Available at: http://www.ictp.trieste.it/~pub_off
IC/98/38

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

THE BRUHAT-TITS BUILDING FOR $PGL(3, K)$, WHERE K IS A TWO-DIMENSIONAL LOCAL FIELD

A.N. Parshin[1]
Steklov Mathematical Institute, Russian Academy of Sciences,
Gubkina str., 8, 117966 Moscow GSP-1, Russian Federation
and
The Abdus Salam International Centre for Theoretical Physics,
Trieste, Italy.

MIRAMARE – TRIESTE
April 1998

[1] E-mail address: an@parshin.mian.su

In [**P1**] the author has introduced a generalization of the Bruhat-Tits buildings for the groups $PGL(V)$ over n-dimensional local fields. The main object of the classical Bruhat-Tits theory is a simplicial complex attached to any reductive algebraic group G defined over a field K. There are two parallel theories for the cases when K has no additional structure and when K is a local field. They are known as the *spherical* and *euclidean* buildings correspondingly (see [**BT1**, **BT2**] for original papers and [**R**, **T1**] for the surveys).

In our theory they correspond to local fields of dimension 0 and of dimension 1. The construction of the Bruhat-Tits building for the group $PGL(2)$ over two-dimensional local field was done in detail in [**P2**]. After that V. Ginzburg and M. Kapranov have developed this theory and extended it to the arbitrary reductive groups over two-dimensional local fields [**GK**]. Their definition coincides with ours for $PGL(2)$ and is different for higher ranks. But it seems that there are closely related (in the case of the groups of type A_l). It remains to develop the theory for arbitrary reductive groups over local fields of dimension > 2.

In this note we describe the structure of the higher building for the group $PGL(3)$ over a two-dimensional local field. We refer to [**P1**, **P2**] for the motivation for our constructions.

The note contains four sections. In the first we recall the main notions and results on local fields of dimension two. The facts about the Weyl group are collected in section 2. Then we describe the building for $PGL(3, K)$ when K has dimension one and in the last section we go to the building over a two-dimensional local field.

1 Local Fields

We begin with the definition of the higher local fields coming from the adelic theory.

DEFINITION 1. Let K and k be fields. We say that K is a *n-dimensional local field* with k as *last residue field* if the field K has the following structure. Either $n = 0$ or K is the quotient field of a (complete) discrete valuation ring $\mathcal{O}_K$ whose residue field is a local field of dimension $n-1$ with last residue field k.

If K', K'', ... are the intermediate residue fields from the definition then we will write $K/K'/K''.../k$ for the structure. The *first residue field* will be denoted mostly as $\bar{K}$.

A typical example is the field of iterated power series

$$K = k((t_1))\dots((t_n))$$

with an obvious inductive local structure on it. In case of dimension two we have

$$K \supset \mathcal{O}_K \xrightarrow{p} \bar{K} \supset \mathcal{O}_{\bar{K}} \to k$$

and for the field of iterated power series with $n = 2$ the local structure is described in the following terms

$$\mathcal{O}_K = k((u))[[t]],$$

$$\bar{K} = k((u)).$$

$$\mathcal{O}_{\bar{K}} = k[[u]]$$

(see [**FP**, ch.2] for other examples and a classification theorem for complete local fields).

Let us mention that the choice of *local parameters* t, u in our example does not follow from the local structure.

We denote by $\wp$ the maximal ideal of the local ring $\mathcal{O}_K$. We also denote by t a generator of $\wp$ and by u a generator of the maximal ideal of $\mathcal{O}_{\bar{K}}$. Let

$$\mathcal{O}'_K \text{ (or simply } \mathcal{O}') \ = p^{-1}(\mathcal{O}_{\bar{K}})$$

be a subring in K. This ring has a maximal ideal m. It is easy to see that for the field $k((u))((t))$

$$\mathcal{O}_K = k[[u]] \oplus k((u))[[t]]t,$$

$$m = k[[u]]u \oplus k((u))[[t]]t.$$

Thus we have a tower of valuation rings for the valuations $\nu^{(i)}$ of rank $i = 0, 1, 2$:

$$\mathcal{O}_{(0)} \supset \mathcal{O}_{(1)} \supset \mathcal{O}_{(2)},$$

where $\mathcal{O}_{(0)} = K,\ \mathcal{O}_{(1)} = \mathcal{O}_K,\ \mathcal{O}_{(2)} = \mathcal{O}'$.

For valuation groups

$$\Gamma_K^{(1)} = K^*/(\mathcal{O}_{(1)})^* = \mathbf{Z},$$

$$\Gamma_K^{(2)} = K^*/(\mathcal{O}_{(2)})^* \cong \mathbf{Z} \oplus \mathbf{Z},$$

there is a canonical exact sequence

$$0 \to \mathbf{Z} \to \Gamma_K^{(2)} \to \Gamma_K^{(1)} \to 0.$$

This extension defines on Γ_K a structure of *ordered* group. If we need to show the local structure we will write $\Gamma_{K/\ldots/k}$ instead of Γ_K. If we choose local parameters t, u of the field K then the exact sequence will split and the order becomes the lexicographical order. Inside the group Γ_K there is a subset Γ_K^+ of non-negative elements.

In our situation we have two valuations (of ranks 1 and 2) $\nu^{(1)} = \nu$ and $\nu^{(2)} = \nu'$ correspondingly.

If $K \supset \mathcal{O}$ is a fraction field of a subring $\mathcal{O}$ we call $\mathcal{O}$-submodules $a \subset K$ fractional $\mathcal{O}$-ideals (or simply fractional ideals).

The ring $\mathcal{O}_K$ have the following properties:

i)

$$\mathcal{O}'/m = k,\ K^* = \{t\}\{u\}(\mathcal{O}')^*,\ (\mathcal{O}')^* = k^*(1+m);$$

ii) every finitely generated fractional $\mathcal{O}'$-ideal a is a principal one and

$$a = m_{i,n} = (u^i t^n), \ i, n \in \mathbf{Z};$$

iii) every infinitely generated fractional $\mathcal{O}'$-ideal a is equal to

$$a = \wp_n = (u^i t^n \mid \text{for all } i \in \mathbf{Z}), \ n \in \mathbf{Z};$$

(see [**FP**] or [**P2**]). The set of these ideals is totally ordered by inclusion.

2 The Weyl group

Let $G = \mathrm{PGL}(n, K)$ where K is a two-dimensional local field. We put

$$B = \begin{pmatrix} \mathcal{O}' & \mathcal{O}' & \dots & \mathcal{O}' \\ m & \mathcal{O}' & \dots & \mathcal{O}' \\ & & \dots & \\ m & m & \dots & \mathcal{O}' \end{pmatrix},$$

We denote in such way the subgroup of G consisting of the matrices whose entries satisfy the written conditions modulo the center of $GL(n)$. Also let N be the subgroup of monomial matrices.

Definition 2. Let

$$T = B \bigcap N = \begin{pmatrix} (\mathcal{O}')^* & \dots & 0 \\ & \ddots & \\ 0 & \dots & (\mathcal{O}')^* \end{pmatrix}$$

The group

$$W_{K/\bar{K}/k)} = N/T.$$

will be called the *Weyl group*

There exists a rich structure of subgroups in G having many common properties with the theory of BN-pairs. In particular, we have the Bruhat, Cartan and Iwasawa decompositions (see [**P2**]).

The Weyl group W contains the following elements of order two

$$s_i = \begin{pmatrix} 1 & \dots & 0 & & & 0 & \dots & 0 \\ & \ddots & & & & & & \\ 0 & \dots & 1 & & & & & 0 \\ 0 & \dots & & 0 & 1 & & \dots & 0 \\ 0 & & & 1 & 0 & & \dots & 0 \\ 0 & \dots & & & & 1 & \dots & 0 \\ & & & & & & \ddots & \\ 0 & \dots & 0 & & & 0 & \dots & 1 \end{pmatrix}, i = 1, ..., n-1;$$

$$w_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & t \\ 0 & 1 & \dots & 0 & 0 \\ & & \dots & & \\ & & \dots & & \\ & & \dots & 1 & 0 \\ t^{-1} & 0 & \dots & 0 & 0 \end{pmatrix}, \ w_2 = \begin{pmatrix} 0 & 0 & \dots & 0 & u \\ 0 & 1 & \dots & 0 & 0 \\ & & \dots & & \\ & & \dots & & \\ & & \dots & 1 & 0 \\ u^{-1} & 0 & \dots & 0 & 0 \end{pmatrix}.$$

The Weyl group W has the following properties:

i) W is generated by the set S of it's elements of order two,

ii) there exists an exact sequence

$$0 \to E(= \text{Ker } \Sigma) \to W_{K/\bar{K}/k} \to W_K \to 1,$$

where

$$\Sigma : \Gamma_K \oplus \overset{n}{\dots} \oplus \Gamma_K \to \Gamma_K$$

is a summation map and W_K is isomorphic to the symmetric group S_n of n elements,

iii) the elements $s_i, \ i = 1, \dots, n-1$ define a splitting of the exact sequence and the subgroup $< s_1, \dots, s_{n-1} >$ acts on E by permutations.

In contrast with the situation in the theory of BN-pairs the pair (W, S) is not a Coxeter group and furthermore there is no subset S of involutions in W such that (W, S) will be a Coxeter group (see [**P2**]).

3 Bruhat-Tits building for PGL(3) over a local field of dimension 1

Let K be a one-dimensional local field, $K \supset \mathcal{O} \supset m, \mathcal{O}/m = k$. Denote by V a vector space over K of dimension 3. We say that $L \subset V$ is a lattice if L is an $\mathcal{O}$-module. The two submodules L and L' belong to one class $< L > = < L' >$, iff $L = aL'$, with $a \in K^*$.

First we define the vertices of our building and then the simplices. The result should be a simplicial set $\Delta_.(G, K/k)$.

Definition 3. The *vertices* of the Bruhat-Tits building:

$$\Delta_0[1](G, K/k) = \{\text{classes of } \mathcal{O}\text{-submodules } L \subset V : L \cong \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}\},$$

$$\Delta_0[0](G, K/k) = \{\text{classes of } \mathcal{O}\text{-submodules } L \subset V :$$
$$L \cong \text{ either } \mathcal{O} \oplus \mathcal{O} \oplus K, \text{or } \mathcal{O} \oplus K \oplus K\},$$

$$\Delta_0(G, K/k) = \Delta_0[1](G, K/k) \bigcup \Delta_0[0](G, K/k).$$

We say that the points from $\Delta_0[1]$ are the *inner* points, the points from $\Delta_0[0]$ are the *boundary* points. Sometimes we will delete G and K/k from our notation if this does not lead to confusion.

We have defined the vertices only. For the simplices of higher dimension we have the following

DEFINITION 4.

Let $\{L_\alpha, \alpha \in I\}$ be a set of $\mathcal{O}$-submodules in V. We say that $\{L_\alpha, \alpha \in I\}$ is a *chain* iff:

i) for any $\alpha \in I$ and for any $a \in K^\star$ there exists an $\alpha' \in I$ such that $aL_\alpha = L_{\alpha'}$,

ii) the set $\{L_\alpha, \alpha \in I\}$ is totally ordered by the inclusion.

$\{L_\alpha, \alpha \in I\}$ is a *maximal chain* iff it cannot be included in a strictly larger set satisfying the same conditions i) and ii).

We say that $< L_0 >, < L_1 >, ..., < L_m >$ belong to a *simplex* of dimension m iff the $L_i, i = 0, 1, ..., m$ belong to a maximal chain of $\mathcal{O}$-submodules in V. The faces and the degeneracies can be defined in a standard way (as a deletion or a repetition of some vertex). A definition of this kind was introduced in [**BT2**].

To describe the structure of the building we first need to determine all types of the maximal chains. Proceeding as it was done in [**P2**] for $PGL(2)$ we get the following result.

Proposition 1 . *The set of all maximal chains of $\mathcal{O}$-submodules in the space V will be exhausted by the following three possibilities:*

i) the chain contains a module L isomorphic to $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$. Then all the modules from the chain will be of that type and the chain will have the following structure:

$$\ldots \supset m_{i,n}L \supset m_{i,n}L' \supset m_{i,n}L'' \supset m_{i+1,n}L \supset m_{i+1,n}L' \supset m_{i+1,n}L'' \supset \ldots$$

$$\ldots \supset m_{i,n+1}L \supset m_{i,n+1}L' \supset m_{i.n+1}L'' \supset m_{i+1,n+1}L \supset \ldots,$$

where $i, n \in \mathbf{Z}$, $< L >, < L' >, < L'' > \in \Delta_0(G, K/k)[1]$ *and* $L \cong \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$, $L' \cong \mathcal{O} \oplus \mathcal{O} \oplus m$, $L'' \cong \mathcal{O} \oplus m \oplus m$.

ii) there exists a module $L \cong \mathcal{O} \oplus \mathcal{O} \oplus K$. Then the chain equals to

$$\ldots \supset m_{i,n}L \supset m_{i,n}L' \supset m_{i+1,n}L \supset \ldots$$

$$\supset m_{i,n+1}L \supset m_{i,n+1}L' \supset m_{i+1,n+1}L \supset \ldots, \quad i, n \in \mathbf{Z},$$

where $< L >, < L' > \in \Delta_0(G, K/k)[0]$ *and* $L \cong \mathcal{O} \oplus \mathcal{O} \oplus K$, $L' \cong m \oplus \mathcal{O} \oplus K$.

iii) if there is a module $L \cong \mathcal{O} \oplus K \oplus K$ then
the chain equals
$\ldots \supset m_{i,n}L \supset m_{i+1,n}L \supset \ldots \supset m_{i,n+1}L \supset m_{i+1,n+1}L \supset \ldots, i, n \in \mathbf{Z}$
where $< L > \in \Delta_0(G, K/k)[0]$.

We see that the chains of the first type correspond to two-simplices, of the second type to edges and the last type represent some vertices. It means that our simplicial set $\Delta_.$ is a disconnected union of it's subsets $\Delta_{.}[m]$, $m = 0, 1$. The dimension of the subset $\Delta_{.}[m]$ equals to 1 for $m = 0$ and 2 for $m = 1$.

Usually the buildings are defined as combinatorial complexes having a system of subcomplexes called apartments (see, for example, [**R**, **T1**, **T2**]). We show how to introduce them in our case.

DEFINITION 5. Let us fix a basis $e_1, e_2, e_3 \in V$. The *apartment*, defined by this basis is the following set

$$\Sigma_{.} = \Sigma_{.}[1] \cup \Sigma_{.}[0],$$

where

$$\Sigma_0[1] = \{< L >|\ L = a_1 e_1 \oplus a_2 e_2 \oplus a_3 e_3,$$

$$\text{where } a_1, a_2, a_3 \text{ are } \mathcal{O}\text{-submodules in } K \text{ isomorphic to } \mathcal{O}\}.$$

$$\Sigma_0[0] = \{< L >|\ L = a_1 e_1 \oplus a_2 e_2 \oplus a_3 e_3,$$

$$\text{where } a_1, a_2, a_3 \text{ are } \mathcal{O}\text{-submodules in } K \text{ isomorphic either to } \mathcal{O}\} \text{or to} K$$

$$\text{and at least one } a_i \cong K.$$

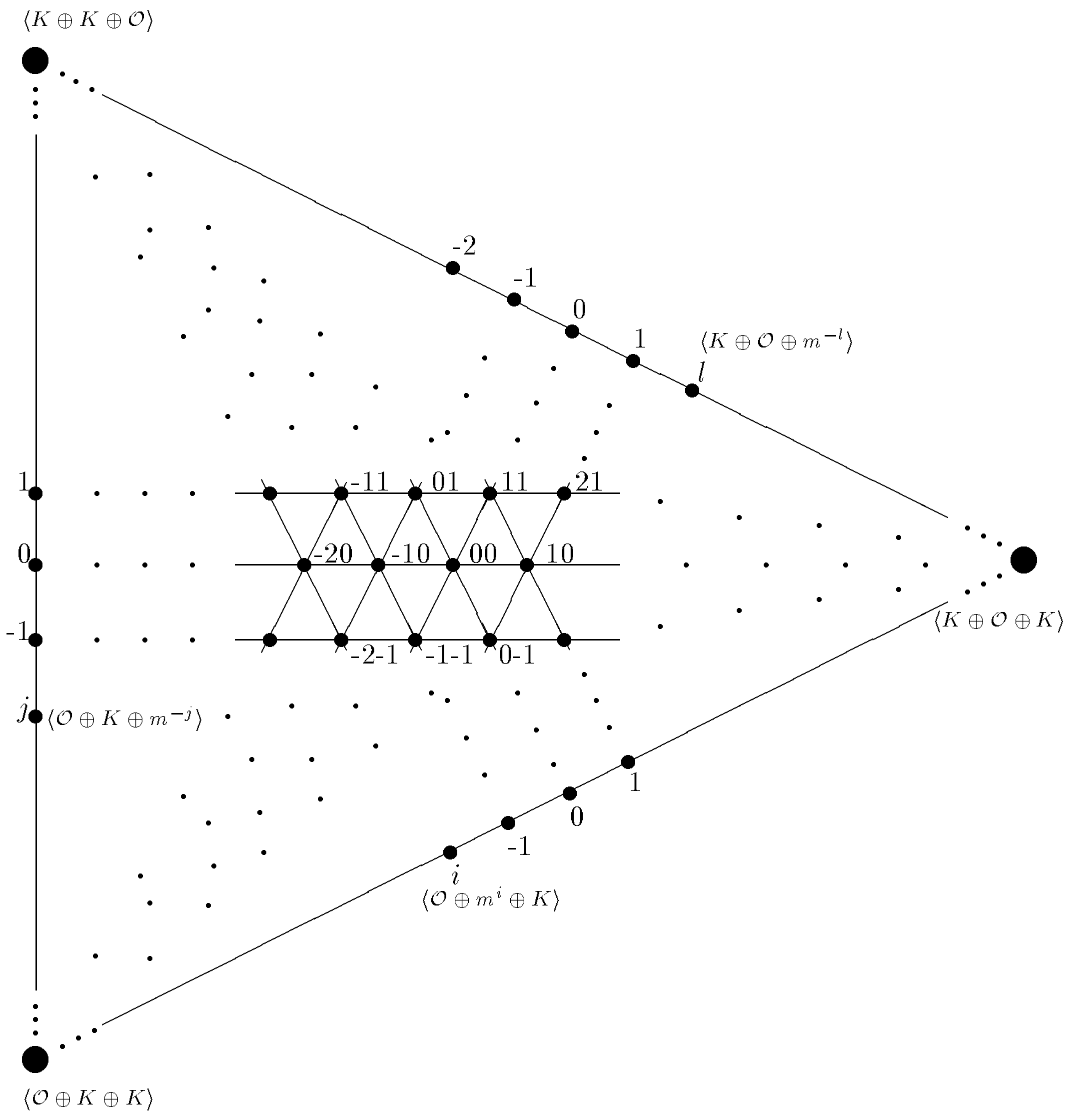
$\Sigma_{.}[m]$ is the minimal subcomplex having $\Sigma_0[m]$ as vertices.

It can be shown that our building is glued from the apartments, namely

$$\Delta_{.}(G, K/k) = \bigsqcup_{\text{all bases from } V} \Sigma_{.} \ / \text{ an equivalence relation}$$

(see [**T2**]).

We can make this description more transparent by drawing all that in the following picture where the dots of different kinds belong to the different parts of the building. In contrast with the case of the group $PGL(2)$ it is not easy to draw the whole building and we restrict ourselves by an apartment.



Here the inner vertices are represented by the lattices

$$ij =< \mathcal{O} \oplus m^i \oplus m^j >, \; i, j \in \mathbf{Z}.$$

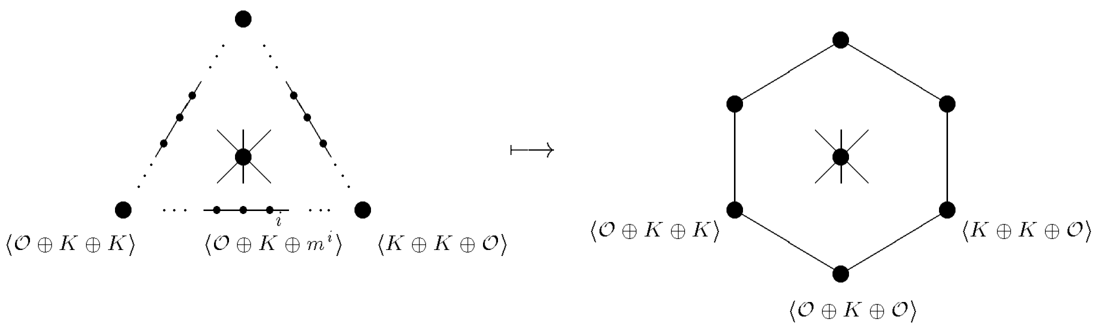
The definition of the boundary gives a topology on $\Delta_0(G, K/k)$ which is discrete on both subsets $\Delta_0[1]$ and $\Delta_0[0]$. The convergence of the inner points to the boundary points is given by the following rules:

$$< \mathcal{O} \oplus m^i \oplus m^j > \rightarrow_{j \rightarrow -\infty} < \mathcal{O} \oplus m^i \oplus K >,$$

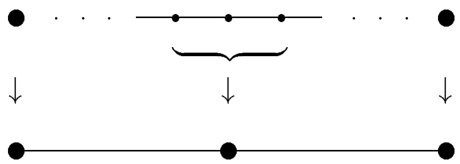
$$< \mathcal{O} \oplus m^i \oplus m^j > \rightarrow_{j \rightarrow \infty} < K \oplus K \oplus \mathcal{O} >,$$

because $< \mathcal{O} \oplus m^i \oplus m^j > = < m^{-j} \oplus m^{-j+i} \oplus \mathcal{O} >$. The convergence in the other two directions can be defined along the same line (and it is shown on the picture). It is easy to extend it to the higher simplices.

Thus we have a structure of a simplicial topological space on the apartment and then we define it on the whole building using the glueing procedure. This topology is stronger than the topology usually introduced to connect the inner part and the boundary together. The connection with standard "compactification" of the building is given by the following map:



This map is bijective on the inner simplices and on a part of the boundary can be described as



We note that our complex is not a CW-complex but only a closure finite complex. This "compactification" was used by G. Mustafin [**M**].

We have two kinds of connections with the buildings for other fields and groups. First, from our local field K/k we get two local fields of dimension 0, namely K and k. Then

$$\text{For any } P \in \Delta_0[1](PGL(V), K/k), \text{Link}(P) = \Delta_.(PGL(V_P), k)$$

where $V_P = L/mL$ if $P = < L >$ and the $\text{Link}(P)$ is a boundary of the $\text{Star}(P)$. As the apartments for the $PGL(3,k)$ are the hexagons we can also see this property on the picture. The analogous relation with the building of $PGL(3,K)$ is more complicated. It was shown in the picture above.

The other relations work if we change the group G but not the field. We see that three different lines will go out from every inner point in the apartment. They will represent the apartments of the group $PGL(2,K/k)$. They correspond to different embeddings of the $PGL(2)$ into $PGL(3)$.

Also we can describe the action of the Weyl group W on an apartment. As we fix a basis, the extension

$$0 \to \Gamma_{K/k} \oplus \Gamma_{K/k} \to W \to S_3 \to 1$$

will split. The elements from $S_3 \subset W$ act either as rotations around the point 00 or as reflections. The elements from $\mathbf{Z} \oplus \mathbf{Z} \subset W$ can be represented as triples of integers (according to property ii), section 2). Then they correspond to translations of the lattice of inner points along the three directions going from the point 00.

If we fix an embedding $PGL(2) \subset PGL(3)$ then the apartments and the Weyl groups will be connected as follows:

$$\Sigma_{.}(PGL(2)) \subset \Sigma_{.}(PGL(3))$$

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{Z} & \to & W' & \to & S_2 & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbf{Z} \oplus \mathbf{Z} & \to & W & \to & S_3 & \to & 1, \end{array}$$

where W' is a Weyl group of the group $PGL(2)$ over the field K/k.

4 Bruhat-Tits building for PGL(3) over a local field of dimension 2

Let K be a two-dimensional local field, $K \supset \mathcal{O} \supset \mathcal{O}' \supset m \supset \wp$ be as above. Then we have

$$\mathcal{O}'/m = k, \quad \mathcal{O}'/\wp = \mathcal{O}_{\bar{K}}, \quad \mathcal{O}/\wp = \bar{K}.$$

Again denote by V a vector space over K of dimension 3. We say that $L \subset V$ is a lattice if L is an $\mathcal{O}'$-module. The classes of submodules are defined as before. We will consider the following *types* of lattices:

$$\begin{array}{lll} 222 & < \mathcal{O}' \oplus \mathcal{O}' \oplus \mathcal{O}' > & \Delta_0[2] \\ 221 & < \mathcal{O}' \oplus \mathcal{O}' \oplus \mathcal{O} > & \multirow{2}{*}{\} \ \Delta_0[1]} \\ 211 & < \mathcal{O}' \oplus \mathcal{O} \oplus \mathcal{O} > & \\ 220 & < \mathcal{O}' \oplus \mathcal{O}' \oplus K > & \} \ \Delta_0[0] \\ 200 & < \mathcal{O}' \oplus K \oplus K > & \end{array}$$

To define the buildings we repeat the procedure from the previous section.

Definition 6. The *vertices* of the Bruhat-Tits building are the elements of the following set:

$$\Delta_0(G, K/\bar{K}/k) = \Delta_0[0] \bigcup \Delta_0[1] \bigcup \Delta_0[0].$$

To define the simplices of higher dimension we can repeat word by word Definition 4 replacing only the ring $\mathcal{O}$ by the ring $\mathcal{O}'$. We call the subset $\Delta[1]$ the *inner boundary* of our building and the subset $\Delta[0]$ the *external boundary*. The points in $\Delta[2]$ are the *inner* points.

To describe the structure of the building we first need to determine all types of the maximal chains. Proceeding as it was done in [**P2**] for $PGL(2)$ we get the following result.

Proposition 2 . *Let* $\{ L_\alpha\}$ *be a maximal chain of* $\mathcal{O}'$*-submodules in the space* V*. Then we have the following opportunities:*

i) if the chain contains a module L isomorphic to $\mathcal{O}' \oplus \mathcal{O}' \oplus \mathcal{O}'$ then all the modules from the chain will be of that type and the chain is uniquely determined by its segment

$$\ldots \supset \mathcal{O}' \oplus \mathcal{O}' \oplus \mathcal{O}' \supset m \oplus \mathcal{O}' \oplus \mathcal{O}' \supset m \oplus m \oplus \mathcal{O}' \supset m \oplus m \oplus m \supset \ldots$$

ii) there exists a module $L \cong \mathcal{O}' \oplus \mathcal{O}' \oplus \mathcal{O}$. Then the chain can be restored from the segment:

$$\mathcal{O} \oplus \underbrace{\mathcal{O} \oplus \mathcal{O} \supset \ldots \supset \mathcal{O}' \oplus \mathcal{O}' \oplus \mathcal{O} \supset m \oplus \mathcal{O}' \oplus \mathcal{O} \supset m \oplus m \oplus \mathcal{O} \supset \ldots \supset \wp \oplus \wp}_{quotient \cong \bar{K} \oplus \bar{K}} \oplus \mathcal{O} =$$

$$\wp \oplus \underbrace{\wp \oplus \mathcal{O} \supset \ldots \supset \wp \oplus \wp \oplus \mathcal{O}' \supset \wp \oplus \wp \oplus m \supset \ldots \supset \wp \oplus \wp}_{\cong \bar{K}} \oplus \wp \supset \ldots$$

*Here the modules $\cong \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$ do not belong to our chain and are inserted as in the proof of Proposition 1 from [**P2**].*

iii) all the modules $L_\alpha \cong \mathcal{O}' \oplus \mathcal{O} \oplus \mathcal{O}$. Then the chain contains a piece

$$\ldots \supset \mathcal{O} \oplus \underbrace{\mathcal{O} \oplus \mathcal{O} \supset \ldots \supset \mathcal{O}' \oplus \mathcal{O} \oplus \mathcal{O} \supset m \oplus \mathcal{O} \oplus \mathcal{O} \supset \ldots \supset \wp \oplus \mathcal{O}}_{\cong \bar{K}} \oplus \mathcal{O} =$$

$$\wp \oplus \underbrace{\mathcal{O} \oplus \mathcal{O} \supset \ldots \supset \wp \oplus \mathcal{O}' \oplus \mathcal{O} \supset \ldots \supset \wp \oplus \wp}_{\cong \bar{K}} \oplus \mathcal{O} =$$

$$\wp \oplus \underbrace{\wp \oplus \mathcal{O} \supset \ldots \supset \wp \oplus \wp \oplus \mathcal{O}' \supset \ldots \supset \wp \oplus \wp}_{\cong \bar{K}} \oplus \wp \supset \ldots$$

and can also be restored from it.

iv) if there is an $L_\alpha \cong \mathcal{O}' \oplus \mathcal{O}' \oplus K$, then we can restore from

$$\ldots \supset \mathcal{O}' \oplus \mathcal{O}' \oplus K \supset m \oplus \mathcal{O}' \oplus K \supset m \oplus m \oplus K \supset \ldots$$

v) if there is an $L_\alpha \cong \mathcal{O}' \oplus K \oplus K$, then the chain can be written down as

$$\ldots \supset m^i \oplus K \oplus K \supset m^{i+1} \oplus K \oplus K \supset \ldots$$

We see that the chains of the first three types correspond to two-simplices, of the fourth type to edges of external boundary and the last type represents a vertex also of the external boundary. As above we can glue the building from apartments. To introduce them we can again repeat the corresponding definition for the building over a local field of dimension one (see Definition 5). Then the apartment $\Sigma_.$ will be a union

$$\Sigma_. = \Sigma_.[2] \cup \Sigma_.[1] \cup \Sigma_.[0],$$

where the pieces $\Sigma_.[i]$ contain the lattices of the types from $\Delta_.[i]$. The combinatorial structure of the apartment can be seen from two pictures at the end of our paper.

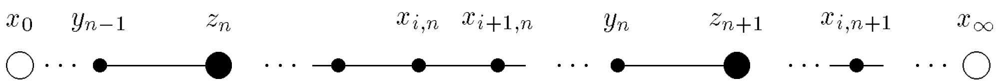
There we removed the external boundary $\Sigma_.[0]$ which is simplicially isomorphic to the external boundary of an apartment of the building
$\Delta_.(PGL(3), K/k)$. The dots in the first picture show a convergence of the vertices inside the apartment. As a result the building will be a simplicial topological space.

We can also describe the relations of the building with buildings of the same group G over local fields $K/\bar{K}$ and $\bar{K}/k$. In the first case there is a projection map

$$\pi : \Delta_.(G, K/\bar{K}/k) \to \Delta_.(G, K/\bar{K}).$$

Under this map the big triangles containing the simplices of type i) will be contracted into points, the triangles containing the simplices of type ii) will go to edges and the simplices of type iii) will be mapped isomorphically to simplices in the target space. The external boundary will not change.

The lines



can easily be visualized inside the apartment. Only the big white dots corresponding to the external boundary are missing. We have three types of lines going from the inner points under the angle $2\pi/3$. They correspond to different embeddings of $PGL(2)$ into $PGL(3)$.

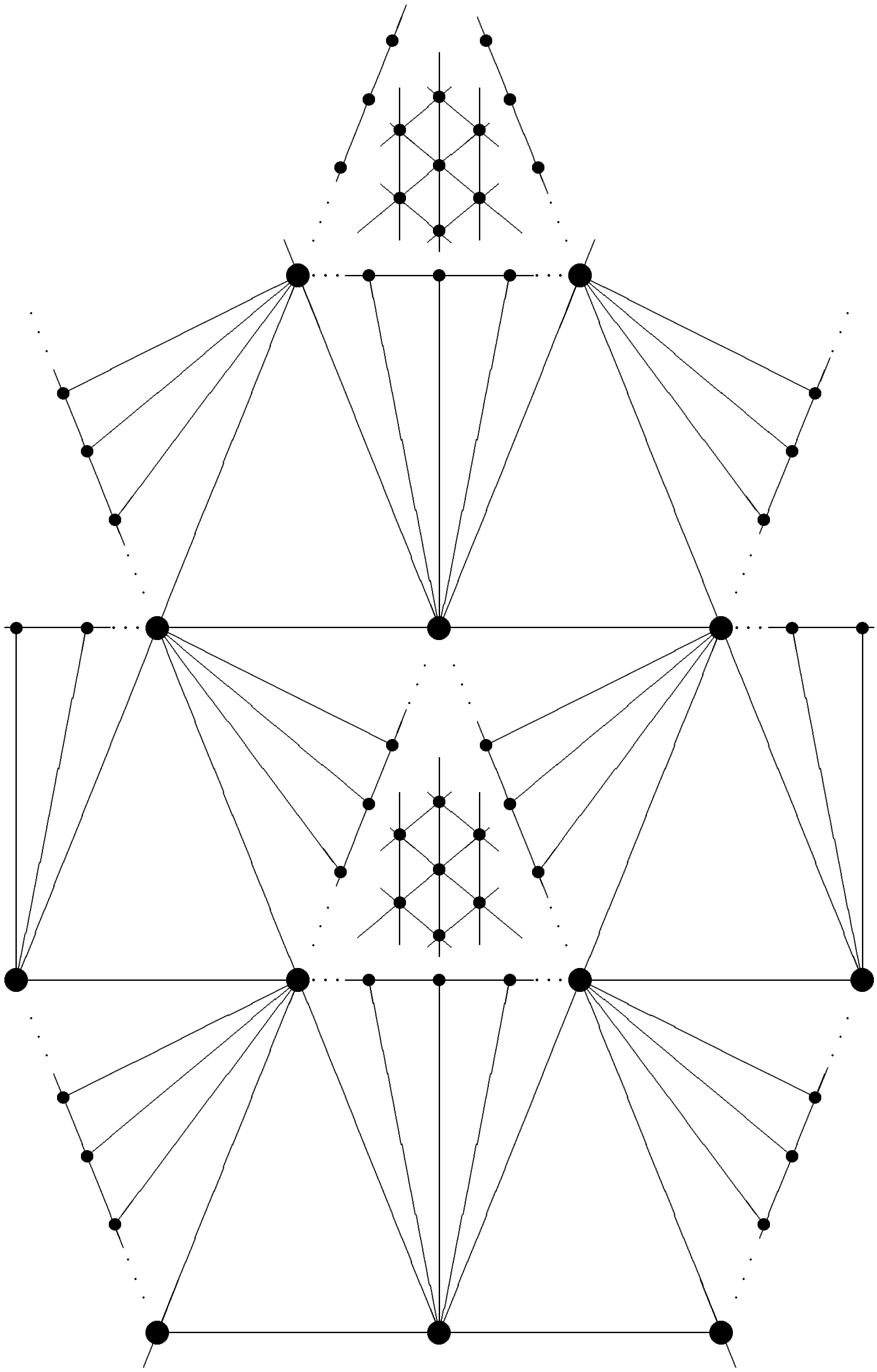
Using the lines we can understand the action of the Weyl group W on an apartment. The subgroup S_3 acts in the same way as in section 3. The free subgroup E (see section 2) has six types of translations along these three directions. Along each line we have two opportunities which were introduced for $PGL(2)$.

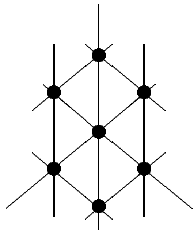
Namely, if $w \in \Gamma_K \cong \mathbf{Z} \oplus \mathbf{Z} \subset W$ then $w = (0, 1)$ acts as a shift of the whole structure to the right: $w(x_{i,n}) = x_{i,n+2}$, $w(y_n) = y_{n+2}$, $w(z_n) = z_{n+2}$, $w(x_0) = x_0$, $w(x_\infty) = x_\infty$.

The element $w = (1, 0)$ acts as a shift on the points $x_{i,n}$ but leaves fixed the points in the inner boundary $w(x_{i,n}) = x_{i+2,n}$, $w(y_n) = y_n$, $w(z_n) = z_n$, $w(x_0) = x_0$, $w(x_\infty) = x_\infty$, (see Theorem 5, v) [**P2**]).

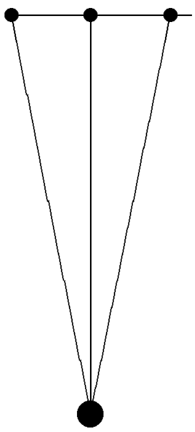
Acknowledgments

These notes were written up during my visit to the Abdus Salam ICTP in the fall of 1997. I am very much grateful to M.S. Narasimhan for the excellent working conditions. My special thanks go to J. Juyumaya who kindly made the drawings.

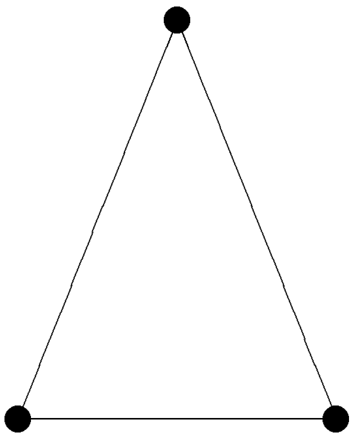




simplices of type i)



simplices of type ii)



simplices of type iii)

References

[**BT1**] Bruhat F., Tits J., *Groupes réductives sur un corps local. I. Données radicielles valuées*, Publ. Math. IHES, **41**(1972), 5-251;*II.Schémas en groupes, Existence d'une donnée radicielle valuée*, Publ. Math. IHES, **60**(1984), 5-184

[**BT2**] Bruhat F., Tits J., *Schémas en groupes et immeubles des groupes classiques sur un corps local*, Bull. Soc. Math. France. **112**(1984), 259-301

[**FP**] Fimmel T., Parshin A. N., *An Introduction into the Higher Adelic Theory* (in preparation)

[**GK**] Ginzburg V. A., Kapranov M. M., *Hecke algebras for p-adique loop groups*, preprint, 1996

[**M**] Mustafin G. A. *On non-archimedian uniformization*, Math. Sbornik, **105**(1978), 207-237

[**P1**] Parshin A. N. *Higher Bruhat-Tits buildings and Vector Bundles on an Algebraic Surface*, Algebra and Number Theory (Proc.Conf. held at the Inst.Exp.Math. , Univ.Essen, Dec. 2-4, 1992), de Gruyter, Berlin, 1994, 165-192

[**P2**] Parshin A. N. *Vector bundles and Arithmetical Groups I.*, Proc. Steklov Math. Institute, **208**(1996), 212-233

[**R**] Ronan M., *Buildings: Main Ideas and Applications.I.*, Bull. London Math.Soc. **24**(1992), 1-51;*II.*, Bull. London Math.Soc. **24**(1992), 97-126

[**T1**] Tits J., *On Buildings and their applications*, Proc. Intern. Congr. Math. (Vancouver 1974), Canad. Math. Congr., Montreal, 1975, Vol. **1**, 209-220

[**T2**] Tits J., *Reductive groups over local fields*, Automorphic Forms, Representations and *L*-functions, Proc. Symp. Pure Math., **33**, part 1, AMS, Providence, 1979, 29-70