (1.6). □

Remark 2.4. Proposition 2.3 implies that as an $SO(2n+1)$ representation, the homology $H_*(\mathcal{N}(2n+1))$ is isomorphic to the direct sum of 2^{n+1} copies of the tensor square of W_a. However, it is less clear how to interpret the even character formula (2.5). Following the final part of the proof of Theorem 1.1 at the level of characters, we may say first that $\det A(n+1)$ is the restriction to $SO(2n+2)$ of the irreducible $SO(2n+3)$-character $\det B(n+1)$. On the other hand, $\det D(n)$ is a virtual character of $SO(2n+2)$, whose restiction to $SO(2n+1)$ is the irreducible character $\det B(n)$. It is unclear how to interpret the product of these two characters.

The manipulations of Giambelli determinants used in the proof of Proposition 2.3 may also be used to prove some refinements of this result, which shed some light on the multiplicity 2^{n+1} occurring in the proposition.

For any set $I \subset \mathbb{N}$, let I_0 denote the set of even numbers in I and I_1 denote the set of odd numbers. We will now write $H_I(\mathcal{N}(r))$ as $H_{[I_1,I_0]}(\mathcal{N}(r))$ and, for any $K \subset \langle r\rangle_1$ and $L \subset \langle r\rangle_0$, will define

$$\begin{aligned} H_{[K,*]}(\mathcal{N}(r)) &= \bigoplus_{J\subset\langle r\rangle_0} H_{[K,J]}(\mathcal{N}(r)) \\ H_{[*,L]}(\mathcal{N}(r)) &= \bigoplus_{J\subset\langle r\rangle_1} H_{[J,L]}(\mathcal{N}(r)) \end{aligned}$$

Proposition 2.5.

$$\begin{aligned} \mathrm{Char}_{SO(2n+1)}\, H_{[K,*]}(\mathcal{N}(2n+1)) &= \det B(n) \det C(n) \\ \mathrm{Char}_{SO(2n+2)}\, H_{[K,*]}(\mathcal{N}(2n+2)) &= \det D(n) \det A(n+1) \\ \mathrm{Char}_{SO(2n+1)}\, H_{[*,L]}(\mathcal{N}(2n+1)) &= 2 \det B(n) \det C(n) \\ \mathrm{Char}_{SO(2n+2)}\, H_{[*,L]}(\mathcal{N}(2n+2)) &= \det D(n) \det A(n+1) \end{aligned}$$

In other words, the partial sums $H_{[K,*]}(\mathcal{N}(r))$ are all isomorphic as representations of $SO(r)$, independent of K. Furthermore, the partial sums $H_{[*,L]}(\mathcal{N}(r))$ are all isomorphic as representations of $SO(r)$, independent of L, and this representation is the same as the one above, when r is even, and twice the one above, when r is odd.

Proof. We work out only the first identity above, since all the others follow in an analogous way. By using (2.1) and the linearity of det in rows, we can write

$$\begin{aligned} \mathrm{Char}_{SO(r)}\, H_{[K,*]}(\mathcal{N}(r)) &= \sum_{J\subset\langle r\rangle_0} (-1)^{\#J} \det G_2([K,J],r) \\ &= \det G_4([K,*],r), \end{aligned}$$

where

$$G_4([K,*],r)_{ij} = \begin{cases} E\binom{r}{j-i} - E\binom{r}{j+i-1}, & \text{if } i \in \langle r\rangle_0; \\ E\binom{r}{j+i-1}, & \text{if } i \in K; \\ E\binom{r}{j-i}, & \text{if } i \in \langle r\rangle_1 - K. \end{cases}$$

Apply to $G_4([K,*],r)$ the column operations

$$\mathrm{Col}_j \mapsto \mathrm{Col}_j + \mathrm{Col}_{r+1-j} \qquad j = 1, \ldots, r-n-1.$$

The first $n+1$ columns are now 0 in the even rows. The first n columns in the odd rows are equal to $E\binom{r}{j+i-1} + E\binom{r}{r-j+i}$, if $i \in K$ and equal to $E\binom{r}{j-i} + E\binom{r}{r-j-i+1}$, if $i \in \langle r \rangle_1 - K$. By the symmetry $E\binom{r}{k} = E\binom{r}{r-k}$ these two values coincide. The $n+1$ column in the odd rows is equal to $E\binom{r}{n+1-i}$, whether $i \in K$ or not.

After collecting the odd rows on top and the even ones at the bottom we may reverse the order of the first $n+1$ columns. As in the proof of Proposition 2.3, this does not affect the determinant and we are left with a block upper triangular matrix rather than a block diagonal one The diagonal blocks are the same as before and so, as before, the determinant is $\det B(n) \det C(n)$. □

We illustrate this result by giving all the representation dimensions for $r = 6$ arranged in a table with rows indexed by I_1 and columns by I_0. As predicted by Proposition 2.5, all rows and columns have the same sum, which is this case is $\beta(2)\beta(3) = 56595$.

	$\{\}$	$\{2\}$	$\{6\}$	$\{4\}$	$\{2,6\}$	$\{2,4\}$	$\{4,6\}$	$\{2,4,6\}$
$\{\}$	1	70	252	720	5760	8064	8960	32768
$\{1\}$	6	105	1050	2430	6000	6804	21000	19200
$\{3\}$	336	1470	11907	6720	17010	4704	11760	2688
$\{5\}$	700	12250	1764	7875	10080	22050	980	896
$\{1,3\}$	896	980	22050	10080	7875	1764	12250	700
$\{1,5\}$	2688	11760	4704	17010	6720	11907	1470	336
$\{3,5\}$	19200	21000	6804	6000	2430	1050	105	6
$\{1,3,5\}$	32768	8960	8064	5760	720	252	70	1

Finally, we note that the power of two dividing $T(r)$ is also reflected in the power of $(1+t)$ dividing the Poincaré polynomial.

Proposition 2.6. *The power of* $(1+t)$ *that divides the Poincaré polynomial* $P(\mathcal{N}(r);t)$ *is* $n+1$*, where* $r = 2n+1$ *or* $r = 2n+2$.

Proof. The Poincaré polynomial $P(\mathcal{N}(r);t) = \det G_3(r;t)$, where

$$G_3(r;t)_{ij} = \binom{r}{j-i} - (-t)^i \binom{r}{j+i-1},$$

After performing the first manipulation in the proof of Proposition 2.3, the first $n+1$ columns are divisible by $(1+t)$ and hence the multiplicity of the factor $(1+t)$ in the determinant is at least $n+1$. To see that this is precisely the power, set $t = 1$ and note that $\beta(n)$ is always odd. Hence the remaining factor is not divisible by $(1+t)$. □

3 Asymptotics of $\beta(n)$ and $T(n)$

The asymptotics of $\beta(n)$ and $T(n)$ may be studied using the refined version of Stirling's formula for the logarithm of the Gamma function

$$\log \Gamma(z) = (z - \tfrac{1}{2}) \log(z) - z + \tfrac{1}{2} \log(2\pi) - E(z), \tag{3.1}$$

where $E(z) = \displaystyle\int_0^\infty \frac{P_1(t)}{z+t}\, dt$ and $P_1(t) = t - [t] - \frac{1}{2}$.

Let

$$\alpha(k) = \frac{(4k)!k!^2}{(2k)!^3} = \frac{1}{2}\frac{(4k-1)!(k-1)!^2}{(2k-1)!^3} = \frac{1}{2}\frac{\Gamma(4k)\Gamma(k)^2}{\Gamma(2k)^3}$$

since $(m-1)! = \Gamma(m)$ for positive integers m. Then from (3.1) we obtain

$$\log\alpha(k) = (2k - \tfrac{1}{2})\log 2 - \Phi(k), \tag{3.2}$$

where $\Phi(k) = E(4k) + 2E(k) - 3E(2k)$.

Lemma 3.1. $\Phi(k) = -\dfrac{1}{16k} + O(k^{-2})$.

Proof. Another refinement of Stirling's formula is

$$\log(n!) = (n + \tfrac{1}{2})\log(n) - n + \tfrac{1}{2}\log(2\pi) + \sum_{j=1}^{M} \frac{B_{j+1}}{j(j+1)} n^{-j} + O\big(n^{-M-1}\big), \tag{3.3}$$

where B_k are the Bernoulli numbers. Using this, with $M = 1$, we obtain

$$\log\alpha(k) = (2k - \tfrac{1}{2})\log 2 + \frac{3}{8}\frac{B_2}{k} + O(k^{-2}).$$

Comparing this with (3.2) gives the result, since $B_2 = \frac{1}{6}$. □

Summing (3.2) for $k = 1, \ldots, n$ we get

$$\log\beta(n) = \big(n^2 + \frac{n}{2}\big)\log 2 - \sum_{k=1}^{n}\Phi(k) \tag{3.4}$$

We may estimate the sum by means of the Euler summation formula

$$\sum_{k=1}^{n}\Phi(k) = \int_1^n \Phi(s)\,ds + \int_1^n P_1(s)\Phi'(s)\,ds + \frac{1}{2}(\Phi(1) + \Phi(n)). \tag{3.5}$$

This yields the following.

Theorem 3.2. *There exists a constant* κ *such that*

$$\beta(n) \sim 2^{(n^2+n/2)} n^{1/16} e^{\kappa}. \tag{3.6}$$

Moreover, $\kappa \simeq 0.0315950$.

Proof. From Lemma 3.1 it follows that the first summand of (3.5) is asymptotic to $-\frac{1}{16}\log n$ and that the second summand converges to a constant κ_1 as $n \to \infty$. Setting $\kappa = \kappa_1 + \frac{1}{2}\Phi(1)$ yields the first part. The estimation of κ must be done by experiment. □

Note that Theorem 1.2 follows immediately from Theorem 1.1 and Theorem 3.2 on setting $\kappa_T = 2^{3/8}e^{2\kappa}$. This asymptotic formula approximates $T(r)$ quite rapidly; for example, for $r = 16$ the error is $\simeq 0.01\%$. In fact one can say a little more about the asymptotic behaviour of $\beta(n)$ and hence $T(r)$, as follows.

Proposition 3.3.

$$(n + 1/2)^{1/16} < \beta(n)/2^{(n^2+n/2)}e^\kappa < (n + 1/2 + 1/24n)^{1/16} \tag{3.7}$$

Proof. We need to show that

$$\log(n + 1/2) < 16\left(\kappa - \sum_{k=1}^{n}\Phi(k)\right) < \log(n + 1/2 + 1/24n) \tag{3.8}$$

This follows from (3.3) and the following known bounds on Euler's constant $\gamma \simeq 0.5772$ (see [1], [6] and also [2]).

$$\log\left(n + \frac{1}{2}\right) + \frac{1}{24(n+1)^2} < \left(\sum_{k=1}^{n}\frac{1}{k}\right) - \gamma < \log\left(n + \frac{1}{2} + \frac{1}{24n}\right) - \frac{1}{48(n+1)^3} \tag{3.9}$$

□

Corollary 3.4. *The ratio $T(r)/2^{r^2/2}r^{1/8}$ is $> \kappa_T$ when r is odd and $< \kappa_T$ when r is even.*

Proof. A straightforward calculation. □

4 Application

Let $\mathfrak{g}$ be any 2-step nilpotent Lie algebra of finite dimension. If $\mathfrak{a}$ is an abelian factor of $\mathfrak{g}$, so that $\mathfrak{g} = \overline{\mathfrak{g}} \oplus \mathfrak{a}$, then $H_*(\mathfrak{g}) = H_*(\overline{\mathfrak{g}}) \otimes \Lambda^*\mathfrak{a}$ and $|H_*(\mathfrak{g})| = |H_*(\overline{\mathfrak{g}})|.2^{|\mathfrak{a}|}$. Assume that $\mathfrak{g}$ has no abelian factors, let $\mathfrak{z} = \text{centre}(\mathfrak{g})$ and let V be any direct complement of $\mathfrak{z}$. The minimum number of generators of $\mathfrak{g}$ is $r = \dim V$ and $\mathfrak{g}$ is a homomorphic image of $\mathcal{N}(r)$. In [4] (Theorem 2.1) it has been proved that $\mathcal{N}(r')$ degenerates to $\mathfrak{g} \times \mathfrak{a}$, where $\mathfrak{a}$ is abelian, $\dim \mathfrak{a} = \dim \mathcal{N}(r') - \dim \mathfrak{g}$ and $r' \geq r$. Since under degeneration the homology can only grow we have that

$$|H_*(\mathfrak{g})|.2^{|\mathfrak{a}|} \geq T(r'). \tag{4.1}$$

Thus, we can improve the lower bounds given in [8] for the total homology of a 2-step nilpotent Lie algebra. Because of Theorem 1.2, in the case r is odd, we obtain what is essentially the best general lower bound.

Proposition 4.1. *Let $\mathfrak{g}$ be any 2-step nilpotent Lie algebra of finite dimension. Let $\mathfrak{z}$ be its centre, $z = \dim(\mathfrak{z})$ and $r = \text{codim}(\mathfrak{z})$. Then*

$$|H_*(\mathfrak{g})| \geq 2^{[(r+1)/2]+z}r^{1/8}k, \tag{4.2}$$

where $k \simeq 0.9768$.

Proof. First assume that $\mathfrak{g}$ has no abelian factors. Let $r' = r$ if r is odd and $r' = r + 1$ if r is even. Then from Corollary 3.4 and (4.1) we obtain

$$|H_*(\mathfrak{g})| \geq \begin{cases} 2^{r/2+z}r^{1/8}\kappa_T & \text{when } r \text{ is odd,} \\ 2^{(r-1)/2+z}(r+1)^{1/8}\kappa_T & \text{when } r \text{ is even,} \end{cases} \tag{4.3}$$

It is clear that the same inequalities hold if $\mathfrak{g}$ has an abelian factor. The result follows setting $k = 2^{-1/2}\kappa_T$. □

5 Miscellaneous

We conclude with several additional observations that may be of interest.

Firstly, (1.6) may be used to find the prime factorisation of $\beta(n)$ and hence, by (1.3) and (1.4), of $T(r)$.

Proposition 5.1. *The exponent of a prime p in the prime factorization of $\beta(n)$ is either 0, if $p = 2$, or otherwise is*

$$\sum_{k=1}^{\infty} f\left(n, p^k\right) \tag{5.1}$$

where

$$f(n,m) = \begin{cases} \left\lfloor \frac{m+1}{4} \right\rfloor - \left| [n]_m - \left\lfloor \frac{m-1}{2} \right\rfloor \right| & \text{if } \frac{m}{4} < [n]_m < \frac{3m}{4}, \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

and $[n]_m$ denotes the remainder on dividing n by m.

Proof. Clearly $f(0, p^k) = 0$, which gives the correct value $\beta(0) = 1$. Now, the exponent of p in the prime decomposition of $n!$ is

$$\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor .$$

Therefore the exponent of p in $(4n)!n!^2/(2n)!^3$ is

$$\sum_{k=1}^{\infty} f'(n, p^k)$$

where

$$f'(n,m) = \left\lfloor \frac{4n}{m} \right\rfloor + 2\left\lfloor \frac{n}{m} \right\rfloor - 3\left\lfloor \frac{2n}{m} \right\rfloor$$

This clearly only depends on $[n]_m$ and one may then readily check that

$$f'(n,m) = \begin{cases} 0 & 0 \le [n]_m < \frac{m}{4} \\ +1 & \frac{m}{4} \le [n]_m < \frac{m}{2} \\ -1 & \frac{m}{2} \le [n]_m < \frac{3m}{4} \\ 0 & \frac{3m}{4} \le [n]_m < m \end{cases}$$

The number of $+1$'s is $\left\lfloor \frac{m+1}{4} \right\rfloor$, which is the same as the number of -1's, except when $m \equiv 2 \mod 4$. Hence, for $p \neq 2$, we have $\sum_{j=1}^{n} f'(j, p^k) = f(n, p^k)$, and the result follows from (1.6) by induction. For $p = 2$, it is clear from (1.5) that the exponent is always 0. ☐

Corollary 5.2. *The exponent of p in the prime factorization of $T(r)$ is either $\left\lfloor \frac{r+1}{2} \right\rfloor$, if $p = 2$, or otherwise is*

$$\sum_{k=1}^{\infty} f_T\left(r, p^k\right) \tag{5.3}$$

where

$$f_T(r,m) = \begin{cases} 2\left\lfloor \frac{m+1}{4} \right\rfloor - \left| [r]_{2m} - m \right| & \text{if } \frac{m}{2} < [r]_{2m} < \frac{3m}{2}, \\ 0 & \text{otherwise} \end{cases} \tag{5.4}$$

Two properties of the representation dimensions for self-conjugate partitions have been observed experimentally for sufficiently many values of r to make them plausible, but they lack rigorous proofs.

Conjecture 5.3. *There is an involution* $\sigma : \langle r\rangle \to \langle r\rangle$ *such that the summand* $H_I(\mathcal{N}(r))$ *has odd dimension if and only if* $\sigma(I) = I$.

The involution is given by dividing $\langle r\rangle$ into certain segments of even length, plus the initial segment $\{1\}$ when r is odd, and reversing each segment. Thus σ interchanges even and odd numbers, while fixing 1 when r is odd. As a consequence, the conjecture may also be formulated as saying that, given any $I_1 \subset \langle r\rangle_1$, there is a unique $I_0 \subset \langle r\rangle_0$ such that $H_I(\mathcal{N}(r))$ has odd dimension for $I = I_0 \cup I_1$. The table at the end of Section 2 is arranged to emphasize this fact.

Conjecture 5.3 is checked for $r \le 20$ and it should follow from careful analysis mod 2 of the Giambelli determinant formula in Proposition 2.1.

Conjecture 5.4. *The largest value of* $\dim H_I(\mathcal{N}(r))$ *is* $2^{r(r-1)/2}$, *which occurs when* $I = \langle r\rangle_0$ *or* $I = \langle r\rangle_1$.

Weyl dimension formula gives the dimension. This is the $SL(r)$-representation whose highest weight is the half sum of positive roots, and this is known to have dimension 2^N, where N is the number of positive roots. However, it is unclear why this is the largest in this case.

6 Appendix

As mentioned in the Introduction, the Lie algebras $\mathcal{N}(r)$ fall into the class of Lie algebras whose cohomology was described by Kostant [5]. We shall show here how Sigg's description of the homology of $\mathcal{N}(r)$ may be derived from Kostant's description.

Let $\mathfrak{g}$ be a (complex) semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra and Δ_+ a set of positive roots. The subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$ is a Borel subalgebra, where $\mathfrak{g}_\alpha$ is the root space corresponding to the root α. A parabolic subalgebra $\mathfrak{p} \supseteq \mathfrak{b}$ has a Levi decomposition $\mathfrak{p} = \mathfrak{g}_1 \ltimes \mathfrak{n}$, where $\mathfrak{g}_1$ is reductive and $\mathfrak{n}$ is the nilpotent radical of $\mathfrak{p}$.

The cohomology group $H^*(\mathfrak{n})$ is a $\mathfrak{g}_1$-module. Let $\Delta(\mathfrak{n})$ be the roots of $\mathfrak{n}$ (cf. §5 [5]) and $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$. We denote by V_λ the $\mathfrak{g}_1$-irreducible representation with highest weight λ.

Theorem 6.1 (Kostant). *Let* W *be the Weyl group of* $\mathfrak{g}$ *and for each* $\sigma \in W$ *consider the set* $\Phi_\sigma = \{\alpha \in \Delta_+ :\ \sigma^{-1}(\alpha) < 0\}$. *Let* $W^1 = \{\sigma \in W :\ \Phi_\sigma \subset \Delta(\mathfrak{n})\}$. *Then*

$$H^*(\mathfrak{n}) = \bigoplus_{\sigma\in W^1} V_{\sigma(\rho)-\rho}.$$

Moreover the cohomology degree of $V_{\sigma(\rho)-\rho}$ *is the number of elements* $\#\Phi_\sigma$.

We recall that if W_1 is the Weyl group of $\mathfrak{g}_1$, then the set W^1 is a cross section of $W_1\backslash W$.

Let $\mathfrak{g} = so(2r+1)$ (type B_r). The set of positive roots of $\mathfrak{g}$ is

$$\Delta_+ = \{e_i - e_j, e_i + e_j\}_{1\le i<j\le r} \cup \{e_i\}_{1\le i\le r}$$

and

$$\rho = \frac{1}{2}\big((2r-1)e_1 + (2r-3)e_2 + \cdots + 3e_{r-1} + e_r\big). \tag{6.1}$$

Consider the parabolic subalgebra $\mathfrak{p} = \sum_{i>j} \mathfrak{g}_{e_i-e_j} \oplus \mathfrak{h} \oplus \sum_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$. One can check that the Levi decomposition $\mathfrak{p} = \mathfrak{g}_1 \ltimes \mathfrak{n}$ is given by

$$\begin{aligned} \mathfrak{g}_1 &= \mathfrak{h} \oplus \sum_{i<j} \mathfrak{g}_{e_i-e_j} \oplus \sum_{i>j} \mathfrak{g}_{e_i-e_j} \\ \mathfrak{n} &= \sum_i \mathfrak{g}_{e_i} \oplus \sum_{i<j} \mathfrak{g}_{e_i+e_j} = V \oplus \Lambda^2 V, \end{aligned}$$

where V is the fundamental representation of $\mathfrak{g}_1 \cong gl(r)$. Thus $\mathfrak{n} \cong \mathcal{N}(r)$.

Recall that the dominant weights of $gl(r)$ are given by $\lambda = \lambda_1 e_1 + \cdots + \lambda_r e_r$, with $\lambda_i \in \mathbb{Z}$ and $\lambda_1 \geq \cdots \geq \lambda_r$. Furthermore, the Weyl group of $\mathfrak{g}_1$ is $W_1 = S_r$, the symmetric group acting on the set $\{e_1, \ldots, e_r\}$. The Weyl group of $\mathfrak{g}$ is $W = S_r \ltimes \mathbb{Z}_2^r$, where $\mathbb{Z}_2^r = \langle \tau_1 \rangle \times \cdots \times \langle \tau_r \rangle$, with $\tau_i(e_i) = -e_i$ and $\tau_i(e_j) = e_j$ if $j \neq i$. In particular, $|W^1| = 2^r$.

We shall first determine W^1. The elements of $\mathbb{Z}_2^r$ are given by $\tau_I = \prod_{i\in I} \tau_i$ for $I \subset \{1, \ldots, r\}$. Since both W^1 and $\mathbb{Z}_2^r$ are cross sections of $W_1\backslash W$ it follows that, for each $\tau_I \in \mathbb{Z}_2^r$, there is a unique permutation $\omega_I \in S_r$ such that $\omega_I \tau_I \in W^1$.

Since $\Delta_+ - \Delta(\mathfrak{n}) = \{e_i - e_j : \ i < j\}$, we see that ω_I is characterized by the condition

$$\tau_I \omega_I^{-1}(e_i - e_j) > 0, \qquad \text{for all } i < j. \tag{6.2}$$

This is equivalent to the conditon that $\omega_I(a) > \omega_I(b)$ whenever

$$(a, b \in I \text{ and } a < b) \text{ or } (a, b \notin I \text{ and } a > b) \text{ or } (a \in I \text{ and } b \notin I). \tag{6.3}$$

In other words, ω_I places all the elements of I after all the elements of the complement I^c, preserving the order of I^c and reversing the order of I. More precisely

$$\omega_I(j) = \begin{cases} r+1-m, & \text{if } j \text{ is the } m\text{-th element of } I, \\ n, & \text{if } j \text{ is the } n\text{-th element of } I^c. \end{cases} \tag{6.4}$$

Let us now compute the number $\#\Phi_\sigma$, for any $\sigma \in W^1$. Note that $\#\Phi_\sigma$ is the number of roots $\alpha \in \Delta(\mathfrak{n}) = \{e_i + e_j : \ i < j\} \cup \{e_i\}$ such that $\sigma^{-1}(\alpha) < 0$. Let $\sigma = \omega_I \tau_I$. Since ω_I is a permutation of $\Delta(\mathfrak{n})$ and $\tau_I^{-1} = \tau_I$, we have

$$\begin{aligned} \#\Phi_\sigma &= \#\{\alpha \in \Delta(\mathfrak{n}) : \tau_I(\alpha) < 0\} \\ &= \#\{(i,j) : i < j \text{ and } i \in I\} + \#I \\ &= \sum_{i\in I} (r+1-i). \end{aligned}$$

Next we determine the highest weight $\lambda_I = \sigma(\rho) - \rho$ for each $\sigma = \omega_I \tau_I \in W^1$. Notice that ω_I is precisely the permutation needed to make $\tau_I(\rho)$ back into a dominant weight and further that $\rho_i \geq \sigma(\rho)_i$ for all i. Some more thought shows that

$$\lambda_I = -(\mu_1 + \cdots + \mu_k). \tag{6.5}$$

where, if $I = \{i_1, \dots, i_k\}$, then

$$\begin{aligned}
\mu_1 &= (0,\dots,0,\overbrace{\ 1\ }^{i_1},\dots\dots\dots\dots,1,r+1-i_1) \\
\mu_2 &= (0,\dots,0,\overbrace{\ 1\ }^{i_2-1},\dots\dots\dots,1,r+1-i_2,0) \\
&\ \vdots \\
\mu_k &= (0,\dots,0,\overbrace{\ 1\ }^{i_k-k+1},\dots,1,r+1-i_k,0,\dots,0)
\end{aligned} \tag{6.6}$$

Kostant's theorem computes cohomology $H^*(\mathcal{N}(r))$, but we were originally interested in the homology $H_*(\mathcal{N}(r))$. However, there is a simple formula relating the two, as follows. Let

$$n = \dim \mathcal{N}(r) = r(r+1)/2 = \sum_{i=1}^{r} (r+1-i) \tag{6.7}$$

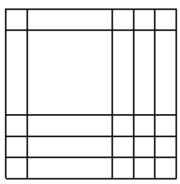
and let det be the 1-dimensional determinant representation of $GL(r)$, which has highest weight $\Delta = (1, \dots, 1)$. Then, as $GL(r)$-representations,

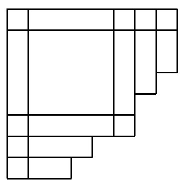
$$H_{n-i}(\mathcal{N}(r)) \cong H^i(\mathcal{N}(r)) \otimes \det{}^r.$$

Thus, we see from above that, for each $I \subset \{1, \dots, r\}$, the homology $H_*(\mathcal{N}(r))$ contains an irreducible $GL(r)$-representation of multiplicity one, of highest weight $r\Delta + \lambda_I$ and in homological degree d_I, where

$$d_I = \sum_{i \notin I} (r+1-i)$$

Now all the weights $r\Delta + \lambda_I$ are non-negative, so can be represented by Young diagrams. For example, if $I = \{r-2, r-4\}$, then

$r\Delta =$ 

$r\Delta + \lambda_I =$ 

In other words, we remove from the $r \times r$ square symmetric hooks of length $r+1-i$ for each $i \in I$. A little thought shows that the Frobenius notation for such a diagram is $(J; J)$, where $J = \{(r+1-i) : i \notin I\}$. Hence the diagrams that appear are precisely the self-conjugate diagrams and, since $d_I = \sum_{j \in J} j$, we have recovered Sigg's results.

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