SPECTRAL SHIFT FUNCTION IN THE LARGE COUPLING CONSTANT LIMIT

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ABSTRACT. Given two selfadjoint operators H_0 and $V = V_+ - V_-$, we study the motion of the spectrum of the operator $H(\alpha) = H_0 + \alpha V$ as α increases. Let λ be a real number. We consider the quantity $\xi(\lambda, H(\alpha), H_0)$ defined as a generalization of Krein's spectral shift function of the pair $H(\alpha)$, H_0. We study the asymptotic behavior of $\xi(\lambda, H(\alpha), H_0)$ as $\alpha \to \infty$. Applications to differential operators are given.

0. INTRODUCTION

Let $H_0 = H_0^*$ be a selfadjoint operator and let $V = V^*$ be of the form

$$V = V_+ - V_-. \tag{1}$$

We put

$$H(\alpha) = H_0 + \alpha V, \quad \alpha > 0$$

and denote by $\xi(\lambda, \alpha) = \xi(\lambda; H(\alpha), H_0)$ the "generalized spectral shift function" of the pair $H(\alpha)$, H_0 which coincides with the Kreins spectral shift function if V is of trace class. For trace class operators V the function $\xi(\cdot, \alpha) \in L_1$ can be defined by the relation

$$\mathrm{Tr}[\phi(H(\alpha)) - \phi(H_0)] = \int_{-\infty}^{+\infty} \xi(\lambda, \alpha)\phi'(\lambda)d\lambda,$$

called the trace formula. (Here ϕ is an arbitrary function on $\mathbb{R}$ of a suitable class.) The main object of our study is the leading term (in the power expansion) of the asymptotics of $\xi(\lambda, \alpha)$ as $\alpha \to \infty$. We find conditions on H_0, V ensuring the stability of such an asymptotics under variations of $\lambda \in \mathbb{R}$. A similar theorem for negative $V = -V_-$ was obtained in [7]. We treat the more difficult case of the perturbations of the form (1). It should be mentioned also that the idea of the stability theorem is presented in [1], where only the discrete spectrum of $H(\alpha)$ was studied.

In §1 we formulate the problem and describe the main result in detail. It should be noted that if $\lambda \in \rho(H_0)$ is a regular point for H_0, then $\xi(\lambda, \alpha)$ coincides with the difference between the number of eigenvalues of $H(\alpha)$ having crossed λ in each of the two directions. This allows us to study the asymptotics of $\xi(\lambda,\alpha)$ for $\lambda \in \rho(H_0)$ with the help of a special perturbation theory developed in [8], [9]. The stability theorem obtained in this paper reduces the case of an arbitrary λ to that of $\lambda \in \rho(H_0)$. In applications to differential operators, this offers a possibility of using known results.

If $V < 0$,then the function $\xi(\lambda,\alpha)$ is monotone decreasing in α (for a fixed $\lambda = \overline{\lambda}$) and coincides with a certain integral of the distribution function of a compact selfadjoint operator. A suitable version of the representation formula can be found in [6], [7]. In [7], the said above forms a basis of the proof of a stability theorem for $V < 0$.

The study of the perturbations (1) requires, fist of all, a new version of representation formula for $\xi(\lambda,\alpha)$. Such a formula is obtained in [4]. Moreover, the technique used in [8], [9] can be also simplified with the help of this representation.

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1. Main result (Theorem 1.1)

1. Notations Throughout the paper formulae and statements with double indices ($\pm$ or $\mp$) are understood independently, as pairs of formulae. Below $\mathcal{H}$ is a separable Hilbert space. By $\rho(T)$, $\sigma(T)$ we denote respectively the resolvent set, the spectrum of a linear operator T. For a selfadjoint operator T let $E_T(\delta)$ be the spectral measure of a Borel set $\delta \subset \mathbb{R}$ and

$$2T_\pm := |T| \pm T.$$

By $\mathbf{S}_\infty$ we denote the space of compact operators. For $T = T^* \in \mathbf{S}_\infty$ and $s > 0$ let $n_\pm(s,T) = \mathrm{rank} E_{T_\pm}(s,+\infty)$, and for $T \in \mathbf{S}_\infty$ and $s > 0$ let $n(s,T) = n_+(s^2; T^*T)$. Recall (see, e.g., [2]) that for a pair of compact operators the following two inequalities hold:

$$n(s_1+s_2, T_1+T_2) \le n(s_1,T_1) + n(s_2,T_2), \quad s_1,s_2 > 0,$$

and (the Horn inequality)

$$n(s_1s_2, T_1T_2) \le n(s_1,T_1) + n(s_2,T_2), \quad s_1,s_2 > 0. \tag{1.1}$$

These inequalities are applicable not only to compact operators but at least to all bounded normal operators for which the right hand side is finite. Moreover one can write the estimates which are equivalent to the Weyl inequalities for selfadjoint operators T_1, T_2:

$$n_\pm(s_1+s_2, T_1+T_2) \le n_\pm(s_1,T_1) + n_\pm(s_2,T_2), \quad s_1,s_2 > 0.$$

For $0 < p < \infty$ a class $\mathbf{S_p}$ is defined as the set of all compact operators T such that the following functional is finite:

$$\|T\|_{\mathbf{S_p}}^p := p \int_0^\infty s^{p-1} n(s,T) ds < \infty.$$

Functional $\|\cdot\|_{\mathbf{S_p}}$ is a norm for $p \geq 1$ and a quasinorm for $p < 1$. For $0 < p < \infty$ a class $\Sigma_{\mathbf{p}}$ is defined as the set of all compact operators T such that the following functional (which is a quasinorm) is finite:

$$\|T\|_{\Sigma_{\mathbf{p}}}^p := \sup_{s>0} s^p n(s,T) < \infty.$$

2. Let $R = R^*$ be a bounded operator and let the spectrum of R in the interval $\delta = (a,b)$ is disctrete or empty. Then for every selfadjoint compact operator K and $\lambda \in \delta$ we introduce the number

$$\eta(\lambda; R+K, R) := \operatorname{ind}(E_{R+K}(-\infty,\lambda), E_R(-\infty,\lambda)),$$

where

$$\operatorname{ind}(P,Q) = \dim \operatorname{Ker}(P-Q-I) - \dim \operatorname{Ker}(P-Q+I).$$

Some properties of the function η should be mentioned here. For example we have the "addition property":

$$\eta(\lambda; R+K_1+K_2, R) =$$
$$= \eta(\lambda; R+K_1+K_2, R+K_1) + \eta(\lambda; R+K_1, R),$$

which holds for $K_j = K_j^* \in \mathbf{S}_\infty$, $j = 1,2$. And for signdefinite perturbations K the following Birman-Schwinger principle holds true.

Proposition 1.1. *Let $\pm K \geq 0$ and $\lambda \in \rho(R)$. Then*

$$\eta(\lambda; R+K, R) = \mp \lim_{s\to 1\mp 0} n_\mp(s,T), \tag{1.2}$$

where

$$T = \sqrt{|K|}(R-\lambda I)^{-1}\sqrt{|K|}.$$

Proof. In fact Proposition 1.1 is a slight modification of Corollary 4.8 of [4]. In particular it was shown that under the conditions $\pm K \geq 0$, $K \in \mathbf{S_1}$, $\lambda \in \rho(R+K)$ the value of $\mp\eta(\lambda; R+K, R)$ coincides with the number of eigenvalues of $R+tK$ which pass λ as t grows from 0 to 1. Thus, according to the classical Birman-Schwinger principle (see for example [1]) the relation (1.2) holds true for $K \in \mathbf{S_1}$. Moreover it was proved in [4] that if $\lambda \in \rho(R+K)$, then the left hand side of (1.2) is continuous with respect to small perturbations of K in the operator norm. Therefore for general operators $K \in \mathbf{S}_\infty$, $\lambda \in \rho(R+K)$, the equality (1.2) is obtained by approximation K by operators from $\mathbf{S_1}$. Now note that both sides of (1.2) are left continuous with respect to λ, therefore we get rid of the assumption $\lambda \in \rho(R+K)$. $\square$

3. Let H_0, V be selfadjoint operators in the Hilbert space $\mathcal{H}$ and $J = J^*$ be the sign of V. Suppose

$$|V|^{1/2}|H_0 + iI|^{-1/2} \in \mathbf{S}_\infty.$$

The family of selfadjoint operators

$$H(\alpha) = H_0 + \alpha V, \ \alpha > 0,$$

is understood in the sense of quadratic forms (see [10]). For a real valued function $f(\cdot)$ (defined on $\mathbb{R}$) such that

$$\pm f(\pm 1) > 0,$$

we introduce the operator

$$V_f = f(J)|V|$$

and define the family of perturbed operators

$$H_f(\alpha) = H_0 + \alpha V_f, \ \alpha > 0.$$

The main question of our paper is how to describe the motion of the spectrum of $H(\alpha)$ as α grows. The family of operators $H_f(\alpha)$ plays only an auxiliary role. We shall always assume that the limit

$$X_\lambda := \lim_{z \to \lambda + i0} \sqrt{|V|}(H_0 - zI)^{-1}\sqrt{|V|} \tag{1.3}$$

exists in the operator norm for a.e. λ and for these λ

$$B_\lambda := \mathrm{Im} X_\lambda \in \mathbf{S}_1. \tag{1.4}$$

Below we also use the notation

$$A_\lambda := \mathrm{Re} X_\lambda, \quad \lambda \in \mathbb{R}.$$

For a.e. $\lambda \in \mathbb{R}$, we define the "generalized spectral shift function" for the pair of selfadjoint operators $H(\alpha), H_0$ as follows

$$\xi(\lambda; H(\alpha), H_0) =$$

$$= \int_{-\infty}^{+\infty} \eta(0; J + \alpha(A_\lambda + tB_\lambda), J) dw(t), \tag{1.5}$$

$$dw(t) = \pi^{-1}(1 + t^2)^{-1} dt.$$

Note that for trace class perturbations V the function ξ coincides with Krein's spectral shift function (see [4]).

If the limit (1.3) exists and the relation (1.4) is fulfilled only for a. e. $\lambda \in \Lambda$, where Λ is a measurable subset of $\mathbb{R}$ then we say that the generalized spectral shift function (1.5) is defined on Λ.

4. Here we present our main result which requires some additional conditions. First of all we shall assume that there exists a point $\mu = \overline{\mu} \in \rho(H_0)$ and a number $p > 0$ such that

$$\xi(\mu; H_f(\alpha), H_0) \sim (C_+|f(1)|^p - C_-|f(-1)|^p)\alpha^p, \qquad (1.6)$$
$$\alpha \to \infty, \ \pm f(\pm 1) > 0,$$

where the constants $C_\pm$ do not depend on f (but may depend on $V_\pm$). Moreover, denoting

$$Q = Q(t) := \alpha(A_\lambda + tB_\lambda - X_\mu),$$

we also require that

$$\int_{-\infty}^{+\infty} n(\epsilon, Q(t))(1+t^2)^{-1} dt = o(\alpha^p), \quad \alpha \to \infty, \ \forall \epsilon > 0. \qquad (1.7)$$

Theorem 1.1. *Let* $\mu = \overline{\mu} \in \rho(H_0)$ *and let the conditions* (1.6), (1.7) *be fulfilled. Then for the function*

$$\psi(\alpha) := \xi(\lambda; H(\alpha), H_0) - \xi(\mu; H(\alpha), H_0)$$

the following relation holds for a.e. $\lambda \in \mathbb{R}$

$$\psi(\alpha) = o(\alpha^p), \ \alpha \to \infty.$$

2. Proof of Theorem 1.1

1. We start with the following auxiliary

Proposition 2.1. *Assume that* $f(-1) < 0$ *and* $f(1) > 0$*. Then*

$$\xi(\lambda; H_f(\alpha), H_0) =$$
$$= \int_{-\infty}^{+\infty} \eta(0; f(J)^{-1} + \alpha(A_\lambda + tB_\lambda), f(J)^{-1}) dw(t), \qquad (2.8)$$
$$dw(t) = \pi^{-1}(1+t^2)^{-1} dt.$$

Proof. For every pair of selfadjoint bounded operators R, K, such that $0 \in \rho(R)$ and $K \in \mathbf{S}_\infty$ we introduce the operators

$$R_s. = SRS, \quad K_s = SKS,$$

where $S = S^*$ is a bounded invertible operator. Then in order to establish (2.8) it is sufficient to prove that

$$\eta(0; R_s + K_s, R_s) = \eta(0; R + K, R). \qquad (2.9)$$

Moreover the substantiation of (2.9) can be reduced to the cases $K \geq 0$ and $K \leq 0$. But if K is of definite sign, the quantity $\eta(0; R + K, R)$ coincides with the number of eigenvalues of the operator $R + tK$ which pass point 0 as t grows from 0 to 1. Therefore (2.9) follows from equivalence of the two statements:

$$0 \in \sigma(R + tK) \Leftrightarrow 0 \in \sigma(R_s + tK_s).$$

Now to complete the proof let us take $S = |f(J)|^{-1/2}$, $R = \mathrm{sign} f(J)$ and $K_s = \alpha(A_\lambda + tB_\lambda)$ in (2.9). □

2. The rest of the section is devoted to the proof of Theorem 1.1. Consider the function

$$\eta(0; J + \alpha(A_\lambda + tB_\lambda), J).$$

In order to compare this function with $\eta(0; J + \alpha X_\mu, J)$, $\mu \in \rho(H_0)$, we use its additivity :

$$\zeta(\alpha) := \eta(0; J + \alpha(A_\lambda + tB_\lambda), J) - \eta(0; J + \alpha X_\mu, J) =$$
$$= \eta(0; J + \alpha(A_\lambda + tB_\lambda), J + \alpha X_\mu).$$

Denoting $\mathcal{Y} := \{\alpha \in \mathbb{R}_+ : \ 0 \in \rho(J + \alpha X_\mu)\}$ and using the Birman-Schwinger principle (see Proposition 1.1) we obtain

$$-n_-(1 - 0, T^{(+)}) \le \zeta(\alpha) \le n_+(1, T^{(-)}), \qquad \alpha \in \mathcal{Y},$$

where

$$T^{(\pm)} := \sqrt{Q}_\pm (J + \alpha X_\mu)^{-1} \sqrt{Q}_\pm$$

and

$$Q = Q(t) = \alpha(A_\lambda + tB_\lambda - X_\mu).$$

By the Horn inequality (1.1),

$$n_\pm(1, T^{(\mp)}) \le 2n(\epsilon, Q) + n(\epsilon^{-1}, (J + \alpha X_\mu)^{-1}).$$

Since

$$\int_{-\infty}^{+\infty} n(\epsilon, Q(t))(1 + t^2)^{-1} dt = o(\alpha^p), \quad \alpha \to \infty,$$

it is sufficient to establish that

$$\limsup_{\alpha \to \infty} \alpha^{-p} n(\epsilon^{-1}, (J + \alpha X_\mu)^{-1}) \le C(\epsilon)$$

and

$$\lim_{\epsilon \to 0} C(\epsilon) = 0.$$

First of all note that

$$n(\epsilon^{-1}, (J + \alpha X_\mu)^{-1}) = \mathrm{rank} E_{(J + \alpha X_\mu)}(-\epsilon, \epsilon) \le$$
$$\le \eta(\epsilon; J + \alpha X_\mu, J) - \eta(-\epsilon; J + \alpha X_\mu, J) =: m(\alpha).$$

It is easy to see from the definition of the function η, that

$$m(\alpha) = \eta(0, (f_+(J) + \alpha X_\mu), f_+(J)) - \eta(0, (f_-(J) + \alpha X_\mu), f_-(J)),$$

where

$$f_{\pm}(J) = J \mp \epsilon I.$$

Denoting

$$H^{\pm}(\alpha) = H_0 + \alpha f_{\pm}(J)^{-1}|V|$$

and using the representation for $\xi(\mu, H^{\pm}(\alpha), H_0)$ (see (2.8)) we obtain

$$m(\alpha) = \xi(\mu, H^{+}(\alpha), H_0) - \xi(\mu, H^{-}(\alpha), H_0).$$

Therefore

$$m(\alpha) \sim (C_{+} + C_{-})[(1-\epsilon)^{-p} - (1+\epsilon)^{-p}]\alpha^{p}, \quad \alpha \to \infty,$$

and

$$\lim_{\epsilon \to 0} \limsup_{\alpha \to \infty} \alpha^{-p} m(\alpha) = 0.$$

The proof is complete.

3. Limits of compact operators

Here we present some conditions on V and H_0 which ensure existence of (1.3) and guarantee (1.4). We begin with the statement which immediately follows from the results of [5]. For $0 < q \leq 1$ we define q^*

$$q^* = q, \qquad \text{if } q < 1;$$

$$q^* > 1, -\text{any number, if } q = 1.$$

Proposition 3.1. *Assume that for every bounded open interval $\delta \subset \mathbb{R}$ the following inclusion holds:*

$$|V|^{1/2} E_{H_0}(\delta) \in \mathbf{S_{2q}}, \quad 0 < q \leq 1. \tag{3.10}$$

Then for a.e. $\lambda \in \mathbb{R}$ the limit

$$X(\lambda, \delta) := \lim_{z \to \lambda + i0} \sqrt{|V|} E_{H_0}(\delta)(H_0 - zI)^{-1} \sqrt{|V|}$$

exists in the $\mathbf{S_{q^*}}$-norm and

$$\mathrm{Im} X(\lambda, \delta) \in \mathbf{S_q}.$$

The following statement is a direct consequence of Proposition 3.1.

Proposition 3.2. [7] *Assume that for every bounded interval $\delta \subset \mathbb{R}$*

$$|V|^{1/2} E_{H_0}(\delta) \in \mathbf{S_2}. \tag{3.11}$$

Then for a.e. $\lambda \in \mathbb{R}$ the limit (1.3) *exists and the condition* (1.4) *is fulfilled.*

The proof of the following statement is a repeat of the proof of Theorem 4.1 in [7].

Proposition 3.3. *Let the condition* (3.10) *be satisfied and*

$$p \geq q, \text{ if } q < 1;$$
$$p > 1, \text{ if } q = 1.$$

Assume that

$$|V|^{1/2}|H_0 + iI|^{-1/2} \in \Sigma_{2p} \tag{3.12}$$

Then for a.e. $\lambda \in \mathbb{R}$ *the condition* (1.7) *holds.*

4. Regular points of H_0

Here we present some sufficient for (1.6) conditions. For a regular point $\mu \in \rho(H_0)$ we introduce the operators

$$X_\mu^\pm := \sqrt{V_\pm}(H_0 - \mu I)^{-1}\sqrt{V_\pm}$$

and

$$X_\mu^0 := \sqrt{V_+}(H_0 - \mu I)^{-1}\sqrt{V_-}.$$

Theorem 4.1. *Let* $\mu = \overline{\mu} \in \rho(H_0)$. *Assume that there exist constants* C_-^0, C_+^0 *such that*

$$n_\mp(s, X_\mu^\pm) \sim C_\pm^0 s^{-p}, \quad s \to 0, \ p > 0, \tag{4.13}$$

and

$$n(s, X_\mu^0) = o(s^{-p}), \quad s \to 0. \tag{4.14}$$

Then (1.6) *is fulfilled with* $C_\pm = C_\pm^0$, *in particular*

$$\xi(\mu; H(\alpha), H_0) \sim (C_+^0 - C_-^0)\alpha^p, \qquad \alpha \to \infty. \tag{4.15}$$

Proof. It is sufficient to establish (4.15). In our case

$$\xi(\mu; H(\alpha), H_0) = \eta(0; J + \alpha X_\mu, J).$$

We are going to compare this function with

$$\eta(0; J + \alpha L, J), \quad L := X_\mu^+ + X_\mu^-.$$

We use its additivity :

$$\zeta_0(\alpha) := \eta(0; J + \alpha X_\mu, J) - \eta(0; J + \alpha L, J) =$$
$$= \eta(0; J + \alpha X_\mu, J + \alpha L).$$

Denoting $\mathcal{Y}_0 := \{\alpha \in \mathbb{R}_+ : \ 0 \in \rho(J + \alpha L)\}$ and using the Birman-Schwinger principle (see Proposition 1.1) we obtain

$$-n_-(1 - 0, T_0^{(+)}) \leq \zeta_0(\alpha) \leq n_+(1, T_0^{(-)}), \qquad \alpha \in \mathcal{Y}_0,$$

where

$$T_0^{(\pm)} := \sqrt{Q_\pm^0}(J + \alpha L)^{-1}\sqrt{Q_\pm^0}$$

and

$$Q^0 := \alpha(X_\mu - L).$$

By the Horn inequality (1.1),

$$n_\pm(1, T^{(\mp)}) \le 2n(\epsilon, Q^0) + n(\epsilon^{-1}, (J + \alpha L)^{-1}).$$

Since by (4.14)

$$n(\epsilon, Q^0) = o(\alpha^p), \quad \alpha \to \infty,$$

it is sufficient to establish that

$$\limsup_{\alpha \to \infty} \alpha^{-p} n(\epsilon^{-1}, (J + \alpha L)^{-1}) \le C(\epsilon)$$

and

$$\lim_{\epsilon \to 0} C(\epsilon) = 0.$$

First of all note that

$$n(\epsilon^{-1}, (J + \alpha L)^{-1}) = \text{rank} E_{(J+\alpha L)}(-\epsilon, \epsilon) \le$$

$$\le \eta(\epsilon; J + \alpha L, J) - \eta(-\epsilon; J + \alpha L, J) =: m_0(\alpha).$$

It is easy to see from the definition of the function η, that

$$m_0(\alpha) = \eta(0, (f_+(J) + \alpha L), f_+(J)) - \eta(0, (f_-(J) + \alpha L), f_-(J)),$$

where

$$f_\pm(J) = J \mp \epsilon I.$$

It follows from (4.13) that

$$\eta(0, f(J) + \alpha L, f(J)) \sim (C^0_+ |f(1)|^{-p} - C^0_- |f(-1)|^{-p}), \ \alpha \to \infty,$$

for real f such that $f(-1) < 0$ and $f(1) > 0$. Therefore

$$m_0(\alpha) \sim (C^0_+ + C^0_-)[(1 - \epsilon)^{-p} - (1 + \epsilon)^{-p}]\alpha^p, \quad \alpha \to \infty,$$

and

$$\lim_{\epsilon \to 0} \limsup_{\alpha \to \infty} \alpha^{-p} m_0(\alpha) = 0.$$

The proof is complete.

5. Applications

In this section we present some applications of Theorem 1.1 to the study of differential operators.

1. Below we write $\int := \int_{\mathbb{R}^d}$ and denote $D_j = -i\frac{\partial}{\partial x_j}$, $D = -i\nabla = (D_1, \dots, D_d)$.

In the first example we deal with the Dirac operator perturbed by a decreasing electric potential. Let $\mathbf{g} = (g_1, g_2, g_3)$ and g_0 be (4×4)-Dirac matrices; $\mathbf{1}$ denotes the unit matrix. The Dirac matrices satisfy the relations

$$g_j g_k + g_k g_j = \delta_{jk}\mathbf{1}, \qquad j, k = 0, 1, 2, 3. \tag{5.16}$$

Let us consider the unperturbed Dirac operator in $\mathcal{H} = L_2(\mathbb{R}^3; \mathbb{C}^4)$

$$H_0 = \mathbf{g} \cdot D + g_0,$$

$$\mathbf{g} \cdot D = -i \sum_{j=1}^{3} g_j \frac{\partial}{\partial x_j},$$

and perturb the operator by a real potential

$$H(\alpha) = H_0 + \alpha V, \quad \alpha > 0, \tag{5.17}$$

$$V \in L_3(\mathbb{R}^3), \quad \overline{V} = V. \tag{5.18}$$

The operator (5.17) needs to be correctly defined. Under the condition (5.18) it is impossible to introduce the operator as the difference of two operators, but it can be understood in a sense of the sum of the sesquilinear forms. This definition could be used not only for semi-bounded operators but for general ones, too. The corresponding scheme for non-semibounded operators is given in [10].

The spectrum of the operator H_0 is absolutely continuous and covers the complement of the interval $\Lambda = (-1, 1)$. The essential spectrum of the operator $A(\alpha)$ coincides with the spectrum of H_0. Besides, the operator $H(\alpha)$ has a discrete spectrum in the gap Λ. It is clear that the generalized spectral shift function of the pair $H(\alpha)$, H_0 exists on the interval Λ in the sense of (1.5).

Theorem 5.1. *Let H_0 be the Dirac operator and $\mu \in \Lambda$. Under the condition* (5.18) *the following asymptotics holds*

$$\xi(\mu; H(\alpha), H_0) \sim \tag{5.19}$$

$$\sim (3\pi)^{-2}\alpha^3\Big(\int V_+^3 dx - \int V_-^3 dx\Big),$$

$$\alpha \to \infty.$$

Proof. It is sufficient to note that the condition (4.13), (4.14) are fulfilled and

$$C^0_\pm = (3\pi)^{-2}\int V^3_\pm dx.$$

For the reference concerning (4.13) see [3]. The relation (4.14) is obtained in [9]. In fact the proof of (5.19) can be found also in [9]. □

Now we are going to apply our abstract theorem to the Dirac operator. Note that the condition (3.11) is fulfilled if and only if

$$V \in L_1(\mathbb{R}^3).$$

The inclusion (3.12) follows from the results of [3], but it can be also found in [9]. Combining Theorem 1.1 with Proposition 3.3 and Theorem 5.1, we obtain

Theorem 5.2. *Let H_0 be the Dirac operator. Under the condition*

$$V \in L_3(\mathbb{R}^3) \cap L_1(\mathbb{R}^3) \tag{5.20}$$

the following asymptotics holds for a.e. $\lambda \in \mathbb{R}$

$$\xi(\lambda; H(\alpha), H_0) \sim \tag{5.21}$$

$$\sim (3\pi)^{-2}\alpha^3\Big(\int V^3_+ dx - \int V^3_- dx\Big),$$
$$\alpha \to \infty.$$

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