Anomalous quantum transport in presence of self-similar spectra

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Abstract

We consider finite-difference Hamiltonians given by Jacobi matrices with self-similar spectra of the Cantor type and prove upper bounds on the diffusion exponents which show that the quantum motion in these models is anomalous diffusive. For Julia matrices, this bound is expressed only in terms of the generalized dimensions of the spectral measures.

1 Introduction

A quantum motion is called anomalous whenever it is neither ballistic nor regular diffusive nor localized, that is, the diffusion exponents β_α defined by

$$\int_0^T \frac{dt}{T} \langle \psi(t)| \, |\vec{X}|^\alpha |\psi(t)\rangle \underset{T\uparrow\infty}{\sim} T^{\alpha\beta_\alpha} , \qquad \alpha \neq 0 ,$$

may take arbitrary values in the interval $[0,1]$ (here $\vec{X}$ is the position operator and ψ some localized state in Hilbert space). There is compelling numerical evidence that the motion in almost periodic structures is anomalous [11, 19] and intermittent [15, 13] in the sense that $\alpha \mapsto \beta_\alpha$ is a strictly increasing function. This is possibly at the origin of the strange transport properties observed experimentally in quasicrystals. The scheme of explanation is based on the anomalous Drude formula [14, 17]: the direct conductivity $\sigma_{//}$ behaves as

$$\sigma_{//}(\tau) \underset{\tau\uparrow\infty}{\sim} \tau^{2\beta_2 - 1} ,$$

where τ is the *relaxation time* due to impurity and electron-phonon scattering. It is hence particularly interesting to calculate the diffusion exponent β_2 from the quasiperiodic Hamiltonian.

Anomalous transport in almost periodic structures is due to delicate quantum interference phenomena. On the spectral level, they lead to a singular continuous local density of states (LDOS) at least in low dimension (two early works are [3, 20], but there are many others), whereas in high dimension, spectral measures are likely to be absolutely continuous even if transport is anomalous. The first results linking spectral and transport properties were established by I. Guarneri [7] and refined by others [2, 12, 17]: the exponents β_α, $\alpha > 0$, are larger than or equal to the Hausdorff dimension of the LDOS devided by the dimension of physical space. Later on, links between diffusion exponents and multifractal dimensions of the density of states (DOS) [15] and the LDOS [13] were derived and numerically verified for some one-dimensional systems.

Here we prove upper bounds on β_α for a restricted class of one-dimensional Hamiltonians given by Jacobi matrices with self-similar spectra. This toy model was suggested and investigated by I. Guarneri and G. Mantica [8, 13] in order to study links between spectral and transport properties. Rigorous proofs of upper bounds on the spreading of wavepackets in these systems were proven by I. Guarneri and one of the authors [10]. However, these results did not allow to deduce bounds on positive moments of the position operator. The present work continues and completes this study and is actually based on one of its central results.

Our upper bound on the diffusion exponents is expressed in terms of three measures supported on the spectrum: the LDOS, the DOS and the maximal entropy measure. For spectral measures supported on self-similar sets with non-trivial thermodynamics, the latter controls the length fluctuations of the bands approximating the spectrum; these fluctuations appear as one reason for intermittency.

On the other hand, the interplay between the position operator and the Hamiltonian is the second reason for intermittency (see [13] where a Hamiltonian having a spectrum with flat thermodynamics was shown to exhibit intermittency). For the analysis of this interplay, we make use of a crucial bound from [10] known only for Jacobi matrices with self-similar spectra. For these Hamiltonians the asymptotic properties of the generalized eigenfunctions are governed by a Herbert-Jones-Thouless formula. However, this bound is in general far from optimal and, in order to obtain tight upper bounds, significant improvements are necessary.

Julia matrices are an exception in this respect due to an exact renormalization property making the links between position operator and Hamiltonian particularly simple. Consequently, in this case our analysis of the thermodynamics as outlined above does lead to a tight upper bound on diffusion exponents in terms of the generalized dimensions $D(q)$ of the LDOS:

Theorem For real Julia sets, $\beta_\alpha \leq D(1-\alpha)$ as long as $0 < \alpha \leq \alpha_c$ for a certain $\alpha_c > 2$.

Appart from this favorable example, our work illustrates that the links between spectral and transport properties is a very intricate one. There is hence need for further numerical and theoretical investigations.

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2 Models and results

2.1 Self-similar sets and measures

The self-similar fractal measures considered in this work are constructed by non-linear, disjoint iterated function systems, sometimes also called cookie cutters or Markov maps. The construction is as follows. Let $I_1^1 < I_2^1 < \ldots < I_L^1$ be a finite sequence of pairwise disjoint closed intervals all contained in a closed interval I^0. Let S be a smooth real function such that, for all $l = 1, \ldots, L$, the restriction S_l of S to I_l^1 is bijective from I_l^1 to I^0 with smooth inverse S_l^{-1} (in particular, we assume the derivative of S to be bounded away from 0 and ∞ on the intervals I_l^1). We call *codes* the one-sided sequences of symbols taken from $\{1, \ldots, L\}$ and denote the set of codes of length N by Σ_L^N and the set of codes of infinite length by Σ_L. For all $N \in \mathbf{N}$, $S^{-N}(I^0)$ consists of L^N closed, disjoint intervals $I_\sigma^N = S_{\sigma_1}^{-1} \circ \ldots \circ S_{\sigma_N}^{-1}(I^0)$, $\sigma = (\sigma_1 \ldots \sigma_N) \in \Sigma_L^N$, which we call the intervals of the Nth generation. We further assume that there exist positive constants $a < 1$ and c so that, for any $N \in \mathbf{N}$, all intervals of the Nth generation satisfy $|I_\sigma^N| \leq c\, a^N$ where $|I|$ denotes the length of the interval I.

Now $J = \bigcap_{N \geq 0} S^{-N}(I^0)$ is a fractal set which is invariant under S, *i.e.* $S(J) = J$. The dynamical system (J, S) is conjugated to the shift on Σ_L by the coding map $E \in J \mapsto \sigma(E) \in \Sigma_L$. Given a shift-invariant, ergodic measure on Σ_L, the pointwise dimensions

$$d_\mu(E) \;=\; \lim_{\epsilon \to 0} \frac{\log(\mu([E-\epsilon, E+\epsilon]))}{\log(\epsilon)} \tag{1}$$

of its pullback measure μ on J exist μ-almost surely and are μ-almost surely equal to the information or Hausdorff dimension $\dim_{\mathrm{H}}(\mu)$ of μ [10]. The latter is furthermore equal to the quotient of the dynamical entropy $\mathcal{E}(\mu)$ and the Lyapunov exponent $\Lambda(\mu)$ of the dynamical system (J, S, μ) [10].

Equilibrium measures introduced now form a special class of invariant and ergodic measures on J. The pressure $P(b)$ at "inverse temperature" b and hölderian "interaction" $\log(|S'(.)|)$ is defined by [5, 16]

$$P(b) \;=\; \sup_{\nu \in \mathcal{M}(J)} \left(\mathcal{E}(\nu) - b \int d\nu(E)\, \log(|S'(E)|) \right) \;,$$

where $\mathcal{E}(\nu)$ is the measure-theoretic entropy of S with respect to ν and $\mathcal{M}(J)$ is the set of S-invariant measures on J. The pressure $P(b)$ is an analytic, convex and decreasing function of b [16]. The maximum of the functional on the right hand side is attained by a unique invariant and ergodic measure μ_b, called the equilibrium measure of $b \log(|S'(.)|)$ [5]. Let us point out three interesting special cases: μ_0 is the measure of maximal entropy, notably the balanced Bernoulli measure; μ_1 is the SRB measure; finally, $\mu_{\dim_H(J)}$ is equivalent to the $\dim_H(J)$-Hausdorff measure on J. Let us further note that, for a linear iterated function system (S_l linear with slope $\pm e^{\Lambda_l}$, $l = 1, \ldots, L$), the equilibrium measure μ_b is the Bernoulli measure with probabilities [18]

$$p_l \;=\; \frac{e^{-b\Lambda_l}}{\sum_{l'=1}^{L} e^{-b\Lambda_{l'}}} \;, \qquad l = 1, \ldots, L \;.$$

A multifractal property of μ, that is finer characteristic than just the Hausdorff dimension $\dim_{\mathrm{H}}(\mu)$, is given by its singularity spectrum

$$f_\mu(\alpha) \;=\; \dim_H\left(\{E \in J \mid d_\mu(E) = \alpha\}\right) \;,$$

where $d_\mu(E) = \alpha$ means that the limit in (1) exists and is equal to α and, by convention, the Hausdorff dimension of an empty set is equal to $-\infty$. For equilibrium measures μ on J, it can be shown that f_μ is a concave function (see, for example, [4]). Its Legendre transform τ_μ allows to define the generalized dimensions $D_\mu(q)$, also called Renyi or Hentschel-Procaccia dimensions, by

$$D_\mu(q) \;=\; \lim_{q' \to q} \frac{\tau_\mu(q')}{q'-1} \;, \qquad \tau_\mu(q) \;=\; \inf_{\alpha \in \mathbf{R}} (\alpha q - f_\mu(\alpha)) \;.$$

2.2 Jacobi matrices

Once the measure μ on J is fixed, we construct the Hamiltonian as the Jacobi matrix of μ. Let P_n, $n \geq 0$, denote the orthogonal and normalized polynomials associated to μ. They form a Hilbert basis $\mathcal{B} = (P_n)_{n\in\mathbf{N}}$ in $L^2(\mathbf{R},\mu)$ and satisfy a three term recurrence relation $EP_n(E) = t_{n+1}P_{n+1}(E) + v_n P_n(E) + t_n P_{n-1}(E)$, $n \geq 0$, where $v_n \in \mathbf{R}$ and $t_n \geq 0$ are bounded sequences, and $P_{-1} = 0$. Therefore the isomorphism of $L^2(\mathbf{R},\mu)$ onto $\ell^2(\mathbf{N})$ associated with the basis $\mathcal{B}$ carries the operator of multiplication by E in $L^2(\mathbf{R},\mu)$ into the self-adjoint finite difference operator H defined on $\ell^2(\mathbf{N})$ by:

$$H|n\rangle \;=\; t_{n+1}|n+1\rangle + v_n|n\rangle + t_n|n-1\rangle \;, \qquad n \geq 1 \;, \tag{2}$$

and $H|0\rangle = t_1|1\rangle + v_0|0\rangle$. Then μ is the spectral measure of H associated to $|0\rangle$, also called its LDOS.

The DOS $\mathcal{N}$ of H is the unique weak limit point of the sequence of pure-point measures $(\sum_{E,\,P_n(E)=0} \delta_E/n)_{n\in\mathbf{N}}$ [21, 10]. It coincides with the Frostman (electrostatic) equilibrium measure on J. Finally the capacity of J can be calculated as $\mathrm{cap}(J) = \exp(\lim_{n\to\infty} \sum_{j=1}^n \log(t_j)/n)$ [21, 10]. The Green's function of J defined by

$$g_J(z) \;=\; \int d\mathcal{N}(E)\, \log(|z-E|) - \log(\mathrm{cap}(J))$$

governs the asymptotic properties of the orthogonal polynomials by means of a Herbert-Jones-Thouless type formula [21, 10]. Both $\mathcal{N}$ and $\mathrm{cap}(J)$ and hence g_J do not depend on the choice of μ, but only on its support J.

2.3 Diffusion exponents

The propagation of wave packets initially localized on the state $|0\rangle$ is characterized by the growth exponents

$$\beta_\alpha^+ \;=\; \limsup_{T\to\infty} \frac{\log(M_\alpha(T))}{\log(T^\alpha)} \;, \qquad \beta_\alpha^- \;=\; \liminf_{T\to\infty} \frac{\log(M_\alpha(T))}{\log(T^\alpha)} \;, \tag{3}$$

of the time-averaged moments of the position operator

$$M_\alpha(T) \;=\; \sum_{n\geq 0} n^\alpha \int_0^T \frac{dt}{T}\, |\langle n|e^{-\imath tH}|0\rangle|^2 \;, \qquad \alpha \neq 0 \;.$$

2.4 Results and comments

In order to state our main result, we need to introduce a constant depending only on the DOS. Let E_c and $R_c = |I^0|/2$ be respectively the center of the spectrum and its radius, and Δ the size of the smallest gap at the first generation. If S is an analytic map, then we set

$$\Upsilon \; = \; \inf_{R>R_c} \frac{\max_{z\in\Gamma_R} \; g_J(S(z))}{\log(R) - \log(R_c)} \; , \tag{4}$$

where Γ_R is the circle of radius R around E_c. If all branches S_l, $l = 1, \ldots, L$, have an analytic continuation $\hat{S}_l$ given by a polynomial of degree D_l, then we pose

$$\Upsilon \; = \; \frac{\max_{l=1\ldots L} \; \sup_{E\in J} \; g_J(\hat{S}_l(E)) + D \, \mathrm{arcsinh}\left(\frac{\Delta}{4R_c}\right)}{\mathrm{arcsinh}\left(\frac{\Delta}{4R_c}\right)} \; , \tag{5}$$

where $D = \max_l D_l$.

Theorem 1 *Let H be the Jacobi matrix of an equilibrium measure μ_b on a self-similar fractal J with L branches constructed with an analytic or piecewise polynomial map S. Let Υ be the corresponding constant given in* (4) *or* (5) *and let us set*

$$\alpha_c \; = \; \frac{P(b) - \log(L)\,(\tau_{\mu_0})^{-1}(-2-b)}{\log(\Upsilon)} \; .$$

Then, for $\alpha \in (0, \alpha_c]$,

$$\beta_\alpha^+ \; \leq \; \frac{\log(L) - P(b) + \alpha\log(\Upsilon)}{\alpha\log(L)} \, D_{\mu_0}\left(\frac{P(b) - \alpha\log(\Upsilon)}{\log(L)}\right) - \frac{b}{\alpha} \; , \tag{6}$$

and, for $\alpha \geq \alpha_c$,

$$\beta_\alpha^+ \; \leq \; 1 - \frac{\alpha_c - 2}{\alpha} \; .$$

Remark 1 The bound depends on the LDOS through the parameter b, on the maximal entropy measure μ_0 through its generalized dimensions D_{μ_0} and on the DOS through the constant Υ. Note that, in the limit $\alpha \to \infty$, our upper bound converges to the *a priori* ballistic bound $\beta_\alpha^+ \leq 1$. In the limit $\alpha \to 0$, we recover the bound obtained in ref. [10]. We know of no theoretical work about negative moments ($\alpha < 0$). We remark, however, that one can show $\beta_\alpha \sim D_\mu(2)/\alpha$ for large negative α using results from [12, 2, 17].

The above result is particularly interesting when applied to the case of Julia matrices which exhibit an exact renormalization property in physical space, so that all intermittency is due to the thermodynamics of the support of the spectral measure.

Theorem 2 *Let H be a Julia matrix, that is, S is a polynomial map and $\mu = \mu_0$ is the balanced measure of maximal entropy. Set $\alpha_c = 1 - (\tau_{\mu_0})^{-1}(-2)$. Then*

$$\beta_\alpha^+ \; \leq \; \begin{cases} D_{\mu_0}(1-\alpha) \; , & 0 < \alpha \leq \alpha_c \; , \\ 1 - \frac{\alpha_c - 2}{\alpha} \; , & \alpha \geq \alpha_c \; . \end{cases} \tag{7}$$

Remark 2 Numerical and theoretical analysis using GM-machinery [13] indicates that the equality $\beta_\alpha = D_\mu(1-\alpha)$ may hold for all $\alpha > 0$. Whether the equality actually holds for all α or $\beta_\alpha \to 1$ as $\alpha \to \infty$ (as is the case in our upper bound) is an interesting question. Two facts, both verified for real Julia sets generated by $S(E) = E^2 - \lambda$ with $\lambda > 2$, indicate that our upper bound is probably not tight for $\alpha > \alpha_c$. First of all, α_c converges to 2 from above as $\lambda \to 2$ (while numerical results give the equality $\beta_\alpha = D_\mu(1-\alpha)$ for much larger value of α if λ is close to 2 [13]); second of all, the curve defined by the upper bound (7) has a discontinuous derivative at α_c. We discuss the problems arising for large α with more technical details in Remark 7 of Section 3.1. Let us finally note that, for quadratic Julia sets, $\alpha_c \sim \log(\lambda)/2$ for large λ.

As already pointed out in the introduction, a second reason for intermittency is due to the interplay between Hamiltonian and position operator. An extreme example of this is given by spectra supported on linear Cantor sets for which the thermodynamics is flat and cannot be at the origin of intermittency, but for which the quantum motion is nevertheless intermittent [13]. Our proof does not allow to exhibit and analyse these fine properties, however, we obtain a sub-ballistic bound on dynamics.

Theorem 3 *Let J be a linear Cantor set, that is, S has two linear branches S_1 and S_2 with slope equal to $\pm e^\Lambda$. Let us set*

$$\Upsilon \;=\; \left(\operatorname{arcsinh}\left(\frac{e^\Lambda - 2}{4}\right)\right)^{-1}\left(\int d\mathcal{N}(E)\,\log(e^\Lambda - E) - \log(\mathrm{cap}(J))\right) + 1\;. \tag{8}$$

Then

$$\beta_\alpha^+ \;\leq\; \begin{cases} \frac{\log(\Upsilon)}{\Lambda}\;, & 0 < \alpha \leq \frac{2\Lambda}{\log(\Upsilon)}\;, \\[2ex] 1 + \frac{2}{\alpha} - \frac{2\Lambda}{\alpha\log(\Upsilon)}\;, & \alpha \geq \frac{2\Lambda}{\log(\Upsilon)}\;. \end{cases} \tag{9}$$

Remark 3 It was shown in ref. [10] that Υ as given in (8) behaves as $\log(\Lambda)$ in the limit $\Lambda \to \infty$. Therefore the bound (9) is strictly better than the ballistic bound for sufficiently large Λ and Theorem 3 shows that the transport is anomalous in these models. Note that the bound (9) does not depend on the measure μ, but only on the DOS. This is, however, an artefact of our proof.

3 Proof of upper bounds

3.1 Resolving the spectrum at different scales

The main result of this section, notably that Proposition 1 holds given Hypothesis I and II, can be directly transposed to other Hamiltonians and other exponentially localized initial states; needed is only the structure of a position operator asigning a number to each element of a given Hilbert basis. For sake of simplicity, we formulate nevertheless everything only for the model described in Sections 2.1 and 2.2.

The following hypothesis allow to separate two reasons for intermittency.

Hypothesis I *Let $\Lambda^*(.,\mu) : \mathbf{R} \to [0,\infty]$ be a convex function such that, for all $\lambda \leq \Lambda(\mu)$,*

$$\mu\left(\left\{E \in J \;\middle|\; \lambda \geq -\frac{\log(|I^N_{\sigma(E)}|)}{N}\right\}\right) \;\leq\; a e^{-N\Lambda^*(\lambda,\mu)} \;, \tag{10}$$

for some constant $a \geq 0$. We further suppose that there is $0 < \Lambda_{\min} \leq \Lambda(\mu)$ such that $\Lambda^*(.,\mu) : [\Lambda_{\min}, \Lambda(\mu)] \to [0,\infty)$ is a C^1 and strictly convex bijection and that $\Lambda^*(\lambda,\mu) = \infty$ for $\lambda < \Lambda_{\min}$.

Hypothesis II *For $\eta > 0$, let $\Upsilon(\eta) > 0$ and $c(\eta) < \infty$ be such that for all n and N verifying*

$$n \;\geq\; \Upsilon(\eta)^N \;,$$

the following bound holds:

$$\sum_{\sigma \in \Sigma_L^N} |\langle \chi_{I^N_\sigma} | n \rangle|^2 \;\leq\; c(\eta) n^{-\eta} \;, \tag{11}$$

where $|\chi_{I^N_\sigma}\rangle = \chi_{I^N_\sigma}(H)|0\rangle$, χ_I denoting the characteristic function of the interval I.

Remark 4 In the next section, the function $\Lambda^*(.,\mu)$ will be determined to be the Legendre transform of the generalized Lyapunov exponents. In Section 3.3, we furthermore show that $\Lambda^*(.,\mu)$ is given in terms of the scaling function of the maximal entropy measure whenever μ is an equilibrium measure. At least in this situation, all the above hypothesis on $\Lambda^*(.,\mu)$ are satisfied. A closer inspection of the proof below shows that weaker results can be obtained under weaker hypothesis (no differentiability, for example). Let us note right away that $\Lambda^*(\Lambda(\mu),\mu) = 0$ and that both $\Lambda^*(.,\mu)$ and $(\Lambda^*)'(.,\mu)$ are bijections from $(\Lambda_{\min}, \Lambda(\mu)]$ to their respective images; further $\Lambda^*(.,\mu)$ is discontinuous at $\Lambda_{\min}$.

Remark 5 Hypothesis II is in a more general form than we can actually prove it. The results from Section 3.2 of ref. [10] show that, for the choice Υ given as in Section 2.4, the bound (11) holds for all $\eta > 0$. Thus Υ is independent of η in this situation. To obtain a smaller, but η-dependent Υ is a tough task for which only numerical results exist [13]. As we cannot prove such a hypothesis for the moment, we restrict ourselves to the case of an η-independent Υ in Proposition 1 below, because it simplifies considerably the proof. However, we cannot obtain any result on intermittency due to Hypothesis II in this way (*cf.* Theorem 3 on linear Cantor sets).

The following proposition is obtained by combining the technique "resolving the spectrum" [10, Proposition 1] and the argument in Section 3.4 of the PhD thesis of one of the authors [1].

Proposition 1 *We suppose* Hypothesis I *and* II *to be verified for a given Υ independent of η. Let $\Lambda_R \in (\Lambda_{\min}, \Lambda(\mu)]$ be the solution of $(\Lambda^*)'(\Lambda_R, \mu) = -2$ if it exists and $\Lambda_R = \Lambda(\mu)$ otherwise. Let us further set*

$$\alpha_c \;=\; \frac{\Lambda^*(\Lambda_R,\mu) + 2\Lambda_R}{\log(\Upsilon)} \;, \tag{12}$$

and let $z_\alpha \in [\Lambda_{\min}, \Lambda(\mu)]$ be the solution of

$$\alpha \log(\Upsilon) \;=\; \Lambda^*(z_\alpha,\mu) - z_\alpha (\Lambda^*)'(z_\alpha,\mu)\ , \tag{13}$$

if it exists, and $z_\alpha = \Lambda(\mu)$ *otherwise. Then, for* $0 < \alpha \le \alpha_c$*,*

$$\beta^+_\alpha \;\le\; \kappa_\alpha \;\equiv\; \frac{\alpha \log(\Upsilon) - \Lambda^*(z_\alpha,\mu)}{\alpha z_\alpha}\ , \tag{14}$$

while, for $\alpha \ge \alpha_c$*,*

$$\beta^+_\alpha \;\le\; \kappa_\alpha \;\equiv\; 1 - \frac{\alpha_c - 2}{\alpha}\ . \tag{15}$$

Proof. Let $0 \le \kappa < 1$, $\rho > 0$ and $\epsilon > 0$ be such that $(1-\kappa)/\epsilon \in \mathbf{N}$. We introduce the monoton sequence $\overline{n}_k(T) = \rho T^{\kappa + k\epsilon}$, $k \ge 0$, as well as the presence probabilities in the rings limited by the radii $\overline{n}_{k-1}(T)$ and $\overline{n}_k(T)$:

$$B_k(T) \;=\; \sum_{\overline{n}_{k-1}(T) < n \le \overline{n}_k(T)} \int_0^T \frac{dt}{T}\, |\langle n| e^{-\imath H t}|0\rangle|^2\ .$$

Then the time-averaged moments of the position operator can be bounded as follows:

$$M_\alpha(T) \;\le\; \rho^\alpha T^{\alpha\kappa} + \sum_{k=1}^{(1-\kappa)/\epsilon} \overline{n}_k(T)^\alpha\, B_k(T) + \sum_{n \ge bT} n^\alpha \int_0^T \frac{dt}{T}\, |\langle n| e^{-\imath H t}|0\rangle|^2\ . \tag{16}$$

We first note that a ballistic bound as given in [9] implies that the last summand is smaller than a constant for any $T > 0$ whenever ρ is sufficiently large. More precisely, let X_γ, $\gamma > 0$, be the Banach space of $\ell^2(\mathbf{N})$-vectors ψ such that $\|\psi\|_\gamma = \sup_{n\ge 0} |\langle \psi|n\rangle| \exp(\gamma n) < \infty$. As H is a bounded operator on X_γ, we have

$$\sum_{n\ge bT} n^\alpha \int_0^T \frac{dt}{T}\, |\langle n|e^{-\imath H t}|0\rangle|^2 \;\le\; \sum_{n \ge bT} n^\alpha \int_0^T \frac{dt}{T}\, e^{-2n\gamma + 2t\|H\|_\gamma} \;\le\; \frac{(\rho\gamma T)^{[\alpha]}}{4\gamma T \|H\|_\gamma}\, e^{-2\rho\gamma T + 2T\|H\|_\gamma}\ ,$$

where $[\alpha]$ is the smallest integer larger than α. Now the latter expression is uniformly bounded in T for any $\rho > \|H\|_\gamma/\gamma$.

In order to bound the second summand in (16), we proceed as in [10, Proposition 1] for each $B_k(T)$ separately. So for each $k \ge 1$, let us suppose N and T to be linked by some relation chosen later on (see equation (20) below) and let us set, for given $\lambda_c(k) \le \Lambda(\mu)$,

$$|\psi_{N,k}(t)\rangle \;=\; \sum_{\sigma \in \mathcal{J}_N(\lambda_c(k))} e^{-\imath E^N_\sigma t} |\chi_{I^N_\sigma}\rangle\ ,$$

where E^N_σ is some point in I^N_σ and $\mathcal{J}_N(\lambda)$ is the set of $\sigma \in \Sigma^N_L$ satisfying $-\log((|I^N_\sigma|)/N \ge \lambda$. Let us further divide the interval $[\lambda_c(k), \Lambda(\mu)]$ into $Q = (\Lambda(\mu) - \lambda_c(k))/\delta \in \mathbf{N}$ intervals of equal length δ. Then the vector $|\psi_{N,k}(t)\rangle$ approximates the time evolution of $|0\rangle$ in Hilbert space norm:

$$\| |\psi_{N,k}(t)\rangle - e^{-\imath H t}|0\rangle\|^2 \;=\; \sum_{\sigma \in \mathcal{J}_N(\lambda_c(k))} \int_{I^N_\sigma} d\mu(E)\, |e^{\imath E t} - e^{\imath E^N_\sigma t}|^2 + \sum_{\sigma \in \mathcal{J}_N(\lambda_c(k))^c} \mu(I^N_\sigma)$$

$$\leq \left(\sum_{j=1}^{Q-1}\sum_{\sigma\in\mathcal{J}_N(\lambda_c(k)+j\delta)^c\setminus\mathcal{J}_N(\lambda_c(k)+(j-1)\delta)^c} + \sum_{\sigma\in\mathcal{J}_N(\Lambda(\mu)-\delta)}\right)\mu(I^N_\sigma)\,t^2\max_{E\in I^N_\sigma}|E-E^N_\sigma|^2$$

$$+\ \mu(\mathcal{J}_N(\lambda_c(k))^c)$$

$$\leq \sum_{j=1}^{Q-1} ae^{-N\Lambda^*(\lambda_c(k)+j\delta,\mu)}e^{-2N(\lambda_c(k)+(j-1)\delta)}t^2 + e^{-N(\Lambda(\mu)-\delta)}t^2 + ae^{-N\Lambda^*(\lambda_c(k),\mu)}\ , \tag{17}$$

where we have used the bound (10). For a given $r \geq T/(2\pi)$, let us first bound $B_k(T)$ by

$$2\int_0^T\frac{dt}{T}\,\||\psi_{N,k}(t)\rangle - e^{-\imath Ht}|0\rangle\|^2 + 2\sum_{\overline{n}_{k-1}(T)\leq n\leq\overline{n}_k(T)}\ \sum_{\sigma,\sigma'\in\mathcal{J}_N(\lambda_c(k))}\langle n|\chi_{I^N_\sigma}\rangle\langle\chi_{I^N_{\sigma'}}|n\rangle\int_0^{2\pi r}\frac{dt}{T}\,e^{\imath(E^N_\sigma-E^N_{\sigma'})t}\ .$$

Now, let $\Lambda_{\max}$ be the minimal λ with the property $|I^N_\sigma| \geq e^{-\lambda N}$ for all $\sigma\in\Sigma_L$ and $N\in\mathbf{N}$. For fixed N, we can therefore choose the E^N_σ's all to be elements of a lattice with spacing $1/r = e^{-\Lambda_{\max}N}$ so that only the diagonal terms $\sigma=\sigma'$ remain in the above sum (at this point, improvements are possible, but not useful if Υ is independent of η). Using (17) we thus obtain

$$\begin{aligned}B_k(T) \ \leq\ & \sum_{j=1}^{Q-1}\frac{2aT^2}{3}e^{-N\Lambda^*(\lambda_c(k)+j\delta,\mu)}e^{-2N(\lambda_c(k)+(j-1)\delta)} + \frac{2T^2}{3}e^{-2N(\Lambda(\mu)-\delta)} \\ & + 2ae^{-N\Lambda^*(\lambda_c(k),\mu)} + \frac{4\pi e^{\Lambda_{\max}N}}{T}\sum_{\overline{n}_{k-1}(T)\leq n\leq\overline{n}_k(T)}\ \sum_{\sigma\in\mathcal{J}_N(\lambda_c(k))}|\langle n|\chi_{I^N_\sigma}\rangle|^2\ .\end{aligned}\tag{18}$$

Putting this and $\overline{n}_k(T) = \rho T^{\kappa+k\epsilon}$ into (16), we obtain for some constant c depending on $a,\rho,\alpha,\epsilon,\Lambda_{\max},\Upsilon$ and γ:

$$\begin{aligned}M_\alpha(T)\ \leq\ & cT^{\kappa\alpha} + cT^{\kappa\alpha}\sum_{k=1}^{(1-\kappa)/\epsilon}T^{k\epsilon\alpha}\left(\sum_{j=1}^{Q}T^2e^{-N\Lambda^*(\lambda_c(k)+j\delta,\mu)}e^{-2N(\lambda_c(k)+(j-1)\delta)}\right. \\ & \left. + e^{-N\Lambda^*(\lambda_c(k),\mu)} + \frac{4\pi e^{\Lambda_{\max}N}}{T}\sum_{\overline{n}_{k-1}(T)\leq n\leq\overline{n}_k(T)}\ \sum_{\sigma\in\mathcal{J}_N(\lambda_c(k))}|\langle n|\chi_{I^N_\sigma}\rangle|^2\right)\ .\end{aligned}\tag{19}$$

Now, for every fixed $k\geq 1$, we choose N and T to be related by

$$\Upsilon^N\ =\ T^{\kappa+k\epsilon-\epsilon}\ , \tag{20}$$

so that the last term in the parenthesis in (19) is uniformly bounded in T by Hypothesis II. We then want to choose κ in such a way that the first and second term in (19) and thus $M_\alpha(T)$ are bounded by $cT^{\alpha\kappa}$. This imposes the two conditions

$$(k\epsilon\alpha+2)\log(\Upsilon)\ \leq\ (\kappa+k\epsilon-\epsilon)\left(\Lambda^*(\lambda_c(k)+j\delta,\mu)+2(\lambda_c(k)+(j-1)\delta)\right)\ , \tag{21}$$

$$k\epsilon\alpha \log(\Upsilon) \;\leq\; (\kappa + k\epsilon)\Lambda^*(\lambda_c(k), \mu) \;, \tag{22}$$

which have to hold for all $j = 1, \ldots, Q = (\Lambda(\mu) - \lambda_c(k))/\delta$ and all $k = 1, \ldots, (1-\kappa)/\epsilon$, the choice of each $\lambda_c(k) \in [0, \Lambda(\mu)]$ still being free.

The problem is now to determine the minimal κ such that these inequalities hold for appropriate choices of $\lambda_c(k)$. For this purpose, we study their continuum limit $\epsilon, \delta \to 0$. Setting $x = k\epsilon$, $\lambda_c(x) = \lambda_c(k)$ and $\lambda = \lambda_c(k) + j\delta$, the following inequalities

$$(x\alpha + 2)\log(\Upsilon) \;\leq\; (\kappa + x)(\Lambda^*(\lambda, \mu) + 2\lambda) \;, \tag{23}$$

$$x\alpha \log(\Upsilon) \;\leq\; (\kappa + x)\Lambda^*(\lambda_c(x), \mu) \;, \tag{24}$$

have to hold for $\lambda \in [\lambda_c(x), \Lambda(\mu)]$ and $x \in (0, 1-\kappa]$.

We first choose $\lambda_c(x)$ to be the biggest λ such that (24) is satisfied:

$$\lambda_c(x) \;=\; \sup\left\{\lambda \leq \Lambda(\mu) \;\middle|\; \Lambda^*(\lambda) \geq \log(\Upsilon)\,\frac{x\alpha}{\kappa + x}\right\} \;. \tag{25}$$

Using the hypothesis on $\Lambda^*(.,\mu)$, it can be verified that $\lambda_c(x)$ is a decreasing function in x. On the other hand, because the minimal value of the function $\lambda \mapsto \Lambda^*(\lambda, \mu) + 2\lambda$ in $[\Lambda_{\min}, \Lambda(\mu)]$ is taken at Λ_R and this function is increasing on $[\Lambda_R, \Lambda(\mu)]$, it follows that (23) is always satisfied for $\lambda \in [\lambda_c(x), \Lambda(\mu)]$ if

$$\kappa + x \;\geq\; \frac{(2 + x\alpha)\log(\Upsilon)}{(\Lambda^*(.,\mu) + 2\,\mathrm{id})(\max\{\Lambda_R, \lambda_c(x)\})} \;. \tag{26}$$

In order to treat the two different values of the maximum in (26) separately, we introduce

$$x_\alpha \;\equiv\; \inf\{x \geq 0 \,|\, \lambda_c(x) \leq \Lambda_R\} \;=\; \frac{\kappa\Lambda^*(\Lambda_R, \mu)}{\alpha \log(\Upsilon) - \Lambda^*(\Lambda_R, \mu)} \;.$$

For $x \in [x_\alpha, 1-\kappa]$, the maximum is equal to Λ_R. For $\alpha \leq \alpha_c$, the inequality (26) is then most difficult to satisfy for the smallest possible x, that is $x = x_\alpha$, whereas, for $\alpha \geq \alpha_c$, this is the case for $x = 1-\kappa$. After a short computation, one therefore obtains that (26) holds for $x \in [x_\alpha, 1-\kappa]$ only if

$$\kappa \;\geq\; \begin{cases} \frac{\alpha\log(\Upsilon) - \Lambda^*(\Lambda_R,\mu)}{\alpha\Lambda_R} \;, & \frac{\Lambda^*(\Lambda_R,\mu)}{(1-\kappa)\log(\Upsilon)} \leq \alpha \leq \alpha_c \;, \\ 1 - \frac{\Lambda^*(\Lambda_R,\mu) + 2\Lambda_R - 2\log(\Upsilon)}{\alpha\log(\Upsilon)} \;, & \alpha \geq \alpha_c \;, \end{cases} \tag{27}$$

while for small α's no condition is imposed on κ because $x_\alpha > 1-\kappa$.

Next we study (26) for $x \in [0, x_\alpha)$. Using the definition of $\lambda_c(x)$, it follows that (26) holds if

$$\kappa + x \;>\; \frac{(2 + x\alpha)\log(\Upsilon)}{\log(\Upsilon)\,\frac{x\alpha}{\kappa + x} + 2\lambda_c(x)} \qquad \Leftrightarrow \qquad \lambda_c(x) \;>\; \frac{\log(\Upsilon)}{\kappa + x} \;.$$

Using the fact that $\Lambda^*(.,\mu)$ is decreasing in $[0, \Lambda(\mu)]$ and again the definition of $\lambda_c(x)$, it is thus sufficient that

$$x\alpha\frac{\log(\Upsilon)}{\kappa+x} \;<\; \Lambda^*\left(\frac{\log(\Upsilon)}{\kappa+x}\right) \tag{28}$$

holds for $x \in [0, x_\alpha)$. Let us set $z = \log(\Upsilon)/(\kappa + x)$, then (28) is equivalent to

$$\alpha\log(\Upsilon) - \kappa\alpha z \;<\; \Lambda^*(z,\mu)\;, \qquad z \in \left[\frac{\log(\Upsilon)}{\kappa+x_\alpha}, \frac{\log(\Upsilon)}{\kappa}\right]\;. \tag{29}$$

The right hand side is convex in z, the left hand side decreasing in κ. The minimal $\kappa = \overline{\kappa}_\alpha$ such that (29) holds for $z \in [0, \Lambda(\mu)]$ or (28) for all $x \in \mathbf{R}$ can be determined by equalizing left and right hand side as well as their derivatives in x. For a given $\alpha > 0$, this shows that

$$\overline{\kappa}_\alpha \;=\; \frac{\alpha\log(\Upsilon) - \Lambda^*(z_\alpha,\mu)}{\alpha z_\alpha}\;, \qquad \overline{x}_\alpha \;=\; \frac{\Lambda^*(z_\alpha,\mu)}{\alpha z_\alpha}\;,$$

where z_α is determined by (13) and $\overline{x}_\alpha$ is the corresponding value of x at which (28) is most difficult to verify. We note that $z_0 = \Lambda(\mu)$ and $z_{\alpha_c} = \Lambda_R$, and further that $\alpha \mapsto z_\alpha$ is well defined and decreasing due to the convexity of $\Lambda^*(.,\mu)$. If $\overline{x}_\alpha \le x_\alpha$ which is equivalent to $\alpha \le \alpha_c$ and $z_\alpha \ge \Lambda_R$, we thus have upon taking into account (27):

$$\kappa \;>\; \max\left\{\frac{\alpha\log(\Upsilon) - \Lambda^*(z_\alpha,\mu)}{\alpha z_\alpha}, \frac{\alpha\log(\Upsilon) - \Lambda^*(\Lambda_R,\mu)}{\alpha\Lambda_R}\right\}\;.$$

Now, for $\alpha = \alpha_c$, the two expressions coincide. Furthermore, if $g_l(\alpha)$ and $g_r(\alpha)$ denote the two expressions as a function of α, then $g_l'(\alpha) = \Lambda^*(z_\alpha,\mu)/(z_\alpha\alpha^2)$ and $g_r'(\alpha) = \Lambda^*(\Lambda_R,\mu)/(\Lambda_R\alpha^2)$, so that $g_l'(\alpha) \le g_r'(\alpha)$; therefore the maximum is equal to $g_l(\alpha)$, which is precisely (14).

For $\overline{x}_\alpha \ge x_\alpha \Leftrightarrow \alpha \ge \alpha_c$, the inequality (28) is satisfied for all $x \in [0, x_\alpha)$ if it is satisfied for x_α, which implies that, using (25),

$$\kappa \;\ge\; \frac{\alpha\log(\Upsilon) - \Lambda^*(\Lambda_R,\mu)}{\alpha\Lambda_R}\;.$$

This bound coincides with the bound (27) at $\alpha = \alpha_c$, but for $\alpha > \alpha_c$ it is less restrictive so that (27) gives (15). $\square$

Remark 6 Using (12), it is straightforward to verify

$$\frac{2\Lambda_R}{\log(\Upsilon)} \;\le\; \alpha_c \;\le\; \frac{2\Lambda(\mu)}{\log(\Upsilon)}\;.$$

For real Julia sets, this allows to deduce the behavior given in Remark 2.

Remark 7 For big α, the main contribution to $M_\alpha(T)$ comes from the part of the wave packet far from the origin. In order to have a better than ballistic bound for arbitrarily large α ($\kappa_\alpha < 1$), one has to show that the presence probability $B_k(T)$ in the growing rings decreases faster than any power in time for all rings ($k = 1, \ldots, 1 - \kappa/\epsilon$). Our bound on $B_k(T)$ is given as the sum of an error term and a main term which comes from the approximate wave function (given by the last term in (19)). Both terms depend on the generation index N designing the order of approximation. As discussed in Remark 5, we can obtain such an almost exponential decay for the main term whenever Υ is chosen by (4) or (5). As the link (20) between N, T and k is thus independent of α, the error term imposes $\kappa_\alpha \to 1$ as $\alpha \to \infty$. More precisely, the factor $T^{(1-\kappa)\alpha}$ corresponding to the largest ring ($k = (1-\kappa)/\epsilon$) can be compensated by the factors in the parenthesis in (19) only if κ is close to 1; the term in the parenthesis with the slowest decrease is determined by the minimum of the function $\lambda \mapsto \Lambda^*(\lambda, \mu) + 2\lambda$ at Λ_R; this gives directly the bound (15). Hence, the only way to obtain a better than ballistic bound for large α is to improve the bound (11) on the main term, leading to an α-dependent relation between T and N.

Proof of Theorem 3. For a linear Cantor set with one contraction factor $e^{-\Lambda}$, there are no fluctuations around the Lyapunov exponent. Then $\Lambda^*(\lambda, \mu) = \infty$ for all $\lambda \leq \Lambda(\mu) = \Lambda$. Hence $\alpha_c = 2\Lambda/\log(\Upsilon)$ and $\kappa_\alpha = \log(\Upsilon)/\Lambda$ for all $\alpha \leq \alpha_c$. Since Υ as given in (5) is equal to (8) (see [10]), the theorem follows directly from Proposition 1.

Note that for linear Cantor sets $\lambda_c(x) = \Lambda(\mu) = \Lambda_R$ for all x, and that $x_\alpha = 0$ for all $\alpha > 0$. Therefore the proof of Proposition 1 would already be concluded by (27). □

3.2 Large deviation bound for Lyapunov exponents

In this section, we determine the rate function $\Lambda^*(., \mu)$, for which the large deviation bound (10) holds. As this is the easier upper one of the two bounds in the Gärtner-Ellis theorem [6], we briefly reproduce how to use Chebychev's inequality in order to obtain it.

Proposition 2 *The bound* (10) *holds with the Legendre transform*

$$\Lambda^*(\lambda, \mu) \;=\; \sup_{q \in \mathbf{R}} (q\lambda - \Lambda(q, \mu)) \;, \qquad \lambda \in \mathbf{R} \;, \tag{30}$$

of the generalized Lyapunov exponents of μ *defined by* [18]

$$\Lambda(q, \mu) \;=\; \limsup_{N \to \infty} \frac{1}{N} \log \left(\int d\mu(E)\, |I^N_{\sigma(E)}|^q \right) \;. \tag{31}$$

Proof. We first note that $\Lambda(q, \mu)$ is a convex function in q because the function appearing in (31) before taking the superior limit is a convex function by Hölder's inequality and because the pointwise superior limit of convex functions is again a convex function. Furthermore $\Lambda(1, \mu) = \Lambda(\mu)$ and $\Lambda(0, \mu) = 0$. The latter implies that $\Lambda^*(\lambda, \mu) \geq 0$ for all $\lambda \in \mathbf{R}$. By Jensen's inequality, $\Lambda(q, \mu) \geq q\Lambda(\mu)$ which implies $\Lambda^*(\Lambda(\mu), \mu) \leq 0$, hence $\Lambda^*(\Lambda(\mu), \mu) = 0$.

We next consider $\Lambda_N(E) = -\log(|I^N_{\sigma(E)}|)/N$ as random variable in the probability space (J, μ). By Chebychev's inequality, we have for any $\lambda \leq \Lambda(\mu)$ and $q \leq 0$:

$$\mu(\{\Lambda_N \leq \lambda\}) \;\leq\; e^{-q\lambda N} \int d\mu(E)\, |(S^{\circ N})'(E)|^q \;\leq\; a e^{-N(q\lambda - \Lambda(q,\mu))} \;,$$

for some constant a. Taking the supremum over all $q \leq 0$ in the exponent leads to the function $\Lambda^*(., \mu)$, because, for positive q, one has $q\lambda - \Lambda(q, \mu) \leq 0$ as long as $\lambda \leq \Lambda(\mu)$. Hence we obtain, for all $\lambda \leq \Lambda(\mu)$, the desired bound (10) □

Remark 8 Let us introduce the random variable $\overline{\Lambda}_N(E) = \log(|(S^{\circ N})'(E)|)/N$. As $\Lambda_N = \overline{\Lambda}_N + \mathcal{O}(1/N)$ (see, for example, eq. (33) in [10]), the generalized Lyapunov exponent can also be calculated as

$$\Lambda(q,\mu) \;=\; \limsup_{N\to\infty} \frac{1}{N} \log\left(\int d\mu(E)\, |(S^{\circ N})'(E)|^q \right) \;.$$

Remark 9 If the limit in (31) exists, then the Gärtner-Ellis theorem also provides a lower bound $\mu(\{\Lambda_N \leq \lambda\}) \geq b e^{-N\Lambda^*(\lambda,\mu)}$ for some constant b so that $\Lambda^*(.,\mu)$ as given in (30) is optimal in (10). This is the case for equilibrium measures discussed in the next section [16].

Remark 10 Let us set

$$\Lambda_{\max} \;=\; \sup_{E\in J} \limsup_{N\to\infty} \Lambda_N(E) \;, \qquad \Lambda_{\min} \;=\; \inf_{E\in J} \limsup_{N\to\infty} \Lambda_N(E) \;.$$

Then $\Lambda(.,\mu)$ is asymptotically affine:

$$\Lambda(q,\mu) \;\underset{q\downarrow-\infty}{\sim}\; \Lambda_{\min} q \;, \qquad \Lambda(q,\mu) \;\underset{q\uparrow\infty}{\sim}\; \Lambda_{\max} q \;.$$

This implies that $\Lambda^*(\lambda,\mu) = \infty$ if $\lambda < \Lambda_{\min}$ or $\lambda > \Lambda_{\max}$.

It is straightforward to see that $\Lambda_{\max} = \log(\max_{E\in J} |S'(E)|)$, but $\Lambda_{\min}$ is more difficult to determine. For example, for quadratic Julia sets, it can be shown that $\Lambda_{\min} = \log(|S'(E_-)|)$ where E_- is the negative fixed point of S.

3.3 Case of equilibrium measures

Here we calculate the generalized Lyapunov exponents and its Legendre transform for the one-parameter family of equilibrium measures constructed in Section 2.1. As it will turn out, the latter is determined by the singularity spectrum of the maximal entropy measure. The argument presented here combines results from [4] and [18].

Proposition 3 $\Lambda^*(.,\mu_b)$ *is convex, analytic in* $[\Lambda_{\min},\Lambda_{\max}]$ *and*

$$\Lambda^*(\lambda,\mu_b) \;=\; -\lambda f_{\mu_0}\left(\frac{\log(L)}{\lambda}\right) + \lambda b + P(b) \;. \tag{32}$$

Proof. For the equilibrium measure μ_b, the generalized Lyapunov exponents can be calculated from the pressure by the formula [18]

$$\Lambda(q,\mu_b) \;=\; P(b-q) - P(b) \;. \tag{33}$$

In order to calculate the Legendre transform, let us introduce the Lyapunov spectrum

$$\ell(\lambda) \;=\; \dim_H\left(\left\{ E \in J \;\middle|\; \lim_{N\to\infty} \Lambda_N(E) = \lambda \right\} \right) \;.$$

Note that, for $\lambda \notin [\Lambda_{\min},\Lambda_{\max}]$, $\ell(\lambda) = -\infty$ because the Hausdorff dimension of an empty set is set to $-\infty$. Now [4, Theorems 1 and 2]

$$|\lambda|\ell(\lambda) \;=\; \inf_{b\in\mathbf{R}} (b\lambda + P(b)) \;. \tag{34}$$

Hence we obtain from (33) and (34) that

$$\Lambda^*(\lambda,\mu_b) \;=\; -|\lambda|(\ell(\lambda)+b)+P(b)\;.$$

As the Lyapunov spectrum is linked to the singularity spectrum f_{μ_0} of the maximal entropy measure μ_0 by $\ell(\lambda)=f_{\mu_0}(\log(L)/\lambda)$, the proof is concluded. □

Proof of Theorem 1. First of all, Proposition 3 and the definition of τ_{μ_0} allow to calculate Λ_c and then α_c. Similarly, for the calculation of κ_α, one obtains from (13) that

$$f'_{\mu_0}\left(\frac{\log(L)}{z_\alpha}\right) \;=\; \frac{-\alpha\log(\Upsilon)+P(b)}{\log(L)}\;, \qquad \kappa_\alpha \;=\; \frac{-\frac{-\alpha\log(\Upsilon)+P(b)}{\log(L)}\frac{\log(L)}{z_\alpha}+f_{\mu_0}\left(\frac{\log(L)}{z_\alpha}\right)-b}{\alpha}\;.$$

By definition of the multifractal dimensions, this implies directly the result. □

Proof of Theorem 2. For Julia sets, $\log(\Upsilon)=\log(L)$ as follows directly from (4) [10], and for the maximal entropy measure, $b=0$ and $P(0)=\mathcal{E}(\mu_0)=\log(L)$. Therefore Theorem 2 follows directly from Theorem 1. □

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