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Beam Transfer Functions, Stability Diagrams, and Schottky Spectra for Unbunched Beams

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Abstract

Beam transfer functions and stability diagrams for unbunched beams can be straightforwardly derived starting from a Hamiltonian which includes collective interactions and using that Hamiltonian in the Vlasov equation. One can start with a very general Hamiltonian which can be applied to longitudinal, transverse, or more complex impedance-like interactions. Furthermore, one can compute Schottky spectra for these same circumstances. Examples are worked out for the standard longitudinal and transverse cases.

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1 Introduction

There are three things that this paper will compute. The first is the response of a distribution to an applied field. The second is the point at which Landau damping ceases to stabilize the beam. Both questions are most straightforwardly answered in the frequency domain, and are in fact intimately related to each other. The third thing that will be computed in this paper is the Schottky spectrum for a beam. This also turns out to be related to the other two items, but through a significantly more complex process. In all cases, the discussion is restricted to storage rings.

The response of a beam distribution at a given frequency to an excitation at that same frequency will be proportional to the field applied at that frequency, and the proportionality constant, which is a function of frequency, is known as a beam transfer function.

The beam transfer function depends on the beam current. This dependence is to be expected: if the current is large enough for the beam to be unstable, then perturbing the beam should cause it to go unstable, and one wouldn't have very much to measure. Thus, one expects that this current dependence should be closely related the instability threshold.

In many cases, an instability threshold can be computed by means of what is known as a stability diagram (see [Zot76] for an early example of these, or [BR96, BR98, BG99] for more modern examples). A stability diagram is a curve drawn in the complex plane such that if one plots current times impedance (or a related quantity) in the complex plane, the stability of a beam is related to where that number falls with relation to the curve. The stability diagram reflects the point at which Landau damping ceases to stabilize the beam. Because of the relationship between beam transfer functions and instability thresholds, one also expects to have a relationship between beam transfer functions and stability diagrams. This will prove to be true.

The concept of a beam transfer function is not in fact new, and has been computed as far back as 1971 [MS71]. The corresponding instabilities were studied even earlier [NS65, LNS65]. This paper simply shows how to compute these beam transfer functions in a general framework which is in principle extensible to more complex phase space dynamics than the simple one-dimensional transverse or longitudinal dynamics one generally considers (although we do not compute more complex cases here). The generality allows us to find expressions which in some cases have terms which were previously unknown. We also use a modern formulation for impedances and single-particle dynamics, in fact adding some generality to the definition of impedances which allows one to begin to consider impedances distributed non-uniformly throughout the ring, although again we do not expand much on this possibility here (see [Chi85, Rug86, Suz87] for some phenomena resulting from this).

The Schottky spectrum of a beam is the signal measured when no external

excitation is applied to the beam. This spectrum results from the fact that the beam is not really a continuous distribution, but a large number of discrete particles.

This paper will start by introducing some definitions for these problems. The definitions will be slightly more general than what is generally done for impedance driven collective effects. This will both allow for more generality in applying the formalism, but will also produce more compact results which are straightforwardly applied to specific cases.

These definitions will involve defining the force on a test charge through a Hamiltonian. This Hamiltonian will then be used in the Vlasov equation to determine the time evolution of perturbations to an equilibrium distribution. The resulting equations will be Fourier analyzed. These results will be used to determine the beam transfer function as well as the threshold for instability. The Hamiltonian will then be used to describe the motion of individual particles when there is no external driving force, and statistical averaging will be applied to determine what the measured signal will be.

1.1   The Vlasov Equation

The Vlasov equation is an equation describing the time evolution of the Hamiltonian. It is true under the assumption that the two-particle distribution function is the product of one particle distribution functions. For a system described by a Hamiltonian, it can be written as

$$\frac{\partial \bar{\Psi}}{\partial s} + [\bar{\Psi}, H] = 0 \tag{1.1}$$

where $\bar{\Psi}$ is the one-particle distribution function in phase space and H is the Hamiltonian describing the force on a single particle. The expression $[f, g]$ is the Poisson bracket of f and g.

Say that one finds a solution Ψ_0 to the Vlasov equation. Consider perturbations Ψ to that distribution such that $\bar{\Psi} = \Psi_0 + \Psi$. Furthermore, assume that H can be written as the sum $H_0 + V(\bar{\Psi})$, where the dependence of V on its argument is linear. Then, ignoring second order terms in Ψ, one gets

$$\frac{\partial \Psi}{\partial s} + [\Psi, H_0 + V(\Psi_0)] + [\Psi_0, V(\Psi)] = 0 \tag{1.2}$$

If, in addition, there is a term in the Hamiltonian H_1 which is considered to be perturbatively small, but is independent of $\bar{\Psi}$, then the Vlasov equation becomes

$$\frac{\partial \Psi}{\partial s} + [\Psi, H_0 + V(\Psi_0)] + [\Psi_0, V(\Psi)] + [\Psi_0, H_1] = 0 \tag{1.3}$$

where again, second order terms are dropped.

1.2 Construction of the Hamiltonian

The force on a test charge will be defined by describing the force through a Hamiltonian. There are three components to that force:

1. Forces coming from elements in the ring which are designed to confine the particles along the design orbit, such as magnets and r.f. cavities (although the latter are not present for unbunched beams).

2. Forces which depend on the beam distribution, in other words, those that come from the beam interacting with itself.

3. External forces applied to the beam in order to measure the beam response and therefore the beam transfer function.

A derivation for the Hamiltonian for the first two parts is given in appendix A.

In describing the Hamiltonian, we will use x, y for the horizontal and vertical transverse planes; their conjugate momenta are p_x and p_y which are made dimensionless by scaling by the momentum of the reference particle (which is $\beta\gamma mc$). The longitudinal coordinate will be $z = -\beta ct$, where t is the arrival time at a given point. Its conjugate momentum is δ, defined such that the particle's energy is $\gamma mc^2(1 + \beta^2\delta)$. s, the position along the ideal orbit, will be the independent variable.

1.2.1 Action-Angle Variables

The first step will be to find an equilibrium solution of the Vlasov equation (i.e., a distribution periodic in s, the arc length along the beam's design orbit) using the first two components in our Hamiltonian, which will be referred to as Ψ_0. The task of this paper is to determine how perturbations to this equilibrium distribution evolve in time. This paper will not discuss how to find such a distribution. Once such a distribution is found, one can then write the Hamiltonian for the first two components using this distribution in the second component.

This Hamiltonian will then be normalized, meaning that a canonical transformation will be found for which the Hamiltonian in the new variables is purely a function of the action variables $\boldsymbol{J}$ and δ. In general, this is not possible. However, one can do so in a power series about the origin [DF76, DF83, FBI89] for regions near the origin (how "near" depends on how many terms in the power series you take). Furthermore, one can argue that for small resonances, there is a normalized Hamiltonian which averages over the resonance [WR92], and that therefore the overall effects on a distribution of such a resonance would be similar to the effect of the averaging Hamiltonian on such a distribution. This normalized Hamiltonian will be called $H_0(\boldsymbol{J}, \delta)$ (note that this H_0 differs from the H_0 introduced earlier). In these variables, Ψ_0 will only be a function of $\boldsymbol{J}$ and δ.

Henceforth, this paper will work in the action-angle variables defined via this approximate normalization. The conjugate variables to the action variables $\boldsymbol{J}$

4

are the angle variables $\boldsymbol{\theta}$; the conjugate variable to δ is z. Due to the time-independence of the elements of the system, this normalized δ will not change from its original definition, and z will only differ from its original definition by a function of $\boldsymbol{J}$, $\boldsymbol{\theta}$, and δ. The relationship between this normalized z and the arrival time is thus given by $-\beta ct = z + z_0(\boldsymbol{J}, \boldsymbol{\theta}, \delta, s)$.

1.2.2 WAKEFIELD TERM IN ACTION-ANGLE VARIABLES

The functions f_α and g_α defined in appendix A, as well as Ψ, can be written in terms of action-angle variables. The result is the wakefield term (see equation A.17)

$$H_W[\Psi] = \frac{q^2 N}{\beta^2 \gamma m_q c^2} \sum_\alpha f_\alpha(\boldsymbol{J}, \boldsymbol{\theta}, \delta, s) \int w_\alpha \Big(z + z_0(\boldsymbol{J}, \boldsymbol{\theta}, \delta, s) - \bar{z} - z_0(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}}, \bar{\delta}, s), s \Big)$$
$$g_\alpha(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}}, \bar{\delta}, s) \Psi(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}}, \bar{z}, \bar{\delta}, s) \, d^2\bar{\boldsymbol{J}} \, d^2\bar{\boldsymbol{\theta}} \, d\bar{z} \, d\bar{\delta} \quad (1.4)$$

1.2.3 EXTERNALLY APPLIED FORCES

There are essentially geometric considerations which determine the relationship between the g_α and the $\bar{f}_\alpha$ which are defined in appendix A. The $\bar{f}_\alpha$ are expected to be able to describe any field pattern which can arise in the vacuum chamber, under the assumption that any field pattern can be generated by a current moving parallel to the ideal orbit. Thus, if a field pattern is generated instead by electrodes with an oscillating voltage, one can expect that the generated field is described by some combination of the $\bar{f}_\alpha$. The generated field will be taken to only depend on the longitudinal position s and the arrival time. This argument is not rigorous, but in practice our assumptions hold. Since the arguments for describing the wakefields as we did (as an extra "potential"-like term in the Hamiltonian) were based only on the effect on a test charge, the term in the Hamiltonian due to an externally applied field can just be added to the wakefield term. The term added to the Hamiltonian can be written as

$$H_V = \frac{q}{\beta^2 \gamma m_q c^2} f_\alpha(\boldsymbol{J}, \boldsymbol{\theta}, \delta, s) V_\alpha \Big(z + z_0(\boldsymbol{J}, \boldsymbol{\theta}, \delta, s), s \Big) \quad (1.5)$$

Note that $f_\alpha V_\alpha$ has units of voltage.

2 Solving the Vlasov Equation

Now, the Hamiltonian, consisting of H_0, plus $H_W[\Psi]$ (equation (1.4)), plus H_V (equation (1.5)) can be put into the Vlasov equation for perturbations (equation (1.3)). The result is an integro-differential equation for Ψ.

2.1 Fourier Transformation

To solve the equation, one Fourier transforms the equation in z and θ. After doing so, the result is

$$
\frac{\partial \Psi_m}{\partial s} + i\boldsymbol{m} \cdot \frac{\partial H_0}{\partial \boldsymbol{J}} \Psi_m(\boldsymbol{J}, \delta, \omega, s) + i\frac{\omega}{\beta c} \frac{\partial H_0}{\partial \delta} \Psi_m(\boldsymbol{J}, \delta, \omega, s)
$$
$$
- \left(\boldsymbol{m} \cdot \frac{\partial \Psi_0}{\partial \boldsymbol{J}} + \frac{\omega}{\beta c} \frac{\partial \Psi_0}{\partial \delta} \right) \frac{q}{\beta^2 \gamma m_q c^2} \sum_\alpha F_{\alpha m}(\boldsymbol{J}, \delta, \omega, s) \bigg[iV_\alpha(\omega, s) +
$$
$$
(2\pi)^2 q N \beta c z_\alpha(\omega, s) \sum_{\bar{m}} \int G^*_{\alpha \bar{m}}(\bar{\boldsymbol{J}}, \bar{\delta}, \omega, s) \Psi_{\bar{m}}(\bar{\boldsymbol{J}}, \bar{\delta}, \omega, s) d^2\bar{\boldsymbol{J}} \, d\bar{\delta} \bigg] = 0 \quad (2.1)
$$

where

$$
\Psi_m(\boldsymbol{J}, \delta, \omega, s) = \frac{1}{(2\pi)^2 \beta c} \int_{-\infty}^{\infty} \int e^{-i\omega z/\beta c} e^{-i\boldsymbol{m} \cdot \boldsymbol{\theta}} \Psi(\boldsymbol{J}, \boldsymbol{\theta}, z, \delta, s) \, d^2\boldsymbol{\theta} \, dz \qquad (2.2)
$$

$$
z_\alpha(\omega, s) = \frac{i}{\beta c} \int_{-\infty}^{\infty} e^{-i\omega z/\beta c} w_\alpha(z, s) \, dz \qquad (2.3)
$$

$$
V_\alpha(\omega, s) = \frac{1}{\beta c} \int_{-\infty}^{\infty} e^{-i\omega z/\beta c} V_\alpha(z, s) \, dz \qquad (2.4)
$$

$$
F_{\alpha m}(\boldsymbol{J}, \delta, \omega, s) = \frac{1}{(2\pi)^2} \int e^{-i\boldsymbol{m} \cdot \boldsymbol{\theta}} f_\alpha(\boldsymbol{J}, \boldsymbol{\theta}, \delta, s) e^{i\omega z_0(\boldsymbol{J}, \boldsymbol{\theta}, \delta, s)/\beta c} \, d^2\boldsymbol{\theta} \qquad (2.5)
$$

$$
G_{\alpha m}(\boldsymbol{J}, \delta, \omega, s) = \frac{1}{(2\pi)^2} \int e^{-i\boldsymbol{m} \cdot \boldsymbol{\theta}} g_\alpha(\boldsymbol{J}, \boldsymbol{\theta}, \delta, s) e^{i\omega z_0(\boldsymbol{J}, \boldsymbol{\theta}, \delta, s)/\beta c} \, d^2\boldsymbol{\theta} \qquad (2.6)
$$

At this point, one must decide how to handle the s-dependence. Note that the choice to do an infinite Fourier analysis in z requires s to implicitly be over the finite interval $[0, L)$, where L is the circumference of the ring. Choosing periodicity in s will cause unstable behavior to show up by ω having a positive imaginary part. This may seem unusual at first, since s is the independent variable. The reason things work out this way is that the fact that a particle passes by a given point turn after turn is expressed by integrating over an infinite interval in z (see equation (1.4)).

Thus, if one assumes periodicity in s, it is desirable to Fourier analyze in s over the interval $[0, L)$ and get an associated Fourier mode number k. For the first 3 terms in (2.1), the Fourier transform is trivial. However, for the last term, there is a complication: the $F_{\alpha m}$ and the z_α all strongly depend on s. Fourier

analyzing in s, one gets the equation

$$\Psi_{mk}(\boldsymbol{J}, \delta, \omega) = -i\frac{q}{\gamma\beta^2 m_q c^2} \frac{\beta c \boldsymbol{m} \cdot \dfrac{\partial \Psi_0}{\partial \boldsymbol{J}} + \omega \dfrac{\partial \Psi_0}{\partial \delta}}{k\omega_0 + \beta c \boldsymbol{m} \cdot \dfrac{\partial H_0}{\partial \boldsymbol{J}} + \omega \dfrac{\partial H_0}{\partial \delta}}$$

$$\sum_\alpha \frac{1}{L} \int_0^L d\bar{s}\, F_{\alpha m}(\boldsymbol{J}, \delta, \omega, \bar{s}) e^{-2\pi i k \bar{s}/L} \Bigg[iV_\alpha(\omega, \bar{s})$$

$$+ (2\pi)^2 q N \beta c z_\alpha(\omega, \bar{s}) \sum_{\bar{k}\bar{\boldsymbol{m}}} e^{2\pi i \bar{k}\bar{s}/L} \int G^*_{\alpha\bar{\boldsymbol{m}}}(\bar{\boldsymbol{J}}, \bar{\delta}, \omega, \bar{s}) \Psi_{\bar{\boldsymbol{m}}\bar{k}}(\bar{\boldsymbol{J}}, \bar{\delta}, \omega)\, d^2\bar{\boldsymbol{J}}\, d\bar{\delta} \Bigg] \quad (2.7)$$

where

$$\Psi_{mk}(\boldsymbol{J}, \delta, \omega) = \frac{1}{L} \int_0^L \Psi_{\boldsymbol{m}}(\boldsymbol{J}, \delta, \omega, s) e^{-2\pi i k s/L} ds \quad (2.8)$$

and $\omega_0 = 2\pi\beta c/L$ is the angular revolution frequency for the ring.

One is not able to measure $\Psi_{mk}(\boldsymbol{J}, \delta, \omega)$ directly. One instead measures the moments of the distribution, as defined in equation (A.7), as a function of time (z). We can Fourier transform those dipole moments in time and define

$$d_\alpha(\omega, s) = \frac{1}{\beta c} \int e^{-i\omega[z + z_0(\boldsymbol{J}, \boldsymbol{\theta}, \delta, s)]/\beta c} g_\alpha(\boldsymbol{J}, \boldsymbol{\theta}, \delta, s) \Psi(\boldsymbol{J}, \boldsymbol{\theta}, z, \delta, s)\, d^2\boldsymbol{J}\, d^2\boldsymbol{\theta}\, d\delta\, dz$$

$$= (2\pi)^2 \sum_{mk} e^{2\pi i k s/L} \int G^*_{\alpha m}(\boldsymbol{J}, \delta, \omega, s) \Psi_{mk}(\boldsymbol{J}, \delta, \omega)\, d^2\boldsymbol{J}\, d\delta \quad (2.9)$$

These are our true *measured* quantities. Writing equation (2.7) in terms of these d_α, we find

$$d_\alpha(\omega, s) = -i(2\pi)^2 \frac{q}{\gamma\beta^2 m_q c^2} \sum_{mk} \Bigg\{ \frac{\beta c \boldsymbol{m} \cdot \dfrac{\partial \Psi_0}{\partial \boldsymbol{J}} + \omega \dfrac{\partial \Psi_0}{\partial \delta}}{k\omega_0 + \beta c \boldsymbol{m} \cdot \dfrac{\partial H_0}{\partial \boldsymbol{J}} + \omega \dfrac{\partial H_0}{\partial \delta}} G^*_{\alpha m}(\boldsymbol{J}, \delta, \omega, s)$$

$$\sum_{\bar{\alpha}} \frac{1}{L} \int_0^L F_{\bar{\alpha} m}(\boldsymbol{J}, \delta, \omega, \bar{s}) e^{2\pi i k(s - \bar{s})/L}$$

$$[q N \beta c z_{\bar{\alpha}}(\omega, \bar{s}) d_{\bar{\alpha}}(\omega, \bar{s}) + iV_{\bar{\alpha}}(\omega, \bar{s})]\, d\bar{s}\, d^2\boldsymbol{J}\, d\delta \quad (2.10)$$

2.2 Simplifications

The s dependence in this equation is still very complex, and it is not obvious how to simplify it to give a more easily soluble system. The difficulty lies primarily with the first term in brackets in the second integral. If the denominator inside the first integral is ignored, then $d_{\bar{\alpha}}$ is multiplied by a term which is perturbatively small; thus, the only case where a significant contribution from that term is

7

expected is when the denominator inside the first integral is small. For a given ω, one expects that this will only be true for one k, assuming that ω is sufficiently small. The extent to which and under what conditions this constraint is met will be discussed later in this paper. Were it not for the s-dependence in $F_{\bar{\alpha}m}$ and $G_{\alpha m}$, this would determine a one-to-one relationship between ω and k, and the problem would be diagonal in k.

It turns out that it is a good approximation in many cases to write

$$G_{\alpha m}(\boldsymbol{J}, \delta, \omega, s) = \mathcal{G}_\alpha(\omega, s)\bar{G}_{\alpha m}(\boldsymbol{J}, \delta, \omega) \tag{2.11}$$

We can now turn the equation for d_α into an equation for

$$\bar{d}_\alpha(\omega, s) = \frac{d_\alpha(\omega, s)}{\mathcal{G}_\alpha^*(\omega, s)} \tag{2.12}$$

which is

$$\bar{d}_\alpha(\omega, s) = -i(2\pi)^2 \frac{q}{\gamma\beta^2 m_q c^2} \sum_{mk} \left\{ \frac{\beta c \boldsymbol{m} \cdot \dfrac{\partial \Psi_0}{\partial \boldsymbol{J}} + \omega \dfrac{\partial \Psi_0}{\partial \delta}}{k\omega_0 + \beta c \boldsymbol{m} \cdot \dfrac{\partial H_0}{\partial \boldsymbol{J}} + \omega \dfrac{\partial H_0}{\partial \delta}} \bar{G}_{\alpha m}^*(\boldsymbol{J}, \delta, \omega) \right.$$

$$\sum_{\bar{\alpha}} \frac{1}{L} \int_0^L F_{\bar{\alpha}m}(\boldsymbol{J}, \delta, \omega, \bar{s}) e^{2\pi i k(s-\bar{s})/L}$$

$$\left[qN\beta c z_{\bar{\alpha}}(\omega, \bar{s}) \mathcal{G}_{\bar{\alpha}}^*(\omega, \bar{s}) \bar{d}_{\bar{\alpha}}(\omega, \bar{s}) + iV_{\bar{\alpha}}(\omega, \bar{s}) \right] d\bar{s} \, d^2\boldsymbol{J} \, d\delta \tag{2.13}$$

Now, Fourier analyze this equation in s, and define

$$\bar{d}_{\alpha k}(\omega) = \frac{1}{L} \int_0^L \bar{d}_\alpha(\omega, s) e^{-2\pi i k s/L} ds \tag{2.14}$$

The result is an equation coupling all the $\bar{d}_{\alpha k}(\omega)$ with different k to each other. There is no analytic expression for the result of inverting that matrix.

However, a useful analytic formulation for an approximate solution can be obtained by considering the denominator inside the integral in (2.13). Note that because both terms in brackets on the right hand side of (2.13) are perturbatively small quantities, to have a significant $\bar{d}_\alpha$ on the left hand side, it is necessary for the denominator inside the integral in (2.13) to be small. As long as the frequency spread in the beam (variation in $\omega[\partial H_0/\partial\delta]$) is small compared to ω_0 and the transverse tune spread in the beam (variation in $\beta c \boldsymbol{m} \cdot [\partial H_0/\partial \boldsymbol{J}]/\omega_0$) is small, the denominator can only be small for one value of k for a given ω and $\boldsymbol{m}$. If the transverse tunes are sufficiently different and if the frequency spread in the beam is sufficiently small, the denominator will only be small for a given $\boldsymbol{m}$. For finding instabilities ($V_\alpha = 0$), this corresponds to saying that second order terms in the beam current are small. For finding beam transfer functions (nonzero V_α), this means that the lines for the beam transfer function do not overlap. When

this is the case, the coupling between $d_{\alpha k}$ with different k is eliminated, and (2.13) becomes

$$\bar{d}_{\alpha k}(\omega) = -i(2\pi)^2 \frac{q}{\gamma\beta^2 m_q c^2} \sum_m \int \frac{\beta c\boldsymbol{m} \cdot \dfrac{\partial \Psi_0}{\partial \boldsymbol{J}} + \omega \dfrac{\partial \Psi_0}{\partial \delta}}{k\omega_0 + \beta c\boldsymbol{m} \cdot \dfrac{\partial H_0}{\partial \boldsymbol{J}} + \omega \dfrac{\partial H_0}{\partial \delta}} \bar{G}^*_{\alpha m}(\boldsymbol{J}, \delta, \omega)$$

$$\sum_{\bar{\alpha}} \left[qN\beta c\bar{d}_{\bar{\alpha} k}(\omega) \frac{1}{L} \int_0^L F_{\bar{\alpha} m}(\boldsymbol{J}, \delta, \omega, \bar{s}) z_{\bar{\alpha}}(\omega, \bar{s}) \mathcal{G}^*_{\bar{\alpha}}(\omega, \bar{s}) d\bar{s} \right.$$

$$\left. + i\frac{1}{L} \int_0^L e^{-2\pi i k\bar{s}/L} F_{\bar{\alpha} m}(\boldsymbol{J}, \delta, \omega, \bar{s}) V_{\bar{\alpha}}(\omega, \bar{s}) d\bar{s} \right] d^2\boldsymbol{J}\, d\delta \quad (2.15)$$

This equation has simplified (2.13) to give an equation which only couples $d_{\alpha k}$ with the same k but potentially different α. It will turn out that for the cases we study, the nature of $G_{\alpha m}$ will eliminate coupling between $d_{\alpha k}$ with different α.

One further simplification is often useful. Just as $G_{\alpha m}$ could often be broken into a product of terms of the form (2.11), a similar operation can be performed on $F_{\alpha m}$:

$$F_{\alpha m}(\boldsymbol{J}, \delta, \omega, s) = \mathcal{F}_\alpha(\omega, s) \bar{F}_{\alpha m}(\boldsymbol{J}, \delta, \omega) \quad (2.16)$$

Using this in (2.15) gives

$$\bar{d}_{\alpha k}(\omega) = -i(2\pi)^2 \frac{q}{\gamma\beta^2 m_q c^2}$$

$$\sum_{\bar{\alpha} m} \int \frac{\beta c\boldsymbol{m} \cdot \dfrac{\partial \Psi_0}{\partial \boldsymbol{J}} + \omega \dfrac{\partial \Psi_0}{\partial \delta}}{k\omega_0 + \beta c\boldsymbol{m} \cdot \dfrac{\partial H_0}{\partial \boldsymbol{J}} + \omega \dfrac{\partial H_0}{\partial \delta}} \bar{G}^*_{\alpha m}(\boldsymbol{J}, \delta, \omega) \bar{F}_{\bar{\alpha} m}(\boldsymbol{J}, \delta, \omega)\, d^2\boldsymbol{J}\, d\delta$$

$$\left[qN\beta c\bar{d}_{\bar{\alpha} k}(\omega) \frac{1}{L} \int_0^L \mathcal{F}_{\bar{\alpha}}(\omega, \bar{s}) z_{\bar{\alpha}}(\omega, \bar{s}) \mathcal{G}^*_{\bar{\alpha}}(\omega, \bar{s}) d\bar{s} \right.$$

$$\left. + i\frac{1}{L} \int_0^L e^{-2\pi i k\bar{s}/L} \mathcal{F}_{\bar{\alpha}}(\omega, \bar{s}) V_{\bar{\alpha}}(\omega, \bar{s}) d\bar{s} \right] \quad (2.17)$$

To analyze equation (2.17), it is useful to examine three quantities. The first is

$$B_{\alpha\bar{\alpha} k}(\omega) =$$

$$(2\pi)^2 \sum_m \int \frac{\beta c\boldsymbol{m} \cdot \dfrac{\partial \Psi_0}{\partial \boldsymbol{J}} + \omega \dfrac{\partial \Psi_0}{\partial \delta}}{k\omega_0 + \beta c\boldsymbol{m} \cdot \dfrac{\partial H_0}{\partial \boldsymbol{J}} + \omega \dfrac{\partial H_0}{\partial \delta}} \bar{G}^*_{\alpha m}(\boldsymbol{J}, \delta, \omega) \bar{F}_{\bar{\alpha} m}(\boldsymbol{J}, \delta, \omega)\, d^2\boldsymbol{J}\, d\delta \quad (2.18)$$

From equation (2.17), this can be seen to give the frequency dependence of the response of the beam to a signal whose distribution in s is proportional to $e^{2\pi k s/L}$

(modulo some s-dependent weighting factors to be discussed shortly). The next quantity of interest is

$$\bar{V}_{\alpha k}(\omega) = \int_0^L e^{-2\pi i k \bar{s}/L} \mathcal{F}_\alpha(\omega, \bar{s}) V_\alpha(\omega, \bar{s}) d\bar{s} \qquad (2.19)$$

This indicates that the signal that drives the beam oscillation is not just the k'th Fourier mode of V_α: V_α must first be weighted by $\mathcal{F}_\alpha$. The practical consequences of this will become clearer when specific examples are considered. The final quantity of interest is

$$Z_\alpha(\omega) = \int_0^L \mathcal{F}_\alpha(\omega, \bar{s}) z_{\bar{\alpha}}(\omega, \bar{s}) \mathcal{G}_\alpha^*(\omega, \bar{s}) d\bar{s} \qquad (2.20)$$

The relevant quantity is not just the average impedance around the ring; the impedance must be weighted by an s-dependent quantity. Putting these quantities together, (2.17) can be written as

$$\bar{d}_{\alpha k}(\omega) = -i\frac{q}{\gamma \beta^2 m_q c^2 L} \sum_{\bar{\alpha}} B_{\alpha \bar{\alpha} k}(\omega) \left[qN\beta c Z_{\bar{\alpha}}(\omega) \bar{d}_{\bar{\alpha} k}(\omega) + i\bar{V}_{\bar{\alpha} k}(\omega) \right] \qquad (2.21)$$

In many of the cases considered here, $B_{\alpha\bar{\alpha}k}(\omega)$ is nonzero only when $\alpha = \bar{\alpha}$. In this case, (2.21) can be solved for $\bar{d}_{\alpha k}$:

$$\bar{d}_{\alpha k}(\omega) = \frac{q}{\gamma \beta^2 m_q c^2 L} \frac{B_{\alpha \alpha k}(\omega) \bar{V}_{\alpha k}(\omega)}{1 + i\dfrac{q^2 N}{\gamma \beta m_q c L} B_{\alpha \alpha k}(\omega) Z_\alpha(\omega)} \qquad (2.22)$$

The homogeneous equation ($\bar{V}_{\alpha k} = 0$) will have solutions only when

$$-i\frac{q^2 N}{\gamma \beta m_q c L} B_{\alpha \alpha k}(\omega) Z_\alpha(\omega) = 1 \qquad (2.23)$$

As usual, one would look for ω which satisfy this equation. If there are any with ω having a positive imaginary part, then unstable oscillations in the beam can occur. Note that there is an intimate relationship between (2.22) and (2.23). The zeros of the denominator of (2.22) occur for the ω which satisfy (2.23).

Equation (2.23) can be used to find a stability diagram by rewriting it as

$$\frac{q^2 N}{\gamma \beta m_q c L} Z_\alpha(\omega) = \frac{i}{B_{\alpha \alpha k}(\omega)} \qquad (2.24)$$

Since instability occurs once ω gets a positive imaginary part, the stability diagram is obtained by plotting $i/B_{\alpha \alpha k}(\omega + i\epsilon)$ for real ω and ϵ an infinitesimal positive quantity. If the left hand side of (2.24) lies inside (on the origin side of) the curve, there is no ω satisfying (2.24) (or at least any such ω has a negative imaginary part), and therefore the motion is stable.

Next, the question arises on how to find the beam transfer function. The obvious answer is to evaluate (2.22) for real ω. However, the difficulty with this is that (2.22) cannot always be evaluated for real ω: one must give ω either small positive imaginary part, or a small negative imaginary part. The reason lies with the denominator of the integrand in (2.18). For real ω, that denominator will become zero. Since the singularity cannot be integrated over, one must make a choice of how to handle it. The choice can be made by examining how the measurement of a beam transfer function is made: at one point in time, one begins making the measurement, and continues to make the measurement for an arbitrarily long time in the future. If one is examining the beam a long time in the future, it is the exponentially growing solutions which are still giving a signal, not the exponentially damping ones. Thus, when one is forced to choose between giving ω a positive imaginary part and a negative imaginary part, the positive imaginary part should be chosen. This can be justified mathematically by examining the inverse Laplace transform:

$$f(t) = \frac{1}{2\pi} \int_{\gamma} f(s)e^{st}ds \tag{2.25}$$

The curve γ goes from $c - i\infty$ to $c + i\infty$ for c real and satisfying certain conditions. From the definition of the Laplace transform

$$f(s) = \int_{0}^{\infty} f(t)e^{-st}dt \tag{2.26}$$

it is expected that $f(s)$ exists for s sufficiently positive. Thus, c can be taken to be positive. The curve γ can be distorted in the negative real direction. If there are any singularities on the imaginary axis, part of the curve must remain to the right of those singularities. Note that the inverse Laplace transform is really just a rotated Fourier transform. The curve γ remaining to the right of the singularities corresponds to the integration curve in ω for our Fourier transform remaining below the singularities, which is equivalent to integrating above the singularities in the complex plane, or in other words giving ω an infinitesimal positive imaginary part. Thus, when finding (2.22), $B_{\alpha\alpha k}(\omega)$ should always be evaluated at $\omega + i\epsilon$ with ω real. Note that this is exactly the same prescription which was used for finding the stability diagrams.

2.3 Specific Examples

Two examples will be done. They are generally referred to as the longitudinal and the transverse cases of the problem. It turns out that for the approximations considered here (which are that the beam pipe is cylindrically symmetric and that the coordinate transforms are those given in appendix B), $B_{\alpha\bar{\alpha}k} = 0$ when $\alpha \neq \bar{\alpha}$.

2.3.1 LONGITUDINAL

This is the case where g_α is 1. If one has a cylindrically symmetric beam pipe, the definition of f_α in appendix A means that $f_\alpha = 1$ as well. By convention, $Z_\alpha(\omega) = \beta c Z_\parallel(\omega)/\omega$. The electric field that the beam sees at time t and position s is given by

$$\frac{1}{\beta c}\frac{\partial}{\partial t}V_\alpha(-\beta ct, s) \tag{2.27}$$

Clearly $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ are both 1. $\bar{F}_{\alpha m}$ is 1 for $\boldsymbol{m} = \boldsymbol{0}$ and zero otherwise; similarly for $\bar{G}_{\alpha m}$. The final quantity one needs is H_0. A form for this is derived in appendix B, and given by equation (B.14).

One can now compute $B_{zzk}(\omega)$ for this case ($\alpha = z$): it is

$$(2\pi)^2\omega \int \frac{\dfrac{\partial \Psi_0}{\partial \delta}}{k\omega_0 - \omega\left(1 + \eta\delta - \dfrac{2\pi\nu_x J_x \xi_x}{L} - \dfrac{2\pi\nu_y J_y \xi_y}{L}\right)} d\delta\, d^2\boldsymbol{J} \tag{2.28}$$

This is a difficult integral to perform because it depends both on δ and $\boldsymbol{J}$, and both are contained in the denominator. The most straightforward way to compute the integral is to neglect one of the terms in the denominator. The terms which depend on $\boldsymbol{J}$ are both roughly the square of the r.m.s. angular spread in the beam times the chromaticity. The chromaticity in a corrected machine is typically much smaller than 1. Here it will be assumed that this is much smaller than η times the r.m.s. δ (relative energy spread). In that case the terms depending on $\boldsymbol{J}$ in the denominator can be neglected.

The integration over $\boldsymbol{J}$ can now be performed. Define

$$\bar{\rho}(x) = (2\pi)^2 L\sigma_\delta \int \Psi_0(\boldsymbol{J}, \bar{\delta} + \sigma_\delta x)d^2\boldsymbol{J} \tag{2.29}$$

where

$$\bar{\delta} = (2\pi)^2 L \int \delta\Psi_0(\boldsymbol{J}, \delta)d^2\boldsymbol{J}\, d\delta \tag{2.30}$$

$$\sigma_\delta^2 = (2\pi)^2 L \int \delta^2\Psi_0(\boldsymbol{J}, \bar{\delta} + \delta)d^2\boldsymbol{J}\, d\delta \tag{2.31}$$

Then B_{zzk} becomes

$$B_{zzk}(\omega) = \frac{\omega}{\sigma_\delta L}\int_{-\infty}^{\infty}\frac{\bar{\rho}'(x)\,dx}{k\omega_0 - \omega(1 + \eta\bar{\delta} + \eta\sigma_\delta x)} \tag{2.32}$$

It is convenient to write this in terms of a dimensionless function $\bar{B}_z(x)$ which has a dimensionless argument x:

$$\bar{B}_z(x) = \int_{-\infty}^{\infty}\frac{\bar{\rho}'(y)}{y - x}dy \tag{2.33}$$

in which case

$$B_{zzk}(\omega) = -\frac{1}{\eta\sigma_\delta^2 L}\bar{B}_z\left(\frac{k\omega_0 - \omega(1+\eta\bar{\delta})}{\eta\sigma_\delta\omega}\right) \qquad (2.34)$$

Note that for x real,

$$\lim_{\epsilon\to 0^+} B_z(x+i\epsilon) = \fint_{-\infty}^{\infty}\frac{\bar{\rho}'(y)}{y-x}dy + i\pi\rho'(x) \qquad (2.35)$$

Thus, the imaginary part of the beam transfer function at low current can be used to find the beam distribution. The first two terms in the asymptotic expansion for $\bar{B}_z(x)$ give

$$\bar{B}_z(x) \sim \frac{1}{y^2} + \frac{3}{y^4} + O\left(\frac{1}{y^5}\right) \qquad (2.36)$$

Now let's compute $\bar{B}_z(x)$ for various distributions. First, consider a Gaussian distribution

$$\bar{\rho}(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \qquad (2.37)$$

Then

$$\bar{B}_{z,g}(x) = -1 - i\sqrt{\frac{\pi}{2}}xe^{-x^2/2}\left[\operatorname{erf}\left(i\frac{x}{\sqrt{2}}\right) + \operatorname{sgn}(\operatorname{Im}\{x\})\right] \qquad (2.38)$$

Next, consider the "parabolic-like" distribution

$$\bar{\rho}(x) = \frac{1}{\sqrt{2\pi\mu}}\frac{\Gamma(\mu)}{\Gamma(\mu-1/2)}\left(1 - \frac{x^2}{2\mu}\right)^{\mu-3/2} \qquad (2.39)$$

This is a distribution with a standard deviation of 1 which goes to zero at $x = \sqrt{2\mu}$. In this case,

$$\bar{B}_{z,p(\mu)} = \frac{1}{x^2}F\left(1,\frac{3}{2};\mu;\frac{2\mu}{x^2}\right) \qquad (2.40)$$

Here F is a (Gauss) hypergeometric function [AS72]. If μ is an integer or half-integer, this can be computed in terms of elementary functions; see appendix C for details.

2.3.2 TRANSVERSE

Let's consider oscillations in the vertical direction. Then

$$f_\alpha(\boldsymbol{J},\boldsymbol{\theta},\delta,s) = g_\alpha(\boldsymbol{J},\boldsymbol{\theta},\delta,s) = \sqrt{2J_y\beta_y(s)}\cos\left(\theta_y + \Delta\psi_y(s)\right) \qquad (2.41)$$

13

The transverse gradient of the electric field is given by

$$\frac{1}{\beta c}\frac{\partial}{\partial t}V_\alpha(-\beta ct, s) \tag{2.42}$$

We can easily find $F_{ym}(\boldsymbol{J}, \delta, \omega, s) = G_{ym}(\boldsymbol{J}, \delta, \omega, s)$ to be

$$F_{y(0,\pm 1)}(\boldsymbol{J}, \delta, \omega, s) = \sqrt{\frac{J_y \beta_y(s)}{2}}e^{\pm i\Delta\psi_y(s)} \tag{2.43}$$

Next, we note that as long as the betatron tune is not too near a half integer and the frequency ω is small enough, the denominator in $B_{yyk}(\omega)$ will be nonzero for only one of $\boldsymbol{m} = (1, 0)$ or $\boldsymbol{m} = (-1, 0)$ (see discussion before Eq. (2.15)). Thus, the sum over $\boldsymbol{m}$ can be eliminated, and we can thus split $F_{\alpha m}$ as

$$\mathcal{F}_\alpha(\omega, s) = \sqrt{\beta_y(s)}e^{\pm i\Delta\psi_y(s)} \qquad \bar{F}_{\alpha(0,\pm 1)}(\boldsymbol{J}, \delta, \omega) = \sqrt{\frac{J_y}{2}} \tag{2.44}$$

and $\mathcal{G}_\alpha$ and $\bar{G}_{\alpha m}$ are exactly the same as $\mathcal{F}_\alpha$ and $\bar{F}_{\alpha m}$ respectively. This implies that

$$Z_y(\omega) = \int_0^L \beta_y(s)z_\perp(\omega, s)\, ds \tag{2.45}$$

where $z_\perp(\omega, s)$ is the transverse impedance per unit length at s. Similarly, V_y must be weighted by $\sqrt{\beta_y(s)}$ to get $\bar{V}_{\alpha k}$, where

$$-\frac{q}{\gamma m_q}V_y(-\beta ct, s) \tag{2.46}$$

is the transverse acceleration in the y direction at time t and position s.

Computing $B_{yyk}(\omega)$, we get

$$2\pi^2 \left| \frac{J_y\left(\pm\beta c\dfrac{\partial\Psi_0}{\partial J_y} + \omega\dfrac{\partial\Psi_0}{\partial\delta}\right)d^2\boldsymbol{J}\,d\delta}{\left[\begin{array}{c} k\omega_0 \pm \omega_y - \omega + (\pm\omega_y\xi_y - \omega\eta)\delta \\ + \left(\pm\omega_0\bar{\alpha}_{xy} + \dfrac{\omega\omega_x\xi_x}{\beta c}\right)J_x + \left(\pm\omega_0\bar{\alpha}_{yy} + \dfrac{\omega\omega_y\xi_y}{\beta c}\right)J_y \end{array}\right]} \right. \tag{2.47}$$

In the numerator, we can ignore the second term as long as $\beta c/\epsilon_y \gg \omega/\sigma_\delta$, which is a safe assumption as long as we're not looking at extremely high frequencies. It is not much more difficult to include that term in the computation; the result will just have an additional term.

We will simplify the denominator by considering two separate cases. The first is when we ignore the terms proportional to J_x and J_y. In that case, we get

$$B_{yyk}(\omega) = \pm 2\pi^2\beta c\left| \frac{J_y\dfrac{\partial\Psi_0}{\partial J_y}d^2\boldsymbol{J}\,d\delta}{k\omega_0 \pm \omega_y - \omega + (\pm\omega_y\xi_y - \omega\eta)\delta} \right. \tag{2.48}$$

This can be rewritten as

$$B_{yyk}(\omega) = \frac{\beta c}{2L} \frac{1}{\sigma_\delta(\omega_y \xi_y \mp \omega\eta)} \bar{B}_\perp \left(\frac{\omega \mp \omega_y - k\omega_0}{(\pm\omega_y\xi_y - \omega\eta)\sigma_\delta} - \frac{\bar{\delta}}{\sigma_\delta} \right) \tag{2.49}$$

where

$$\bar{B}_\perp(y) = \int_{-\infty}^{\infty} \frac{\bar{\rho}(x)\,dx}{y - x} \tag{2.50}$$

As before, for y real we can compute

$$\lim_{\epsilon \to 0^+} \bar{B}_\perp(y) = \fint_{-\infty}^{\infty} \frac{\bar{\rho}(x)\,dx}{y - x} - i\pi\rho(y) \tag{2.51}$$

Also note that $\bar{B}'_\perp(y) = -\bar{B}_z(y)$. The first two terms in the asymptotic expansion of $\bar{B}_\perp(y)$ for large y give

$$\bar{B}_\perp(y) \sim \frac{1}{y} + \frac{1}{y^3} + O\left(\frac{1}{y^4}\right) \tag{2.52}$$

For a Gaussian distribution,

$$\bar{B}_\perp(y) = -i\sqrt{\frac{\pi}{2}} e^{-y^2/2} \left[\mathrm{erf}\left(i\frac{y}{\sqrt{2}} \right) + \mathrm{sgn}(\mathrm{Im}\{y\}) \right] \tag{2.53}$$

and for the parabolic-like distribution (2.39),

$$\bar{B}_\perp(y) = \frac{1}{y} F\left(1, \frac{1}{2}; \mu; \frac{2\mu}{y^2} \right) \tag{2.54}$$

The second case is when we ignore terms proportional to δ in the denominator. This case is important for frequencies $\omega \approx \pm\omega_y\xi_y/\eta$, when the frequency spread resulting from energy spread becomes small. In this case,

$$B_{yyk}(\omega) =$$
$$\pm \int \frac{2\pi^2 \beta c J_y \dfrac{\partial\Psi_0}{\partial J_y} d^2\boldsymbol{J}\,d\delta}{k\omega_0 \pm \omega_y - \omega + \left(\pm\omega_0\bar{\alpha}_{xy} + \dfrac{\omega\omega_x\xi_x}{\beta c} \right) J_x + \left(\pm\omega_0\bar{\alpha}_{yy} + \dfrac{\omega\omega_y\xi_y}{\beta c} \right) J_y} \tag{2.55}$$

We now assume that the distribution satisfies

$$\Lambda(x + y) = (2\pi)^2 \epsilon_x \epsilon_y L \int_{-\infty}^{\infty} \Psi_0(\epsilon_x x, \epsilon_y y, \delta) d\delta \tag{2.56}$$

15

where

$$\epsilon_x = \int J_x \Psi_0(\boldsymbol{J}, \delta) d^2\boldsymbol{J} \, d\delta \qquad\qquad \epsilon_y = \int J_y \Psi_0(\boldsymbol{J}, \delta) d^2\boldsymbol{J} \, d\delta \qquad (2.57)$$

This is not necessary for the computation to be tractable, but gives relatively nice results and is also a realistic assumption (the beam shape is elliptically symmetric). With this assumption, we can write

$$B_{yyk}(\omega) =$$

$$\pm \frac{\beta c}{2L} \frac{1}{\omega \mp \omega_y - k\omega_0} \bar{T}\left(\frac{\omega \mp \omega_y - k\omega_0}{\epsilon_x \left[\pm\omega_0\bar{\alpha}_{xy} + \dfrac{\omega\omega_x\xi_x}{\beta c} \right]}, \frac{\omega \mp \omega_y - k\omega_0}{\epsilon_y \left[\pm\omega_0\bar{\alpha}_{yy} + \dfrac{\omega\omega_y\xi_y}{\beta c} \right]} \right) \qquad (2.58)$$

where

$$\bar{T}(X, Y) = -\int_0^\infty \int_0^\infty \frac{y\,\Lambda'(x+y)\,dx\,dy}{1 - x/X - y/Y} \qquad (2.59)$$

This can be rewritten as

$$\bar{T}(X, Y) = \begin{cases} XY^2 \dfrac{\hat{T}'(Y) - \dfrac{\hat{T}(Y) - \hat{T}(X)}{Y - X}}{Y - X} & X \neq Y \\[4mm] \dfrac{1}{2} X^3 \hat{T}''(X) & X = Y \end{cases} \qquad (2.60)$$

where

$$\hat{T}(z) = -\int_0^\infty \bar{\Lambda}(u) \ln\left(1 - \frac{u}{z}\right) du \qquad (2.61)$$

For nearly real z,

$$\lim_{\epsilon \to 0^+} \hat{T}(z + i\epsilon) = -\int_0^\infty \bar{\Lambda}(u) \ln\left|1 - \frac{u}{z}\right| du - i\pi\bar{\lambda}(z) \qquad (2.62)$$

where

$$\bar{\lambda}(z) = \int_z^\infty \bar{\Lambda}(u)\,du \qquad (2.63)$$

is the normalized distribution in action in one variable. The first two terms in the asymptotic expansion of $\hat{T}(z)$ for large z can be computed, giving

$$\hat{T}(z) \sim \frac{1}{z} + \frac{1}{z^2} + O\left(\frac{1}{z^3}\right) \qquad (2.64)$$

In particular, this asymptotic expansion does not give the imaginary contribution for real z.

16

The function $\bar{\Lambda}(u) = e^{-u}$ corresponds to the Gaussian distribution (2.37), and gives

$$\hat{T}(z) = -e^{-z}E_1(-z) \tag{2.65}$$

where E_1 is the exponential integral function [AS72]. The $\bar{\Lambda}$ corresponding to a parabolic-like distribution (2.39) in each coordinate is

$$\bar{\Lambda}(u) = \frac{(\mu-1)(\mu-2)}{\mu^2}\left(1-\frac{u}{\mu}\right)^{\mu-3} \tag{2.66}$$

and the corresponding $\hat{T}$ is

$$\hat{T}(z) = \frac{1}{z}F\left(1,1;\mu;\frac{\mu}{z}\right) \tag{2.67}$$

3 Schottky Spectra

The treatment of Schottky spectra will not be as detailed as the treatment of beam transfer functions was, for reasons explained at the end of this section. The computation is basically identical to the computation in [BBH+74], except greater generality is allowed in the description of the motion of particles.

3.1 Spectra without Interactions

Begin by finding the Schottky spectra for non-interacting particles. There are several steps involved in this:

1. Give an expression for the motion for a single particle by finding and solving the equations of motion for the Hamiltonian (B.14).

2. Describe the measured signal at a pickup for N particles with different initial conditions.

3. Find the Fourier transform of that signal.

4. Square that Fourier transform, separating terms into coherent (proportional to N^2) and incoherent (proportional to N) terms.

5. Find the resulting incoherent signal when the initial conditions are distributed according to some distribution function.

3.1.1 EQUATIONS OF MOTION

In this paper, dependence of particle motion on action will be ignored. Thus, solving for particle motion for the Hamiltonian (B.14) gives

$$z = -(s-s_0)(1+\eta\delta_0) \qquad \theta_{x,y} = \theta_{x,y;0} + \frac{2\pi\nu_y}{L}(1+\xi_{x,y}\delta_0)(s-s_0) \tag{3.1}$$

17

where the 0-subscripted symbols are the initial conditions at "time" $z = 0$. All other variables will retain their initial conditions. The specification of a fixed value for a dependent variable in the initial conditions may seem backwards, but in reality it is the only sensible way to do things (due to issues involving wrapping around the ring).

3.1.2 MEASURED SIGNALS

We treat the pickup just like the source term for a wakefield, using (A.7). Thus, the moment-weighted current corresponding to a set of N particles with initial conditions $(\boldsymbol{\theta}_l, \boldsymbol{J}_l, \delta_l, s_l)$ and charges q_l is

$$\sum_l q_l g_\alpha \left(\boldsymbol{J}_l, \boldsymbol{\theta}_l + \frac{2\pi\boldsymbol{\nu}}{L}(1 + \xi\delta_l)(s - s_l), \delta_l, s \right) \delta\Big(t - (s - s_l)(1 + \eta\delta_l)/\beta c \Big) \quad (3.2)$$

where the ξ is the chromaticity corresponding to the vector component in question. This is the current that will be measured at a given position s. The variable s incorporates two physical realities: the position along the ideal orbit of the ring, and the number of times that the particle has passed that position in the ring. Thus, if you want the signal at a given position for all times, there is an implicit summation in (3.2) over all turns, which is incorporated into s. Let's say that $s = 0$ corresponds to the observation point in the ring. Then that summation gives

$$\sum_{kl} q_l g_\alpha \left(\boldsymbol{J}_l, \boldsymbol{\theta}_l + \frac{2\pi\boldsymbol{\nu}}{L}(1 + \xi\delta_l)(kL - s_l), \delta_l, 0 \right) \delta\Big(t - (kL - s_l)(1 + \eta\delta_l)/\beta c \Big)$$

$$(3.3)$$

3.1.3 FOURIER TRANSFORMATION

It is trivial to Fourier transform (3.3): the result is

$$d_\alpha(\omega) = \sum_{kl} q_l g_\alpha \left(\boldsymbol{J}_l, \boldsymbol{\theta}_l + \frac{2\pi\boldsymbol{\nu}}{L}(1 + \xi\delta_l)(kL - s_l), \delta_l, 0 \right) e^{i\omega(kL - s_l)(1 + \eta\delta_l)/\beta c} \quad (3.4)$$

Let's say that we are taking a data sample from the time $-T$ to T (we will eventually let T go to infinity, but we need it for now). The k summation will extend from $-K_l^-$ to K_l^+, where

$$K_l^\pm = \left\lfloor \frac{T}{T_0} \frac{1}{1 + \eta\delta_l} \mp \frac{s_l}{L} \right\rfloor \quad (3.5)$$

where $T_0 = 2\pi/\omega_0$ is the revolution period of the ring. There are $K_l^- + K_l^+ + 1$ terms in the summation , which for large T is approximately $2T/[T_0(1 + \eta\delta_l)]$.

18

From the discussion in appendix D, we can compute the incoherent contribution to $\langle |d_\alpha(\omega)|^2 \rangle$:

$$Nq^2 \int \Psi_0(\boldsymbol{J}, \delta) \sum_{k\bar{k}} g_\alpha \left(\boldsymbol{J}, \boldsymbol{\theta} + \frac{2\pi\boldsymbol{\nu}}{L}(1 + \xi\delta)(kL - s), \delta, 0 \right)$$

$$g_\alpha \left(\boldsymbol{J}, \boldsymbol{\theta} + \frac{2\pi\boldsymbol{\nu}}{L}(1 + \xi\delta)(\bar{k}L - s), \delta, 0 \right) e^{i\omega(k-\bar{k})L(1+\eta\delta)/\beta c} d\boldsymbol{J} \, d\boldsymbol{\theta} \, d\delta \, ds \quad (3.6)$$

3.1.4 LONGITUDINAL CASE

When $g_\alpha = 1$, the summation in the above integral can be computed using the results of appendix E. Using the result that $K_l^- + K_l^+ + 1 \approx 2T/[T_0(1 + \eta\delta_l)]$, we get

$$4\pi \frac{TNq^2}{T_0} \sum_k \int \frac{\bar{\rho}(x)}{1 + \eta(\bar{\delta} + \sigma_\delta x)} \delta \left(\frac{\omega L[1 + \eta(\bar{\delta} + \sigma_\delta x)]}{\beta c} - 2\pi k \right) dx =$$

$$\frac{2TNq^2}{T_0 |\eta| \sigma_\delta} \operatorname{sgn}(\omega) \sum_k \frac{1}{k} \bar{\rho} \left(\frac{k\omega_0 - \omega}{\omega\eta} - \frac{\bar{\delta}}{\sigma_\delta} \right) \quad (3.7)$$

using the definition of $\bar{\rho}$ in (2.29).

3.1.5 TRANSVERSE CASE

In this case, g_α is given by (2.41), and is the transverse coordinate. The same computation can be performed, giving

$$\frac{2\pi Nq^2 \sigma_y^2 T}{T_0^2 \sigma_\delta} \sum_\pm \operatorname{sgn}(\omega\eta \mp \omega_y \xi) \sum_k \frac{1}{\eta k\omega_0 \mp \omega_y(\xi - \eta)} \bar{\rho} \left(\frac{k\omega_0 \pm \omega_y - \omega}{\sigma_\delta(\omega\eta \mp \omega_y\xi)} - \frac{\bar{\delta}}{\sigma_\delta} \right)$$

$$(3.8)$$

where y is the direction in which we are measuring the transverse displacement.

3.2 Adding Interactions

The mathematics behind adding wakefield interactions to this picture to obtain the Schottky spectra at higher currents is difficult, involving the computation of two-particle correlation functions (the author does not claim to understand this derivation ...). For the details see [Bis80, BL82, Bis83].

The net result is that each term in the above expressions in each case should be multiplied by the quantity

$$\left| 1 + i\frac{q^2 N}{\gamma\beta m_q cL} B_{\alpha\alpha k}(\omega) Z_\alpha(\omega) \right|^{-2} \quad (3.9)$$

where the k here corresponds to the k in the above equations. Note that this quantity is the square of the absolute value of what appears in the denominator of (2.22). Thus, there is a close relationship between what happens to the Schottky spectrum at high currents and what happens to the beam transfer function at high currents. Note that the $\pm$ in (2.47) corresponds to the $\pm$ appearing in (3.8), in addition to the k's corresponding to each other.

4   Acknowledgements

The initial inspiration for these studies was an interesting computation of transverse stability diagrams by Francesco Ruggiero, which we later collaborated on [BR96] and generalized [BR98]. Jacques Gareyte pointed out the connection of these studies to beam transfer functions, and resulted in our collaboration for articles in [BG99], for which this document is intended to be an expanded reference.

Work on this document was conducted while the author was at CERN and Indiana University.

A   Wakefield Hamiltonian

The derivation in this appendix is similar to that appearing in [Ber96], but is slightly more general, and applies to unbunched beams instead of bunched beams.

A.1   Force on a Charged Particle

Begin with the Lorentz force equation

$$\boldsymbol{F} = q \left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B} \right) \tag{A.1}$$

Here $\boldsymbol{F}$ is the force felt by a test charge with charge q and velocity $\boldsymbol{v}$ due to electric fields $\boldsymbol{E}$ and magnetic fields $\boldsymbol{B}$. c is the speed of light. Gaussian units are used throughout. Take the curl of (A.1):

$$\boldsymbol{\nabla} \times \boldsymbol{F} = q \left[\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\boldsymbol{v}}{c} (\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \frac{1}{c} (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{B} \right] \tag{A.2}$$

Using Maxwell's equations, this is

$$\boldsymbol{\nabla} \times \boldsymbol{F} = -\frac{q}{c} \left[\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{B} \right] = -\frac{q}{c} \left. \frac{d\boldsymbol{B}}{dt} \right|_{\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{x}_0, \boldsymbol{p}_0, t)} \qquad \boldsymbol{v} = \frac{\partial \boldsymbol{X}}{\partial t} \tag{A.3}$$

where the function $\boldsymbol{X}(\boldsymbol{x}_0, \boldsymbol{p}_0, t)$ gives the trajectory of the test charge in question with initial conditions $\boldsymbol{X}(\boldsymbol{x}_0, \boldsymbol{p}_0, 0) = \boldsymbol{x}_0$, and initial momentum $\boldsymbol{p}_0$. This implies

that

$$\int_{t_0}^{t_1} (\boldsymbol{\nabla} \times \boldsymbol{F})_{\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{x}_0, \boldsymbol{p}_0, t)} dt =$$

$$- \frac{q}{c} \Big[\boldsymbol{B}\big(\boldsymbol{X}(\boldsymbol{x}_0, \boldsymbol{p}_0, t_1), t_1\big) - \boldsymbol{B}\big(\boldsymbol{X}(\boldsymbol{x}_0, \boldsymbol{p}_0, t_0), t_0\big) \Big] \quad \text{(A.4)}$$

In particular, if a particle goes from a field-free region to another field-free region, the integral is zero.

Now, assume that the particle's trajectory is given by $\boldsymbol{X}(\boldsymbol{x}_0, \boldsymbol{p}_0, t) = \boldsymbol{x}_0 + \boldsymbol{x}_1(\boldsymbol{p}_0, t)$. Thus, the particle may undergo a change in velocity, but the velocity cannot depend on spatial coordinates. Then

$$(\boldsymbol{\nabla} \times \boldsymbol{F})_{\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{x}_0, \boldsymbol{p}_0, t)} = \boldsymbol{\nabla}_0 \times \boldsymbol{F}\big(\boldsymbol{X}(\boldsymbol{x}_0, \boldsymbol{p}_0, t), t\big) \quad \text{(A.5)}$$

where $\boldsymbol{\nabla}_0$ means to take the derivative with respect to $\boldsymbol{x}_0$. Defining

$$\int_{t_0}^{t_1} \boldsymbol{F}\big(\boldsymbol{X}(\boldsymbol{x}_0, \boldsymbol{p}_0, t), t\big) \, dt = I(\boldsymbol{x}_0, \boldsymbol{p}_0, t_0) \quad \text{(A.6)}$$

then our assumption about $\boldsymbol{X}$ means that $\boldsymbol{\nabla}_{\boldsymbol{x}} \times I = \boldsymbol{0}$ if we move from one field-free region to another. Thus, we can write $I(\boldsymbol{x}, \boldsymbol{p}, t) = -\boldsymbol{\nabla}_{\boldsymbol{x}} S$.

A.2 Fields Created by a Source Distribution

Imagine now that the particles are moving down a tube. The tube is translationally invariant in one direction, except for a section which occupies a finite length along the translationally invariant axis which may have some arbitrary shape (the "object"). The integrations in the previous section go from a point well away from the object, through the object, to another point well away from the object. Call the direction of the translationally invariant axis $\hat{s}$.

Imagine one particle (labeled 1) passing through an object, followed by another particle (labeled 2). If the object is sufficiently short such that one can neglect

- the change in velocity of particle 2 as it passes through the object, and

- the difference in times required to pass through the object for the two particles,

then the momentum kick received by particle 2 can only depend on the difference in arrival times of particle 1 and particle 2 at some point s inside or very near the object. We will therefore assume that the $\hat{s}$ component of the velocity of all particles is βc. In this case, the s and t dependence of the force on particle 2 due to particle 1 will be $s_2 - s_1 - \beta c(t_2 - t_1)$.

We characterize our distribution by the moment of each slice at a given s. If we label the moments with the index α, then the moment at a given point s is

$$\int g_\alpha(\boldsymbol{z}, s) \Psi(\boldsymbol{z}, s, t) \, d^5 \boldsymbol{z} \quad \text{(A.7)}$$

where the vector $\boldsymbol{z}$ refers to $\boldsymbol{x}_\perp$, $\boldsymbol{p}_\perp$, and p_s. The $\perp$ subscript refers to the directions perpendicular to $\hat{s}$. We define $g_\alpha(\boldsymbol{z}, s)$ to have little variation in s over the length of the object in question. Ψ is the single-particle distribution function, and is normalized to 1 when integrating over everything except t. Each of these moments will generate an integrated force (and therefore a momentum kick if the velocity is constant) generated by an S which is proportional to a function $\bar{f}_\alpha(\boldsymbol{z}, s)$, where s is really just a coordinate referring to the position of the object in question. The variation of $\bar{f}_\alpha$ in s should be small over the length of the object.

Putting all of this together, we can write

$$S(\boldsymbol{z}, s, t) = q_1 q_2 N_2 \frac{1}{\beta c} \bar{f}_\alpha(\boldsymbol{z}, s) \int W_\alpha(s - \bar{s}) g_\alpha(\bar{\boldsymbol{z}}, \bar{s}) \Psi(\bar{\boldsymbol{z}}, \bar{s}, t) \, d^5\bar{\boldsymbol{z}} \, d\bar{s} \qquad (A.8)$$

Since we're taking the distribution at time t and are considering the force at time t, the expression $s_2 - s_1 - \beta c(t_2 - t_1)$ simply turned out to be $s - \bar{s}$. The W_α are called wakefields: the argument of the wakefield is really $-\beta c$ times the difference in arrival times of the particles at the object (this will be important later). Various combinations of s, $\bar{s}$, and t can be used for the arguments of Ψ. This is probably the most useful for unbunched beams (for bunched beams, $\bar{s} - \beta ct$ would be better, since if all particles are moving with velocity $\beta c\hat{s}$, then the distribution must only be a function of $s - \beta ct$, and thus would lead to Ψ being a slowly varying function of its last argument (t), at least on the scale of passing through a single object).

A.3 Hamiltonian

The Hamiltonian doesn't give the integrated force, but the force. However, if the coordinates change very little with time (except for s, in which case $s - \beta ct$ changes little with time), then the integrated force will be a gradient of the integrated Hamiltonian, with s replaced by $z + \beta ct$. We approximate this by assuming that the time (t) dependence of Ψ is negligible, and defining a function $w_\alpha(z, s)$ such that

$$W_\alpha(z) = \int w_\alpha(z, s) \, ds = \beta c \int w_\alpha(z, \beta ct) \, dt \qquad (A.9)$$

where the integral is over the length of the object in question. There are in general several objects placed in various locations around the ring, and so the second function of the second argument to w_α is to indicate which object is being looked at. Therefore the term in the Hamiltonian which gives an integrated force defined by S is

$$V_\alpha(\boldsymbol{z}, s, t) = q_1 q_2 N_2 \bar{f}_\alpha(\boldsymbol{z}, s) \int w_\alpha(s - \bar{s}, s) g_\alpha(\bar{\boldsymbol{z}}, \bar{s}) \Psi(\bar{\boldsymbol{z}}, \bar{s} - \beta ct, t) \, d^5\bar{\boldsymbol{z}} \, d\bar{s} \quad (A.10)$$

A.4 Expansion of Hamiltonian

Now, begin with the Hamiltonian

$$H = c\sqrt{\left[\boldsymbol{p}_\perp - \frac{q}{c}\boldsymbol{A}_\perp(\boldsymbol{x}_\perp, s, t)\right]^2 + \left[\frac{p_s}{1 + x/\rho(s)} - \frac{q}{c}A_s(\boldsymbol{x}_\perp, s, t)\right]^2 + (m_q c)^2}$$
$$+ qV(\boldsymbol{x}_\perp, \boldsymbol{p}_\perp, p_s, s, t) \quad \text{(A.11)}$$

Here $\rho(s)$ is the local radius of curvature of the design trajectory for particles with mass m_q and charge q. The $\hat{s}$ direction is chosen to lie along the design trajectory, and s is the arc length along that trajectory. This $\hat{s}$ direction is assumed to coincide with the $\hat{s}$ direction defined in section A.2. Converting to s as the independent variable,

$$\mathcal{H} = -\frac{q}{c}\left(1 + \frac{x}{\rho(s)}\right)A_s(\boldsymbol{x}_\perp, s, t) - \left(1 + \frac{x}{\rho}\right)$$
$$\sqrt{\left(\frac{H - V}{c}\right)^2 - \left(\boldsymbol{p}_\perp - \frac{q}{c}\boldsymbol{A}_\perp(\boldsymbol{x}_\perp, s, t)\right)^2 - (m_q c)^2} \quad \text{(A.12)}$$

The sign of the square root has been chosen such that particles with positive H have $dt/ds > 0$. Note that the arguments of V have been dropped for simplicity. Scaling $\boldsymbol{p}_\perp$ and the Hamiltonian $\mathcal{H}$ by $p = \beta\gamma m_q c$ and defining $z = -\beta ct$ and $H = \gamma m_q c^2(1 + \beta^2\delta)$ the Hamiltonian can be written as

$$\mathcal{H} = -\frac{q}{pc}\left(1 + \frac{x}{\rho(s)}\right)A_s(\boldsymbol{x}_\perp, s, t) - \left(1 + \frac{x}{\rho(s)}\right)$$
$$\sqrt{(1 + \delta)^2 - \frac{\delta^2}{\gamma^2} - \left(\boldsymbol{p}_\perp - \frac{q}{pc}\boldsymbol{A}_\perp(\boldsymbol{x}_\perp, s, t)\right)^2 - \frac{2V}{\beta pc}(1 + \beta^2\delta) + \left(\frac{V}{pc}\right)^2} \quad \text{(A.13)}$$

Expanding the square root and keeping terms only to first order in V,

$$\mathcal{H} = -\frac{q}{pc}\left(1 + \frac{x}{\rho(s)}\right)A_s(\boldsymbol{x}_\perp, s, t)$$
$$- \left(1 + \frac{x}{\rho(s)}\right)\sqrt{(1 + \delta)^2 - \frac{\delta^2}{\gamma^2} - \left(\boldsymbol{p}_\perp - \frac{q}{pc}\boldsymbol{A}_\perp(\boldsymbol{x}_\perp, s, t)\right)^2}$$
$$+ \left(1 + \frac{x}{\rho(s)}\right)\left[(1 + \delta)^2 - \frac{\delta^2}{\gamma^2} - \left(\boldsymbol{p}_\perp - \frac{q}{pc}\boldsymbol{A}_\perp(\boldsymbol{x}_\perp, s, t)\right)^2\right]^{-1/2}\frac{V}{\beta pc}(1 + \beta^2\delta)$$
$$\text{(A.14)}$$

Writing out V,

$$
\mathcal{H} = -\frac{q}{pc}\left(1 + \frac{x}{\rho(s)}\right) A_s(\boldsymbol{x}_\perp, s, t)
$$

$$
- \left(1 + \frac{x}{\rho(s)}\right)\sqrt{(1+\delta)^2 - \frac{\delta^2}{\gamma^2} - \left(\boldsymbol{p}_\perp - \frac{q}{pc}\boldsymbol{A}_\perp(\boldsymbol{x}_\perp, s, t)\right)^2} + \frac{q^2 N}{\beta pc}\left(1 + \frac{x}{\rho(s)}\right)
$$

$$
\left[(1+\delta)^2 - \frac{\delta^2}{\gamma^2} - \left(\boldsymbol{p}_\perp - \frac{q}{pc}\boldsymbol{A}_\perp(\boldsymbol{x}_\perp, s, t)\right)^2\right]^{-1/2} (1 + \beta^2\delta)\sum_\alpha \bar{f}_\alpha(\boldsymbol{x}_\perp, \boldsymbol{p}_\perp, \delta, s)
$$

$$
\int w_\alpha(z - \bar{z}, s)g_\alpha(\bar{\boldsymbol{x}}_\perp, \bar{\boldsymbol{p}}_\perp, \bar{\delta}, s)\Psi(\bar{\boldsymbol{x}}_\perp, \bar{\boldsymbol{p}}_\perp, \bar{z}, \bar{\delta}, s)\, d^2\bar{\boldsymbol{x}}_\perp d^2\bar{\boldsymbol{p}}_\perp\, d\bar{z}\, d\bar{\delta} \quad \text{(A.15)}
$$

Note that the arguments of $\bar{f}_\alpha$, g_α, and Ψ have been changed freely. Conversion from one to the other is accomplished by applying the transformation from one set of coordinates to the other.

At this point, it is important to clarify what is meant by the distribution function when s is the independent variable. Since our interactions are local to a given point in the ring s, we really want the particle density as a function of time at that given point s, and that is how the distribution function Ψ will be defined. If this is the case, then integrating the distribution over z will in fact give an infinite result, since the same particles repeatedly come back to that same point s on subsequent turns. Thus, the range of integration in $\bar{z}$ in (A.15) is infinite.

Assuming that $\boldsymbol{A}_\perp$ is independent of t, and therefore defining

$$
f_\alpha(\boldsymbol{x}_\perp, \boldsymbol{p}_\perp, \delta, s) = \left[(1+\delta)^2 - \frac{\delta^2}{\gamma^2} - \left(\boldsymbol{p}_\perp - \frac{q}{pc}\boldsymbol{A}_\perp(\boldsymbol{x}_\perp, s, t)\right)^2\right]^{-1/2}
$$

$$
\left(1 + \frac{x}{\rho(s)}\right)(1 + \beta^2\delta)\bar{f}_\alpha(\boldsymbol{x}_\perp, \boldsymbol{p}_\perp, \delta, s) \quad \text{(A.16)}
$$

the last term in the Hamiltonian simply becomes

$$
\frac{q^2 N}{\beta pc}\sum_\alpha f_\alpha(\boldsymbol{x}_\perp, \boldsymbol{p}_\perp, \delta, s)\int w_\alpha(z - \bar{z}, s)g_\alpha(\bar{\boldsymbol{x}}_\perp, \bar{\boldsymbol{p}}_\perp, \bar{\delta}, s)
$$

$$
\Psi(\bar{\boldsymbol{x}}_\perp, \bar{\boldsymbol{p}}_\perp, \bar{z}, \bar{\delta}, s)\, d^2\bar{\boldsymbol{x}}_\perp\, d^2\bar{\boldsymbol{p}}_\perp\, d\bar{z}\, d\bar{\delta} \quad \text{(A.17)}
$$

Note that near the origin in phase space, $f_\alpha \approx \bar{f}_\alpha$.

B   Single-Particle Hamiltonian for Standard Accelerator Lattices

An approximate expansion, valid for many accelerators, of the first two terms of the Hamiltonian (A.15) is derived in this appendix. The derivation assumes that

$\boldsymbol{A}_\perp = \boldsymbol{0}$. Terms in the Hamiltonian are taken out to third order in the canonical variables. Magnet edge effects are ignored, but they can often be treated through thin elements which fall into this framework.

Expanding A_s out to fourth order, the first term in (A.15) becomes

$$\frac{x}{\rho(s)} + \frac{1}{2}\frac{1}{\rho^2(s)}x^2 + \frac{K(s)}{2}(y^2 - x^2) - \frac{S(s)}{6}(x^3 - 3xy^2) - \frac{O(s)}{24}(x^4 - 6x^2y^2 + y^4) \tag{B.1}$$

In doing this, it has been assumed that the fields in bending magnets were exactly the value that they need to be to achieve a radius of curvature ρ for a particle following the ideal orbit. Upright quadrupoles, sextupoles, and octupoles have been considered, and they are located in straight sections. This is really meant as an example: the subsequent analysis (and the analysis in this paper) will require that the first three terms are of the form given, but the analysis can easily be extended to more general linear/quadratic terms in the Hamiltonian. The subsequent approximations will apply to higher order terms of a more general form than the ones given in (B.1).

The second term in (A.15) can be expanded out to second order in the canonical variables, giving

$$-1 - \frac{x}{\rho(s)} - \delta - \frac{\delta x}{\rho(s)} + \frac{\delta^2}{2\gamma^2} + \frac{p_x^2}{2} + \frac{p_y^2}{2} \tag{B.2}$$

There are also higher order terms in the expansion of (A.15). For the purposes of this analysis, it is only necessary to acknowledge their existence, since their exact form will not be important.

B.1 Transformation to Action-Angle Variables

Next, the Hamiltonian consisting of the sum of (B.1) and (B.2) is converted to the action-angle variables for the linear Hamiltonian. Using the usual accelerator physics convention, the transformations from the old variables $(x, p_x, y, p_y, z, \delta)$

to the new variables $(\theta_x, J_x, \theta_y, J_y, Z, \delta)$ are (δ doesn't change)

$$x = \delta D_x(s) + \sqrt{2J_x\beta_x(s)} \cos[\theta_x + \Delta\psi_x(s)] \tag{B.3}$$

$$p_x = \delta D_x'(s) - \sqrt{\frac{2J_x}{\beta_x(s)}}\{\sin[\theta_x + \Delta\psi_x(s)] + \alpha_x(s)\cos[\theta_x + \Delta\psi_x(s)]\} \tag{B.4}$$

$$y = \sqrt{2J_y\beta_y(s)} \cos[\theta_y + \Delta\psi_y(s)] \tag{B.5}$$

$$p_y = -\sqrt{\frac{2J_y}{\beta_y(s)}}\{\sin[\theta_y + \Delta\psi_y(s)] + \alpha_y(s)\cos[\theta_y + \Delta\psi_y(s)]\} \tag{B.6}$$

$$z = Z + D_x(s)\sqrt{\frac{2J_x}{\beta_x(s)}}\{\sin[\theta_x + \Delta\psi_x(s)] + \alpha_x(s)\cos[\theta_x + \Delta\psi_x(s)]\} \tag{B.7}$$

$$+ D_x'(s)\sqrt{2J_x\beta_x(s)} \cos[\theta_x + \Delta\psi_x(s)] + \int_0^s \left[\eta - \frac{D_x(s)}{\rho(s)} + \frac{1}{\gamma^2}\right] ds \tag{B.8}$$

where the betatron functions $\beta_x(s)$, $\beta_y(s)$, $\alpha_x(s)$, $\alpha_y(s)$ and the dispersion function $D_x(s)$ are all defined in the usual fashion,

$$\Delta\psi_x(s) = \int_0^s \frac{ds}{\beta_x(s)} - \frac{2\pi\nu_x s}{L} \tag{B.9}$$

and analogously for $\Delta\psi_y(s)$, and

$$\eta = \frac{1}{L}\int_0^L \left[\frac{D_x(s)}{\rho(s)} - \frac{1}{\gamma^2}\right] ds \tag{B.10}$$

ν_x is defined as usual to be the tune of the accelerator. Note that by definition $\Delta\psi_x(s)$ is periodic in s, with period L, where L is the circumference of the ring.

When these transformations are made, the Hamiltonian becomes

$$-\delta - \frac{\eta\delta^2}{2} + \frac{2\pi\nu_x J_x}{L} + \frac{2\pi\nu_y J_y}{L} \tag{B.11}$$

plus some terms which are third order in the original phase space coordinates. There are several types of terms:

1. Terms which only depend on δ. These terms give a more accurate (higher order) representation of the purely longitudinal motion. They will be ignored in this paper.

2. Terms which are proportional to a power of δ and one of $\sqrt{J_x}$ or $\sqrt{J_y}$ (but not both). Such terms correspond to a closed orbit distortion which depends on δ. They are the equivalent of the dispersion, but to higher order in δ. They should have been eliminated before transforming into action-angle variables. Their effect will be ignored in this paper.

26

3. Terms which are proportional to $J_x\delta$ or $J_y\delta$ and are independent of $\boldsymbol{\theta}$. These terms have an s-dependent coefficient which can be made independent of s by a simple transformation of $\boldsymbol{\theta}$ and Z, giving a contribution to the Hamiltonian of

$$\frac{2\pi\nu_x\xi_x J_x\delta}{L} + \frac{2\pi\nu_y\xi_y J_y\delta}{L} \tag{B.12}$$

The $\xi_{x,y}$ are the chromaticities. The transformation of $\boldsymbol{\theta}$ and Z will be ignored in this paper.

4. Terms which are proportional to δ^2 and J_x or J_y, but are independent of $\boldsymbol{\theta}$. They give a contribution to the Hamiltonian similar to the contribution from the chromaticity terms. Their effects will be ignored in this paper.

5. Terms which are purely a function of $\boldsymbol{J}$ and s. Again, these terms can be made independent of s via a simple transformation of $\boldsymbol{\theta}$. Once this is done, they add a term to the Hamiltonian which is of the form

$$\frac{2\pi}{L}\left(\frac{1}{2}\bar{\alpha}_{xx}J_x^2 + \bar{\alpha}_{xy}J_x J_y + \frac{1}{2}\bar{\alpha}_{yy}J_y^2\right) \tag{B.13}$$

Again, the transformation of $\boldsymbol{\theta}$ will be ignored in this paper.

6. Other nonlinear terms which will depend on $\boldsymbol{\theta}$. In second order, they may give contributions to the pure-$\boldsymbol{J}$ terms, but otherwise will have no effect on the Hamiltonian to the order given above. They will introduce coordinate transformations and contributions to higher order terms in the Hamiltonian which will be ignored in this paper.

For the purposes of this paper, the Hamiltonian will be taken to be

$$-\delta - \frac{\eta\delta^2}{2} + \frac{2\pi\nu_x J_x}{L}(1 + \xi_x\delta) + \frac{2\pi\nu_y J_y}{L}(1 + \xi_y\delta)$$
$$+ \frac{2\pi}{L}\left(\frac{1}{2}\bar{\alpha}_{xx}J_x^2 + \bar{\alpha}_{xy}J_x J_y + \frac{1}{2}\bar{\alpha}_{yy}J_y^2\right) \tag{B.14}$$

and the coordinate transformations from the canonical coordinates used here to the original canonical coordinates are given by (B.3)–(B.8).

Many of the terms that have been ignored here can be included in this formalism straightforwardly. Others would require the use of some of the more general forms of the formalism, and might require different approximations to be made than those that are made in this paper. It appears that in most cases, however, these corrections would be of the same order as the relative size of the perturbation that they generate. One possible exception will occur when the chromaticity is extremely small, in which case certain symmetries of the problem may be broken by the inclusion of these extra terms.

C   Evaluation of Hypergeometric Functions

The hypergeometric functions which appear in this paper can all be evaluated in the case where μ is an integer or half integer, where μ is the parameter in the parabolic-like distribution that corresponds to the hypergeometric function in question. First of all, one should be aware that this is often not the optimal method for evaluating the hypergeometric function numerically; it often leads to numerical instability. For large indices, the formulas here tend to be good for large z, but fail for smaller z. For very small z, one should just use the series expansion about the origin. For medium sizes of z (near 1), one should use the transformation formulas in [AS72] to transform to a hypergeometric function where the argument is either small or large.

We begin with

$$F\left(1, \frac{1}{2}; \frac{3}{2}; z^2\right) = \frac{1}{2z} \ln\left(\frac{1+z}{1-z}\right) \tag{C.1}$$

$$F\left(1, \frac{1}{2}; 2; z\right) = \frac{2}{z}(1 - \sqrt{1-z}) \tag{C.2}$$

$$F(1, 1; 2; z) = -\frac{1}{z} \ln(1-z) \tag{C.3}$$

$$F\left(1, 1; \frac{3}{2}; z^2\right) = \begin{cases} \dfrac{1}{2z}\sqrt{\dfrac{z}{z-1}} \ln\left(\dfrac{1 - \sqrt{\dfrac{z}{z-1}}}{1 + \sqrt{\dfrac{z}{z-1}}}\right) & z \notin [0, \infty) \\ \dfrac{1}{\sqrt{z(1-z)}} \tan^{-1}\sqrt{\dfrac{z}{1-z}} = \dfrac{\cos^{-1}(1-2z)}{2\sqrt{z(1-z)}} & 0 \leqslant z < 1 \end{cases} \tag{C.4}$$

The branch cuts for ln and the square root are taken to be the negative real axis. We then use the recursion relation

$$F(1, b+m; c+n; z) =$$

$$\frac{\Gamma(b)\Gamma(c+n)\Gamma(c-b)}{\Gamma(b+m)\Gamma(c)\Gamma(c-b+n-m)} \frac{(z-1)^{n-m}}{z^n} F(1, b; c; z)$$

$$+ \frac{\Gamma(b)}{\Gamma(b+m)} \sum_{k=0}^{n-1} \frac{\Gamma(c+n)\Gamma(c-b+n-k-1)}{\Gamma(c+n-k-1)\Gamma(c-b+n-m)} \frac{(z-1)^{k-m}}{z^{k+1}}$$

$$- (c+n-1) \sum_{k=0}^{m-1} \frac{\Gamma(b+k)\Gamma(c-b+n-k-1)}{\Gamma(b+m)\Gamma(c-b+n-m)} (z-1)^{k-m} \tag{C.5}$$

The case where $c - b + n - m \leqslant 0$ and is an integer is covered by

$$F(1, c+m; c; z) = (c-1) \sum_{k=0}^{m} \frac{\Gamma(c+m-k-1)m!}{\Gamma(c+m)(m-k)!} (1-z)^{-k-1} \tag{C.6}$$

which holds for $c > 1$.

These formulas can be derived from the relations found in [AS72].

D   Coherent and Incoherent Terms in Averages

Consider a function $f(z)$. We wish to consider the average of the summation

$$F = \sum_{l=0}^{N-1} f(z_l) \tag{D.1}$$

and its square, where each of the N z_l are independently distributed according to the distribution function $\rho(z)$. First of all, it is straightforward to compute

$$\langle F \rangle = N \langle f \rangle \qquad\qquad \langle f \rangle = \int f(z) \rho(z) \, dz \tag{D.2}$$

Now, what about $\langle |F|^2 \rangle$? First of all, write the summation for $\langle |F|^2 \rangle$ as

$$\langle |F|^2 \rangle = \sum_{l=0}^{N-1} \sum_{\bar{l}=0}^{N-1} f(z_l) f^*(z_{\bar{l}}) = \sum_{l=0}^{N-1} \sum_{\substack{\bar{l}=0 \\ \bar{l} \neq l}}^{N-1} f(z_l) f^*(z_{\bar{l}}) + \sum_{l=0}^{N-1} |f(z_l)|^2 \tag{D.3}$$

Assuming that each z_l is distributed according to $\rho(z)$ and averaging, we find

$$\langle |F|^2 \rangle = N(N-1) \langle f \rangle^2 + N \int |f(z)|^2 \, \rho(z) \, dz \tag{D.4}$$

The first term term provides no information than $\langle F \rangle$ did; it is essentially (for large N) $\langle F \rangle^2$. However, the second term is different: it gives a different value, which is proportional to N instead of N^2. The first term we will call the "coherent term" (it comes from all the particles acting in the same way), and the second term will be called the "incoherent term" (it comes from all the particles acting independently).

Schottky spectra are the result of the incoherent term. They provide a signal over a range of frequencies that the coherent term (i.e., the average signal) does not, and also give access to information that cannot be obtained from the coherent signal.

E   Exponential Summations

The double summation

$$\sum_{k=-K^-}^{K^+} \sum_{\bar{k}=-K^-}^{K^+} e^{i\alpha(k-\bar{k})} \tag{E.1}$$

appears in our computation of Schottky spectra. We are generally interested in what happens to this summation when K^+ and K^- become large. The summation can be computed in closed form: it is

$$\frac{\sin^2 \dfrac{K^+ + K^- + 1}{2}\alpha}{\sin^2 \dfrac{\alpha}{2}} \tag{E.2}$$

This expression is not a particularly useful form. However, note several facts about the expression:

1. It is periodic in α with period 2π.

2. Its integral from $-\pi$ to π is $2\pi(K^+ + K^- + 1)$.

3. The peak value of the expression is at $\alpha = 0$, and is $(K^+ + K^- + 1)^2$.

4. For large $(K^+ + K^- + 1)$, the function oscillates rapidly, with an approximate period of $2\pi/(K^+ + K^- + 1)$

5. The height of the non-central peaks is $\csc^2(\alpha/2)$; for a given α, the ratio of the peak height to the central peak height decreases as $1/(K^+ + K^- + 1)^2$

6. For large $(K^+ + K^- + 1)$, the area under the central peak is approximately $\mathrm{Si}(2\pi)(K^+ + K^- + 1) \approx 1.4(K^+ + K^- + 1)$.

Thus, we have a function with period 2π, which is strongly peaked near $\alpha = 0$ and its periodic counterparts, and the area under which is $2\pi(K^+ + K^- + 1)$. The area under the narrow central peak (it narrows as $(K^+ + K^- + 1) \to \infty$) is a substantial fraction of the total area of the function. Thus, the function behaves much like a δ-function. Thus, we can write the summation in the limit of large $(K^+ + K^- + 1)$ as

$$2\pi(K^+ + K^- + 1) \sum_{k=-\infty}^{\infty} \delta(\alpha - 2\pi k) \tag{E.3}$$

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