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Masses of the Goldstone modes in the CFL phase of QCD at finite density

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Abstract

We construct the $U_L(3) \times U_R(3)$ effective lagrangian which encodes the dynamics of the low energy pseudoscalar excitations in the Color-Flavor-Locking superconducting phase of QCD at finite quark density. We include the effects of instanton-induced interactions and study the mass pattern of the pseudoscalar mesons. A tentative comparison with the analytical estimate for the gap suggests that some of these low energy momentum modes are not stable for moderate values of the quark chemical potential μ.

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I. INTRODUCTION

The fascinating possibility that the ground state of QCD at finite quark density can behave as a color superconductor has attracted much attention recently [1–14]. Of particular interest is the symmetry breaking pattern that may arise for $N_f = 3$ light quark flavors. As first shown by Alford, Rajagopal and Wilczek [4], the diquark condensates can lock the color and flavor symmetry transformations (Color-Flavor-Locking or CFL for brief). In the limit of massless quarks, these condensates spontaneously break both color and chiral symmetries. The eight gluons become massive through the Higgs mechanism while eight plus two Goldstone bosons are leftover. The two octets of states are analogous to the octets of vector and light pseudoscalar mesons in vacuum and Schäfer and Wilczek [5] have conjectured that there might exist some sort of continuity between the properties of QCD at zero density and in the CFL phase. The two other massless excitations are the Goldstone modes associated with the spontaneous breaking of the baryon number ($U(1)_B$) and axial ($U(1)_A$) symmetries. Because of the axial anomaly, the latter is not a true symmetry of QCD. However, because instantons effects are small at large densities, the associated mode can be treated as a true Goldstone mode as a first approximation. Although there are significant differences, to which we shall come back later, the situation at high density is analogous to considering the limit of large numbers of colors N_c in vacuum [15], in which $U(1)_A$ breaking effects are $1/N_c$ suppressed.

For energies which are small compared to the gap of the superconducting CFL phase, the dynamics of these Goldstone modes is most conveniently described with the help of an effective lagrangian. As suggested by the symmetry breaking pattern in the CFL phase at large densities, this effective theory is analogous to Chiral Perturbation Theory (χPT) in the large N_c limit of QCD in vacuum [16]. For large densities, the leading order effective lagrangian invariant under $U(3)_L \times U(3)_R$ flavor symmetry has been constructed in some recent works [17–20]. In particular, Son and Stephanov [20] have shown how the parameters of the lagrangian can be computed at large densities by matching to the underlying microscopic theory. In the present note we make a first attempt to extend these works to lower density regimes, taking into account the effects of instanton-induced interactions. By using the power counting rules of χPT we construct the effective lagrangians up to order E^4 in an energy expansion. We make a particular emphasis on the meson spectrum and compute their masses as function of the quark chemical potential μ. For moderate densities, $\mu \lesssim 10^4$ MeV, the gauge coupling grows large and instanton effects become important. We give numerical values for the masses of the mesons simply assuming that the analytical expressions for the gap and condensates, which have been computed at weak gauge coupling, also hold at strong coupling. Taking into account the running of the gauge coupling constant at one-loop, and using $\Lambda = 200$ MeV, our calculations suggests that the pions can exist as low energy excitations in a narrow window of quark chemical potentials, $600 \lesssim \mu \lesssim 2500$ MeV, but that their masses are very close to the instability threshold, $M_\pi \gtrsim 0.75 \times (2\Delta)$, where Δ is the gap in the CFL phase. The kaons are always unstable, except maybe at asymptotically large densities. These conclusions depend on the value of the color superconducting gap as estimated in the literature, and of a number of approximations that we have made. It seems nevertheless that if the gap is substantially smaller than current estimates indicate, there can be no stable pseudoscalar excitations in the CFL phase. If on the other hand the gap is larger, there is probably a spontaneously broken approximate $SU(2) \times SU(2)$ symmetry at low densities.

In the next section, we write down the general low energy effective action for the pseudoscalar Goldstone modes up to E^4 in a low energy expansion and estimate the couplings that are relevant for the meson mass spectrum. In Sec. III we give some numbers for the meson masses at low densities, $\mu \lesssim 4000$ MeV and finally draw some conclusions.

II. EFFECTIVE CHIRAL LAGRANGIAN IN THE CFL PHASE

We follow Gatto and Casualboni [18] and Son and Stephanov [20] to construct the effective lagrangian for the low energy excitations of the CLF phase of QCD. The ground state is characterized by the two diquark condensates

$$X^{ia} \sim \epsilon^{ijk}\epsilon^{abc}\langle\psi_L^{bj}\psi_L^{ck}\rangle^* \ , \qquad Y^{ia} \sim \epsilon^{ijk}\epsilon^{abc}\langle\psi_R^{bj}\psi_R^{ck}\rangle^* \ , \tag{2.1}$$

where a, b, c denote color indices, while i, j, k refer to flavor ones. Under an $SU(3)_c \times SU(3)_L \times SU(3)_R$ the condensates transform as

$$X \to U_L X U_c^\dagger \ , \qquad Y \to U_R Y U_c^\dagger \ , \tag{2.2}$$

and as

$$X \to e^{2i\alpha}\, e^{2i\beta} X \ , \qquad Y \to e^{-2i\alpha}\, e^{2i\beta} Y \ , \tag{2.3}$$

under $U(1)_A$ and $U(1)_B$ transformations defined as

$$\psi_L \to e^{i(\alpha+\beta)}\psi_L \ , \qquad \psi_R \to e^{i(-\alpha+\beta)}\psi_R \ . \tag{2.4}$$

One can factor out the norm of the condensates and consider the unitary matrices X and Y. The slow variations of the phases of these matrices then correspond to the low energy excitations. Altogether these are $9 + 9 = 18$ degrees of freedom: 8 will be absorbed by the gluons through the Higgs mechanism, which leaves 10 true low energy excitations. At low energies the gluons decouple from the theory, as they are heavy degrees of freedom. It is convenient to collect all the Goldstone modes in the unitary matrix

$$\Sigma = XY^\dagger \ , \tag{2.5}$$

which is singlet of $SU(3)_c$ and $U(1)_B$ and transforms as

$$\Sigma \to e^{4i\alpha}\, U_L \Sigma U_R^\dagger \ , \tag{2.6}$$

under $SU(3)_L \times SU(3)_R \times U(1)_A$. This leaves apart the low energy excitation that emerges from the spontaneous breaking of baryon number symmetry $U(1)_B$. Because baryon number is an exact global symmetry of QCD, this Goldstone mode is always massless in the CFL phase, independent of the quark masses. As our focus here is on the effect of chiral symmetry breaking by finite quark masses, to simplify our discussion we will simply drop this degree of freedom in the sequel.

To proceed we parametrise the unitary matrix Σ as

$$\Sigma = \exp\left(i\,\frac{\Phi}{f_\pi}\right) \ , \tag{2.7}$$

where $\Phi = \phi^A T^A$, $A = 1, \ldots 9$. The T^A for $A = 1, \ldots, 8$ are the Gell-Mann generators of $SU(3)$ and $T^9 = \sqrt{2/3}\,\mathbf{1}_3$, all normalized to $\mathrm{Tr}(T^A T^B) = 2\delta^{AB}$. We use the same nomenclature as in vacuum for the nonet of Goldstone modes,

$$\Phi = \begin{pmatrix} \pi_0 + \frac{1}{\sqrt{3}}\left(\eta_8 + \sqrt{2}\frac{f_\pi}{f_{\eta_0}}\eta_0\right) & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi_0 + \frac{1}{\sqrt{3}}\left(\eta_8 + \sqrt{2}\frac{f_\pi}{f_{\eta_0}}\eta_0\right) & \sqrt{2}K_0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}_0 & \frac{1}{\sqrt{3}}\left(-2\eta_8 + \sqrt{2}\frac{f_\pi}{f_{\eta_0}}\eta_0\right) \end{pmatrix} . \tag{2.8}$$

In (2.8), f_π and f_{η_0} are the decay constants at finite density respectively of the octet of mesons and of the singlet η^0. *A priori* there is no reason for these to be equal. By matching to the microscopic theory, Son and Stephanov [20] have found

$$f_\pi^2 = \frac{21 - 8\log 2}{18}\,\frac{\mu^2}{2\pi^2} \ , \qquad f_{\eta_0}^2 = \frac{3}{4}\,\frac{\mu^2}{2\pi^2} \ , \tag{2.9}$$

where μ is the quark chemical potential. These expressions are valid at large densities (*i.e.* small gauge coupling). From (2.9), the ratio $f_\pi/f_{\eta_0} \approx 1.07$ is close to one. This result is reminiscent of the OZI rule often invoked to hold in vacuum. At smaller densities, the gauge coupling grows large and the ratio could significantly depart from unity because of instanton effects.

The quark mass term in the microscopic lagrangian

$$\Delta\mathcal{L} = -\bar{\psi}_L \mathcal{M} \psi_R + h.c. \tag{2.10}$$

breaks chiral symmetry. Its effects can be introduced in the effective lagrangian by treating the mass matrix as an external field with vacuum expectation value

$$\mathcal{M} = \mathrm{diag}(m_u, m_d, m_s) \ , \tag{2.11}$$

where m_u, m_d and m_s refer to the up, down and strange quark masses, respectively. The spurion field then transforms as

$$\mathcal{M} \to e^{-2i\alpha} U_L^\dagger \mathcal{M} U_R \ , \tag{2.12}$$

under $SU(3)_L \times SU(3)_R \times U(1)_A$.

To take into account the effects of the $U(1)_A$ anomaly, we also allow for the presence of the θ term in the microscopic lagrangian,

$$\Delta\mathcal{L}_\theta = \theta \frac{g^2}{32\pi^2} F^A_{\mu\nu} \tilde{F}^{\mu\nu,A} \ . \tag{2.13}$$

As for the quark mass term, we will treat θ as an external field with vanishing expectation value which, to account for the variation of the quark measure, transforms as

$$\theta \to \theta - 2N_f\alpha \equiv \theta - 6\,\alpha \tag{2.14}$$

under $U(1)_A$ transformations. Then any arbitrary function of the combination

$$X = \theta - \frac{i}{2}\mathrm{Tr}\log\Sigma \equiv \theta + \sqrt{\frac{3}{2}}\frac{\eta_0}{f_{\eta_0}}, \tag{2.15}$$

is invariant under $SU(3)_L \times SU(3)_R \times U(1)_A$ transformations.

With these ingredients, we are almost ready to construct the low energy effective lagrangian. One last issue is that of power counting. In QCD in vacuum, because $M_\pi^2 \sim m_q$, the expansion is in powers of the pion external energy or momenta and quark masses and $E^2 \sim p^2 \sim m_q$. Similarly, at large N_c, $U(1)_A$ breaking effects are counted as $E^2 \sim 1/N_c$, because $M_{\eta_0}^2 \sim 1/N_c$. In the CFL phase of QCD on the other hand, the power counting depends very much on the density. At finite densities, instantons effects are screened $\propto (\Lambda/\mu)^\alpha$, where $\Lambda \approx 200$ MeV is the scale of QCD, μ is the quark chemical potential and α is some positive constant which depends on the process under consideration. This has two consequences at large densities. The first is that $U(1)_A$ is essentially a good symmetry and the η_0 is approximately massless in the chiral limit. Then, because the two condensates break chiral symmetry only through the intermediate of color transformations, there is an approximate $Z_2^L \times Z_2^R$ symmetry which acts independently on the left and right-handed quarks [4]. At high densities, this implies that the leading contribution to the mass of the Goldstone modes is $\mathcal{O}(m_q^2)$. The natural power counting rule at high densities is then $E^2 \sim m_q^2$ because $M_\pi^2 \sim m_q^2$. At the moderate densities that could eventually be of interest for heavy ion collisions or for neutron stars, instantons effects are likely to be non-negligible. As $Z_2^L \times Z_2^R$ symmetry is broken to the diagonal Z_2 by instantons, a meson mass term $M_\pi^2 \sim m_q$ is allowed [4]. If instanton effects are dominant, the power counting is that of vacuum, $E^2 \sim m_q$. For convenience, we will adopt the counting rules of χPT as in vacuum and comment when necessary about the differences that arise in the CFL phase of QCD.

A final remark concerning the region of applicability of the energy expansion. In vacuum, the low energy expansion is valid for $E^2 \lesssim f_\pi^2$. In the CFL phase however, $f_\pi \sim \mu$, which is independent and much larger than the gap ($\Delta \ll \mu$) at large densities (see (2.9)). However, the effective theory must break down for $E \sim 2\Delta$, which is the energy necessary to excite one quasiparticle pair out of the superconducting ground state, and the low energy expansion is only valid for the more restrictive range $E^2 \sim M_\pi^2 \lesssim 4\Delta^2$ ‡.

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‡There is a formal analogy between this behavior and that of χPT in vacuum in the large N_c limit. *A priori* the expansion is valid for $E < f_\pi$. But because $f_\pi^2 \sim N_c\Lambda^2$ grows large as $N_c \to \infty$ and as something is bound to happen for $E \sim \Lambda \sim 200$ MeV, which is the scale of quark confinement, the low energy expansion is limited to $E \lesssim \Lambda$.

At order E^2, the most general lagrangian compatible with the symmetries of the condensates is

$$\begin{aligned}\mathcal{L}_2 &= V_1(X)\mathrm{Tr}\left(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right) + \left\{V_2(X)\mathrm{Tr}\left(\mathcal{M}\Sigma^\dagger\right)e^{i\theta} + h.c.\right\} \\ &\quad + V_3(X)\left(\mathrm{Tr}\,\Sigma\partial_\mu\Sigma^\dagger\right)^2 + V_4(X)\left(\mathrm{Tr}\,\Sigma\partial_\mu\Sigma^\dagger\right)\partial^\mu\theta + V_5(X)\partial_\mu\theta\partial^\mu\theta\ . \end{aligned} \tag{2.16}$$

We have written (2.16) using a compact notation. Since Lorentz invariance is broken at finite density (only a $O(3)$ symmetry is preserved) all the four-vectors should be split into temporal and spatial components, and the functions V_i multiplying those spatial or temporal components are not forced to be the same by symmetry arguments. For example, the first term in (2.16) should read

$$V_1(X)\mathrm{Tr}\left(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right) \to V_{1,t}(X)\mathrm{Tr}\left(\partial_0\Sigma\partial_0\Sigma^\dagger\right) - V_{1,s}(X)\mathrm{Tr}\left(\partial_i\Sigma\partial_i\Sigma^\dagger\right)\ . \tag{2.17}$$

The same situation occurs for the remaining terms and functions V_3, V_4, V_5 of X. All the couplings in (2.16) can be *a priori* arbitrary functions of X (2.15). At large densities, $U(1)_A$ symmetry breaking effects are exponentially suppressed and the couplings only depend on the chemical potential μ.

To reproduce the standard normalization of the meson kinetic terms, we impose

$$V_{1,t}(0) = \frac{f_\pi^2}{4}\ . \tag{2.18}$$

The ratio $V_{1,s}(0)/V_{1,t}(0) = v^2$ is the velocity squared of the Goldstone bosons. At large densities, Son and Stephanov [20] have found that v is equal to the speed of sound $1/\sqrt{3}$ for all the low energy modes, including the baryon Goldstone mode.

The operators in (2.16) are not all independent. The last three terms can be transformed into each other with a field redefinition, $\eta_0/f_{\eta_0} \to \eta_0/f_{\eta_0} + \kappa\theta$. Using this freedom, we can choose to set $V_4(0)$ to zero. With this choice, the last operator becomes irrelevant for the meson spectrum and can be discarded. The operator that is left, with coupling $V_3(X)$, contributes to the difference between f_π and f_{η_0}. At high densities, using (2.9),

$$V_{3,t}(0) = \frac{f_\pi^2 - f_{\eta_0}^2}{12} \approx 0.01 f_\pi^2, \tag{2.19}$$

which is small compared to f_π^2. At moderate densities, $V_{3,t}$ could receive large contributions from instantons as is manifest from the mixing with V_4 and V_5.

Consider now the mass term in (2.16). This term is analogous to the leading mass term in χPT in vacuum. The only difference is the occurrence of the phase θ, which is absent in vacuum. This is because at zero density the condensate that breaks chiral symmetry is $\langle\bar\psi_L\psi_R\rangle$ which transforms like $\mathcal{M}^\dagger$ under $SU(3)_L \times SU(3)_R \times U(1)_A$. In (2.16) the presence of the θ in the effective lagrangian is the trademark of a one instanton effect; in the CFL phase, a one instanton process can be saturated by closing its six external quark legs with the insertion of one left-handed diquark, one right-handed diquark and one chiral condensate, $\sim (\psi_L\psi_L)(\bar\psi_R\bar\psi_R)(\bar\psi_R\psi_L)$. As in vacuum, a non-zero chiral condensate $\langle\psi\psi\rangle$

leads to $M_\pi^2 \propto m_q\langle\bar\psi\psi\rangle$. Instanton effects are small and can be reliably computed at high densities $\mu \gg \Lambda$. In particular, Schäfer [13] has obtained

$$\begin{aligned}\langle\bar\psi\psi\rangle &= \langle\bar u u + \bar d d + \bar s s\rangle \\ &\approx -2\left(\frac{\mu^2}{2\pi^2}\right)\frac{3\sqrt{2}\pi}{g(\mu)}\,\frac{18}{5}G(\mu)\phi_A^2(\mu)\ ,\end{aligned} \tag{2.20}$$

where $g(\mu)$ is the gauge coupling estimated at the scale μ using the one-loop beta function. The factor $G(\mu)$ is the one instanton weight integrated over all instanton sizes ρ, which are peaked around $\rho \sim \mu^{-1}$ at finite density,

$$G(\mu) \approx 0.26\,\Lambda^{-5}\ \left((\beta_0\log(\mu/\Lambda)\right)^{2N_c}\left(\frac{\Lambda}{\mu}\right)^{\frac{\beta_0+5}{2}}\ . \tag{2.21}$$

We have used the running of $g(\mu)$ at one-loop, $\Lambda = 200$ MeV and $\beta_0 = 11/3\,N_c - 2/3\,N_f \equiv 9$ is the first coefficient of the QCD beta function. Finally, $\phi_A(\mu) \sim \langle\psi_L\psi_L\rangle \sim \langle\psi_R\psi_R\rangle$ is the diquark condensate in the CFL phase of QCD,

$$\phi_A(\mu) \approx 2\left(\frac{\mu^2}{2\pi^2}\right)\left(\frac{3\sqrt{2}\pi}{g(\mu)}\right)\Delta(\mu)\ , \tag{2.22}$$

and [9,5,11]

$$\Delta(\mu) = b_0' 512\pi^4(2/N_f)^{5/2}\,g(\mu)^{-5}\,\mu\,\exp\left(-\frac{3\pi^3}{\sqrt{2}g(\mu)}\right)\ , \tag{2.23}$$

is the color superconducting gap. The factor b_0' is unknown but expected to be $\mathcal{O}(1)$. [11] At large densities $\mu \gg \Lambda$, instantons are suppressed, $\langle\bar\psi\psi\rangle$ goes to zero and the mass term of (2.16) vanishes. At the moderate densities that could be of interest for heavy ion collisions or neutron stars, this instanton effect is however not negligible. Using the Gell-Mann-Oakes-Renner relation, and $\langle\bar\psi\psi\rangle = \langle\bar u u + \bar d d + \bar s s\rangle \equiv 3\langle\bar u u\rangle$, we find

$$V_2(0) = -\frac{1}{6}\langle\bar\psi\psi\rangle\ . \tag{2.24}$$

At order E^4, there are a few more operators[§],

$$\begin{aligned}\mathcal{L}_4 = {} & K_1(X)\left[\mathrm{Tr}\left(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right)\right]^2 + K_2(X)\mathrm{Tr}\left(\partial_\mu\Sigma\partial_\nu\Sigma^\dagger\right)\mathrm{Tr}\left(\partial^\mu\Sigma\partial^\nu\Sigma^\dagger\right) \\ & + K_3(X)\mathrm{Tr}\left(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\partial_\nu\Sigma\partial^\nu\Sigma^\dagger\right) \\ & + \left\{K_4(X)\mathrm{Tr}\left(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right)\mathrm{Tr}\left(\mathcal{M}\Sigma^\dagger\right)e^{i\theta} + h.c.\right\} + \left\{K_5(X)\mathrm{Tr}\left(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\mathcal{M}\Sigma^\dagger\right)e^{i\theta} + h.c.\right\} \\ & + \left\{K_6(X)\det(\Sigma)\,\mathrm{Tr}\left(\mathcal{M}\Sigma^\dagger\right)\mathrm{Tr}\left(\mathcal{M}\Sigma^\dagger\right) + h.c.\right\} + K_7(X)\mathrm{Tr}\left(\mathcal{M}\Sigma^\dagger\right)\mathrm{Tr}\left(\mathcal{M}^\dagger\Sigma\right) \\ & + \left\{K_8(X)\det(\Sigma)\,\mathrm{Tr}\left(\mathcal{M}\Sigma^\dagger\mathcal{M}\Sigma^\dagger\right) + h.c.\right\}\ .\end{aligned} \tag{2.25}$$

[§] At this order, one could also consider adding a Wess-Zumino-Witten term [17].

Again, we have used a compact notation for the terms in K_i, $i = 1 \ldots 5$ which should be split into temporal and spatial components.

Like the term V_2 in (2.16), the terms K_4 and K_5 depend explicitly on θ and so are exponentially suppressed at high densities. Similarly, at large chemical potential μ, the remaining functions $K_i(X)$ reduce to constants $K_i(X = 0; \mu)$. In this limit, the mass pattern of the Goldstone bosons will be determined by the couplings K_6, K_7 and K_8 in (2.25). As first shown by Son and Stephanov [20], at large densities, these coupling constants can be computed by matching to the underlying microscopic theory.

In principle, a systematic and non-ambiguous strategy to compute the coefficients of the effective theory is to use the background field technique [22]. This consists in introducing external sources and symmetry breaking order parameters, and integrating out the quark fields. That is, one has to compute $\mathrm{Tr} \log D$, where D at finite density is the inverse of the Gorkov-Nambu quark propagator. At very high densities or weak gauge coupling, the exchange of gluons is suppressed and this amounts to a "simple" quark one-loop calculation. This computation would fix the coefficient of all the operators to arbitrary order in the meson fields. The strategy followed in [20] is simpler. They set the meson fields to zero, $\Sigma = \mathbf{1}_3$ in the operators of (2.25), and compared the shift in ground state energy induced by non-zero quark masses in both the effective and microscopic theories. *A priori* this approach is ambiguous, as different operators could contribute to the shift in ground state energy but in practice, to order E^4, enough constraints can be derived to completely fix the couplings K_6, K_7 and K_8. The diagrams in the microscopic theory are those of Fig.1. The *rhs* diagram can only contribute to K_6 and K_8. The *lhs* diagram can contribute to K_7. These couplings can be fixed by considering two different quark mass patterns,

$$\mathcal{M}_1 = m\mathbf{1}_3 \quad \text{and} \quad \mathcal{M}_2 = \mathrm{diag}(0, 0, m_s) = -\frac{m}{\sqrt{3}}\,\lambda^8 + \frac{m}{\sqrt{6}}\,\lambda^9 \ . \tag{2.26}$$

In the effective theory, these respective choices lead to the following shifts in ground state energy density

$$\Delta\varepsilon_1 = -\left\{(9m^2K_6 + h.c.) + 9m^2K_7 + (3m^2K_8 + h.c.)\right\} \ , \tag{2.27}$$

$$\Delta\varepsilon_2 = -\left\{(m_s^2K_6 + h.c.) + m_s^2K_7 + (m_s^2K_8 + h.c.)\right\} \ . \tag{2.28}$$

Comparing with the first diagram, Son and Stephanov [20] have computed

$$K_6 = -\left(\frac{11}{36} - \frac{\Delta_9\Delta_8}{9(\Delta_9^2 - \Delta_8^2)}\log\left|\frac{\Delta_9}{\Delta_8}\right|\right)\frac{\mu^2}{2\pi^2} = -\left(\frac{11}{36} + \frac{2}{27}\ln 2\right)\frac{\mu^2}{2\pi^2} = -0.357\frac{\mu^2}{2\pi^2} \ , \tag{2.29}$$

$$K_8 = \left(\frac{1}{6} - \frac{\Delta_9\Delta_8}{3(\Delta_9^2 - \Delta_8^2)}\ln\left|\frac{\Delta_9}{\Delta_8}\right|\right)\frac{\mu^2}{2\pi^2} = \left(\frac{1}{6} + \frac{2}{9}\ln 2\right)\frac{\mu^2}{2\pi^2} = 0.321\frac{\mu^2}{2\pi^2} \tag{2.30}$$

where we have used the relation $\Delta_9 = -2\Delta_8$ between the singlet (Δ_9) and octet (Δ_8) gaps. The second diagram gives

$$\frac{\Delta\varepsilon_1}{\Delta\varepsilon_2} = 3\frac{m^2}{m_s^2} \ , \tag{2.31}$$

which cannot be accomodated by K_7. The second diagram contributes to a trivial shift of the vacuum energy, which in the effective theory corresponds to the constant (independent of Σ) invariant operator

$$\Delta\mathcal{L} \propto \mathrm{Tr}(\mathcal{M}^\dagger \mathcal{M}) \tag{2.32}$$

and consequently $K_7 = 0$ to this order**. A non-zero K_7 probably requires a two-instanton process. To see this, consider a meson one-loop diagram with two insertions of the mass operator in (2.16). Such a diagram generates a divergence which can be absorbed by a counter-term $K_7^{c.t.}$.

The surprise is that K_6 and K_8 are of the same order and opposite in sign††. This implies in particular that the dominant meson mass operator at large densities is the invariant combination

$$\det(\mathcal{M})Tr(\mathcal{M}^{-1}\Sigma) \equiv \frac{1}{2}\left[\mathrm{Tr}(\mathcal{M}\Sigma^\dagger)\mathrm{Tr}(\mathcal{M}\Sigma^\dagger) - \mathrm{Tr}(\mathcal{M}\Sigma^\dagger\mathcal{M}\Sigma^\dagger)\right]\det\Sigma\ . \tag{2.33}$$

This term is quite interesting as it leads to an inversed meson mass pattern at large densities [20].

To be complete we should also include the contribution of instantons to the mass of the singlet meson η_0, which enters the effective lagrangian through a function of X alone,

$$\mathcal{L}_0 = -V_0(X)\ , \tag{2.34}$$

This piece is E^0 as it involves no derivatives of the meson fields. Expanding to second order in X gives

$$M_{\eta_0}^2 = \frac{3}{2}\frac{V_0^{''}(0)}{f_{\eta_0}^2}\ . \tag{2.35}$$

The constant $V_0^{''}$ (and more generally the whole function $V(X)$) could in principle be computed at large densities using instanton calculus. Unfortunately, the result is not known. However, at large to moderate densities we might expect this contribution to be suppressed. For three flavors, the two remaining legs of a one instanton contribution to $M_{\eta_0}^2$ can only be closed with the insertion of either a quark mass term or a chiral condensate $\langle\bar\psi\psi\rangle$. In the chiral limit, the latter only arises through another instanton process. On dimensional grounds we would expect

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**The fit to K_6, K_7 and K_8 is non-ambiguous because there is no constant invariant operator involving to quark mass matrices $\propto \mathcal{M}\,\mathcal{M}$.

††That K_6 is non-vanishing at the one-loop quark level even though the operator in the effective theory has two traces over the quark flavors is because the condensates are off-diagonal in flavor. This is very different from the large N_c limit in vacuum where extra quark loops or traces over flavor indices are $1/N_c$ suppressed.

$$M_{\eta_0}^2 \propto \mu^4\, G(\mu) |\langle \bar\psi\psi \rangle| \tag{2.36}$$

in the chiral limit, but from such a rough estimate, it is not reasonable to infer whether M_{η_0} ever grows large at low densities.

III. MESON MASS PATTERN

In this section we give numerical estimates of the pseudoscalar mesons as function of the quark chemical potential.

At very large densities, instanton effects are suppressed and the meson masses are linear in the quark masses. The mass pattern that arises in this regime has been discussed by Son and Stephanov [20]. There are two things which are particular at very high densities. The first one is that the meson masses are constant at leading order, that is, independent of the chemical potential. This is essentially for dimensional reasons, as $M^2 \sim m_q^2$.[‡‡] The second peculiarity is that the mass pattern is somehow reversed compared to the one in vacuum as the η' is lighter than the neutral pion. If instanton effects can be neglected, the meson masses are [20]

$$M_{\pi^\pm} \approx 53\,\mathrm{MeV}\ ,\ M_{K^0,\bar K^0} \approx 76\,\mathrm{MeV}\ ,\ M_{K^\pm} \approx 72\,\mathrm{MeV}\ , \tag{3.1}$$

using $m_u = 4$ MeV, $m_d = 7$ MeV and $m_s = 150$ MeV while in the neutral mesons sector,

$$M_{\eta} \approx 117\,\mathrm{MeV}\ ,\ M_{\pi_0} \approx 53\,\mathrm{MeV}\ ,\ M_{\eta'} \approx 30\,\mathrm{MeV}\ , \tag{3.2}$$

Mixing is important only between the η_8 and η_0 states. For these particles to be stable, the condition $M \lesssim 2\Delta$ must be satisfied, which, expect for the η', necessitates an extremely large chemical potential *e.g.* $\mu \gtrsim 10^7$ MeV for the pions.

At lower densities, it has been argued that the gap can be as large as $\mathcal{O}(100$ MeV$)$, but at the same time a non-zero $\langle\bar\psi\psi\rangle$ condensate induced by instantons introduce contributions to the meson masses which are proportional to $m_q^{1/2}$. In order to *estimate* this effect, we will use the expression of the chiral condensate and the gap as computed at weak coupling (2.20) and will assume that the corrections are small even at moderate densities $\mu \sim 600$ MeV. Because the expression for the chiral condensate involves the square of the gap, which is only known up to a factor b_0' assumed to be $\mathcal{O}(1)$ [9,5,11], there is some further theoretical uncertainty in the meson masses at low densities. The values quoted for the meson masses are thus at most indicative. If the instanton effect was dominant, this uncertainty could be cancelled by considering the ratios of meson masses over the gap, which is *independent* of the gap and would also tell us whether the mesons can exist as low energy excitations in the CFL phase at

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[‡‡] That the gap does not appear explicitly in the meson masses is because of the use of an effective theory to describe the properties of the low energy excitations; the dependence in the gap is only implicit in that the effective theory breaks down at energies or mesons masses that are of the order of two times the gap.

chemical potential μ. Because both contributions (linear and quadratic in the quark mass) are comparable at low densities, some uncertainty unfortunately subsists. For definitiveness, we will consider three distincts values, $b_0' = 0.5, 1$ and 2. Finally, we will assume that the contribution from the quark masses and the chiral condensate is parametrically larger than the two-instantons contribution to the mass of η_0. As the latter effect, if important, would increase the mass of η_0, our estimates should be considered as lower bounds on the mass of η'.

The expressions of the masses in the neutral meson sector are a bit involved and the mass matrix has to be diagonalized numerically. For the other mesons, we simply add the contribution that are linear and quadratic in the quark masses from (2.16) and (2.25)[§§],

$$M_{\pi^\pm}^2 \approx \frac{2V_2}{f_\pi^2}\,(m_u+m_d) - 4\frac{K_6+K_8}{f_\pi^2}(m_d+m_u)^2 - \frac{4K_6}{f_\pi^2}m_s(m_d+m_u)\ , \tag{3.3}$$

$$M_{K^\pm}^2 \approx \frac{2V_2}{f_\pi^2}\,(m_u+m_s) - 4\frac{K_6+K_8}{f_\pi^2}(m_s+m_u)^2 - \frac{4K_6}{f_\pi^2}m_d(m_s+m_u)\ , \tag{3.4}$$

$$M_{K^0,\bar K^0}^2 \approx \frac{2V_2}{f_\pi^2}\,(m_d+m_s) - 4\frac{K_6+K_8}{f_\pi^2}(m_d+m_s)^2 - \frac{4K_6}{f_\pi^2}m_u(m_d+m_s)\ , \tag{3.5}$$

where we have used (2.24) and (2.29-2.30). Because $K_6 \approx -K_8 < 0$, the second term in (3.3) is small for the charged pions, which leads to $M_{\pi^\pm}^2 \sim m_s(m_u+m_d)$ at large densities. This suppression is compensated for the kaons because the strange quark is not light. If instanton effects are negligible (*i.e.* $V_2 \sim 0$) these expressions give the estimates (3.1). At lower densities, the masses increase. For instance, at the chemical potential $\mu = 600$ MeV and for $b_0' = 1$ we get

$$M_{\pi^\pm} \approx 60\,\mathrm{MeV}\ ,\ M_{K^0,\bar K^0} \approx 134\,\mathrm{MeV}\ ,\ M_{K^\pm} \approx 135\,\mathrm{MeV}\ , \tag{3.6}$$

while for the neutral mesons we get

$$M_\eta \approx 230\,\mathrm{MeV}\ ,\ M_{\pi_0} \approx 107\,\mathrm{MeV}\ ,\ M_{\eta'} \approx 67\,\mathrm{MeV}\ , \tag{3.7}$$

to be compared with (3.1-3.2).

The ratio of the charged pions and kaons masses over two times the gap is shown in Fig.4 for $b_0' = 1$. For the kaons, the ratio is always larger than one, which suggests that these particles can exist as low energy excitations only at asymptotically large (and presumably physically irrelevant) densities. The situation for the charged pions seems better as there is a narrow window between 600 and 2500 MeV within which these excitations are below the instability threshold. Fig.5 shows a similar plot for the neutral mesons. One noticeable feature is that level crossing between the three states takes place at $\mu \sim 1000$ MeV. At high densities, the lighest state is the η' which is essentially the η_0 with some admixture of η_8. At low densities, in the absence of strong $U(1)_A$ symmetry breaking, mixing tends to be ideal

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[§§]In principle we should also take into account loop effects from the quadratic mass terms. For simplicity we assume that these corrections are small.

but then the $\bar{s}s$ becomes the heaviest state with $M^2 \sim m_s$ while the lightest is essentially a pure $\bar{u}u$ state. To call the latter η' is essentially a matter of convention. The other feature is that one of the states is always too heavy to be stable. Like for the charged pions, π_0 can exist below $\mu \sim 2500$ MeV.

This conclusions depend very much on the value of b_0' [11]. If b_0' increases, the gap is larger and the states have more room to exist. This is confirmed by the plot of Fig.6. On the other hand, if b_0' was only slightly below one, then not even the pions are stable (See Fig.7). For definitiveness we have used throughout $\Lambda = 200$ MeV, taken from [23]. At low densities, $\mu \lesssim 2$ GeV, the value of Λ could be significantly larger, $\Lambda \sim 300$ MeV. Roughly speaking, a larger Λ can compensate the effect of a smaller value of b_0'.

IV. CONCLUSIONS

We have considered the effective lagrangian for the Goldstone modes in the Color-Flavor-Locking phase of QCD at high densities, up to E^4 in a low energy expansion $E \lesssim 2\Delta$. A tentative analysis of the meson mass pattern that emerges from this lagrangian, including instanton effects, suggests that the kaons and, depending on $U(1)_A$ breaking effects, one of the neutral mesons may be absent from the spectrum. Their masses are significantly larger than two times the gap for any, but asymptotically large densities. For the pions, the situation is better as there is a window at low densities, $600 \lesssim \mu \lesssim 2500$ MeV, where these states are stable, even though their masses are dangerously close to the threshold, as $M_\pi \gtrsim 0.75(2\Delta)$ for a conservative $b_0' = 1$ and $\Lambda = 200$ MeV. Within the approximations we have made, the η' meson is the only mode whose mass is significantly below the threshold for all densities. At low densities, $\mu \sim 1000$ MeV, this could change if $U(1)_A$ breaking effects become important.

These conclusions depend sensitively on the unknown coefficient b_0' in the analytical expression for the gap. If b_0' is larger than one, things improve a lot. If however b_0' is smaller than one, then most presumably there are no Goldstone pseudoscalar excitations in the CFL phase of QCD at finite density. The only Goldstone mode would then be that of spontaneously broken baryon number, like in superfluidity.

Obviously there is much room for improvements. Our approach has been far from systematic and we have made some *ad hoc* assumptions. The sensitivity to b_0', in particular, suggests that corrections in α_s could also drastically affect our estimates. A calculation of the contribution of instantons to the mass of η_0 in the chiral limit would also prove to be most useful. If a spontaneously broken approximate $U(2)_L \times U(2)_R$ flavor symmetry exists in the CFL phase of QCD at moderate densities, it would be most interesting to determine the potential for the η_0, $V_0(X)$. The effective theory for the low energy excitations should be very much analogous to the large N_c lagrangian of Di Vecchia and Veneziano [24] and Witten [25].

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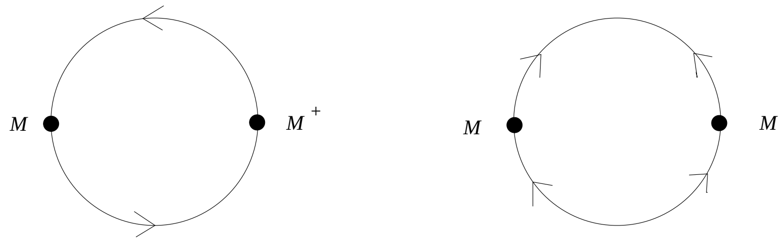

FIG. 1. Quark loops in the microscopic theory. $\mathcal{M}$ ($\mathcal{M}^{\dagger}$) stands for one insertion of the (conjugate) quark mass matrix. The *lhs* diagram is like in vacuum. The *rhs* diagram arises only in presence of a diquark condensate.

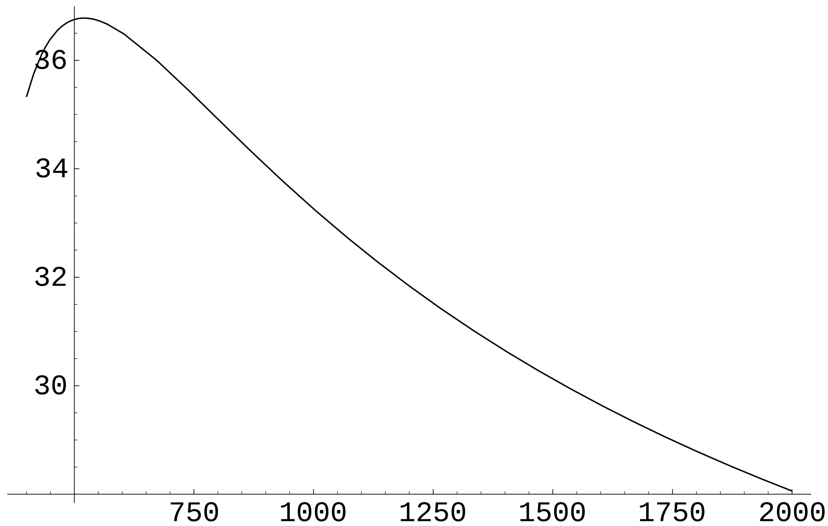

FIG. 2. Color gap (in MeV) as a function of the quark chemical potential μ (in MeV) for $b_0' = 1$.

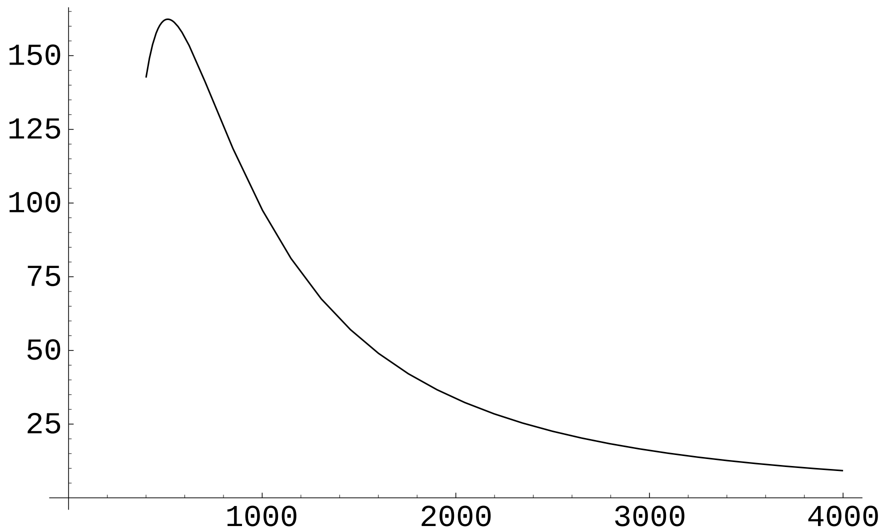

FIG. 3. Plot of $(-\langle\bar{\psi}\psi\rangle)^{1/3}$ (in MeV) as function of the quark chemical potential μ (in MeV), where $\langle\bar{\psi}\psi\rangle = \langle\bar{u}u + \bar{d}d + \bar{s}s\rangle$ is the chiral quark condensate induced by instantons.

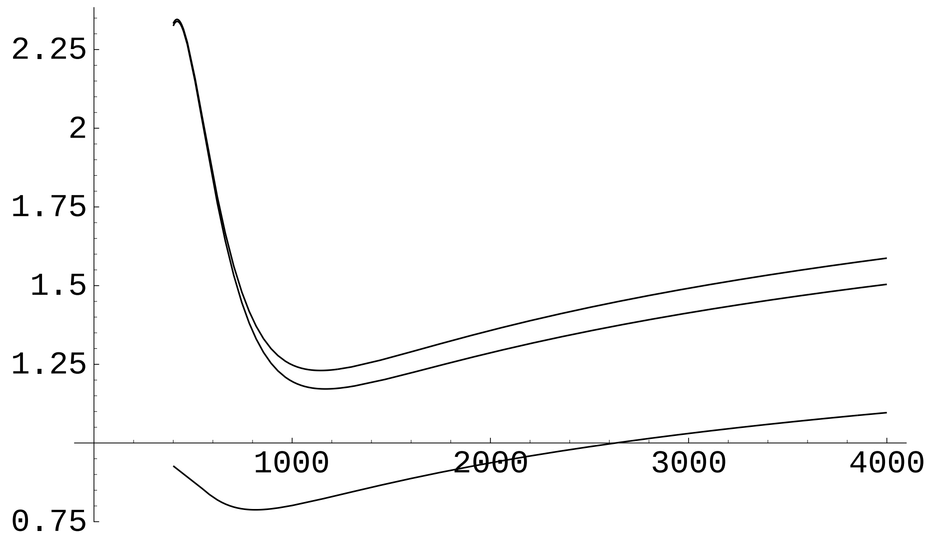

FIG. 4. Ratios of charged pions (lower curve) and kaons (two upper curves) masses over two times the gap as function of the quark chemical potential (in MeV) for $b_0' = 1$.

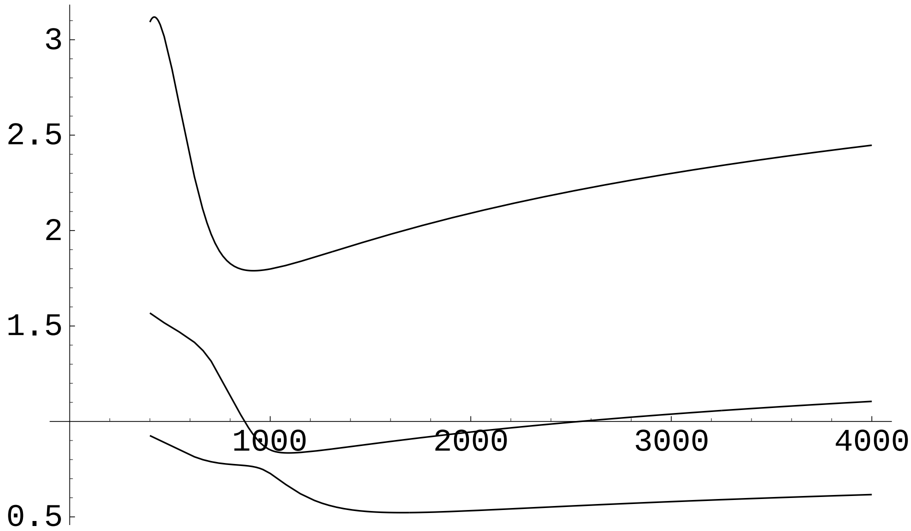

FIG. 5. Ratios of neutral meson masses ($M_{\eta'} < M_{\pi_0} < M_\eta$) over two times the gap as functions of the quark chemical potential (in MeV) for $b'_0 = 1$.

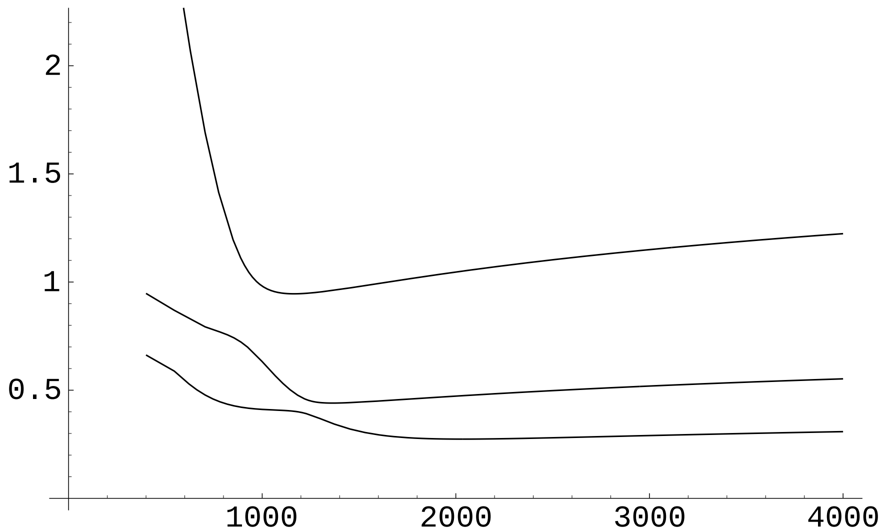

FIG. 6. Same as Fig.5 with $b'_0 = 2$.

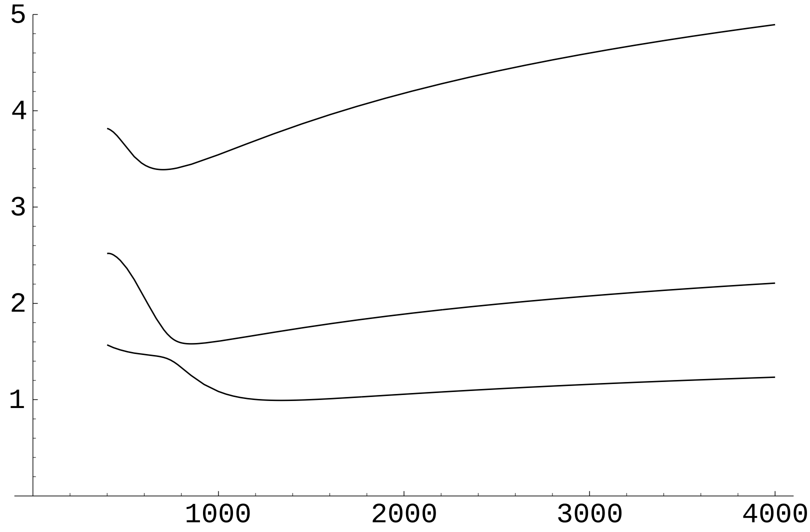

FIG. 7. Same as Fig.5 with $b'_0 = 0.5$.