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**CLASSICAL ALGEBRAIC  $K$ -THEORY**  
i.e. **THE FUNCTORS  $K_0, K_1, K_2$** <sup>1</sup>

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# INTRODUCTION

Algebraic  $K$ -theory has grown phenomenally in various directions in the last three decades as a multidisciplinary subject whose methods and contents span many areas of mathematics – notably – algebra, number theory, topology, geometry (algebraic, differential, non-commutative) and functional analysis. As such, it has grown to become one of the most unifying forces in mathematical research. The subject has also recorded outstanding success in the investigations and solutions of many famous problems (see [6]).

It is generally accepted that Algebraic  $K$ -theory actually started with Grothendieck's construction of an Abelian group  $K(\mathcal{A})$  (now denoted  $K_0(\mathcal{A})$ ) associated to a suitable subcategory of an Abelian category (e.g. for a scheme  $X$ ,  $\mathcal{A} = \mathcal{P}(X)$  the category of locally free sheaves of  $O_X$ -modules or  $\mathcal{A} = \mathcal{M}(X)$ , the category of coherent sheaves of  $O_X$ -modules). This construction was done by A. Grothendieck during his reformulation and proof of his generalised Riemann-Roch theorem see [15] or [31]. However, there were earlier works which were later recognized as proper constituents of the subject e.g. J.H.C. Whitehead's construction of  $Wh(\pi_1(X))$  ( $X$  a topological space) [119] or even much earlier work of Dedekind and Weber [19] on ideal class groups.

Meanwhile, M.F. Atiyah and F. Hirzebruch [4], [5], studied for any finite CW-complex  $X$ , the Abelian group  $K_0(\mathcal{A})$  where for  $k = \mathbb{R}$  or  $\mathbb{C}$ ,  $\mathcal{A} = Vect_k(X)$  the category of finite dimensional  $k$ -vector bundles on  $X$  in what became known as topological  $K$ -theory. Now, the realisation by R.G. Swan [94] that when  $X$  is a compact space, the category  $Vect_{\mathbb{C}}(X)$  is equivalent to the category  $\mathcal{P}(\mathbb{C}(X))$  of finitely generated projective modules over the ring  $\mathbb{C}(X)$  of complex-valued continuous functions on  $X$ , provided the initial connection between topological  $K$ -theory and Algebraic  $K$ -theory. Moreover, the fact that when  $X$  is affine (i.e.  $X = spec(A)$ ,  $A$  a commutative ring), the category  $\mathcal{P}(X)$  is equivalent to  $\mathcal{P}(A)$ , the category of finitely generated projective  $A$ -modules, also confirms the appropriateness of  $K_0(\mathcal{P}(A))$  ( $A$  any ring with identity) as a good definition of  $K_0$  of a ring  $A$ , usually written  $K_0(A)$ .

The groups  $K_0(A)$  for various types of rings  $A$  (e.g. Dedekind domains, number fields, group-rings, orders,  $C^*$ -algebras etc.) have been subjected to intense studies over the years especially because the groups  $K_0(A)$  for relevant  $A$ 's are replete with applications at first in several areas of mathematics and later in some areas of applied mathematics and physics. For example, C.T.C. Wall [113] showed that if  $X$  is a connected space dominated by a finite CW-complex, then there is a well defined obstruction  $\omega$  in  $K_0(\mathbb{Z}\pi_1(X))$  such that  $X$  has the homotopy type of a finite complex if and only if  $\omega = 0$  (see 2.2.12). Moreover,  $K_0$  of  $C^*$ -algebras is connected with non-commutative geometry (see [17] or 1.4.3 (iv));  $K_0$  of Dedekind domains with class groups of number theory (see §2);  $K_0$  of orders and group-rings with representation theory (see [18] or §3) etc. Furthermore,  $K_0(\mathcal{C})$  is also well defined for other types of category  $\mathcal{C}$  (e.g. symmetric monoidal categories, see §1.4).

The definition of  $K_1$ , due to H. Bass was inspired by Atiyah-Hirzebruch topological  $K$ -theory

$K^{-n}(X) := \tilde{K}(S^n(X))$  where  $S(X)$  is the suspension of  $X$  and  $\tilde{K}(Y) := \text{Ker}(K^0(Y) \rightarrow K(*))$  for any paracompact space  $Y$  and  $*$  a point of  $Y$  (see [3] or [38]). H. Bass defined  $K_1$  of ring  $A(K_1(A))$ , modelled on the description of bundles on  $SX$  by clutching. Because for any finite group  $G$ ,  $Wh(G)$  (the Whitehead group of  $G$ ), defined as a quotient of  $K_1(\mathbb{Z}G)$  (see §6.10) houses some topological invariants known as “Whitehead torsion” when  $G = \pi_1(X)$ , ( $X$  a finite CW-complex), computations of  $K_1(\mathbb{Z}G)$  and also of  $SK_1(\mathbb{Z}G) := \text{Ker}(K_1(\mathbb{Z}G) \rightarrow K_1(\mathbb{Q}G))$ , became of interest in topology since  $\text{rank } K_1(\mathbb{Z}G) = \text{rank } Wh(G)$  and  $SK_1(\mathbb{Z}G)$  is the full torsion subgroup of  $Wh(G)$  (see §6.10 or [63]). Also of interest are computations of  $SK_1(\hat{\mathbb{Z}}_p G)$  (see e.g. [63] or [42]). Moreover, stability considerations of  $K_1(A)$  yielded results of finite generation of  $K_1(\mathbb{Z}G)$  and finiteness of  $SK_1(\mathbb{Z}G)$  ([7], [8]) as well as a key step to the solution of the congruence subgroup problem for  $SL_n(A)$  where  $A$  is the ring of integers of a number field  $F$  [3].

The definition of  $K_2^M(A)$ ,  $A$  any ring is due to J. Milnor [62]. As will be seen in a forthcoming chapter on Higher  $K$ -theory,  $K_2^M(A)$  coincides with the Quillen  $K$ -groups  $K_2(\mathcal{P}(A)) = K_2(A)$  but for  $n \neq 2$ ,  $K_n^M(A)$ , defined only for commutative rings  $A$ , are in general different from  $K_n(A)$  even though there are maps between them, as well as connections with other theories e.g. Galois and etale cohomology theories, Brauer groups etc. yielding famous conjectures – e.g. Milnor, Bloch-Kato conjectures etc.

We now briefly review the contents of this chapter. §1 introduces the Grothendieck group associated to a semigroup  $A$  and the ring associated to a semi-ring (1.1) leading to discussions on  $K_0$  of rings and  $K_0$  of symmetric monoidal categories with copious examples – Topological  $K$ -groups  $K_0(X)$ ,  $KU(X)$ ,  $K_G^0(X)$ , Burnside rings, Representation rings, Witt rings, Picard groups – ( $\text{Pic}(R)$  ( $R$ , a commutative ring with identity),  $\text{Pic}(X)$ , ( $X$  a locally ringed space);  $K_0$  of Azamaya algebras etc. We also briefly indicate how to realise  $K_0$  as “Mackey” functors yielding induction theory for  $K_0$  of group-rings (see [48]) – a topic that will be a subject of another chapter in more generality (see [24], [25] or [46]).

§2 deals with class groups of Dedekind domains, orders and group rings, and we also briefly discuss Wall’s finiteness obstruction as an application. In §3, we discuss  $K_0$  of an exact category with copious examples while §4 exposes some fundamental properties and examples of  $K_0$  of exact categories e.g. Devissage, Resolution and localisation theorems – that will be seen in more generality in a forthcoming chapter on Higher  $K$ -theory. We also discuss  $K_0$  of the category of nilpotent endomorphisms with consequent fundamental theorems for  $K_0$ ,  $G_0$  of rings and schemes.

In §5 we discuss  $K_1(A)$  with the observation that the definition due to Bass coincides with Quillen’s  $K_1(\mathcal{P}(A))$  or  $\pi_1(BGL(A)^+)$ . We also briefly discuss  $K_1$  of local rings and skew fields; Menicke symbols and some stability results for  $K_1$ . In §6, which deals with  $SK_1$  of orders and group rings, we, among other things, call attention to the fact that when  $R$  is the ring of integers in a number field,  $\Lambda$  is an  $R$ -order,  $K_1(\Lambda)$  is a finitely generated Abelian group,  $SK_1(\Lambda)$ ,  $SK_1(\hat{\Lambda}_p)$

are finite groups, see [63], with the observation that these results have since been generalised i.e. for all  $n \geq 1$ ,  $K_n(\Lambda)$  is finitely generated and  $SK_n(\Lambda)$ ,  $SK_n(\hat{\Lambda}_p)$  are finite for all prime ideals  $p$  of  $R$  (see [49], [50]). Similarly, for a maximal order  $\Gamma$  in a  $p$ -adic semi-simple algebra  $\Sigma$ , the result that  $SK_1(\Gamma) = 0$  iff  $\Sigma$  is unramified over its centre [41] has been extended for all  $n \geq 1$  (i.e.  $SK_{2n}(\Gamma) = 0$  and  $SK_{2n-1}(\Gamma) = 0$  iff  $\Sigma$  is unramified over its centre) (see [44]). We refer to copious computations of  $SK_1$  of orders and group rings in [63] and discuss Whitehead torsion in §6.10.

§7 is devoted to discussing some  $K_1 - K_0$  exact sequences – Mayer-Vietoris, localisation sequences and the exact sequence associated to an ideal. The localisation sequence leads to the introduction of the fundamental theorem for  $K_1$ .

The last section (§8) deals with a rather brief review of the functor  $K_2$  due to J. Milnor [62]. We observe that  $K_2(A) = H_2(E(A), \mathbb{Z})$  for any ring  $A$  with identity and that when  $A$  is a field, division ring, local or semi-local ring,  $K_2(A)$  is generated by symbols. We then briefly discuss the connections between  $K_2$ , Brauer group of fields and Galois cohomology leading to Merkurjev-Suslin theorem (8.2.4) which we discuss in the context of Bloch-Kato conjecture for higher-dimensional  $K$ -theory of fields with a brief review of the current situation with the conjecture. We end §8 with applications of  $K_2$  to pseudo-isotopy of manifolds and Bloch's formula for Chow groups.

In anticipation of the forthcoming chapter on Higher  $K$ -theory we have included references to some results on Higher  $K$ -theory that applies to lower  $K$ -groups as well as constitute generalisations of such known results for  $K_0, K_1$  or  $K_2$ .

### Notes on Notation

If  $R$  is a Dedekind domain with quotient field  $F$ ,  $\mathfrak{p}$  any prime ideal of  $R$ , we write  $R_{\mathfrak{p}}$  for the localisation of  $R$  at  $\mathfrak{p}$ ,  $\hat{R}_{\mathfrak{p}}(\hat{F}_{\mathfrak{p}})$  for the completion of  $R$  (resp  $F$ ) at  $\mathfrak{p}$ . If  $\Lambda$  is an  $R$ -order in a semi-simple  $F$ -algebra  $\Sigma$ , we write  $\Lambda_{\mathfrak{p}}$  for  $R_{\mathfrak{p}} \otimes_R \Lambda$ ,  $\hat{\Lambda}_{\mathfrak{p}}$  for  $\hat{R}_{\mathfrak{p}} \otimes_R \Lambda$ , and  $\hat{\Sigma}_{\mathfrak{p}} = \hat{F}_{\mathfrak{p}} \otimes_F \Sigma$ . For all  $n \geq 0$ , we write  $SK_n(\Lambda) = \text{Ker}(K_n(\Lambda) \rightarrow K_n(\Sigma))$  and  $SG_n(\Lambda) = \text{Ker}(G_n(\Lambda) \rightarrow G_n(\Sigma))$ .

We shall write  $\mathcal{G}\text{Set}$  for the category of  $G$ -sets ( $G$  a group),  $\mathcal{R}\text{ings}$  for the category of rings with identity and homomorphisms preserving identity,  $\mathcal{C}\mathcal{R}\text{ings}$  for the category of commutative rings and ring homomorphisms preserving identity.

Many other notations used are defined in the text.

# 1 SOME BASIC GROTHENDIECK GROUP CONSTRUCTIONS AND EXAMPLES

## 1.1 Grothendieck group associated with a semi-group

**1.1.1** Let  $(A, +)$  be an Abelian semi-group. Define a relation ' $\sim$ ' on  $A \times A$  by  $(a, b) \sim (c, d)$  if there exists  $u \in A$  such that  $a + d + u = b + c + u$ . One can easily check that ' $\sim$ ' is an equivalence relation. Let  $\bar{A}$  denote the set of equivalence classes of ' $\sim$ ', and write  $[a, b]$  for the class of  $(a, b)$  under ' $\sim$ '. We define addition ( $\dot{+}$ ) on  $\bar{A}$  by  $[a, b] \dot{+} [c, d] = [a + c, b + d]$ . Then  $(\bar{A}, \dot{+})$  is an Abelian group in which the identity element is  $[a, a]$  and the inverse of  $[a, b]$  is  $[b, a]$ .

Moreover, there is a well-defined additive map  $f : A \rightarrow \bar{A} : a \rightarrow [a + a, a]$  which is, in general, neither injective nor surjective. However,  $f$  is injective iff  $A$  is a cancellation semi-group i.e. iff  $a + c = b + c$  implies that  $a = b$  for all  $a, b, c \in A$ , see [48] or [38].

**1.1.2** It can be easily checked that  $\bar{A}$  possesses the following universal property with respect to the map  $f : A \rightarrow \bar{A}$ . Given any additive map  $h : A \rightarrow B$  from  $A$  to an Abelian group  $B$ , then there exists a unique map  $g : \bar{A} \rightarrow B$  such that  $h = gf$ .

**Definition 1.1.3**  $\bar{A}$  is usually called the Grothendieck group of  $A$  or the group completion of  $A$  and denoted by  $K(A)$ .

### Remarks 1.1.4

- (i) The construction of  $K(A) = \bar{A}$  above can be shown to be equivalent to the following:
  - Let  $(F(A), \dot{+})$  be the free Abelian group freely generated by the elements of  $A$ , and  $R(A)$  the subgroup of  $F(A)$  generated by all elements of the form  $a \dot{+} b - (a + b)$   $a, b \in A$ . Then  $K(A) \simeq F(A)/R(A)$ .
- (ii) If  $A, B, C$  are Abelian semi-groups together with bi-additive map  $f : A \times B \rightarrow C$ , then  $f$  extends to a unique bi-additive map  $\bar{f} : \bar{A} \times \bar{B} \rightarrow \bar{C}$  of the associated Grothendieck groups. If  $A$  is a semi-ring i.e. an additive Abelian group together with a bi-additive multiplication  $A \times A \rightarrow A$   $(a, b) \rightarrow ab$ , then the multiplication extends uniquely to a multiplication  $\bar{A} \times \bar{A} \rightarrow \bar{A}$  which makes  $\bar{A}$  into a ring (commutative if  $A$  is commutative) with identity  $1 = [1 + 1, 1]$  in  $\bar{A}$  if  $1 \in A$ .
- (iii) If  $B$  is a semi-module over a semi-ring  $A$  i.e. if  $B$  is an Abelian semi-group together with a bi-additive map  $A \times B \rightarrow B : (a, b) \rightarrow a \cdot b$  satisfying  $a'(ab) = (a'a)b$  for  $a, a' \in A, b \in B$ , then the associated Grothendieck group  $\bar{B}$  is an  $\bar{A}$ -module.
- (iv) If  $A = \{1, 2, 3, \dots\}$ ,  $\bar{A} = K(A) = \mathbb{Z}$ . Hence the construction in 1.1.1 is just a generalisation of the standard procedure of constructing integers from the natural numbers.

- (v) A sub-semi-group  $A$  of an Abelian semi-group  $B$  is said to be cofinal in  $B$  if for any  $b \in B$ , there exists  $b' \in B$  such that  $b + b' \in A$ . It can be easily checked that  $K(A)$  is a subgroup of  $K(B)$  if  $A$  is cofinal in  $B$ .

## 1.2 $K_0$ of a ring

**1.2.1** For any ring  $\Lambda$  with identity, let  $\mathcal{P}(\Lambda)$  be the category of finitely generated projective  $\Lambda$ -modules. Then the isomorphism classes  $IP(\Lambda)$  of objects of  $\mathcal{P}(\Lambda)$  form an Abelian semi-group under direct sum ' $\oplus$ '. We write  $K_0(\Lambda)$  for  $K(IP(\Lambda))$  and call  $K_0(\Lambda)$  the Grothendieck group of  $\Lambda$ . For any  $P \in \mathcal{P}(\Lambda)$ , we write  $(P)$  for the isomorphism class of  $P$  (i.e. an element of  $IP(\Lambda)$ ) and  $[P]$  for the class of  $(P)$  in  $K_0(\Lambda)$ .

If  $\Lambda$  is commutative, then  $IP(\Lambda)$  is a semi-ring with tensor product  $\otimes_\Lambda$  as multiplication which distributes over ' $\oplus$ '. Hence  $K_0(\Lambda)$  is a ring by 1.1.4 (ii).

### 1.2.2 Remarks

- (i)  $K_0 : Rings \rightarrow Ab : \Lambda \rightarrow K_0(\Lambda)$  is a functor – since any ring homomorphism  $f : \Lambda \rightarrow \Lambda'$  induces a semi-group homomorphism  $IP(\Lambda) \rightarrow IP(\Lambda') : P \rightarrow P \otimes \Lambda'$  and hence a group homomorphism  $K_0(\Lambda) \rightarrow K_0(\Lambda')$ .
- (ii)  $K_0$  is also a functor:  $CRings \rightarrow CRings$ .
- (iii)  $[P] = [Q]$  in  $K_0(\Lambda)$  iff  $P$  is stably isomorphic to  $Q$  in  $\mathcal{P}(\Lambda)$  i.e. iff  $P \oplus \Lambda^n \simeq Q \oplus \Lambda^n$  for some integer  $n$ . In particular  $[P] = [\Lambda^n]$  for some  $n$  iff  $P$  is stably free, see [8] or [7].

### 1.2.3 Examples

- (i) If  $\Lambda$  is a field or a division ring or a local ring or a principal ideal domain, then  $K_0(\Lambda) \simeq \mathbb{Z}$ .

**Note.** The proof in each case is based on the fact that any finitely generated  $\Lambda$ -module is free and  $\Lambda$  satisfies invariant basis property (i.e.  $\Lambda^r \simeq \Lambda^s \Rightarrow r = s$ ). So  $IP(\Lambda) \simeq \{1, 2, 3, \dots\}$  and so,  $K_0(\Lambda) \simeq \mathbb{Z}$  by 1.1.4 (iv), see [8] or [78].

- (ii) Any element of  $K_0(\Lambda)$  can be written as  $[P] - r[\Lambda]$  for some integer  $r > 0$ ,  $P \in \mathcal{P}(\Lambda)$  or as  $s[\Lambda] - [Q]$  for some  $s > 0$ ,  $Q \in \mathcal{P}(\Lambda)$  (see [8] or [104]). If we write  $\tilde{K}_0(\Lambda)$  for the quotient of  $K_0(\Lambda)$  by the subgroup generated by  $[\Lambda]$ , then every element of  $\tilde{K}_0(\Lambda)$  can be written as  $[P]$  for some  $P \in \mathcal{P}(\Lambda)$ , see [104] or [8].
- (iii) If  $\Lambda \simeq \Lambda_1 \times \Lambda_2$  is a direct product of two rings  $\Lambda_1, \Lambda_2$  then  $K_0(\Lambda) \simeq K_0(\Lambda_1) \times K_0(\Lambda_2)$ , (see [104] for a proof).
- (iv) Let  $G$  be a semi-simple simply connected affine algebraic group over an algebraically closed field. Let  $A$  be the coordinate ring of  $G$ . Then  $K_0(A) \simeq \mathbb{Z}$ .

**Remarks** See [54] for a proof of this result which says that all algebraic vector bundles on  $G$  are stably trivial. The result is due to A. Grothendieck.

(v)  $K_0(k[x_0, x_1, \dots, x_n]) \simeq \mathbb{Z}$ . This result is due to J.P. Serre, see [82].

### 1.3 $K_0$ of a ring via idempotents

**1.3.1** For any ring  $\Lambda$  with identity, let  $M_n(\Lambda)$  be the set of  $n \times n$  matrices over  $\Lambda$ , and write  $M(\Lambda) = \bigcup_{n=1}^{\infty} M_n(\Lambda)$ . Also let  $GL_n(\Lambda)$  be the group of invertible  $n \times n$  matrices over  $\Lambda$  and write  $GL(\Lambda) = \bigcup_{n=1}^{\infty} GL_n(\Lambda)$ . For  $P \in \mathcal{P}(\Lambda)$  there exists  $Q \in \mathcal{P}(\Lambda)$  such that  $P \oplus Q \simeq \Lambda^n$  for some  $n$ . So, we can identify with each  $P \in \mathcal{P}(\Lambda)$  an idempotent matrix  $p \in M_n(\Lambda)$  (i.e.  $p : \Lambda^n \rightarrow \Lambda^n$ ) which is an identity on  $P$  and '0' on  $Q$ .

Note that if  $p, q$  are idempotent matrices in  $M(\Lambda)$ , say  $p \in M_r(\Lambda)$ ,  $q \in M_s(\Lambda)$ , corresponding to  $P, Q \in \mathcal{P}(\Lambda)$ , then  $P \simeq Q$  iff it is possible to enlarge the sizes of  $p, q$  (by possibly adding zeros in the lower right-hand corners) such that  $p, q$  have the same size ( $t \times t$ , say) and are conjugate under the action of  $GL_t(\Lambda)$ , see [78].

Let  $\text{Idem}(\Lambda)$  be set of idempotent matrices in  $M(\Lambda)$ . It follows from the last paragraph that  $GL(\Lambda)$  acts by conjugation on  $\text{Idem}(\Lambda)$ , and so, we can identify the semi-group  $\mathcal{IP}(\Lambda)$  with the semi-group of conjugation orbits  $(\text{Idem}(\Lambda))^{\wedge}$  of the action of  $GL(\Lambda)$  on  $\text{Idem}(\Lambda)$  where the semi-group operation is induced by  $(p, q) \rightarrow \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ .  $K_0(\Lambda)$  is the Grothendieck group of this semi-group  $(\text{Idem}(\Lambda))^{\wedge}$ .

#### Remarks 1.3.2

- (i) Computing  $K_0$ -groups via idempotents is particularly useful when  $\Lambda$  is an involutive Banach algebra or  $C^*$ -algebra (see [17] or [21] for example).
- (ii) Also the methods of computing  $K_0$ -groups via idempotents are used to prove the following results 1.3.2. and 1.3.3. below.

**Theorem 1.3.3** [78] If  $\{\Lambda_i\}_{i \in I}$  is a direct system of rings (with identity), then

$$K_0(\Lambda) = \varinjlim_{i \in I} K_0(\Lambda_i)$$

For proof see [78].

#### Theorem 1.3.4 Morita equivalence for $K_0$ of rings

For any ring  $\Lambda$  and any natural number  $n > 0$ ,  $K_0(\Lambda) \simeq K_0(M_n(\Lambda))$ .

**Proof:** Follows from 1.3.3 since  $\text{Idem}(M_n(\Lambda)) = \text{Idem}(\Lambda)$  and  $GL(M_n(\Lambda)) \simeq GL(\Lambda)$ .

**Corollary 1.3.4** If  $\Lambda$  is a semi-simple ring, then  $K_0(\Lambda) \simeq \mathbb{Z}^r$  for some positive integer  $r$ .

**Proof:** (Sketch). Let  $V_1, \dots, V_r$  be simple  $\Lambda$ -modules. By Wedderburn's theorem,  $\Lambda \simeq \prod_{i=1}^r M_{n_i}(D_i)$  where  $D_i = \text{Hom}_{\Lambda}(V_i, V_i)$  and  $\dim_{D_i}(V_i) = n_i$ . Hence  $K_0(\Lambda) \simeq \prod_{i=1}^r K_0(M_{n_i}(D_i)) \simeq \prod_{i=1}^r K_0(D_i) \simeq \mathbb{Z}^r$  by 1.2.3 (i) and (iii) as well as 1.3.4.



## 1.4 $K_0$ of Symmetric Monoidal Categories

**Definition 1.4.1** A symmetric monoidal category is a category  $\mathcal{C}$  equipped with a functor  $\perp : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  and a distinguished object “0” such that  $\perp$  is “coherently associative and commutative” in the sense of MacLane, that is,

$$(i) A \perp 0 \simeq A \simeq 0 \perp A$$

$$(ii) A \perp (B \perp C) \simeq (A \perp B) \perp C$$

$$(iii) A \perp B \simeq B \perp A \text{ for all } A, B, C \in \mathcal{C}$$

Moreover, the following diagrams commute.

$$\begin{array}{ccc}
 (A \perp (0 \perp B)) & \cong & (A \perp 0) \perp B \\
 (i) \quad \parallel & & \parallel \\
 A \perp B & \approx & B \perp A \\
 A \perp (B \perp (C \perp D)) & \approx & (A \perp B) \perp (C \perp D) \\
 (iii) \quad \parallel & & \parallel \\
 A \perp ((B \perp C) \perp D) & & ((A \perp B) \perp C) \perp D \\
 \parallel & & \cong \\
 (A \perp (B \perp C)) \perp D & & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 A \perp 0 & \simeq & 0 \perp A \\
 (ii) \quad \cong & & \cong \\
 & & A
 \end{array}$$

Let  $IC$  be the set of isomorphism classes of object of  $\mathcal{C}$ . Clearly, if  $\mathcal{C}$  is small, then  $(IC, \perp)$  is an Abelian semi-group, (in fact a monoid) and we write  $K_0^\perp(\mathcal{C})$  for  $K(IC, \perp)$  or simply  $K_0(\mathcal{C})$  when the context is clear.

In other words,  $K_0^\perp(\mathcal{C}) = F(\mathcal{C})/R(\mathcal{C})$  where  $F(\mathcal{C})$  is the free Abelian group on the isomorphism classes  $(C)$  of  $\mathcal{C}$ -objects, and  $R(\mathcal{C})$  the subgroup of  $F(\mathcal{C})$  generated by  $(C' \perp C'') - (C') - (C'')$  for all  $C', C''$  in  $ob(\mathcal{C})$ .

### Remarks 1.4.2

(i)  $K_0^\perp(\mathcal{C})$  satisfies universal property as in 1.1.

(ii) If  $\mathcal{C}$  has another composition ‘0’ that is associative and distributive with respect to  $\perp$ , then  $K_0^\perp(\mathcal{C})$  can be given a ring structure through ‘0’ as multiplication and we shall sometimes denote this ring by  $K_0^\perp(\mathcal{C}, \perp, 0)$  or  $K_0(\mathcal{C}, \perp, 0)$  or just  $K_0(\mathcal{C})$  if the context is clear.

### Examples 1.4.3

(i) If  $\Lambda$  is any ring with identity, then  $(\mathcal{P}(\Lambda), \oplus)$  is a symmetric monoidal category (s.m.c.) and  $K_0^\oplus(\Lambda) = K_0(\Lambda)$  as in 1.2.1.

(ii) If  $\Lambda$  is commutative, then  $K_0^\oplus(\Lambda)$  is a ring where  $(\mathcal{P}(\Lambda), \oplus)$  has a further composition ‘ $\otimes$ ’.

(iii) Let  $X$  be a compact topological space and for  $F = \mathbb{R}$  or  $\mathbb{C}$ , let  $\mathbf{VB}_F(X)$  be the (symmetric monoidal) category of (finite dimensional) vector bundles on  $X$ . Then  $I\mathbf{VB}_F(X)$  is an Abelian monoid under Whitney sum ‘ $\oplus$ ’. It is usual to write  $KO(X)$  for  $K_0^\oplus(\mathbf{VB}_{\mathbb{R}}(X))$  and  $KU(X)$  for  $K_0^\oplus(\mathbf{VB}_{\mathbb{C}}(X))$ . Note that if  $X, Y$  are homotopy equivalent, then  $KO(X) = KO(Y)$  and  $KU(X) = KU(Y)$ . Moreover, if  $X$  is contractible, we have  $KO(X) = KU(X) = \mathbb{Z}$  (see [3] or [38]).

(iv) Let  $X$  be a compact space,  $\mathbb{C}(X)$  the ring of  $\mathbb{C}$ -valued functions on  $X$ . By a theorem of R.G. Swan [94], there exists an equivalence of categories  $\Gamma: \mathbf{VB}_{\mathbb{C}}(X) \rightarrow \mathcal{P}(\mathbb{C}(X))$  taking a vector bundle:  $E \xrightarrow{p} X$  to  $\Gamma(E)$ , where  $\Gamma(E) = \{\text{sections } s: X \rightarrow E \mid ps = 1\}$ . This equivalence induces a group isomorphism  $KU(X) \simeq K_0(\mathbb{C}(X))$  (I).

This isomorphism (I) provides the basic initial connection between Algebraic  $K$ -theory (r.h.s. of I) and topological  $K$ -theory (l.h.s. of I) since the  $K$ -theory of  $\mathcal{P}(\Lambda)$  for an arbitrary ring  $\Lambda$  could be studied instead of the  $K$ -theory of  $\mathcal{P}(\mathbb{C}(X))$ .

Now,  $\mathbb{C}(X)$  is a commutative  $C^*$ -algebra and Gelfand-Naimak theorem [17] says that any commutative  $C^*$ -algebra  $\Lambda$  has the form  $\Lambda = \mathbb{C}(X)$  for some locally compact space  $X$ . Indeed, for any commutative  $C^*$ -algebra  $\Lambda$ , we could take  $X$  as the spectrum of  $\Lambda$  i.e. the set of all non-zero homomorphisms from  $\Lambda$  to  $\mathbb{C}$  with topology of pointwise convergence. Non-commutative geometry is concerned with the study of non-commutative  $C^*$ -algebras associated with “non-commutative” spaces and  $K$ -theory (algebraic and topological) of such  $C^*$ -algebras have been extensively studied and connected to some (co)homology theories (e.g. Hochschild and cyclic (co)homology theories) of such algebras through Chern characters (see e.g. [21], [53], [17], [22]).

(v) Let  $G$  be a group acting continuously on a topological space  $X$ . The category  $\mathbf{VB}_G(X)$  of complex  $G$ -vector bundles on  $X$  is symmetric monoidal under Whitney sum ‘ $\oplus$ ’ and we write  $K_G^0(X)$  for the Grothendieck group  $K_0(\mathbf{VB}_G(X))$ . If  $X$  is a point,  $\mathbf{VB}_G(X)$  is the category of representations of  $G$  in  $\mathcal{P}(\mathbb{C})$  and  $K_G^0(X) = R(G)$ , the representation ring of  $G$ .

If  $G$  acts trivially on  $X$ , then  $K_G^0(X) \simeq KU(X) \otimes_{\mathbb{Z}} R(G)$  (see [80] or [81]).

(vi) Let  $F\text{Set}$  be the category of finite sets,  $\dot{\cup}$  the disjoint union. Then  $(F\text{Set}, \dot{\cup})$  is a symmetric monoidal category and  $K_0^{\dot{\cup}}(F\text{Set}) \simeq \mathbb{Z}$  (see [48]).

(vii) Let  $R$  be a commutative ring with identity. Then  $\text{Pic}(R)$ , the category of finitely generated projective  $R$ -modules of rank one (or equivalently the category of algebraic line bundles  $L$  over  $R$ ) is a symmetric monoidal category and  $K_0^\otimes(\text{Pic}(R)) = \text{Pic}(R)$ , the Picard group of  $R$ .

(viii) The category  $\text{Pic}(X)$  of line bundles on a locally ringed space is a symmetric monoidal category under ‘ $\otimes$ ’ and  $K_0^\otimes(\text{Pic}(X)) := \text{Pic}(X)$  is called the Picard group of  $X$ . Observe that when  $X = \text{Spec}(R)$ , we recover  $\text{Pic}(R)$  in (vii). It is well known that  $\text{Pic}(X) \simeq H^1(X, \mathcal{O}_X^*)$  see [33] or [67].

(ix) Let  $R$  be a commutative ring with identity. An  $R$ -algebra  $\Lambda$  is called an Azumaya algebra if there exists another  $R$ -algebra  $\Lambda'$  such that  $\Lambda \otimes_R \Lambda' \simeq M_n(R)$  for some positive integer  $n$ . Let  $\text{Az}(R)$  be the category of Azumaya algebras. Then  $(\text{Az}(R), \otimes_R)$  is a symmetric monoidal category. Moreover, the category  $F\mathcal{P}(R)$  of faithfully projective  $R$ -modules is symmetric monoidal with respect to  $\perp = \otimes_R$  if the morphisms in  $F\mathcal{P}(R)$  are restricted to isomorphisms. There is a monoidal functor  $F\mathcal{P}(R) \rightarrow \text{Az}(R) : P \rightarrow \text{End}_R(P)$  inducing a group homomorphism  $K_0(F\mathcal{P}(R)) \xrightarrow{\varphi} K_0(\text{Az}(R))$ . The cokernel of  $\varphi$  is called the Brauer group of  $R$  and denoted by  $Br(R)$ . Hence  $Br(R)$  is the Abelian group generated by isomorphism classes  $[\Lambda]$  with relations  $[\Lambda \otimes_R \Lambda'] = [\Lambda] + [\Lambda']$  and  $[\text{End}_R(P)] = 0$ .

If  $R$  is a field  $F$ , then  $\text{End}_R(P) \simeq M_n(F)$  for some  $n$  and  $Br(F)$  is the Abelian group generated by isomorphism classes of central simple  $F$ -algebras with relations  $[\Lambda \otimes \Lambda'] = [[\Lambda] + [\Lambda']]$  and  $[M_n(F)] = 0$  (see [78]).

(x) Let  $G$  be a finite group,  $\mathcal{C}$  any small category. Let  $\mathcal{C}_G$  be the category of  $G$ -objects in  $\mathcal{C}$  or equivalently, the category of  $G$ -representations in  $\mathcal{C}$  i.e. objects of  $\mathcal{C}_G$  are pairs  $(X, U : G \rightarrow \text{Aut}(X))$  where  $X \in \text{ob}(\mathcal{C})$  and  $U$  is a group homomorphism from  $G$  to the group of  $\mathcal{C}$ -automorphisms of  $X$ . If  $(\mathcal{C}, \perp)$  is a symmetric monoidal category, so is  $(\mathcal{C}_G, \dot{\perp})$  where for

$$(X, U : G \rightarrow \text{Aut}(X)), \quad (X', U' : G \rightarrow \text{Aut}(X'))$$

in  $\mathcal{C}_G$ , we define

$$(X, U) \dot{\perp} (X', U') := (X \perp X', U \perp U' : G \rightarrow \text{Aut}(X \perp X'))$$

where  $U \perp U'$  is defined by the composition

$$G \xrightarrow{U \times U'} \text{Aut}(X) \times \text{Aut}(X') \longrightarrow \text{Aut}(X \perp X').$$

So we obtain the Grothendieck group  $K_0^{\dot{\perp}}(\mathcal{C}_G)$ .

If  $\mathcal{C}$  possesses a further associative composition ‘ $\circ$ ’ such that  $\mathcal{C}$  is distributive with respect to  $\perp$  and ‘ $\circ$ ’, then so is  $\mathcal{C}_G$ , and hence  $K_0^{\dot{\perp}}(\mathcal{C}_G)$  is a ring.

**For example** (a) If  $\mathcal{C} = \mathcal{P}(R)$ ,  $\perp = \oplus$ , ‘ $\circ$ ’ =  $\otimes_R$  where  $R$  is a commutative ring with identity, then  $\mathcal{P}(R)_G$  is the category of  $RG$ -lattices (see [48] or [18] or [47] and  $K_0(\mathcal{P}(R)_G)$  is a ring usually denoted by  $G_0(R, G)$ . Observe that when  $R = \mathbb{C}$ ,  $G_0(\mathbb{C}, G)$  is the usual representation ring of  $G$  denoted in the literature by  $R(G)$ . Also see 3.1.4 (iv).

(b) If  $\mathcal{C} = F\text{Sets}$ , ‘ $\perp$ ’ = disjoint union, ‘ $\circ$ ’ - cartesian product. Then  $K_0(\mathcal{C}_G)$  is the Burnside ring of  $G$  usually denoted by  $\Omega(G)$ . See [48].

(xi) Let  $G$  be a finite group,  $S$  a  $G$ -set. We can associate with  $S$  a category  $\mathbf{S}$  as follows: -  
 $ob(\mathbf{S}) = \{s | s \in S\}$ . For  $s, t \in S$ ,  $Hom_{\mathbf{S}}(s, t) = \{(g, s) | s \in G, gs = t\}$ , where composition is defined for  $t = gs$  by  $(h, t) \cdot (g, s) = (hg, s)$  and the identity morphism  $s \rightarrow s$  is given by  $(e, s)$  where  $e$  is the identity element of  $G$ . Now let  $(\mathcal{C}, \perp)$  be a symmetric monoidal category and let  $[\mathbf{S}, \mathcal{C}]$  be the category of covariant functors  $\zeta : \mathbf{S} \rightarrow \mathcal{C}$ . The  $([\mathbf{S}, \mathcal{C}], \dot{\perp})$  is also a symmetric monoidal category where  $(\zeta \dot{\perp} \eta)_{(g, s)} = \zeta_s \perp \eta_s \rightarrow \zeta_{gs} \perp \eta_{gs}$ . We write  $K_0^G(S, \mathcal{C})$  for the Grothendieck group of  $[\mathbf{S}, \mathcal{C}]$ .

If  $(\mathcal{C}, \perp)$  possesses an additional composition ‘0’ that is associative and distributive with respect to ‘ $\perp$ ’, then  $K_0^G(S, \mathcal{C})$  can be given a ring structure (see [48]).

Note that for any symmetric monoidal category  $(\mathcal{C}, \perp)$ ,  $K_0^G(-, \mathcal{C}) : GSet \rightarrow \mathcal{A}b$  is a ‘Mackey’ functor (see [48]), and that when  $\mathcal{C}$  possesses an additional composition ‘0’ discussed above, then  $K_0^G(-, \mathcal{C}) : GSet \rightarrow \mathcal{A}b$  is a ‘Green’ functor (see [48]). We shall discuss these matters in further details under Abstract Representation theory – a forthcoming chapter.

(xii) Let  $A$  be an involutive Banach algebra and  $Witt(A)$  the group generated by isomorphism classes  $[Q]$  of invertible Hermitian forms  $Q$  on  $P \in \mathcal{P}(A)$  with relations  $[Q_1 \oplus Q_2] = [Q_1] + [Q_2]$  and  $[Q] + [-Q] = 0$ . Define a map  $\varphi : K_0(A) \rightarrow Witt(A)$  by  $[P] \rightarrow$  class of  $(P, Q)$  with  $Q$  positive. If  $A$  is a  $C^*$ -algebra with 1, then there exists on any  $P \in \mathcal{P}(A)$  an invertible form  $Q$  satisfying  $Q(x, x) \geq 0$  for all  $x \in P$  and in this case  $\varphi : K_0(A) \rightarrow Witt(A)$  is an isomorphism. However,  $\varphi$  is not an isomorphism in general for arbitrary involutive Banach algebras. See [17].

(xiii) Let  $F$  be a field and  $Sym B(F)$  the category of symmetric inner product spaces  $(V, \beta) - V$  a finite dimensional vector space over  $F$  and  $\beta : V \otimes V \rightarrow F$  a symmetric bilinear form. Then  $(Sym B(F), \perp)$  is a symmetric monoidal category where  $(V, \beta) \perp (V^1, \beta^1)$  is the orthogonal sum of  $(V, \beta)$  and  $(V^1, \beta^1)$  defined as the vector space  $V \oplus V^1$  together with a bilinear form  $\beta^* : (V \oplus V^1, V \oplus V^1) \rightarrow F$  given by  $\beta^*(V \oplus V^1, V_1 \oplus V_1^1) = \beta(V, V_1) + \beta^1(V^1, V_1^1)$ .

If we define composition  $(V, \beta) \odot (V^1, \beta^1)$  as the tensor product  $V \otimes V^1$  together with a bilinear form  $\beta^*(V \otimes V^1, V_1 \otimes V_1^1) = \beta(V, V_1)\beta^1(V^1, V_1^1)$ , then  $K_0(Sym B(F), \perp, \odot)$  is a commutative ring with identity.

The Witt ring  $W(F)$  is defined as the quotient of  $K_0(Sym B(F))$  by the subgroup  $\{nH\}$  generated by the hyperbolic plane  $H = \left( F^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$ .

For more details about  $W(F)$  see [79].

## 2 $K_0$ AND CLASS GROUPS OF DEDEKIND DOMAINS, ORDERS AND GROUP-RINGS

### 2.1 $K_0$ and class groups of Dedekind domains

**2.1.1** An integral domain  $R$  with quotient field  $F$  is called a Dedekind domain if it satisfies any of the following equivalent conditions

- (i) Every ideal in  $R$  is projective (i.e.  $R$  is hereditary).
- (ii) Every non-zero ideal  $\mathfrak{a}$  of  $R$  is invertible (that is  $\mathfrak{a}\mathfrak{a}^{-1} = R$  where  $\mathfrak{a}^{-1} = \{x \in F \mid x\mathfrak{a} \subset R\}$ ).
- (iii)  $R$  is Noetherian, integrally closed and every non-zero prime ideal is maximal.
- (iv)  $R$  is Noetherian, and  $R_{\mathfrak{m}}$  is a discrete valuation ring for all maximal ideals  $\mathfrak{m}$  of  $R$ .
- (v) Every non-zero ideal is uniquely a product of prime ideals.

**Examples 2.1.2**  $\mathbb{Z}, F[x]$ , are Dedekind domains. So is the ring of integers in a number field.

**Definition 2.1.3** A fractional ideal of a Dedekind domain (with quotient field  $F$ ) is an  $R$ -submodule  $\mathfrak{a}$  of  $F$  such that  $s\mathfrak{a} \subset R$  for some  $s \neq 0$  in  $F$ . Then  $\mathfrak{a}^{-1} = \{x \in F \mid x\mathfrak{a} \subset R\}$  is also a fractional ideal. Say that  $\mathfrak{a}$  is invertible if  $\mathfrak{a}\mathfrak{a}^{-1} = R$ . The invertible fractional ideals form a group which we denote by  $I_R$ . Also each element  $u \in F^*$  determines a principal fractional ideal  $Ru$ . Let  $P_R$  be the subgroup of  $I_R$  consisting of all principal fractional ideals. The (ideal) class group of  $R$  is defined as  $I_R/P_R$  and denoted by  $Cl(R)$ .

It is well known that if  $R$  is the ring of integers in a number field, then  $Cl(R)$  is finite see [18].

**Definition 2.1.4** Let  $R$  be a Dedekind domain with quotient field  $F$ . An  $R$ -lattice is a finitely generated torsion free  $R$ -module. Note that any  $R$ -lattice  $M$  is embeddable in a finite dimensional  $F$ -vector space  $V$  such that  $F \otimes_R M = V$ . Moreover, every  $R$ -lattice  $M$  is  $R$ -projective (since  $R$  is hereditary and  $M$  can be written as a direct sum of ideals) (see 2.1.5 below – Steinitz’s theorem). For  $P \in \mathcal{P}(R)$  define the  $R$ -rank of  $P$  as the dimension of the vector space  $F \otimes_R P$  and denote this number by  $rk(P)$ .

**Theorem 2.1.5** [18] Steinitz’s theorem.

Let  $R$  be a Dedekind domain. Then

- (i) If  $M \in \mathcal{P}(R)$ , then  $M = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_n$  where  $n$  is the  $R$ -rank of  $M$  and each  $\mathfrak{a}_i$  is an ideal of  $R$ .
- (ii) Two direct sums  $\mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_n$  and  $\mathfrak{b}_1 \oplus \mathfrak{b}_2 \oplus \cdots \oplus \mathfrak{b}_m$  of non-zero ideals of  $R$  are  $R$ -isomorphic if and only if  $n = m$  and the ideal class of  $\mathfrak{a}_1\mathfrak{a}_2 \cdots \mathfrak{a}_n =$  ideal class of  $\mathfrak{b}_1\mathfrak{b}_2 \cdots \mathfrak{b}_m$ .

**Definition 2.1.6** The ideal class associated to  $M$  as in 2.1.5, is called the Steinitz class and is denoted by  $St(M)$ .

**Theorem 2.1.7** Let  $R$  be a Dedekind domain. Then

$$K_0(R) \simeq \mathbb{Z} \oplus Cl(R)$$

**Sketch of proof:** Define a map

$$Q = (rk, st) : K_0(R) \longrightarrow \mathbb{Z} \times Cl(R)$$

by

$$(rk, st)[P] = (rkP, st(P))$$

where  $rkP$  is the  $R$ -rank of  $P$  (2.1.4) and  $st(P)$  is the Steinitz class of  $P$ . We have  $rk(P \oplus P^1) = rk(P) + rk(P^1)$  and  $st(P \oplus P^1) = st(P) \cdot st(P^1)$ . So  $\varphi$  is a homomorphism that can easily be checked to be an isomorphism, the inverse being given by  $\eta : \mathbb{Z} \times Cl(R) \rightarrow K_0(R), (n, (\mathbf{a})) \rightarrow n[R] + [\mathbf{a}]$ .

**Remarks 2.1.8**

- (i) It follows easily from Steinitz's theorem that  $\text{Pic}(R) \simeq Cl(R)$  for any Dedekind domain  $R$ .
- (ii) Let  $R$  be a commutative ring with identity,  $\text{Spec}(R)$  the set of prime ideals of  $R$ . For  $P \in \mathcal{P}(R)$  define  $r_P : \text{Spec}(R) \rightarrow \mathbb{Z}$  by  $r_P(\mathfrak{p}) = \text{rank of } P_{\mathfrak{p}} \text{ over } R_{\mathfrak{p}} = \text{dimension of } P_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}P_{\mathfrak{p}}$ . Then  $r_P$  is continuous where  $\mathbb{Z}$  is given the discrete topology (see [8] or [100]). Let  $H_0(R) := \text{group of continuous functions } \text{Spec}(R) \rightarrow \mathbb{Z}$ . Then we have a homomorphism  $r : K_0(R) \rightarrow H_0(R) : r([P]) = r_P$  (see [8]). One can show that if  $R$  is a one-dimensional commutative Noetherian ring then  $(rk, \det) : K_0(R) \rightarrow H_0(R) \oplus \text{Pic}(R)$  is an isomorphism – a generalisation of 2.1.7 which we recover by seeing that for Dedekind domains  $R$ ,  $H_0(R) \simeq \mathbb{Z}$ . Note that  $\det : K_0(R) \rightarrow \text{Pic}(R)$  is defined by  $\det(P) = \Lambda^n P$  if the  $R$ -rank of  $P$  is  $n$ . (See [8].)
- (iii) Since a Dedekind domain is a regular ring,  $K_0(R) \simeq G_0(R)$ .

## 2.2 Class groups of orders and group rings

**Definition 2.2.1** Let  $R$  be a Dedekind domain with quotient field  $F$ . An  $R$ -order  $\Lambda$  in a finite dimensional semi-simple  $F$ -algebra  $\Sigma$  is a subring of  $\Sigma$  such that (i)  $R$  is contained in the centre of  $\Lambda$ , (ii)  $\Lambda$  is a finitely generated  $R$ -module and (iii)  $F \otimes_R \Lambda = \Sigma$ .

**Example** For a finite group  $G$ , the group ring  $RG$  is an  $R$ -order in  $FG$ .

**Definition 2.2.2** Let  $R, F, \Sigma$  be as in 3.2.1. A maximal  $R$ -order  $\Gamma$  in  $\Sigma$  is an order that is not contained in any other  $R$ -order in  $\Sigma$ .

**Examples** (i)  $R$  is a maximal  $R$ -order in  $F$ .

(ii)  $M_n(R)$  is a maximal  $R$ -order in  $M_n(F)$ .

**Remarks 2.2.3** Let  $R, F, \Sigma$  be as in 3.2.1. Then

(i) Any  $R$ -order  $\Lambda$  is contained in at least one maximal  $R$ -order in  $\Sigma$  (see [18]).

(ii) Every semi-simple  $F$ -algebra  $\Sigma$  contains at least one maximal order. However, if  $\Sigma$  is commutative, then  $\Sigma$  contains a unique maximal order, namely, the integral closure of  $R$  in  $\Sigma$  (see [18] or [72]).

(iii) If  $\Lambda$  is an  $R$ -order in  $\Sigma$ , then  $\Lambda_{\mathfrak{p}}$  is an  $R_{\mathfrak{p}}$ -order in  $\Sigma$  for any prime= maximal ideal  $\mathfrak{p}$  of  $R$ . Moreover,  $\Lambda = \bigcap_{\mathfrak{p}} \Lambda_{\mathfrak{p}}$  (intersection within  $\Sigma$ ).

(iv) In any  $R$ -order  $\Lambda$ , every element is integral over  $R$  (see [18] or [74]).

**Definition 2.2.4** Let  $R, F, \Sigma, \Lambda$  be as in 2.2.1. A left  $\Lambda$ -lattice is a left  $\Lambda$ -module which is also an  $R$ -lattice (i.e. finitely generated and projective as an  $R$ -module).

A  $\Lambda$ -ideal in  $\Sigma$  is a left  $\Lambda$ -lattice  $M \subset \Sigma$  such that  $FM \subset \Sigma$ .

Two left  $\Lambda$ -lattices  $M, N$  are said to be in the same genus if  $M_{\mathfrak{p}} \simeq N_{\mathfrak{p}}$  for each prime ideal  $\mathfrak{p}$  of  $R$ . A left  $\Lambda$ -ideal is said to be locally free if  $M_{\mathfrak{p}} \simeq \Lambda_{\mathfrak{p}}$  for all  $\mathfrak{p} \in \text{Spec}(R)$ . We write  $M \vee N$  if  $M$  and  $N$  are in the same genus.

**Definition 2.2.5** Let  $R, F, \Sigma$  be as in 2.2.1,  $\Lambda$  an  $R$ -order in  $\Sigma$ . Let  $S(\Lambda) = \{\mathfrak{p} \in \text{Spec}(R) \mid \hat{\Lambda}_{\mathfrak{p}}$  is not a maximal  $\hat{R}_{\mathfrak{p}}$ -order in  $\hat{\Sigma}\}$ . Then  $S(\Lambda)$  is a finite set and  $S(\Lambda) = \emptyset$  iff  $\Lambda$  is a maximal  $R$ -order. Note that the genus of a  $\Lambda$ -lattice  $M$  is determined by isomorphism classes of modules  $\{M_{\mathfrak{p}} \mid \mathfrak{p} \in S(\Lambda)\}$  (see [18] or [72]).

**Theorem 2.2.6** Let  $L, M, N$  be lattices in the same genus. Then  $M \oplus N \simeq L \oplus L'$  for some lattice  $L'$  in the same genus. Hence, if  $M, M'$  are locally free  $\Lambda$ -ideals in  $\Sigma$ , then  $M \oplus M' = \Lambda \oplus M''$  for some locally free ideal  $M''$ .

**Definition 2.2.7** Let  $R, F, \Sigma$  be as in 2.2.1. The idèle group of  $\Sigma$ , denoted  $J(\Sigma)$  is defined by  $J(\Sigma) := \{(\alpha_{\mathfrak{p}}) \in \prod (\hat{\Sigma}_{\mathfrak{p}})^* \mid \alpha_{\mathfrak{p}} \in \hat{\Lambda}_{\mathfrak{p}}^* \text{ almost everywhere }\}$ . For  $\alpha = (\alpha_{\mathfrak{p}}) \in J(\Sigma)$ , define

$$\Lambda \alpha := \Sigma \cap \left\{ \bigcap_{\mathfrak{p}} \hat{\Lambda}_{\mathfrak{p}} \alpha_{\mathfrak{p}} \right\} = \bigcap_{\mathfrak{p}} \left\{ \Sigma \cap \hat{\Lambda}_{\mathfrak{p}} \alpha_{\mathfrak{p}} \right\}$$

The group of principal idèles, denoted  $u(\Sigma)$  is defined by  $u(\Sigma) = \{\alpha = (\alpha_{\mathfrak{p}}) \mid \alpha_{\mathfrak{p}} = x \in \Sigma^* \text{ for all } \mathfrak{p} \in \text{Spec}(R)\}$ . The group of unit idèles is defined by

$$U(\Lambda) = \prod_{\mathfrak{p}} (\Lambda_{\mathfrak{p}})^* \subseteq J(\Sigma)$$

**Remarks** (i)  $J(\Sigma)$  is independent of the choice of the  $R$ -order  $\Lambda$  in  $\Sigma$  since if  $\Lambda'$  is another  $R$ -order, then  $\Lambda_{\mathfrak{p}} = \Lambda'_{\mathfrak{p}}$  a.e.

(ii)  $\Lambda\alpha$  is isomorphic to a left ideal of  $\Lambda$  and  $\Lambda\alpha$  is the same genus as  $\Lambda$ . Call  $\Lambda\alpha$  a locally free (rank 1)  $\Lambda$ -lattice or a locally free fractional  $\Lambda$ -ideal in  $\Sigma$ . Note that any  $M \in g(\Lambda)$  can be written in the form  $M = \Lambda\alpha$  for some  $\alpha \in J(\Sigma)$  (see [18]).

(iii) If  $\Sigma = F$  and  $\Lambda = R$ , we also have  $J(F)$ ,  $u(F)$  and  $U(R)$  as defined above.

(iv) For  $\alpha, \beta \in J(\Sigma)$ ,  $\Lambda\alpha \oplus \Lambda\beta \cong \Lambda \oplus \Lambda\alpha\beta$  (see [18]).

**Definition 2.2.8** Let  $F, \Sigma, R, \Lambda$  be as in 2.2.1. Two left  $\Lambda$ -modules  $M, N$  are said to be stably isomorphic if  $M \oplus \Lambda^{(k)} \simeq N \oplus \Lambda^{(k)}$  for some positive integer  $k$ . If  $F$  is a number field, then  $M \oplus \Lambda^{(k)} \simeq N \oplus \Lambda^{(k)}$  iff  $M \oplus \Lambda \simeq N \oplus \Lambda$ . We write  $[M]$  for the stable isomorphism class of  $M$ .

**Theorem 2.2.9** [18] The stable isomorphism classes of locally free ideals form an Abelian group  $\mathcal{Cl}(\Lambda)$  called the locally free class group of  $\Lambda$  where addition is given by  $[M] + [M'] = [M'']$  whenever  $M \oplus M' \simeq \Lambda \oplus M''$ . The zero element is  $(\Lambda)$  and inverses exist since  $(\Lambda\alpha) \oplus (\Lambda\alpha^{-1}) \simeq \Lambda \oplus \Lambda$  for any  $\alpha \in J(\Sigma)$ .

**Theorem 2.2.10** Let  $R, F, \Lambda, \Sigma$  be as in 2.2.1. If  $F$  is an algebraic number field, then  $\mathcal{Cl}(\Lambda)$  is a finite group.

**Proof:** (Sketch) If  $L$  is a left  $\Lambda$ -lattice, then there exists only a finite number of isomorphism classes of left  $\Lambda$ -lattices  $M$  such that  $FM \simeq FL$  as  $\Sigma$ -modules. In particular, there exists only a finite number of isomorphism classes of left  $\Lambda$  ideals in  $\Sigma$  (see [18] or [74]).

**Remarks 2.2.11** Let  $R, F, \Lambda, \Sigma$  be as in 3.2.1.

(i) If  $\Lambda = R$ , then  $\mathcal{Cl}(\Lambda)$  is the ideal class group of  $R$ .

(ii) If  $\Gamma$  is a maximal  $R$ -order in  $\Sigma$ , then every left  $\Lambda$ -ideal in  $\Sigma$  is locally free. So,  $\mathcal{Cl}(\Gamma)$  is the group of stable isomorphism classes of all left  $\Gamma$ -ideals in  $\Sigma$ .

(iii) Define a map  $J(\Sigma) \rightarrow \mathcal{Cl}(\Lambda); \alpha \rightarrow [\Lambda\alpha]$ . Then one can show that this map is surjective and that the kernel is  $J_0(\Sigma)\Sigma^*U(\Lambda)$  where  $J_0(\Sigma)$  is the kernel of the reduced norm acting on  $J(\Sigma)$ . So  $J(\Sigma)/(J_0(\Sigma)\Sigma^*U(\Lambda)) \simeq \mathcal{Cl}(\Lambda)$  (see [18]).

(iv) If  $G$  is a finite group such that no proper divisor of  $|G|$  is a unit in  $R$ , then  $\mathcal{Cl}(RG) \simeq SK_0(RG)$ . Hence  $\mathcal{Cl}(\mathbb{Z}G) \simeq SK_0(\mathbb{Z}G)$  for every finite group  $G$  (see [18] or [98]).

For computations of  $\mathcal{Cl}(RG)$  for various  $R$  and  $G$  see [18].

### 2.2.12 An Application – Wall's finiteness obstruction theorem

Let  $R$  be a ring. A bounded chain complex  $C = (C_*, d)$  of  $R$ -modules is said to be of finite type if all the  $C_j$ 's are finitely generated. The Euler characteristic of  $C = (C_*, d)$  is given by:  $\chi(C) = \sum_{i=-\infty}^{\infty} (-1)^i [C_i]$ , and we write  $\bar{\chi}(C)$  for the image of  $\chi(C)$  in  $\tilde{K}_0(R)$ .

The initial motivation for Wall's finiteness obstruction theorem stated below was the desire to find out when a connected space has the homotopy type of a CW-complex. If  $X$  is homotopically



equivalent to a CW-complex, the singular chain complex  $S_*(X)$  with local coefficients is said to be finitely dominated if it is chain homotopic to a complex of finite type. Let  $R = \mathbb{Z}\pi_1(X)$ , the integral group-ring of the fundamental group of  $X$ . Wall's finite obstruction theorem stated below implies that a finitely dominated complex has a finiteness obstruction in  $\tilde{K}_0(R)$  and is chain homotopic to a complex of finite type of free  $R$ -modules if and only if the finiteness obstruction vanishes. More precisely we have the following

**Theorem** [113] Let  $(C_*, d)$  be a chain complex of projective  $R$ -modules which is homotopic to a chain complex of finite type of projective  $R$ -modules. Then  $(C_*, d)$  is chain homotopic to a chain complex of finite type of free  $R$ -modules if and only if  $\bar{\chi}(C) = 0$  in  $\tilde{K}_0(R)$ .

**Note:** For further applications in this direction see [114], [104], [84].

### 3 $K_0$ OF EXACT AND ABELIAN CATEGORIES – DEFINITIONS AND EXAMPLES

#### 3.1 $K_0$ of exact categories and examples

**Definition 3.1.1** An exact category is an additive category  $\mathcal{C}$  embeddable as a full subcategory of an Abelian category  $\mathcal{A}$  such that  $\mathcal{C}$  is equipped with a class  $\mathcal{E}$  of short exact sequences  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  ( $I$ ) satisfying (i)  $\mathcal{E}$  is the class of all sequences ( $I$ ) in  $\mathcal{C}$  that are exact in  $\mathcal{A}$ .

(ii)  $\mathcal{E}$  is closed under extensions in  $\mathcal{A}$  i.e. if ( $I$ ) is an exact sequence in  $\mathcal{A}$  and  $M', M'' \in \mathcal{C}$ , then  $M \in \mathcal{C}$ .

**Definition 3.1.2** For a small exact category  $\mathcal{C}$ , define the Grothendieck group  $K_0(\mathcal{C})$  of  $\mathcal{C}$  as the Abelian group generated by isomorphism classes ( $C$ ) of  $\mathcal{C}$ -objects subject to the relation  $(C') + (C'') = (C)$  whenever  $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$  is an exact sequence in  $\mathcal{C}$ .

#### Remarks 3.1.3

- (i)  $K_0(\mathcal{C}) \simeq \mathcal{F}/\mathcal{R}$  where  $\mathcal{F}$  is the free Abelian group on the isomorphism classes ( $C$ ) of  $\mathcal{C}$ -objects and  $\mathcal{R}$  the subgroup of  $\mathcal{F}$  generated by all  $(C) - (C') - (C'')$  for each exact sequence  $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$  in  $\mathcal{C}$ . Denote by  $[C]$  the class of ( $C$ ) in  $K_0(\mathcal{C}) = \mathcal{F}/\mathcal{R}$ .
- (ii) The construction satisfies the following property: If  $\chi : \mathcal{C} \rightarrow A$  is a map from  $\mathcal{C}$  to an Abelian group  $A$  given that  $\chi(C)$  depends only on the isomorphism class of  $C$  and  $\chi(C) = \chi(C') + \chi(C'')$  for any exact sequence  $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ , then there exists a unique  $\chi' : K_0(\mathcal{C}) \rightarrow A$  such that  $\chi(C) = \chi'([C])$  for any  $\mathcal{C}$ -object  $C$ .
- (iii) Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be an exact functor between two exact categories  $\mathcal{C}, \mathcal{D}$  (i.e.  $F$  is additive and takes short exact sequences in  $\mathcal{C}$  to such sequences in  $\mathcal{D}$ ). Then  $F$  induces a group homomorphism  $K_0(\mathcal{C}) \rightarrow K_0(\mathcal{D})$ .
- (iv) Note that an Abelian category  $\mathcal{A}$  is also an exact category and the definition of  $K_0(\mathcal{A})$  is the same as in 2.1.2.

#### Examples 3.1.4

- (i) Any additive category is an exact category as well as a symmetric monoidal category under ‘ $\oplus$ ’, and  $K_0(\mathcal{C})$  is a quotient of the group  $K_0^\oplus(\mathcal{C})$  defined in 1.4.1.  
If every short exact sequence in  $\mathcal{C}$  splits, then  $K_0(\mathcal{C}) = K_0^\oplus(\mathcal{C})$ . For example,  $K_0(\Lambda) = K_0(\mathcal{P}(\Lambda)) = K_0^\oplus(\mathcal{P}(\Lambda))$  for any ring  $\Lambda$  with identity.
- (ii) Let  $\Lambda$  be a (left) Noetherian ring. Then the category  $\mathcal{M}(\Lambda)$  of finitely generated (left)- $\Lambda$ -modules is an exact category and we denote  $K_0(\mathcal{M}(\Lambda))$  by  $G_0(\Lambda)$ . The inclusion functor

$\mathcal{P}(\Lambda) \rightarrow \mathcal{M}(\Lambda)$  induces a map  $K_0(\Lambda) \rightarrow G_0(\Lambda)$  called the Cartan map. For example  $\Lambda = RG$ ,  $R$  a Dedekind domain,  $G$  a finite group, yields a Cartan map  $K_0(RG) \rightarrow G_0(RG)$ .

If  $\Lambda$  is left Artinian, then  $G_0(\Lambda)$  is free Abelian on  $[S_1], \dots, [S_r]$  where the  $S_i$  are distinct classes of simple  $\Lambda$ -modules while  $K_0(\Lambda)$  is free Abelian on  $[I_1], \dots, [I_\ell]$  and the  $I_i$  are distinct classes of indecomposable projective  $\Lambda$ -modules (see [18]). So, the map  $K_0(\Lambda) \rightarrow G_0(\Lambda)$  gives a matrix  $a_{ij}$  where  $a_{ij}$  = the number of times  $S_j$  occurs in a composition series for  $I_i$ . This matrix is known as the Cartan matrix.

If  $\Lambda$  is left regular (i.e. every finitely generated left  $\Lambda$ -module has finite resolution by finitely generated projective left  $\Lambda$ -modules), then it is well known that the Cartan map is an isomorphism (see [18]).

For example, if  $R$  is a Dedekind domain with quotient field  $F$  and  $\Lambda$  is a maximal  $R$ -order in a semi-simple  $F$ -algebra,  $\Sigma$ , then  $K_0(\Lambda) \simeq G_0(\Lambda)$  since  $\Lambda$  is regular. (See [18] or [25] for further information on Cartan maps.)

(iii) Let  $R$  be a commutative ring with identity,  $\Lambda$  an  $R$ -algebra. Let  $\mathcal{P}_R(\Lambda)$  be the category of left  $\Lambda$ -lattices i.e.  $\Lambda$ -modules which are finitely generated and projective as  $R$ -modules. Then  $\mathcal{P}_R(\Lambda)$  is an exact category and we write  $G_0(R, \Lambda)$  for  $K_0(\mathcal{P}_R(\Lambda))$ . If  $\Lambda = RG$ ,  $G$  a finite group, we write  $\mathcal{P}_R(G)$  for  $\mathcal{P}_R(RG)$  and also write  $G_0(R, G)$  for  $G_0(R, RG)$ . If  $M, N \in \mathcal{P}_R(\Lambda)$ , then, so is  $M \otimes_R N$  and hence the multiplication given in  $G_0(R, G)$  by  $[M][N] = (M \otimes_R N)$  makes  $G_0(R, G)$  a commutative ring with identity.

(iv) If  $R$  is a commutative regular ring and  $\Lambda$  is an  $R$ -algebra that is finitely generated and projective as an  $R$ -module (e.g.  $\Lambda = RG$ ,  $G$  a finite group or  $R$  is a Dedekind domain with quotient field  $F$  and  $\Lambda$  is an  $R$ -order in a semi-simple  $F$ -algebra) then  $G_0(R, \Lambda) \simeq G_0(\Lambda)$ .

**Sketch of proof:** Define a map  $\varphi : G_0(R, \Lambda) \rightarrow G_0(\Lambda)$  by  $\varphi[M] = [M]$ . Then  $\varphi$  is a well defined homomorphism. Now for  $M \in \mathcal{M}(\Lambda)$ , there exists an exact sequence  $0 \rightarrow L \rightarrow P_{n-1} \xrightarrow{\varphi_{n-1}} P_{n-2} \rightarrow \dots \rightarrow P_0 \rightarrow M \rightarrow 0$  where  $P_i \in \mathcal{P}(\Lambda)$   $L \in \mathcal{M}(\Lambda)$ . Now, since  $\Lambda \in \mathcal{P}(R)$ , each  $P_i \in \mathcal{P}(R)$  and hence  $L \in \mathcal{P}(R)$ . So  $L \in \mathcal{P}_R(\Lambda)$ . Now define  $\delta[M] = [P_0] - [P_1] + \dots + (-1)^{n-1}[P_{n-1}] + (-1)^n[L] \in G_0(R, \Lambda)$ . One easily checks that  $\delta f = 1 = f \delta$ .

(v) Let  $X$  be a scheme (see [33]),  $\mathcal{P}(X)$  the category of locally free sheaves of  $O_X$ -modules of finite rank (or equivalently the category of finite dimensional (algebraic) vector bundles on  $X$ ). Then  $\mathcal{P}(X)$  is an exact category and we write  $K_0(X)$  for  $K_0(\mathcal{P}(X))$  (see [69]).

If  $X = \text{Spec}(A)$  for some commutative ring  $A$ , then we have an equivalence of categories  $\mathcal{P}(X) \rightarrow \mathcal{P}(A) : E \rightarrow \Gamma(X, E) = \{A\text{-module of global sections}\}$ , with the inverse equivalence  $\mathcal{P}(A) \rightarrow \mathcal{P}(X)$  given by  $P \rightarrow \tilde{P} : U \rightarrow O_X(U) \otimes_A P$ . Hence  $K_0(X) \simeq K_0(A)$ .

(vi) Let  $X$  be a Noetherian scheme (i.e.  $X$  can be covered by affine open sets  $\text{Spec}(A_i)$ ) where

each  $A_i$  is Noetherian), then the category  $\mathcal{M}(X)$  of coherent sheaves of  $O_X$ -modules is exact. We write  $G_0(X)$  for  $K_0(\mathcal{M}(X))$ . If  $X = \text{Spec}(A)$  then we have an equivalence of categories  $\mathcal{M}(X) \simeq \mathcal{M}(A)$  and  $G_0(X) \simeq G_0(A)$ .

- (vii) Let  $G$  be a finite group,  $S$  a  $G$ -set,  $\mathbf{S}$  the category associated to  $S$  (see 1.4.3 (xi)),  $\mathcal{C}$  an exact category, and  $[\mathbf{S}, \mathcal{C}]$  the category of covariant functors  $\zeta : \mathbf{S} \rightarrow \mathcal{C}$ . We write  $\zeta_s$  for  $\zeta(s), s \in S$ . Then  $[\mathbf{S}, \mathcal{C}]$  is an exact category where a sequence  $0 \rightarrow \zeta' \rightarrow \zeta \rightarrow \zeta'' \rightarrow 0$  in  $[\mathbf{S}, \mathcal{C}]$  is defined to be exact if  $0 \rightarrow \zeta'_s \rightarrow \zeta_s \rightarrow \zeta''_s \rightarrow 0$  is exact in  $\mathcal{C}$  for all  $s \in S$ . Denote by  $K_0^G(\mathbf{S}, \mathcal{C})$  the  $K_0$  of  $[\mathbf{S}, \mathcal{C}]$ . Then  $K_0^G(-, \mathcal{C}) : \mathbf{GSet} \rightarrow \mathbf{Ab}$  is a functor that can be proved to be a ‘Mackey’ functor (see [24] or [48]).

It can also be shown (see [48] or [47]) that if  $S = G/G$ , the  $[G/G, \mathcal{C}] \simeq \mathcal{C}_G$  in the notation of 1.4.3 (x). Also, constructions analogous to the one above can be done for  $G$  a profinite group, (see [46]) or compact Lie groups ([51A]).

Now if  $R$  is a commutative Noetherian ring with identity, we have  $[G/G, \mathbf{P}(R)] \simeq \mathcal{P}(R)_G \simeq \mathcal{P}_R(RG)$  (see [48] or [47]), and so,  $K_0^G[G/G, \mathcal{P}(R)] \simeq K_0(\mathcal{P}(R)_G) \simeq G_0(R, G)$  and that if  $R$  is regular  $K_0(\mathcal{P}(R)_G) \simeq G_0(R, G) \simeq G_0(RG)$ . This provides an initial connection between  $K$ -theory of representations of  $G$  in  $\mathcal{P}(R)$  and  $K$ -theory of the group ring  $RG$ .

In particular, when  $R = \mathbb{C}$ ,  $\mathcal{P}(\mathbb{C}) = \mathcal{M}(\mathbb{C})$  and  $K_0(\mathcal{P}(\mathbb{C})_G) \simeq G_0(\mathbb{C}, G) = G_0(\mathbb{C}G) =$  the Abelian group of characters  $\chi : G \rightarrow \mathbb{C}$  (see [18]), as already observed in §1.

- (viii) Let  $X$  be a compact topological space and  $F = \mathbb{R}$  or  $\mathbb{C}$ . Then the category  $VB_F(X)$  of vector bundles over  $X$  is an exact category. We had earlier observed (see §1) that  $VB_F(X)$  is also a symmetric monoidal category. Since every short exact sequence in  $VB_F(X)$  splits, we have  $K_0(VB_F(X)) \simeq K_0^{\oplus}(VB_F(X))$ .

## 4 SOME FUNDAMENTAL RESULTS ON $K_0$ OF EXACT AND ABELIAN CATEGORIES

In this section, we discuss some of the results that will be seen in more generality when Higher  $K$ -groups are treated in a forthcoming chapter.

### 4.1 Devissage theorem

**Definition 4.1.1** Let  $\mathcal{C}_0 \subset \mathcal{C}$  be exact categories. The inclusion functor  $\mathcal{C}_0 \rightarrow \mathcal{C}$  is exact and hence induces a homomorphism  $K_0(\mathcal{C}_0) \rightarrow K_0(\mathcal{C})$ . A  $\mathcal{C}_0$ -filtration of an object  $A$  in  $\mathcal{C}$  is a finite sequence of the form:  $0 = A_0 \subset A_1 \subset \dots \subset A_n = A$  where each  $A_i/A_{i-1} \in \mathcal{C}_0$ .

**Lemma 4.1.2** If  $0 \subset A_0 \subset A_1 \subset \dots \subset A_n = A$  is a  $\mathcal{C}_0$ -filtration, then  $[A] = \sum [A_i/A_{i-1}] \quad 1 \leq i \leq n$  in  $K_0(\mathcal{C})$ .

**Theorem 4.1.3** (Devissage theorem). Let  $\mathcal{C}_0 \subset \mathcal{C}$  be exact categories such that  $\mathcal{C}_0$  is Abelian. If every  $A \in \mathcal{C}$  has a  $\mathcal{C}_0$ -filtration, then  $K_0(\mathcal{C}_0) \rightarrow K_0(\mathcal{C})$  is an isomorphism.

**Proof:** Since  $\mathcal{C}_0$  is Abelian, any refinement of a  $\mathcal{C}_0$ -filtration is also a  $\mathcal{C}_0$ -filtration. So, by Zassenhaus lemma, any two finite filtrations have equivalent refinements, that is, refinements such that the successive factors of the first refinement are, up to a permutation of the order of their occurrences, isomorphic to those of the second.

So, if  $0 \subset A_0 \subset A_1 \subset \dots \subset A_n = A$  is any  $\mathcal{C}_0$ -filtration of  $A$  in  $\mathcal{C}$ , then

$$J(A) = \sum [A_i/A_{i-1}] \quad (1 \leq i \leq n)$$

is well defined, since  $J(A)$  is unaltered by replacing the given filtration with a refinement.

Now let  $0 \rightarrow A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \rightarrow 0$  be an exact sequence in  $\mathcal{C}$ . Obtain a filtration for  $A$  by  $0 = A_0 \subset A_1 \subset \dots \subset A_n = A'$  for  $A'$  and  $\beta^{-1}(A^0) \subset \beta^{-1}(A^1) \subset \dots \subset \beta^{-1}(A'')$  if  $A^0 \subset A^1 \subset \dots \subset A''$  is a  $\mathcal{C}_0$ -filtration of  $A''$ . Then  $0 = A_0 \subset A_1 \subset \dots \subset A_k \subset \beta^{-1}(A^0) \subset \beta^{-1}(A^1 \subset \dots \subset \beta^{-1}(A^n))$  is a filtration of  $A$ .

So,  $J(A) = J(A') + J(A'')$ . Hence  $J$  induces a homomorphism  $K_0(\mathcal{C}) \rightarrow K_0(\mathcal{C}_0)$ . We also have a homomorphism  $i : K_0(\mathcal{C}_0) \rightarrow K_0(\mathcal{C})$  induced by the inclusion functor  $i : \mathcal{C}_0 \rightarrow \mathcal{C}$ . Moreover,  $i \circ J = 1_{K_0(\mathcal{C})}$  and  $J \circ i = 1_{K_0(\mathcal{C}_0)}$ . Hence  $K_0(\mathcal{C}_0) \simeq K_0(\mathcal{C})$ .

**Corollary 4.1.4** Let  $\mathfrak{a}$  be a nilpotent two-sided ideal of a Noetherian ring  $R$ . Then  $G_0(R/\mathfrak{a}) \simeq G_0(R)$ .

**Proof:** If  $M \in \mathcal{M}(R)$ , then  $M \supset \mathfrak{a}M \supset \dots \supset \mathfrak{a}^k M = 0$  is an  $\mathcal{M}(R/\mathfrak{a})$  filtration of  $M$ . Result follows from 3.3.

**Example 4.1.5**

- (i) Let  $R$  be an Artinian ring with maximal ideal  $\mathfrak{m}$  such that  $\mathfrak{m}^r = 0$  for some  $r$ . Let  $k = R/\mathfrak{m}$  (e.g.  $R = Z/p^r, k = F_p$ ).

In 4.1.3, put  $\mathcal{C}_0 =$  category of finite dimensional  $k$ -vector spaces and  $\mathcal{C}$ , the category of finitely generated  $R$ -modules. Then, we have a

$$0 = \mathfrak{m}^r M \subset \mathfrak{m}^{r-1} M \subset \cdots \subset \mathfrak{m} M \subset M \text{ of } M$$

where  $M \in \text{ob}\mathcal{C}$ . Hence by 4.1.3,  $K_0(\mathcal{C}_0) \simeq K_0(\mathcal{C})$ .

- (ii) Let  $X$  be a Noetherian scheme,  $\mathcal{M}(X)$  the category of coherent sheaves of  $O_X$ -modules,  $i : Z \subset X$  the inclusion of a closed subscheme. Then  $\mathcal{M}(Z)$  becomes an Abelian subcategory of  $\mathcal{M}(X)$  via the direct image  $i : \mathcal{M}(Z) \subset \mathcal{M}(X)$ . Let  $\mathcal{M}_Z(X)$  be the Abelian category of coherent sheaves of  $O_X$ -modules supported on  $Z$ ,  $\mathfrak{a}$  an ideal sheaf in  $O_X$  such that  $O_X/\mathfrak{a} \simeq O_Z$ . Then every  $M \in \mathcal{M}_Z(X)$  has a finite filtration  $M \supset M\mathfrak{a} \supset M\mathfrak{a}^2 \supset \cdots$  and so, by Devissage  $K_0(\mathcal{M}_Z(X)) \simeq K_0(\mathcal{M}(Z)) \simeq G_0(Z)$ . See §4.3.4 for more examples of applications of Devissage.

**4.2 Resolution theorem and examples**

**Resolution theorem 4.2.1** [8] or [67]. Let  $\mathcal{A}_0 \subset \mathcal{A}$  be an inclusion of exact categories. Suppose that every object of  $\mathcal{A}$  has a finite resolution by objects of  $\mathcal{A}_0$  and that if  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is an exact sequence in  $\mathcal{A}$ , then  $M \in \mathcal{A}_0$  implies that  $M', M'' \in \mathcal{A}_0$ . Then  $K_0(\mathcal{A}_0) \simeq K_0(\mathcal{A})$ .

**Examples 4.2.2** (i) Let  $R$  be a regular ring. Then, for any  $M \in \text{ob}\mathcal{M}(R)$ , there exists  $P_i \in \mathcal{P}(R)$   $i = 0, 1, \dots, n$  such that the sequence  $0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow M \rightarrow 0$  is exact. Put  $\mathcal{A}_0 = \mathcal{P}(R), \mathcal{A} = \mathcal{M}(R)$  in 4.6. Then we have  $K_0(R) \simeq G_0(R)$  (see [8]).

(ii) Let  $\mathbf{H}(R)$  be the category of all  $R$ -modules having finite homological dimension i.e., having a finite resolution by finitely generated projective  $R$ -modules:  $\mathbf{H}_n(R)$  the subcategory of modules having resolutions of length  $\leq n$ . Then by resolution theorem 4.2.1, applied to  $\mathbf{P}(R) \subseteq \mathbf{H}(R)$  we have  $K_0(R) \simeq K_0\mathbf{H}(R) \simeq K_0\mathbf{H}_n(R)$  for all  $n \geq 1$  (see [8] or [100]).

(iii) Let  $\mathcal{C}$  be an exact category and  $\text{Nil}(\mathcal{C})$  the category whose objects are pairs  $(M, \nu)$  where  $M \in \mathcal{C}$  and  $\nu$  is a nilpotent endomorphism of  $M$  i.e.  $\nu \in \text{End}_{\mathcal{C}}(M)$ . Let  $\mathcal{C}_0 \subset \mathcal{C}$  be an exact subcategory of  $\mathcal{C}$  such that every object of  $\mathcal{C}$  has a finite  $\mathcal{C}_0$ -resolution. Then every object of  $\text{Nil}(\mathcal{C})$  has a finite  $\text{Nil}(\mathcal{C}_0)$ -resolution and so, by 4.2.1,  $K_0(\text{Nil}(\mathcal{C}_0)) \simeq K_0(\text{Nil}(\mathcal{C}))$ .

(iv) In the notation of (iii), we have two functors  $Z : \mathcal{C} \rightarrow \text{Nil}(\mathcal{C}) : Z(M) = (M, 0)$  (where ‘0’ denotes zero endomorphism) and  $F : \text{Nil}(\mathcal{C}) \rightarrow \mathcal{C} : F(M, \nu) = M$  satisfying  $FZ = 1_{\mathcal{C}}$  and hence a split exact sequence  $0 \rightarrow K_0(\mathcal{C}) \xrightarrow{Z} K_0(\text{Nil}(\mathcal{C})) \rightarrow \text{Nil}_0(\mathcal{C}) \rightarrow 0$  which defines  $\text{Nil}_0(\mathcal{C})$  as cokernel of  $Z$ .

If  $\Lambda$  is a ring, and  $\mathbf{H}(\Lambda)$  is the category defined in (ii) above, then we denote  $\text{Nil}_0(\mathcal{P}(\Lambda))$  by  $\text{Nil}_0(\Lambda)$ . If  $S$  is a central multiplicative system in  $\Lambda$ ,  $\mathbf{H}_S(\Lambda)$  the category of  $S$ -torsion objects

of  $\mathbf{H}(\Lambda)$  and  $\mathcal{M}_S(\Lambda)$  the category of finitely generated  $S$ -torsion  $\Lambda$ -modules, one can show that if  $S = T_+ = \{t^i\}$  – a free Abelian monoid on one generator  $t$ , then there exists isomorphisms of categories  $\mathcal{M}_{T_+}(\Lambda[t]) \simeq Nil(\mathcal{M}(\Lambda))$  and  $\mathbf{H}_{T_+}(\Lambda[t]) \simeq Nil(\mathbf{H}(\Lambda))$  and an isomorphism of groups:  $K_0(\mathbf{H}_{T_+}(\Lambda[t])) \simeq K_0(\Lambda) \oplus Nil_0(\Lambda)$ . Hence  $K_0(Nil\mathbf{H}(\Lambda)) \simeq K_0(\Lambda) \oplus Nil_0(\Lambda)$ . See [8] or [100] or [67] or [32] for further information.

(v) The fundamental theorem for  $K_0$  says that:

$$K_0(\Lambda[t, t^{-1}]) \simeq K_0(\Lambda) \oplus K_{-1}(\Lambda) \oplus NK_0(\Lambda) \oplus NK_0(\Lambda)$$

where  $NK_0(\Lambda) := \text{Ker}(K_0(\Lambda[t] \xrightarrow{\tau_+} K_0(\Lambda)))$  where  $\tau_+$  is induced by augmentation  $t = 1$ , and  $K_{-1}$  is the negative  $K$ -functor  $K_{-1}: \mathbf{Rings} \rightarrow \text{Abelian groups}$  defined by H. Bass in [8]. For generalisation of this fundamental theorem to Higher  $K$ -theory, see [67].

### 4.3 $K_0$ and localisation in Abelian categories

We close this section with a discussion leading to a localisation short exact sequence 4.3.2 and then give copious examples to illustrate the use of the sequence.

**4.3.1** A full subcategory  $\mathcal{B}$  of an Abelian category  $\mathcal{A}$  is called a Serre subcategory if whenever  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is an exact sequence in  $\mathcal{A}$ , then  $M \in \mathcal{B}$  if and only if  $M', M'' \in \mathcal{B}$ . We now construct a quotient Abelian category  $\mathcal{A}/\mathcal{B}$  whose objects are just objects of  $\mathcal{A}$ .  $\text{Hom}_{\mathcal{A}/\mathcal{B}}(M, N)$  is defined as follows: If  $M' \subseteq M, N' \subseteq N$  are subobjects such that  $N/N' \in ob(\mathcal{B}), M' \in ob(\mathcal{B})$ , then there exists a natural isomorphism  $\text{Hom}_{\mathcal{B}}(M, N) \rightarrow \text{Hom}_{\mathcal{B}}(M', N/N')$ . As  $M', N'$  range over such pairs of objects, the group  $\text{Hom}_{\mathcal{B}}(M', N/N')$  forms a direct system of Abelian groups and we define  $\mathcal{A}/\mathcal{B}(M, N) = \varinjlim_{(M', N')} \mathcal{B}(M', N/N')$ .

A quotient functor  $T: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{B}$  defined by  $M \rightarrow T(M) = M$  is such that

(i)  $T: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{B}$  is an additive functor.

(ii) If  $\mu \in \text{Hom}_{\mathcal{A}}(M, N)$ , then  $T(\mu)$  is null if and only if  $\text{Ker}(\mu) \in ob(\mathcal{B})$ . Also  $T(\mu)$  is epimorphism if and only if  $\text{coker } \mu \in ob(\mathcal{B})$ . Hence  $T(\mu)$  is an isomorphism if and only if  $\mu$  is a  $\mathcal{B}$ -isomorphism.

**Remarks 4.3.2** Note that  $\mathcal{A}/\mathcal{B}$  satisfies the following universal property: If  $T': \mathcal{A} \rightarrow \mathcal{D}$  is an exact functor such that  $T'(M) \simeq 0$  for all  $M \in \mathcal{B}$ , then there exists a unique exact functor  $U: \mathcal{A}/\mathcal{B} \rightarrow \mathcal{D}$  such that  $T' = U \circ T$ .

**Theorem 4.3.3** [8] or [35]. Let  $\mathcal{B}$  be a Serre subcategory of an Abelian category  $\mathcal{A}$ . Then there exists an exact sequence

$$K_0(\mathcal{B}) \rightarrow K_0(\mathcal{A}) \rightarrow K_0(\mathcal{A}/\mathcal{B}) \rightarrow 0$$

### Examples 4.3.4

- (i) Let  $\Lambda$  be a Noetherian ring,  $S \subset \Lambda$  a central multiplicative subset of  $\Lambda$ ,  $\mathcal{M}_S(\Lambda)$  the category of finitely generated  $S$ -torsion  $\Lambda$ -modules. Then  $\mathcal{M}(\Lambda)/\mathcal{M}_S(\Lambda) \simeq \mathcal{M}(\Lambda_S)$  see [8] or [35] or [100] and so the exact sequence in 4.3.3 becomes

$$K_0(\mathcal{M}_S(\Lambda)) \rightarrow G_0(\Lambda) \rightarrow G_0(\Lambda_S) \rightarrow 0 \quad (I)$$

- (ii) If  $\Lambda$  in (i) is a Dedekind domain  $R$  with quotient field  $F$ , and  $S = R - 0$ , then  $K_0(\mathcal{M}_S(R)) \simeq \bigoplus_{\mathfrak{m}} G_0(R/\mathfrak{m}) = \bigoplus_{\mathfrak{m}} K_0(R/\mathfrak{m})$  where  $\mathfrak{m}$  runs through the maximal ideals of  $R$ . Now, since  $K_0(R/\mathfrak{m}) \simeq \mathbb{Z}$  and  $K_0(R) \simeq \mathbb{Z} \oplus Cl(R)$  the sequence (I) yields the exactness of

$$\bigoplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus Cl(R) \rightarrow \mathbb{Z} \rightarrow 0$$

- (iii) Let  $\Lambda$  be a Noetherian ring,  $S = \{s^i\}$  for some  $s \in \Lambda$ . Then  $K_0(\mathcal{M}_S(R)) \simeq G_0(R/sR)$  (by Devissage) yielding the exact sequence

$$G_0(\Lambda/s\Lambda) \rightarrow G_0(\Lambda) \rightarrow G_0\left(\Lambda \left(\frac{1}{s}\right)\right) \rightarrow 0$$

- (iv) Let  $R$  be the ring of integers in a  $p$ -adic field  $F$ ,  $\Gamma$  a maximal  $R$ -order in a semi-simple  $F$ -algebra  $\Sigma$ ,  $S = R - 0$ , then  $K_0(\mathcal{M}_S(\Gamma)) \simeq G_0(\Gamma/\pi\tau) \simeq K_0(\Gamma/rad \Gamma)$  (see [18] or [42]) where  $\pi R$  is the unique maximal ideal of  $R$ .

- (v) If  $R$  is the ring of integers in a number field  $F$ ,  $\Gamma$  and  $R$ -order in a semi-simple  $F$ -algebra  $\Sigma$ , let  $S = R - 0$ . Then  $K_0(\mathcal{M}_S(\Lambda)) \simeq \bigoplus G_0(\Lambda/p\Lambda)$  (see [18] or [45]) where  $p$  runs through all the prime ideals of  $R$ .

- (vi) Let  $X$  be a Noetherian scheme,  $U$  an open subscheme of  $X$ ,  $Z = X - U$ , let  $\mathcal{A} = \mathcal{M}(X)$  the category of coherent (sheaves of)  $O_X$ -modules,  $\mathcal{B}$  the category of  $O_X$ -modules whose restriction to  $U$  is zero (i.e. the category of coherent modules supported on  $Z$ ). Then  $\mathcal{A}/\mathcal{B}$  is the category of coherent  $O_U$ -modules and so, (I) becomes  $G_0(Z) \rightarrow G_0(X) \rightarrow G_0(U) \rightarrow 0$  (see 4.1.5 (ii) or [67]).

- (vii) Let  $\Lambda$  be a (left) Noetherian ring,  $\Lambda[t]$  the polynomial ring in the variable  $t$ ,  $\Lambda[t, t^{-1}]$  the Laurent polynomial ring. Then  $\Lambda[t, t^{-1}] = \Lambda[t]_S$  where  $S = \{t^i\}$ . Now, the map  $\varepsilon : \Lambda[t] \rightarrow \Lambda$   $t \rightarrow 0$  induces an inclusion  $\mathcal{M}(\Lambda) \subset \mathcal{M}(\Lambda[t])$  and the canonical map  $i : \Lambda[t] \rightarrow \Lambda[t]_S = \Lambda[t, t^{-1}]$   $t \rightarrow t/1$  yields an exact functor  $\mathcal{M}(\Lambda[t]) \rightarrow \mathcal{M}(\Lambda[t, t^{-1}])$ . So from 4.3.3, we have the localisation sequence

$$G_0(\Lambda) \xrightarrow{\varepsilon_*} G_0(\Lambda[t]) \rightarrow G_0(\Lambda[t, t^{-1}]) \rightarrow 0 \quad (II)$$

Now  $\varepsilon_* = 0$  since for any  $\Lambda$ , the exact sequence of  $\Lambda[t]$ -modules  $0 \rightarrow N[t] \xrightarrow{t} N[t] \rightarrow N \rightarrow 0$  yields

$$\varepsilon_*[N] = [N[t]] - [N[t]] = 0$$



So,  $G_0(\Lambda[t]) \simeq G_0(\Lambda[t, t^{-1}])$  from (II) above. This proves the first part of the fundamental theorem for  $G_0$  of rings 4.3.5 below.

**Theorem 4.3.5 Fundamental theorem for  $G_0$  of rings**

If  $\Lambda$  is a left Noetherian ring, then the inclusions  $\Lambda \xrightarrow{i} \Lambda[t] \xrightarrow{j} \Lambda[t, t^{-1}]$  induce isomorphisms

$$G_0(\Lambda) \cong G_0(\Lambda[t]) \cong G_0(\Lambda[t, t^{-1}])$$

**Proof:** See [8] or [100] for the proof of the second part.

**Remarks 4.3.6** (i) The fundamental theorem 4.3.5 above can be generalised to schemes (see [67]). If  $X$  is a scheme, write  $X[s]$  for  $X \times \text{Spec}(\mathbb{Z}[s])$  and  $X[s, s^{-1}]$  for  $X \times \text{Spec}(\mathbb{Z}[s, s^{-1}])$ . When  $X$  is Noetherian, the map  $\varepsilon : X \rightarrow X[s]$  defined by  $s = 0$  induces an inclusion  $\mathcal{M}(X) \subset \mathcal{M}(X[s])$  and hence a transfer map  $\varepsilon_* : G_0(X) \rightarrow G_0(X[s])$ . So we have a localisation exact sequence

$$G_0(X) \xrightarrow{\varepsilon_*} G_0(X[s]) \longrightarrow G_0(X[s, s^{-1}]) \longrightarrow 0$$

We also have a fundamental theorem similar to 4.3.5 as follows

**Theorem 4.3.7 Fundamental theorem for  $G_0$  of schemes**

If  $X$  is a Noetherian scheme, then the flat maps  $X[s, s^{-1}] \xrightarrow{j} X[s] \xrightarrow{i} X$  induce isomorphisms  $G_0(X) \cong G_0(X[[s]]) \cong G_0(X[s, s^{-1}])$ .

**Remarks 4.3.8**

- (i) If we put  $X = \text{Spec}(\Lambda)$  in 4.3.7,  $\Lambda$  is Noetherian ring, we recover 4.3.5.
- (ii) For all  $n \geq 0$ , there are fundamental theorems for  $G_n$  of rings and schemes (see [67] or [85]) and these will be discussed in a forthcoming chapter on Higher  $K$ -theory.
- (iii) There is a generalisation of 4.3.5 due to A. Grothendieck as follows: Let  $R$  be a commutative Noetherian ring,  $\Lambda$  a finite  $R$ -algebra,  $T$  a free Abelian group or monoid with a finite basis. Then  $G_0(\Lambda) \rightarrow G_0(\Lambda[T])$  is an isomorphism, see [8].
- (iv) If  $\Lambda$  is a (left) Noetherian regular ring, so are  $\Lambda[t]$  and  $\Lambda[t, t^{-1}]$ . Since  $K_0(R) \cong G_0(R)$  for any Noetherian regular ring  $R$ , we have from 4.3.5 that  $K_0(\Lambda) \simeq K_0(\Lambda[t]) \simeq K_0(\Lambda[t, t^{-1}])$ . Furthermore, if  $T$  is a free Abelian group or monoid with a finite basis, then  $K_0(\Lambda) \rightarrow K_0(\Lambda[T])$  is an isomorphism (see [8]).

## 5 $K_1$ OF RINGS

### 5.1 Definitions and basic properties

**5.1.1** Let  $R$  be a ring with identity,  $GL_n(R)$  the group of invertible  $n \times n$  matrices over  $R$ . Note that  $GL_n(R) \subset GL_{n+1}(R) : A \rightarrow \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ . Put  $GL(R) = \varinjlim GL_n(R) = \bigcup_{n=1}^{\infty} GL_n(R)$ . Let  $E_n(R)$  be the subgroup of  $GL_n(R)$  generated by elementary matrices  $e_{ij}(a)$  where  $e_{ij}(a)$  is the  $n \times n$  matrix with 1's along the diagonal,  $a$  in the  $(i, j)$ -position and zeros elsewhere. Put  $E(R) = \varinjlim E_n(R)$ .

**Note:** The  $e_{ij}(a)$  satisfy the following.

- (i)  $e_{ij}(a)e_{ij}(b) = e_{ij}(a+b)$  for all  $a, b \in R$
- (ii)  $[e_{ij}(a), e_{jk}(b)] = e_{ik}(ab)$  for all  $i \neq k, a, b \in R$
- (iii)  $[e_{ij}(a), e_{k\ell}(b)] = 1$  for  $j \neq i \neq \ell \neq k$

**Lemma 5.1.2** If  $A \in GL_n(R)$ , then  $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \in E_{2n}(R)$ .

**Proof:** First observe that for any  $C \in M_n(R)$ ,  $\begin{pmatrix} I_n & 0 \\ C & I_n \end{pmatrix}$  and  $\begin{pmatrix} I_n & C \\ 0 & I_n \end{pmatrix}$  are in  $E_{2n}(R)$ , where  $I_n$  is the identity  $n \times n$  matrix. Hence  $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} I_n & A \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & A \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \in E_{2n}(R)$  since  $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \begin{pmatrix} I_n & -I_n \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix} \begin{pmatrix} I_n & -I_n \\ 0 & I_n \end{pmatrix}$

**Theorem 5.1.3** (Whitehead Lemma)

- (i)  $E(R) = [E(R), E(R)]$  i.e.  $E(R)$  is perfect
- (ii)  $E(R) = [GL(R), GL(R)]$

**Proof:** (Sketch) (i) It follows from properties (ii) of elementary matrices that  $[E(R), E(R)] \subset E(R)$ . Also,  $E_n(R)$  is generated by elements of the form  $e_{ij}(a) = [e_{ik}(a), e_{kj}(1)]$  and so  $E(R) \subset [E(R), E(R)]$ . So,  $E(R) = [E(R), E(R)]$ .

(ii) For  $A, B \in GL_n(R)$ ,

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & (AB)^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix} \in E_{2n}(R)$$

Hence  $[GL(R), GL(R)] \subset E(R)$  by 5.1.2. Also, from (i) above,  $E(R) \subset [E(R), E(R)] \subset [GL(R), GL(R)]$ . Hence  $E(R) = [GL(R), GL(R)]$ .

**Definition 5.1.4**

$$\begin{aligned} K_1(R) &:= GL(R)/E(R) = GL(R)/[GL(R), GL(R)] \\ &= H_1(GL(R), \mathbb{Z}) \end{aligned}$$

### Remarks 5.1.5

- (i) For an exact category  $\mathcal{C}$ , the Quillen definition of  $K_n(\mathcal{C})$ ,  $n \geq 0$  coincides with the above definition of  $K_1(R)$  when  $\mathcal{C} = \mathcal{P}(R)$  (see [66] or [67]). We hope to discuss Quillen construction in a forthcoming chapter.
- (ii) The above definition 4.1.4 is functorial i.e. any ring homomorphism  $R \rightarrow R'$  induces an Abelian group homomorphism  $K_1(R) \rightarrow K_1(R')$ .
- (iii)  $K_1(R) = K_1(M_n(R))$  for any positive integer  $n$  and any ring  $R$ .
- (iv)  $K_1(R)$ , as defined above, coincides with  $K_{\det}(\mathcal{P}(R))$  where  $K_{\det}(\mathcal{P}(R))$  is a quotient of the additive group generated by all isomorphism classes  $[P, \mu]$ ,  $P \in \mathcal{P}(R)$ ,  $\mu \in \text{Aut}(P)$  (see [18] or [8]).

**5.1.6** If  $R$  is a commutative, the determinant map  $\det : GL_n(R) \rightarrow R^*$  commutes with  $GL_n(R) \rightarrow GL_{n+1}(R)$  and hence defines a map  $\det : GL(R) \rightarrow R^*$  which is surjective since given  $a \in R^*$ , there exists  $A = \begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix}$  such that  $\det A = a$ . Now  $\det$  induces a map  $\det : GL(R)/[GL(R), GL(R)] \rightarrow R^*$  i.e.  $\det : K_1(R) \rightarrow R^*$ . Moreover,  $\alpha(a) = \begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix}$  for all  $a \in R^*$  defines a map  $\alpha : R^* \rightarrow K_1(R)$  and  $\det \alpha = 1_R$ . Hence  $K_1(R) \simeq R^* \oplus SK_1(R)$  where  $SK_1(R) := \text{Ker}(\det : K_1(R) \rightarrow R^*)$ . Note that  $SK_1(R) = SL(R)/E(R)$  where  $SL(R) = \lim_{n \rightarrow \infty} SL_n(R)$  and  $SL_n(R) = \{A \in GL_n(R) \mid \det A = 1\}$ . Hence  $SK_1(R) = 0$  if and only if  $K_1(R) \simeq R^*$ .

### Examples 5.1.7

- (i) If  $F$  is a field, then  $K_1(F) \simeq F^*$ ,  $K_1(F[x]) \simeq F^*$ .
- (ii) If  $R$  is a Euclidean domain (for example  $\mathbb{Z}$ ,  $\mathbb{Z}[i] = \{a + bi; a, b \in \mathbb{Z}\}$ , polynomial ring  $F[x]$ ,  $F$  a field) then  $SK_1(R) = 0$  i.e.  $K_1(R) \simeq R^*$  (see [62] [78]).
- (iii) If  $R$  is the ring of integers in a number field  $F$ , then  $SK_1(R) = 0$  (see [13] or [78]).
- (iv) If  $R$  is a Noetherian ring of Krull dimension  $\leq 1$  with finite residue fields and all maximal ideals, then  $SK_1(R)$  is torsion [8].

## 5.2 $K_1$ of local rings and skew fields

**Theorem 5.2.1** [18] or [78]. Let  $R$  be a non-commutative local ring. Then there exists a homomorphism  $\det : GL_n(R) \rightarrow R^*/[R^*, R^*]$  for each positive integer  $n$  such that

- (i)  $E_n(R) \subset \text{Ker}(\det)$

(ii)

$$\det \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & 0 \\ & & \ddots & \\ & 0 & & \ddots \\ & & & & \alpha_n \end{pmatrix} = \bar{\alpha}_1 \bar{\alpha}_2 \cdots \bar{\alpha}_n \quad \text{where } \alpha_i \in R^*$$

for all  $i$  and  $\alpha \rightarrow \bar{\alpha}$  is the natural map

$$R^* \rightarrow (R^*)^{ab} = R^*/[R^*, R^*]$$

(iii)

$$\begin{array}{ccc} GL_n(R) & \longrightarrow & GL_{n+1}(R) \\ & \searrow & \swarrow \\ & (R^*)^{ab} & \end{array} \quad \text{commutes}$$

**Note:** The homomorphism ‘det’ above is usually called Dieudonne determinant because it was J. Dieudonne who first introduced the ideas in 4.2.1 for skew fields (see [23]).

**Theorem 5.2.2** [78]. Let  $R$  be a non-commutative local ring. Then the natural map  $GL_1(R) = R^* \hookrightarrow GL(R)$  induces a surjection  $R^*/[R^*, R^*] \rightarrow K_1(R)$  whose kernel is the subgroup generated by the images of all elements  $(1 - xy)/(1 - yx)^{-1} \in R^*$  for all  $x, y$  in the unique maximal ideal  $\mathfrak{m}$  of  $R$ .

**Theorem 5.2.3** [78]. If  $R$  is a skew field then  $K_1(R) \cong R^*/[R^*, R^*]$ .

### 5.3 Menicke symbols

**5.3.1** Let  $R$  be a commutative ring with identity,  $a, b \in R$ . Choose  $c, d \in R$  such that  $ad - bc = 1$  i.e. such that  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(R)$ . Define Menicke symbols  $[a, b] \in SK_1(R)$  as the class of  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SK_1(R)$ . Then

(i)  $[a, b]$  is well defined

(ii)  $[a, b] = [b, a]$  if  $a \in R^*$

(iii)  $[a_1 a_2, b] = [a_1, b][a_2, b]$  if  $a_1 a_2 R + bR = R$

(iv)  $[a, b] = [a + rb, b]$  for all  $r \in R$

We have the following result

**Theorem 5.3.2** [8]. If  $R$  is a commutative ring of Krull dimension  $\leq 1$ , then the Menicke symbols generate  $SK_1(R)$ .

**Remarks** See [8] or [13] for further details on Menicke symbols.

## 5.4 Stability for $K_1$

**5.4.1** Stability results are very useful for reducing computations of  $K_1(R)$  to computations of matrices over  $R$  of manageable size.

Let  $A$  be any ring with identity. An integer  $n$  is said to satisfy stable range condition  $(SR_n)$  for  $GL(A)$  if whenever  $r > n$ , and  $a_1, a_2, \dots, a_r$  is a unimodular row, then there exists  $b_1, b_2, \dots, b_{r-1} \in A$  such that  $(a_1 + a_r b_1, a_2 + a_r b_2, \dots, a_{r-1} + a_r b_{r-1})$  is unimodular. Note that  $(a_1, a_2, \dots, a_r) \in A^r$  unimodular implies that  $(a_1, a_2, \dots, a_r)$  generates a unit ideal i.e.  $\sum Aa_i = A$  (see [8]). For example, any semi-local ring satisfied  $SR_2$  (see [100] or [8]).

**Theorem 5.4.2** [8],[100]. If  $SR_n$  is satisfied, then

- (i)  $GL_m(A)/E_m(A) \rightarrow GL(\infty)/E(\infty)$  is onto for all  $m \geq n$
- (ii)  $E_m(A) \triangleleft GL_r(A)$  if  $m \geq n + 1$
- (iii)  $GL_m(A)/E_m(A)$  is Abelian for  $m \geq 2n$

For further information on  $K_1$ -stability, see [8] or [100] or [105].

## 6 $K_1, SK_1$ OF ORDERS AND GROUP-RINGS; WHITEHEAD TORSION

**6.1** Let  $R$  be the ring of integers in a number field  $F$ ,  $\Lambda$  an  $R$ -order in a semi-simple  $F$ -algebra  $\Sigma$ . First we have the following result (see [8]).

**Theorem 6.2**  $K_1(\Lambda)$  is a finitely generated Abelian group.

**Proof:** The proof relies on the fact that  $GL_n(\Lambda)$  is finitely generated and also that  $GL_n(\Lambda) \rightarrow K_1(\Lambda)$  is surjective (see [8]).

**Remarks 6.3** Let  $R$  be a Dedekind domain with quotient field  $F$ ,  $\Lambda$  an  $R$ -order in a semi-simple  $F$ -algebra  $\Sigma$ . The inclusion  $\Lambda \hookrightarrow \Sigma$  induces a map  $K_1(\Lambda) \rightarrow K_1(\Sigma)$ . Putting  $SK_1(\Lambda) = \text{Ker}(K_1(\Lambda) \rightarrow K_1(\Sigma))$ , it means that understanding  $K_1(\Lambda)$  reduces to understanding  $K_1(\Sigma)$  and  $SK_1(\Lambda)$ . Since  $\Sigma$  is semi-simple,  $\Sigma = \bigoplus \Sigma_i$  where  $\Sigma_i = M_{n_i}(D_i)$ ,  $D_i$  a skew field. So  $K_1(\Sigma) = \bigoplus K_1(D_i)$ .

One way of studying  $K_1(\Lambda)$  and  $SK_1(\Lambda)$ ,  $K_1(\Sigma)$  is via reduced norms. We consider the case where  $R$  is the ring of integers in a number field or  $p$ -adic field  $F$ .

Let  $R$  be the ring of integers in a number field or  $p$ -adic field  $F$ . Then there exists a finite extension  $E$  of  $F$  such that  $E \otimes \Sigma$  is a direct sum of full matrix algebras over  $E$ , i.e.  $E$  is a splitting field of  $\Sigma$ . If  $a \in \Sigma$ , the element  $1 \otimes a \in E \otimes \Sigma$  may be represented by a direct sum of matrices and the reduced norm of  $a$ , written  $nr(a)$  is defined as the product of their determinants. We then have  $nr : GL(\Sigma) \rightarrow C^*$  where  $C = \text{centre of } \Sigma$  (if  $\Sigma = \bigoplus_{i=1}^m \Sigma_i$  and  $C = \bigoplus_{i=1}^m C_i$  we could compute  $nr(a)$  component-wise via  $GL(\Sigma_i) \rightarrow C_i^*$ ). Since  $C^*$  is Abelian we have  $nr : K_1(\Sigma) \rightarrow C^*$ . Composing this with  $K_1(\Lambda) \rightarrow K_1(\Sigma)$  we have a reduced norm map  $nr : K_1(\Lambda) \rightarrow K_1(\Sigma) \rightarrow C^*$ .

From the discussion below, it will be clear that an alternative definition of  $SK_1(\Lambda) = \{x \in K_1(\Lambda) | nr(x) = 1\}$ .

**Theorem 6.4** Let  $R$  be the ring of integers in a number field  $F$ ,  $\Lambda$  an  $R$ -order in a semi-simple  $F$ -algebra  $\Sigma$ . In the notation of 6.3, let  $U_i$  be the group of all non-zero elements  $a \in C_i$  such that  $\beta(a) > 0$  for each embedding  $\beta : C_i \rightarrow \mathbb{R}$  at which  $\mathbb{R} \otimes_{C_i} \Sigma_i$  is not a full matrix algebra over  $\mathbb{R}$ . Then (i) the reduced norm map yields an isomorphism  $nr : K_1(\Sigma) \cong \prod_{i=1}^m U_i$  (ii)  $nr : K_1(\Lambda) \subset \prod_{i=1}^m (U_i \cap R_i^*)$  where  $R_i$  is the ring of integers in  $C_i$ .

**Proof:** See [18].

**Remarks 6.5**

(i) If  $\Gamma$  is a maximal  $R$ -order in  $\Sigma$ , then we have equality in (ii) of 6.4 i.e.  $nr(K_1(\Gamma)) = \prod_{i=1}^m (U_i \cap R_i^*)$ . (See [18].) Hence  $\text{rank } K_1(\Gamma) = \text{rank } \prod_{i=1}^m (U_i \cap R_i^*)$ .

(ii) If  $\Lambda$  is any  $R$ -order in  $\Sigma$ , then  $nr(K_1(\Lambda))$  is of finite index in  $S^*$  (see [18]).

(iii) For all  $n \geq 1$ ,  $K_n(\Lambda)$  is finitely generated and  $SK_n(\Lambda)$  is finite (see [49] or [50]).

**Theorem 6.6** Let  $R$  be the ring of integers in a number field  $F$ ,  $\Lambda$  any  $R$ -order in a semi-simple  $F$ -algebra  $\Sigma$ . Then  $SK_1(\Lambda)$  is a finite group.

**Proof:** See [8]. The proof involves showing that  $SK_1(\Lambda)$  is torsion and observing that  $SK_1(\Lambda)$  is also finitely generated as a subgroup of  $K_1(\Lambda)$  see 6.2.

The next results are local versions of 6.4 and 6.6.

**Theorem 6.7** Let  $R$  be the ring of integers in a  $p$ -adic field  $F$ ,  $\Gamma$  a maximal  $R$ -order in a semi-simple  $F$ -algebra  $\Sigma$ . In the notation of 6.3, we have (i)  $nr : K_1(\Sigma) \simeq C^*$ ; (ii)  $nr : K_1(\Gamma) \cong S^*$  where  $S = \bigoplus R_i$  and  $R_i$  is the ring of integers in  $C_i$ .

**Theorem 6.8**

(i) Let  $F$  be a  $p$ -adic field (i.e. any finite extension of  $\hat{Q}_p$ ),  $R$  the ring of integers of  $F$ ,  $\Lambda$  any  $R$ -order in a semi-simple  $F$ -algebra  $\Sigma$ . Then  $SK_1(\Lambda)$  is finite.

(ii) Let  $R$  be the ring of integers in a  $p$ -adic field  $F$ ,  $\mathfrak{m}$  the maximal ideal of  $R$ ,  $q = |R/\mathfrak{m}|$ . Suppose that  $\Gamma$  is a maximal order in central division algebra over  $F$ . Then  $SK_1(\Gamma)$  is a cyclic group of order  $(q^n - 1)/q - 1$ .  $SK_1(\Gamma) = 0$  iff  $D = F$ .

**Remarks 6.9**

(i) For the proof of 6.8, see [41] and [63].

(ii) It follows from 6.7 that  $\text{rank } K_1(\Gamma) = \text{rank}(S^*)$  for any maximal order  $\Gamma$  in a  $p$ -adic semi-simple  $F$ -algebra.

(iii) If in 6.4 and 6.6  $R = \mathbb{Z}$ ,  $F = \mathbb{Q}$ ,  $G$  a finite group, we have that  $\text{rank of } K_1(\mathbb{Z}G) = s - t$  where  $s$  is the number of real representations of  $G$ , and  $t$  is the corresponding number of rational representations of  $G$ . (See [63].)

(iv) Computations of  $SK_1(\mathbb{Z}G)$  for various groups has attracted extensive attention because of its applicability in topology. For details of such computations, (see [63]).

(v) That for all  $n \geq 1$ ,  $SK_n(\mathbb{Z}G)$ ,  $SK_n(\hat{\mathbb{Z}}_p G)$  are finite groups are proved in [49], [50].

(vi) It also is known that if  $\Gamma$  is a maximal order in a semi-simple  $F$ -algebra  $\Sigma$ , then  $SK_{2n}(\Gamma) = 0$  and  $SK_{2n-1}(\Gamma) = 0$  for all  $n \geq 1$  iff  $\Sigma$  is unramified over its centre, (see [44]). These generalisations will be discussed in a forthcoming chapter on Higher  $K$ -theory.

**6.10 Whitehead Torsion.** J.H.C. Whitehead (see [119]) observed that if  $X$  is a topological space with fundamental group  $G$ , and  $R = \mathbb{Z}G$ , then the elementary row and column transformations of matrices over  $R$  have some natural topological meaning. To enable him to study homotopy between spaces, he introduced the group  $Wh(G) = K_1(\mathbb{Z}G)/\omega(\pm G)$  where  $\omega$  is the map  $G \rightarrow GL_1(\mathbb{Z}G) \rightarrow GL(\mathbb{Z}G) \rightarrow K_1(\mathbb{Z}G)$ , such that if  $f : X \rightarrow Y$  is a homotopy equivalence, then there exists an invariant  $\tau(f)$  in  $Wh(G)$  such that  $\tau(f) = 0$  if and only if  $f$  is a simple homotopy equivalence i.e.  $\tau(f) = 0$  iff  $f$  is induced by elementary deformations transforming  $X$  to  $Y$ . The invariant  $\tau(f)$  is known as Whitehead torsion. (See [60]).

Now, it follows from 6.1 that  $Wh(G)$  is finitely generated when  $G$  is a finite group. Moreover, it is also well known that  $\text{Tor}(K_1(\mathbb{Z}G) = (\pm 1) \times G^{ab} \times SK_1(\mathbb{Z}G)$  where  $SK_1(\mathbb{Z}G) = \text{Ker}(K_1(\mathbb{Z}G) \rightarrow K_1(\mathbb{Q}G))$  see [63]. So  $\text{rank } K_1(\mathbb{Z}G) = \text{rank } Wh(G)$  and it is well known that  $SK_1(\mathbb{Z}G)$  is the full torsion subgroup of  $Wh(G)$  (see [63]). So, computations of  $\text{Tor}(K_1(\mathbb{Z}G))$  reduce essentially to computations of  $SK_1(\mathbb{Z}G)$ . The last two decades have witnessed extensive research on computations of  $SK_1(\mathbb{Z}G)$  for various groups  $G$  (see [63]). More generally, if  $R$  is the ring of integers in a number field or a  $p$ -adic field  $F$ , there has been extensive effort in understanding the groups  $SK_n(RG) = \text{Ker}(K_n(RG) \rightarrow K_n(FG))$  for all  $n \geq 1$ . (See [49], [50], [51]) for all  $n \geq 1$ . More generally still, if  $\Lambda$  is an  $R$ -order in a semi-simple  $F$ -algebra  $\Sigma$  (i.e.  $\Lambda$  is a subring of  $\Sigma$ , finitely generated as an  $R$ -module and  $\Lambda \otimes_R F = \Sigma$ ), there has been extensive effort to compute  $SK_n(\Lambda) = \text{Ker}(K_n(\Lambda) \rightarrow K_n(\Sigma))$ . (See [49], [50], [51]) the results of which apply to  $\Lambda = RG$ . We shall discuss these computations further in the forthcoming chapter on Higher  $K$ -theory.

Note also that Whitehead torsion is useful in the classifications of manifolds (see [63] or [60]).



## 7 SOME $K_1 - K_0$ EXACT SEQUENCES

### 7.1 Mayer-Vietoris sequence

**7.1.1** Let  $\begin{array}{ccc} A & \xrightarrow{f_1} & A_1 \\ \downarrow f_2 & & \downarrow g_1 \\ A_2 & \xrightarrow{g_1} & A' \end{array}$ ,  $(I)$  be a commutative square of ring homomorphisms satisfying

(i)  $A = A_1 \times_A A_2 = \{(a_1, a_2) \in A_1 \times A_2 | g_1(a_1) = g_2(a_2)\}$  i.e. given  $a_1 \in A_1, a_2 \in A_2$  such that  $g_1 a_1 = g_2 a_2$ , then there exists one and only one element  $a \in A$  such that  $f_1(a) = a_1, f_2(a) = a_2$ .

(ii) At least one of the two homomorphisms  $g_1, g_2$  is surjective. The square  $(I)$  is then called a Cartesian square of rings.

**Theorem 7.1.2** Given a Cartesian square of rings as in 7.1.1, then there exists an exact sequence  $K_1(A) \xrightarrow{\alpha_1} K_1(A_1) \oplus K_1(A_2) \xrightarrow{\beta_1} K_1(A') \xrightarrow{\delta} K_0(A) \xrightarrow{\alpha_0} K_0(A_1) \oplus K_0(A_2) \xrightarrow{\beta_0} K_0(A')$ .

**Note:** Call this sequence the Mayer-Vietoris sequence associated to the Cartesian square  $(I)$ . For details of the proof of 7.1.2, see [62].

**Sketch of proof:** The maps  $\alpha_i, \beta_i (i = 0, 1)$  are defined as follows: For  $x \in K_i(A), \alpha_i(x) = (f_{1*}(x), f_{2*}(x))$  and for  $(y, z) \in K_i(A_1) \oplus K_i(A_2) i = 0, 1, \beta_i(y, z) = g_{1*}y - g_{2*}z$ . The boundary map  $\delta : K_1(A') \rightarrow K_0(A)$  is defined as follows: Represent  $x \in K_1(A')$  by a matrix  $\gamma = (a_{ij})$  in  $GL_r(A')$ . This matrix determines an automorphism  $\gamma : A'^n \rightarrow A'^n$ . Let  $\gamma(z_j) = \sum a_{ij} z_j$  where  $\{z_j\}$  is a standard basis for  $A'^n$ . Let  $P(\gamma)$  be the subgroup of  $A_1^n \times A_2^n$  consisting of  $\{(x, y) | \gamma g_1^n(x) = g_2^n(y)\}$  where  $g_1^n : A_1^n \rightarrow A'^n, g_2^n : A_2^n \rightarrow A'^n$  are induced by  $g_1, g_2$  respectively. We need the following

#### Lemma 7.1.3

- (i) If there exists  $(b_{ij}) \in GL_n(A_2)$  which maps to  $\gamma = (a_{ij})$ , then  $P(\gamma) \simeq A^n$ .
- (ii) If  $g_2$  is surjective, then  $P(\gamma)$  is a finitely generated projective  $A$ -module.

For the proof of 7.1.3 see [63]. *Conclusion of definition of  $\delta$ :* Now define

$$\delta[\gamma] = [P(\gamma)] - [A^n] \in K_0(A)$$

and verify exactness of the sequence 7.1.2 as an exercise.

**Corollary 7.1.4** If  $A$  is a ring and  $\mathfrak{a}_1, \mathfrak{a}_2$  ideals of  $A$  such that  $\mathfrak{a}_1 \cap \mathfrak{a}_2 = 0$ , then there exists an exact sequence

$$\begin{aligned} K_1(A) &\rightarrow K_1(A/\mathfrak{a}_1) \oplus K_1(A/\mathfrak{a}_2) \rightarrow K_1(A/(\mathfrak{a}_1 + \mathfrak{a}_2)) \\ \xrightarrow{\delta} K_0(A) &\rightarrow K_0(A/\mathfrak{a}_1) \oplus K_0(A/\mathfrak{a}_2) \rightarrow K_0(A/(\mathfrak{a}_1 + \mathfrak{a}_2)) \end{aligned}$$

**Proof:** Follows by applying 7.1.2 to the Cartesian square:

$$\begin{array}{ccc}
A & \xrightarrow{f_1} & A/\mathfrak{a}_1 \\
f_2 \downarrow & & \downarrow g_1 \\
A/\mathfrak{a}_2 & \xrightarrow{g_2} & A/(\mathfrak{a}_1 + \mathfrak{a}_2)
\end{array} \tag{II}$$

**Example 7.1.5** Let  $G$  be a finite group of order  $n$ ,  $A = ZG$ . Let  $\mathfrak{a}_1$  be the principal ideal of  $A$  generated by  $b = \sum_{g \in G} g$ ,  $\mathfrak{a}_2$  the augmentation ideal  $= \{\sum r_g g \mid \sum r_g = 0\}$ . Then  $\mathfrak{a}_1 \cap \mathfrak{a}_2 = 0$ . So,  $A_2 = A/\mathfrak{a}_2 \simeq \mathbb{Z}$ ,  $A' = A/(\mathfrak{a}_1 + \mathfrak{a}_2) \simeq \mathbb{Z}/n\mathbb{Z}$  from the Cartesian squares (I) and (II) above.

Now suppose that  $|G| = p$ , a prime. Let  $G = \langle x \rangle$ . Put  $t = f_1(x)$ . Then,  $A_1$  has the form  $\mathbb{Z}[t]$  with a single relation  $\sum_{i=0}^{p-1} t^i = 0$ . So,  $A_1$  may be identified with  $Z[\xi]$  where  $\xi$  is the primitive  $p^{\text{th}}$  root of unity.

We now have the following:

**Theorem 7.1.6** If  $|G| = p$ , then  $F_1 : K_0(ZG) \cong K_0(Z[\xi])$  is an isomorphism. Hence  $K_0(ZG) \simeq Z \oplus C\ell(\mathbb{Z}[\xi])$ .

**Proof:** From 7.1.2, we have an exact sequence

$$K_1(Z[\xi]) \oplus K_1(Z) \rightarrow K_1(Z/pZ) \xrightarrow{\delta} K_0(\mathbb{Z}G) \rightarrow K_0(Z[\xi]) \oplus K_0(Z) \rightarrow K_0(Z/pZ)$$

Now since  $g_{2*} : K_0(\mathbb{Z}) \simeq K_0(Z/pZ)$  is an isomorphism, the result will follow once we show that  $\delta = 0$ . To show that  $\delta = 0$ , it suffices to show that  $K_1(Z[\xi]) \rightarrow K_1(Z/pZ)$  is onto. Let  $r$  be a positive integer prime to  $p$ . Put  $u = 1 + \xi + \cdots + \xi^{r-1} \in Z[\xi]$ . Let  $\xi^r = \eta$ ,  $\eta^s = \xi$ , for some  $s > 0$ . Then  $\nu = 1 + \eta + \cdots + \eta^{s-1} \in Z[\xi]$ . In  $Q(\xi)$ , we have

$$\nu = (\nu^s - 1)/(\eta - 1) = (\xi - 1)/(\xi^r - 1) = 1/u$$

So,  $u \in (Z[\xi])^*$ , i.e. given  $r \in (Z/pZ)^* \simeq K_1(Z/pZ)$ , there exists  $u \in (Z[\xi])^*$  such that  $g_{1*}(u) = r$ . That  $K_0(\mathbb{Z}G) \simeq Z \oplus C\ell(Z[\xi])$  follows from 2.1.7.

**Remarks 7.1.7** (i) The Mayer-Vietoris sequence 7.1.2 can be extended to the right to negative  $K$ -groups defined by H. Bass in [8]. More precisely, there exists functors  $K_{-n}$   $n \geq 1$  from rings to Abelian groups such that the sequence

$$\cdots \rightarrow K_0(A') \rightarrow K_{-1}(A) \rightarrow K_{-1}(A_1) \oplus K_{-1}(A_2) \rightarrow K_{-1}(A') \rightarrow \cdots$$

is exact.

(ii) The Mayer-Vietoris sequence 7.1.2 can be extended beyond  $K_2$  under special circumstances that will be discussed in the forthcoming chapter on Higher  $K$ -theory,

## 7.2 Exact sequence associated to an ideal of a ring

**7.2.1** Let  $A$  be a ring,  $\mathfrak{a}$  any ideal of  $A$ . The canonical map  $f : A \rightarrow A/\mathfrak{a}$  induces  $f_* : K_i(A) \rightarrow K_i(A/\mathfrak{a})$   $i = 0, 1$ . We write  $\bar{A}$  for  $A/\mathfrak{a}$  and for  $M \in \mathcal{P}(A)$  we put  $\bar{M} = M/\mathfrak{a}M \simeq \bar{A} \otimes_A M$ . Let  $K_0(A, \mathfrak{a})$  be the Abelian group generated by expressions of the form  $[M, f, N]$ ,  $M, N \in \mathcal{P}(A)$  where  $f : \bar{A} \otimes_A M \simeq \bar{A} \otimes_A N$  with relations defined as follows:

For  $L, M, N \in \mathcal{P}(A)$  and  $\bar{A}$ -isomorphisms  $f : \bar{L} \simeq \bar{M}$ ,  $g : \bar{M} \simeq \bar{N}$ , we have

$$[L, gf, N] = [L, f, M] + [M, g, N]$$

(ii) Given exact sequences

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0; \quad 0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$$

where  $M_i, N_i \in \mathcal{P}(A)$ , and given  $\bar{A}$ -isomorphisms  $f_i : \bar{M}_i \simeq \bar{N}_i$  ( $i = 1, 2, 3$ ) which commute with the maps associated with the given sequences, we have  $[M_2, f_2, N_2] = [M_1, f_1, N_1] + [M_3, f_3, N_3]$ .

**Theorem 7.2.2** There exists an exact sequence

$$K_1(A) \rightarrow K_1(\bar{A}) \xrightarrow{\delta} K_0(A, \mathfrak{a}) \xrightarrow{\eta} K_0(A) \rightarrow K_0(\bar{A})$$

### Remarks 7.2.3

- (i) We shall not prove the above result in detail but indicate how the maps  $\delta, \eta$  are defined leaving the rest as an exercise. It is clear how the maps  $K_i(A) \rightarrow K_i(\bar{A})$   $i = 0, 1$  are defined. The map  $\delta$  assigns to each  $f \in GL_n(\bar{A})$  the triple  $[A^n, f, A^n] \in K_0(A, \mathfrak{a})$  while the map  $\eta$  takes  $[M, f, N]$  onto  $[M] - [N]$  for  $M, N \in \mathcal{P}(A)$  such that  $f : \bar{M} \simeq \bar{N}$ .
- (ii) The exact sequence 7.2.2 could be extended to  $K_2$  and beyond with appropriate definitions of  $K_i(A, \mathfrak{a})$   $i \geq 1$ . We shall discuss this in the context of Higher  $K$ -theory in a forthcoming chapter, see [67].

## 7.3 Localisation sequences

**7.3.1** Let  $S$  be a central multiplicative system in a ring  $A$ ,  $\mathbf{H}_S(A)$  the category of finitely generated  $S$ -torsion  $A$ -modules of finite projective dimension. Note that an  $A$ -module  $M$  is  $S$ -torsion if there exists  $s \in S$  such that  $sM = 0$ , and that an  $A$ -module has finite projective dimension if there exists a finite  $\mathbf{P}(A)$ -resolution i.e. there exists an exact sequence.  $(I)0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$  where  $P_i \in \mathbf{P}(A)$ . Then we have the following theorem.

**Theorem 7.3.2** With notation as in 7.3.1, there exist natural homomorphisms  $\delta, \varepsilon$  such that the following sequence is exact:

$$K_1(A) \rightarrow K_1(A_S) \xrightarrow{\delta} K_0(\mathbf{H}_S(A)) \xrightarrow{\varepsilon} K_0(A) \rightarrow K_0(A_S)$$

where  $A_S$  is the ring of fractions of  $A$  with respect to  $S$ .

**Proof:** We shall not prove exactness in detail but indicate how the maps  $\delta$  and  $\varepsilon$  are defined leaving details of proof of exactness at each point as an exercise.

Let  $M \in \mathbf{H}_S(A)$  have a finite  $\mathbf{P}(A)$ -resolution as in 7.3.1 above. Define  $\varepsilon([M]) = \Sigma(-1)^i [P_i] \in K_0(A)$ . We define  $\delta$  as follows: If  $\alpha \in GL_m(A_S)$ , let  $s \in S$  be a common denominator for all entries of  $\alpha$  such that  $\beta = s\alpha$  has entries in  $A$ . We claim that  $A^n/\beta A^n \in \mathbf{H}_S(A)$  and  $A^n/sA^n \in \mathbf{H}_S(A)$ . That they have finite  $\mathbf{P}(A)$ -resolutions follow from the exact sequences

$$0 \rightarrow A^n \xrightarrow{\beta} A^n \rightarrow A^n/\beta A^n \rightarrow 0 \quad \text{and}$$

$$0 \rightarrow A^n \xrightarrow{s} A^n \rightarrow A^n/sA^n \rightarrow 0$$

To see that  $A/\beta A^n$  is  $S$ -torsion, let  $t \in S$  be such that  $\alpha^{-1}t = \gamma$  has entries in  $A$ . Then  $\gamma A^n \subset A^n$  implies that  $tA^n \subset \alpha A^n$  and hence that  $stA^n \subset s\alpha A^n = \beta A^n$ . Then  $st \in S$  annihilates  $A^n/\beta A^n$ .

We now define

$$\begin{aligned} \delta[\alpha] &= [A^n/\beta A^n] - [A^n/sA^n] \\ \text{So } \varepsilon\delta[\alpha] &= \varepsilon[A^n/\beta A^n] - \varepsilon[A^n/sA^n] \\ &= ([A^n] - [A^n]) - ([A^n] - [A^n]) = 0 \end{aligned}$$

### Remarks 7.3.3

(i) Putting  $A = \Lambda[t]$  and  $S = \{t^i\}_{i \geq 0}$  in 6.3.2, we obtain an exact sequence

$$K_1(\Lambda[t]) \rightarrow K_1(\Lambda[t, t^{-1}]) \xrightarrow{\partial} K_0(H_{\{t^i\}}(A[t])) \rightarrow K_0(\Lambda[t]) \rightarrow K_0(\Lambda[t, t^{-1}])$$

which is an important ingredient in the proof of the following result called the fundamental theorem for  $K_1$  (see [8]).

(ii) **Fundamental theorem for  $K_1$**

$$K_1(\Lambda[t, t^{-1}]) \simeq K_1(\Lambda) \oplus K_0(\Lambda) \oplus NK_1(\Lambda) \oplus NK_1(\Lambda)$$

where  $NK_1(\Lambda) = \text{Ker}(K_1(\Lambda[t]) \xrightarrow{\tau} K_1(\Lambda))$  and  $\tau$  is induced by the augmentation  $\Lambda[t] \rightarrow \Lambda : (t = 1)$ .

(iii) In the forthcoming chapter on Higher  $K$ -theory, we shall discuss the extension of the localisation sequence 7.3.2 to the left for all  $n \geq 1$  as well as some further generalisations of the sequence.

## 8 THE FUNCTOR $K_2$ - BRIEF REVIEW

In this section we provide a brief review of the functor  $K_2$  due to J. Milnor, see [62]. A more comprehensive treatment of this topic is envisaged for a chapter to be written by F. Keune.

### 8.1 $K_2$ of a ring – Definitions and basic properties

**8.1.1** Let  $A$  be a ring. The Steinberg group of order  $n(n \geq 3)$  over  $A$ , denoted  $St_n(A)$  is the group generated by  $x_{ij}(a), i \neq j, 1 \leq i, j \leq n, a \in A$ , with relations

- (i)  $x_{ij}(a)x_{ij}(b) = x_{ij}(a + b)$
- (ii)  $[x_{ij}(a), x_{k\ell}(b)] = 1 \quad j \neq k, i \neq \ell$
- (iii)  $[x_{ij}(a), x_{jk}(b)] = x_{ik}(ab) \quad i, j, k \text{ distinct}$
- (iv)  $[x_{ij}(a), x_{ki}(b)] = x_{kj}(-ba) \quad j \neq k.$

Since generators  $e_{ij}(a)$  of  $E_n(A)$  satisfy relations (i)-(iv), above, we have a unique surjective homomorphism  $\varphi_n : St_n(A) \rightarrow E_n(A)$  given by  $\varphi_n(x_{ij}(a)) = e_{ij}(a)$ . Moreover, the relations for  $St_{n+1}(A)$  include those of  $St_n(A)$  and so, there are maps  $St_n(A) \rightarrow St_{n+1}(A)$ . Let  $St(A) = \varinjlim_n St_n(A)$ ,  $E(A) = \varinjlim_n E_n(A)$ , then we have a canonical map  $\varphi : St(A) \rightarrow E(A)$ .

**Definition 8.1.2** Define  $K_2^M(A)$  as the kernel of the map  $\varphi : St(A) \rightarrow E(A)$ .

**Theorem 8.1.3**  $K_2^M(A)$  is Abelian and is the centre of  $St(A)$ . So  $St(A)$  is a central extension of  $E(A)$ .

**Proof:** See [62].

**Definition 8.1.4** An exact sequence of groups of the form  $1 \rightarrow A \rightarrow E \xrightarrow{\varphi} G \rightarrow 1$  is called a central extension of  $G$  by  $A$  if  $A$  is central in  $E$ . Write this extension as  $(E, \varphi)$ . A central extension  $(E, \varphi)$  of  $G$  by  $A$  is said to be universal if for any other central extension  $(E', \varphi')$  of  $G$ , there is a unique morphism  $(E, \varphi) \rightarrow (E', \varphi')$ .

**Theorem 8.1.5**  $St(A)$  is the universal central extension of  $E(A)$ . Hence there exists a natural isomorphism  $K_2^M(A) \simeq H_2(E(A), \mathbb{Z})$ .

**Proof:** The last statement follows from the fact that for a perfect group  $G$  (in this case  $E(A)$ ), the kernel of the universal central extension  $(E, \varphi)$  (in this case  $(St(A), \varphi)$  is naturally isomorphic to  $H_2(G, \mathbb{Z})$  (in this case  $H_2(E(A), \mathbb{Z})$ ).

For the proof of the first part see [62].

**Definition 8.1.6** Let  $A$  be a commutative ring,  $u \in A^*$

$$w_{ij}(u) := x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)$$

Define  $h_{ij}(u) := w_{ij}(u)w_{ij}(-1)$ . For  $u, v \in A^*$ , one can easily check that  $\varphi([h_{12}(u), h_{13}(v)]) = 1$ .

So  $[h_{12}(u), h_{13}(v)] \in K_2(A)$ . It can be shown that  $h_{12}(u), h_{13}(v)$  is independent of the indices 1,2,3. We write  $\{u, v\}$  for  $h_{12}(u), h_{13}(v)$  and call this the Steinberg symbol.

**Theorem 8.1.7** Let  $A$  be a commutative ring. The Steinberg symbol  $\{, \} : A^* \times A^* \rightarrow K_2(A)$  is skew symmetric and bilinear, i.e.

$$\{u, v\} = \{v, u\}^{-1} \quad \text{and} \quad \{u_1 u_2, v\} = \{u_1, v\} \{u_2, v\}$$

**Proof:** See [62].

**Theorem 8.1.8** Let  $A$  be a field, division ring, local ring or a semi-local ring. Then  $K_2^M(A)$  is generated by symbols.

**Proof:** See [20] or [108] or [26].

**Theorem 8.1.9** (Matsumoto)

If  $F$  is a field, then  $K_2^M(F)$  is generated by  $\{u, v\}, u, v \in F^*$  with relations

- (i)  $\{uu', v\} = \{u, v\} \{u', v\}$
- (ii)  $\{u, vv'\} = \{u, v\} \{u, v'\}$
- (iii)  $\{u, 1 - u\} = 1$

i.e.  $K_2^M(F)$  is the quotient of  $F^* \otimes_{\mathbb{Z}} F^*$  by the subgroup generated by the elements  $x \otimes (1 - x), x \in F^*$ .

**Examples 8.1.10**

- (i)  $K_2(\mathbb{Z})$  is cyclic of order 2. See [62].
- (ii)  $K_2(\mathbb{Z}(i)) = 1$ , so is  $K_2(\mathbb{Z}\sqrt{-7})$  see [62].
- (iii)  $K_2(\mathbb{F}_q) = 1$  if  $F_q$  is a finite field with  $q$  elements. See [62].
- (iv) If  $F$  is a field  $K_2(F[t]) \simeq K_2(F)$  see [62].

More generally  $K_2(R[t]) \simeq K_2(R)$  if  $R$  is a commutative regular ring.

**Remarks 8.1.11**

- (i) There is a definition by J. Milnor of Higher  $K$ -theory of fields  $K_n^M(F)$   $n \geq 1$  which coincides with  $K_2(F)$  above for  $n = 2$ . More precisely,

$$K_n^M(F) := \underbrace{F^* \otimes F^* \otimes \cdots \otimes F^*}_{n \text{ times}} / \left\{ a_1 \otimes \cdots \otimes a_n \mid \begin{array}{l} a_i + a_j = 1 \text{ for some } i \neq j \\ a_i \in F^* \end{array} \right\} \text{ i.e. } K_n^M(F)$$

is the quotient of  $F^* \otimes F^* \otimes \cdots \otimes F^*$  ( $n$  times) by the subgroup generated by all  $a_1 \otimes a_2 \otimes \cdots \otimes a_n$   $a_i \in F$  such that  $a_i + a_j = 1$  for some  $i \neq j$ . Note that  $K_*^M(F) = \bigoplus_{n \geq 0} K_n^M(F)$  is a ring.

- (ii) The Higher  $K$ -groups defined by D. Quillen [66], [67], namely  $K_n(\mathcal{C})$ ,  $\mathcal{C}$  an exact category  $n \geq 0$  and  $K_n(A) = \pi_n(BGL(A)^+)$   $n \geq 1$  coincides with  $K_2^M(A)$  above when  $n = 2$  and  $\mathcal{C} = \mathcal{P}(A)$ .

## 8.2 Connections with Brauer group of fields and Galois cohomology

**8.2.1** Let  $F$  be field and  $B_r(F)$  the Brauer group of  $F$ , i.e. the group of stable isomorphism classes of central simple  $F$ -algebras with multiplication given by tensor product of algebras. See [57].

A central simple  $F$ -algebra  $A$  is said to be split by an extension  $E$  of  $F$  if  $E \otimes A$  is  $E$ -isomorphic to  $M_r(E)$  for some positive integer  $r$ . It is well known (see [57]) that such  $E$  could be taken as some finite Galois extension of  $F$ . Let  $B_r(F, E)$  be the group of stable isomorphism classes of  $E$ -split central simple algebras. Then  $B_r(F) := B_r(F, F_s)$  where  $F_s$  is the separable closure of  $F$ .

**Theorem 8.2.2** [57] Let  $E$  be a Galois extension of a field  $F$ ,  $G = Gal(E/F)$ . Then there exists an isomorphism  $H^2(G, E^*) \cong B_r(F, E)$ . In particular  $B_r(F) \cong H^2(G, F_s^*)$  where  $G = Gal(F_s/F) = \varprojlim Gal(E_i/F)$ , where  $E_i$  runs through finite Galois extensions of  $F$ .

**8.2.3** Now, for any  $m > 0$ , let  $\mu_m$  be the group of  $m^{th}$  roots of 1,  $G = Gal(F_s/F)$ , we have the Kummer sequence of  $G$ -modules

$$0 \rightarrow \mu_m \rightarrow F_s^* \rightarrow F_s^* \rightarrow 0$$

from which we obtain an exact sequence of Galois cohomology groups.

$$F^* \xrightarrow{m} F^* \rightarrow H^1(F, \mu_m) \rightarrow H^1(F, F_s^*) \rightarrow \dots$$

where  $H^1(F, F_s^*) = 0$  by Hilbert theorem 90. So we obtain isomorphism  $\chi_m : F^*/mF^* \cong F^* \otimes \mathbb{Z}/m \rightarrow H^1(F, \mu_m)$ .

Now, the composite

$$F^* \otimes_{\mathbb{Z}} F^* \rightarrow (F^* \otimes_{\mathbb{Z}} F^*) \otimes \mathbb{Z}/m \rightarrow H^1(F, \mu_m) \otimes H^1(F, \mu_m) \rightarrow H^2(F, \mu_m^{\otimes 2})$$

is given by  $a \otimes b \rightarrow \chi_m(a) \cup \chi_m(b)$  (where  $\cup$  is cup product) which can be shown to be a Steinberg symbol inducing a homomorphism  $g_{2,m} : K_2(F) \otimes \mathbb{Z}/m\mathbb{Z} \rightarrow H^2(F, \mu_m^{\otimes 2})$ .

We then have the following result due to A.S. Merkurjev and A.A. Suslin, see [58].

**Theorem 8.2.4** [58]. Let  $F$  be a field,  $m$  an integer  $> 0$  such that the characteristic of  $F$  is prime to  $m$ . Then the map

$$g_{2,m} : K_2(F)/m K_2(F) \rightarrow H^2(F, \mu_m^{\otimes 2})$$

is an isomorphism where  $H^2(F, \mu_m^{\otimes 2})$  can be identified with the  $m$ -torsion subgroup of  $Br(F)$ .

**Remarks 8.2.5** By generalising the process outlined in 8.2.3 above, we obtain a map

$$g_{n,m} : K_n^M(F)/m K_n^M(F) \rightarrow H^n(F, \mu_m^{\otimes 2}) \quad (I)$$

It is a conjecture of Bloch-Kato that  $g_{n,m}$  is an isomorphism for all  $F, m, n$ . So, 8.2.4 is the  $g_{2,m}$  case of the Bloch-Kato conjecture when  $m$  is prime to the characteristic of  $F$ . Furthermore, A. Merkurjev proved in [57], that 8.2.4 holds without any restriction on  $F$  with respect to  $m$ .

It is also a conjecture of Milnor that  $g_{n,2}$  is an isomorphism. In 1996, V. Voevodsky proved that  $g_{n,2^r}$  is an isomorphism for any  $r$ . See [110].

### 8.3 Some applications in algebraic topology and algebraic geometry

**8.3.1  $K_2$  and pseudo-isotopy** Let  $R = \mathbb{Z}G, G$  a group. For  $u \in R^*$ , put  $w_{ij}(u) := x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)$ . Let  $W_G$  be the subgroup of  $St(R)$  generated by all  $w_{ij}(g), g \in G$ . Define  $Wh_2(G) = K_2(R)/(K_2(R) \cap W_G)$ .

Now let  $M$  be a smooth  $n$ -dimensional compact connected manifold without boundary. Two diffeomorphisms  $h_0, h_1$  of  $M$  are said to be isotopic if they lie in the same path component of the diffeomorphism group.  $h_0, h_1$  are said to be pseudo isotopic if there is a diffeomorphism of the cylinder  $M \times [0, 1]$  restricted to  $h_0$  on  $M \times (0)$  and to  $h_1$  on  $M \times (1)$ . Let  $P(M)$  be the pseudo-isotopy space of  $M$  i.e. the group of diffeomorphism  $h$  of  $M \times [0, 1]$  restricting to the identity on  $M \times (0)$ . Computation of  $\pi_0(P(M^n))$  helps to understand the differences between isotopies and we have the following result due to Hatcher and Wagoner.

**Theorem [33].** Let  $M$  be an  $n$ -dimensional ( $n \geq 5$ ) smooth compact manifold with boundary. Then there exists a surjective map

$$\pi_0(P(M)) \rightarrow Wh_2(\pi_1(X))$$

where  $\pi_1(X)$  is the fundamental group of  $X$ .

#### 8.3.2 Bloch's formula for Chow groups

Let  $X$  be a regular scheme of finite type over a field  $F$ ,  $CH^r(X)$  the Chow group of codimension  $r$  cycles on  $X$  modulo rational equivalence (see [33]). The functors  $K_n, n \geq 0$ , are contravariant functors from the category of schemes to the category of graded commutative rings, see [67]. Now we can sheafify the presheaf  $U \rightarrow K_r(U)$  for  $r \geq 0$  to obtain a sheaf  $\mathcal{K}_{r,X}$ . The stalk of  $\mathcal{K}_{r,X}$  at  $x \in X$  can be shown to be  $K_r(\mathcal{O}_{X,x})$ . The following result, known as Bloch's formula, provides a  $K_2$ -theoretic formula for  $CH^2(X)$ .

**Theorem** Let  $X$  be a regular scheme of finite type over a field  $F$ . Then there is a natural isomorphism

$$H^2(X, \mathcal{K}_{r,X}) \simeq CH^2(X)$$



**Remark D.** Quillen proved a generalisation of the above result i.e.  $H^2(X, \mathcal{K}_{r,X}) \simeq CH^r(X)$  for all  $r > 1$  in [67].

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