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**THIRD HOMOLOGY GROUPS  
OF UNIVERSAL CENTRAL EXTENSIONS OF A LIE ALGEBRA**

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**Abstract**

Leibniz algebras are a non-commutative generalization of usual Lie algebras. We study universal central extensions of a perfect Leibniz algebra  $\mathcal{L}$  and we prove that the kernel is isomorphic to the second homology group  $HL_2(\mathcal{L})$ . Next, given a perfect Lie (hence Leibniz) algebra  $\mathfrak{g}$ , we compute and compare the homology groups  $HL_3(\mathfrak{u})$ ,  $HL_3(\mathfrak{u})$  and  $H_3(\mathfrak{u})$ , where  $\mathfrak{u}$  (resp.  $\mathfrak{u}$ ) is the universal central extension of  $\mathfrak{g}$  in the category of Leibniz (resp. Lie) algebras, and where  $HL_*$  (resp.  $H_*$ ) stands for the Leibniz (resp. the classical Chevalley-Eilenberg) homology theory.

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## 0. Introduction

Discovered by Jean-Louis Loday (see [7]), Leibniz algebras are a non-commutative variation of usual Lie algebras. There is an homology theory  $\mathrm{HL}_*$  for these new algebraic objects (see [8]), whose properties are similar to those of the classical Chevalley-Eilenberg homology theory  $H_*$  for Lie algebras (see [2]).

In this paper, we study universal central extensions of a Leibniz algebra. Mimicking an article by H. Garland (see [3]), we give a criterion for a central extension to be universal. Then we deduce a criterion for a Leibniz algebra to admit a universal central extension (perfectness). We show that the kernel of the universal central extension is canonically isomorphic to the second homology group of the initial object. Since Lie algebras are examples of Leibniz algebras, any perfect Lie algebra  $\mathfrak{g}$  admits a universal central extension  $\mathfrak{U}$  (resp.  $\mathfrak{u}$ ) in the category **(Leib)** (resp. **(Lie)**) of Leibniz (resp. Lie) algebras. It turns out that the Leibniz algebra  $\mathfrak{U}$  is the universal central extension of  $\mathfrak{u}$  in the category **(Leib)**, and that  $\mathfrak{u} \cong \mathfrak{U}_{Lie}$  that is, the Lie algebra canonically associated to  $\mathfrak{U}$ . These universal central extensions are homologically characterized by the following isomorphisms

$$\begin{aligned} \mathrm{HL}_1(\mathfrak{U}) = \mathrm{HL}_2(\mathfrak{U}) = 0 = H_1(\mathfrak{u}) = H_2(\mathfrak{u}), \\ \ker(\mathrm{HL}_2(\mathfrak{g}) \rightarrow H_2(\mathfrak{g})) \cong \mathrm{HL}_2(\mathfrak{u}) \cong \ker(\mathfrak{U} \rightarrow \mathfrak{u} \cong \mathfrak{U}_{Lie}). \end{aligned}$$

Next, we compute the homology groups  $\mathrm{HL}_3(\mathfrak{U})$ ,  $\mathrm{HL}_3(\mathfrak{u})$  and  $H_3(\mathfrak{u})$  in terms of the homology groups  $\mathrm{HL}_*(\mathfrak{g})$  and  $H_*(\mathfrak{g})$ . This is done by using the Hochschild-Serre and its Leibniz version spectral sequences (see [4], [5]). We give an interpretation of the natural maps  $\mathrm{HL}_3(\mathfrak{U}) \rightarrow \mathrm{HL}_3(\mathfrak{u})$  and  $\mathrm{HL}_3(\mathfrak{u}) \rightarrow H_3(\mathfrak{u})$  as follows

**Theorem.** *Let  $\mathfrak{U}$  (resp.  $\mathfrak{u}$ ) be the universal central extension in the category **(Leib)** (resp. **(Lie)**) of a perfect Lie algebra  $\mathfrak{g}$ . Then we have a (non-natural) commutative diagram*

$$\begin{array}{ccccc} \mathrm{HL}_3(\mathfrak{U}) & \xrightarrow{\cong} & \mathrm{HL}_2(\mathfrak{g})^{\otimes 2} & \oplus & \mathrm{HL}_3(\mathfrak{g}) \\ \downarrow & & \downarrow & & \parallel \\ \mathrm{HL}_3(\mathfrak{u}) & \xrightarrow{\cong} & \mathrm{HL}_2(\mathfrak{g})^{\otimes 2}/K^{\otimes 2} & \oplus & \mathrm{HL}_3(\mathfrak{g}) \\ \downarrow & & \downarrow & & \downarrow \\ H_3(\mathfrak{u}) & \xrightarrow{\cong} & S^2(H_2(\mathfrak{g})) & \oplus & H_3(\mathfrak{g}) \end{array}$$

where

$$K := \ker(\mathrm{HL}_2(\mathfrak{g}) \rightarrow H_2(\mathfrak{g})) \cong \ker(\mathfrak{U} \rightarrow \mathfrak{u} \cong \mathfrak{U}_{Lie}) \cong \mathrm{HL}_2(\mathfrak{u})$$

and  $S^2$  is the symmetric functor.

The symbol  $\mathbb{K}$  denotes a fixed commutative ring with unit. All modules, linear maps and tensor products involved here are over  $\mathbb{K}$ . In order to simplify computations, we will assume that  $\mathbb{K}$  is a field in the section 3.

## 1. Prerequisites on Leibniz algebras

**1.1. Leibniz algebras.** A *Leibniz algebra* is a  $\mathbb{K}$ -module  $\mathcal{L}$  equipped with a bilinear map  $[-, -] : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ , called *bracket* and satisfying only the *Leibniz identity*

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

for any  $x, y, z \in \mathcal{L}$ . In the presence of the condition  $[x, x] = 0$ , the Leibniz identity is equivalent to the so-called *Jacobi identity*; therefore Lie algebras are examples of Leibniz algebras.

A *morphism* of Leibniz algebras is a linear map  $f : \mathcal{L}_1 \rightarrow \mathcal{L}_2$  such that

$$f([x, y]) = [f(x), f(y)]$$

for any  $x, y \in \mathcal{L}_1$ . It is clear that Leibniz algebras and their morphisms form a category that we denote by **(Leib)**.

A *two-sided ideal* of a Leibniz algebra  $\mathcal{L}$  is a submodule  $\mathcal{H}$  such that  $[x, y] \in \mathcal{H}$  and  $[y, x] \in \mathcal{H}$  for any  $x \in \mathcal{H}$  and any  $y \in \mathcal{L}$ . For any two-sided ideal  $\mathcal{H}$  in  $\mathcal{L}$ , the quotient module  $\mathcal{L}/\mathcal{H}$  inherits a structure of Leibniz algebra induced by the bracket of  $\mathcal{L}$ . In particular, let  $([x, x])$  denotes the two-sided ideal in  $\mathcal{L}$  generated by all brackets  $[x, x]$ ; then the Leibniz algebra  $\mathcal{L}/([x, x])$  is in fact a Lie algebra, said *canonically associated to  $\mathcal{L}$*  and is denoted by  $\mathcal{L}_{Lie}$ .

Let  $\mathcal{L}$  be a Leibniz algebra. Denote by  $\mathcal{L}' := [\mathcal{L}, \mathcal{L}]$  the submodule generated by all brackets  $[x, y]$ . The Leibniz algebra  $\mathcal{L}$  is said to be *perfect* when  $\mathcal{L}' = \mathcal{L}$ . It is clear that any submodule of  $\mathcal{L}$  containing  $\mathcal{L}'$  is a two-sided ideal in  $\mathcal{L}$ .

**1.2. Non-trivial examples.** i) If  $(\mathfrak{g}, [-, -], d)$  is a differential Lie algebra, then the bracket defined by  $[x, y]_d := [x, d(y)]$  satisfies the Leibniz identity (but obviously it is not skew-symmetric).

ii) Let  $M$  be a representation of a Lie algebra  $\mathfrak{g}$  (the action of  $\mathfrak{g}$  on  $M$  being denoted by  $m^x$  for  $m \in M$  and  $x \in \mathfrak{g}$ ). For any  $\mathfrak{g}$ -equivariant map  $f : M \rightarrow \mathfrak{g}$ , the bracket given by  $[m, m'] := m^{f(m')}$  induces a structure of Leibniz (non-Lie) algebra on  $M$ .

**1.3. Semi-representations.** Let  $\mathcal{L}$  be a Leibniz algebra. A *semi-representation* of  $\mathcal{L}$  is a  $\mathbb{K}$ -module  $M$  equipped with an action of  $\mathcal{L}$ ,  $[-, -] : M \times \mathcal{L} \rightarrow M$ , satisfying the rule

$$[m, [x, y]] = [[m, x], y] - [[m, y], x]$$

for any  $m \in M$  and any  $x, y \in \mathcal{L}$ . It turns out that a semi-representation of a Leibniz algebra  $\mathcal{L}$  is equivalent to a representation the Lie algebra  $\mathcal{L}_{Lie}$  in the classical sense. It is clear that a Leibniz algebra is a semi-representation over itself by the *adjoint action*.

In [10], there are notions of (co)representations with a suitable notion of universal enveloping algebra of a Leibniz algebra.

**1.4. Leibniz homology.** Let  $\mathcal{L}$  be a Leibniz algebra and let  $M$  be a semi-representation of  $\mathcal{L}$ . There is a well-defined complex  $(T^*(\mathcal{L}, M) := M \otimes \mathcal{L}^{\otimes *}, d)$  where the boundary map  $d : T^n(\mathcal{L}, M) \rightarrow T^{n-1}(\mathcal{L}, M)$  is given by the formula (see [7])

$$d(x_0, \dots, x_n) := \sum_{0 \leq i < j \leq n} (-1)^{j+1} (x_0, \dots, x_{i-1}, [x_i, x_j], x_{i+1}, \dots, \widehat{x}_j, \dots, x_n)$$

for any  $x_0 \in M$  and  $x_1, \dots, x_n \in \mathcal{L}$ ; here  $(x_0, \dots, x_n)$  stands for  $x_0 \otimes \dots \otimes x_n \in T^n(\mathcal{L}, M)$ . The homology of this complex is denoted  $\mathrm{HL}_*(\mathcal{L}, M)$ , and simply  $\mathrm{HL}_*(\mathcal{L})$  if  $M = \mathbb{K}$  equipped with the trivial action of  $\mathcal{L}$ . One easily checks that

$$\begin{aligned} \mathrm{HL}_0(\mathcal{L}) &\cong \mathbb{K}, & \mathrm{HL}_1(\mathcal{L}) &\cong \mathcal{L}_{ab} := \mathcal{L}/[\mathcal{L}, \mathcal{L}], \\ \mathrm{HL}_0(\mathcal{L}, M) &\cong M_{\mathcal{L}} := M/[\mathcal{L}, M], \\ \mathrm{HL}_*(\mathcal{L}, \mathcal{L}) &\cong \mathrm{HL}_{*+1}(\mathcal{L}). \end{aligned}$$

Remark that if  $\mathfrak{g}$  is a Lie algebra, then the complex  $(T^*(\mathfrak{g}, M), d)$  is nothing but a lifting of the classical Chevalley-Eilenberg complex  $(M \otimes \Lambda^*(\mathfrak{g}), d)$  defining the homology  $H_*(\mathfrak{g}, M)$  of Lie algebras (see [2]). Moreover the canonical projection  $can_* : M \otimes \mathcal{L}^{\otimes * *} \rightarrow M \otimes \Lambda^*(\mathcal{L})$  is a morphism of complexes which induces in homology isomorphisms in degrees 0 and 1, and an epimorphism in degree 2.

Furthermore, J.-L. Loday and T. Pirashvili define general (co)homology theories with coefficients in (co)representations, and they give a Tor-Ext interpretation for Leibniz (co)homology.

## 2. Universal central extensions of a Leibniz algebra

**2.1. Central extensions.** An exact sequence of Leibniz algebras

$$(\mathcal{E}) \quad 0 \rightarrow \mathcal{H} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{L} \rightarrow 0$$

is called *extension of  $\mathcal{L}$  by  $\mathcal{H}$* . The morphism  $i$  is a Leibniz algebra isomorphism from  $\mathcal{H}$  onto the *kernel*  $\ker(p)$  of the extension. Therefore we merely write  $p : \mathcal{E} \rightarrow \mathcal{L}$  the extension  $(\mathcal{E})$ , and by abuse of language, we say that the Leibniz algebra  $\mathcal{E}$  is an extension of  $\mathcal{L}$ .

An extension  $p : \mathcal{E} \rightarrow \mathcal{L}$  is said to be *split* in the category **(Leib)** if there exists a Leibniz algebra morphism  $s : \mathcal{L} \rightarrow \mathcal{E}$  such that  $ps = \mathrm{id}_{\mathcal{L}}$ ; the map  $s$  is called a *section* of  $p$ .

A *central extension* of a Leibniz algebra  $\mathcal{L}$  is an extension  $p : \mathcal{C} \rightarrow \mathcal{L}$  whose kernel satisfies  $[\ker(p), \mathcal{C}] = [\mathcal{C}, \ker(p)] = 0$ .

A central extension  $\alpha : \mathcal{U} \rightarrow \mathcal{L}$  is said to be *universal* if for any central extension  $p : \mathcal{C} \rightarrow \mathcal{L}$  there exists a unique Leibniz algebra morphism  $\phi : \mathcal{U} \rightarrow \mathcal{C}$  such that  $p\phi = \alpha$ . It is clear that a universal central extension, when it exists, is unique (up to a unique isomorphism).

**2.2. Universality criterion.** Here are some properties which characterize the universality of a central extension.

**Proposition 2.1.** *If a central extension  $\alpha : \mathcal{U} \rightarrow \mathcal{L}$  is universal, then the Leibniz algebra  $\mathcal{U}$  is perfect.*

**Proof.** Assume that the Leibniz algebra  $\mathcal{U}$  is not perfect; then there is a non-trivial map  $f : \mathcal{U}/[\mathcal{U}, \mathcal{U}] \rightarrow \mathbb{K}$ .

Equip the direct sum  $\mathcal{U} \oplus \mathbb{K}$  with the Leibniz algebra structure given by

$$[(x, \lambda), (x', \lambda')] := ([x, x'], 0)$$

where  $x, x' \in \mathcal{U}$  and  $\lambda, \lambda' \in \mathbb{K}$ . Clearly the map  $\tilde{\alpha} : \mathcal{U} \oplus \mathbb{K} \rightarrow \mathcal{L}$ ,  $(x, \lambda) \mapsto \alpha(x)$  is a surjective morphism of Leibniz algebras whose kernel (which is nothing but  $\ker(\alpha) \oplus \mathbb{K}$ ) is central in  $\mathcal{U} \oplus \mathbb{K}$ . Thus the extension  $\tilde{\alpha} : \mathcal{U} \oplus \mathbb{K} \rightarrow \mathcal{L}$  is central.

One checks that the maps  $\phi_1, \phi_2 : \mathcal{U} \rightarrow \mathcal{U} \oplus \mathbb{K}$ , given by  $\phi_1(x) := (x, 0)$  and  $\phi_2(x) := (x, f(\bar{x}))$ , are two distinct morphisms of Leibniz algebras such that  $\tilde{\alpha}\phi_i = \alpha$ , ( $i = 1, 2$ ); which contradicts the universality of the extension  $\mathcal{U}$ .  $\square$

**Proposition 2.2.** *If a central extension  $\alpha : \mathcal{U} \twoheadrightarrow \mathcal{L}$  is universal, then any central extension of  $\mathcal{U}$  splits in the category **(Leib)**.*

**Proof.** This is done in two steps.

i) Let  $p : \mathcal{L} \twoheadrightarrow \mathcal{U}$  be a central extension such that  $\mathcal{L}$  is perfect. It is clear that the map  $\beta := \alpha p : \mathcal{L} \rightarrow \mathcal{L}$  is a surjective morphism of Leibniz algebras. Let us show that its kernel is central in  $\mathcal{L}$ . Firstly remark that an element  $z$  is in  $\ker(\beta)$  if, and only if,  $p(z)$  is in  $\ker(\alpha)$ . Since  $\ker(\alpha)$  is central in  $\mathcal{U}$ , if  $x$  or  $y$  is in  $\ker(\beta)$ , then  $[x, y]$  is in  $\ker(p)$ . Since  $\ker(p)$  is central in  $\mathcal{L}$ , one has  $[[x, y], z] = 0$  (resp.  $[x, [y, z]] = 0$ ) if  $x$  or  $y$  (resp.  $y$  or  $z$ ) is in  $\ker(\beta)$ . Consequently, by the Leibniz identity, we get

$$\begin{aligned} [x, [y_1, y_2]] &= [[x, y_1], y_2] - [[x, y_2], y_1] = 0, \\ [[y_1, y_2], x] &= [y_1, [y_2, x]] + [[y_1, x], y_2] = 0 \end{aligned}$$

for any  $x \in \ker(\beta)$ ,  $y_1, y_2 \in \mathcal{L}$ . Therefore  $\ker(\beta)$  is central in  $[\mathcal{L}, \mathcal{L}] = \mathcal{L}$ .

By the universality of the extension  $\alpha : \mathcal{U} \twoheadrightarrow \mathcal{L}$ , there exists a Leibniz algebra morphism  $s : \mathcal{U} \rightarrow \mathcal{L}$  such that  $\beta s = \alpha$  i.e.,  $\alpha p s = \alpha$ . Consequently the morphism  $\psi := p s - \text{id}_{\mathcal{U}}$  takes value in  $\ker(\alpha)$ . Since  $\ker(\alpha)$  is central in  $\mathcal{U}$ , one has

$$\psi([x, y]) = [p s(x), p s(y)] - [x, y] = [\psi(x), p s(y)] + [x, \psi(y)] = 0.$$

It follows that  $\psi$  is trivial on  $[\mathcal{U}, \mathcal{U}]$ ; from whence  $p s = \text{id}_{\mathcal{U}}$  since  $\mathcal{U} = [\mathcal{U}, \mathcal{U}]$  (cf. Proposition 2.1).

ii) Let  $p : \mathcal{L} \twoheadrightarrow \mathcal{U}$  be any central extension. Denote by  $p'$  the restriction of  $p$  to the subalgebra  $\mathcal{L}' = [\mathcal{L}, \mathcal{L}]$ . Since  $\mathcal{U}$  is perfect, it is clear that the morphism  $p' : \mathcal{L}' \rightarrow \mathcal{U}$  is still surjective with a central kernel. Let us show that  $\mathcal{L}'$  is perfect. Let  $[x, y]$  be a generator of  $\mathcal{L}'$ . Since  $\mathcal{U}$  is perfect and  $p$  is surjective, one can successively write

$$p(x) = \sum [l_i, l'_i] = \sum [p(x_i), p(x'_i)] = \sum p([x_i, x'_i]), \quad l_i, l'_i \in \mathcal{U}, \quad x_i, x'_i \in \mathcal{L}.$$

Therefore the element  $h := x - \sum [x_i, x'_i]$  is in  $\ker(p)$ . By the same way, there exist elements  $(y_j, y'_j)$  in  $\mathcal{L}$  such that  $h' := y - \sum [y_j, y'_j]$  is in  $\ker(p)$ . Since  $\ker(p)$  is central in  $\mathcal{L}$ , we get

$$[x, y] = [h + \sum [x_i, x'_i], h' + \sum [y_j, y'_j]] = \sum \sum [[x_i, x'_i], [y_j, y'_j]].$$

Then the subalgebra  $\mathcal{L}'$  is perfect. And by the step i) there exists an algebra morphism  $s' : \mathcal{U} \rightarrow \mathcal{L}'$  splitting  $p'$ . Denoting by  $\iota$  the inclusion map  $\mathcal{L}' \hookrightarrow \mathcal{L}$ , the composed map  $s := \iota s' : \mathcal{U} \rightarrow \mathcal{L}$  is an algebra morphism splitting  $p$ .  $\square$

Now we can state the following

**Theorem 2.3.** *Let  $\mathcal{L}$  be a Leibniz algebra. A central extension  $\mathcal{U}$  of  $\mathcal{L}$  is universal if, and only if, the Leibniz algebra  $\mathcal{U}$  is perfect and any central extension of  $\mathcal{U}$  splits.*

**Proof of the converse.** Let  $p : \mathcal{C} \twoheadrightarrow \mathcal{L}$  be a central extension of  $\mathcal{L}$ . We have to show that there is a unique algebra morphism  $\phi : \mathcal{U} \rightarrow \mathcal{C}$  such that  $p\phi = \alpha$ . The uniqueness follows from

**Lemma 2.4.** *If  $\alpha : \mathcal{U} \twoheadrightarrow \mathcal{L}$  is a central extension and if the Leibniz algebra  $\mathcal{U}$  is perfect, then for any central extension  $p : \mathcal{C} \twoheadrightarrow \mathcal{L}$  there exists at most one algebra morphism  $\phi : \mathcal{U} \rightarrow \mathcal{C}$  such that  $p\phi = \alpha$ .*

**Proof of the Lemma.** In fact, suppose that there are two such morphisms  $\phi$  and  $\phi'$ . Then for all  $x, y \in \mathcal{U}$  one has

$$\begin{aligned} (\phi - \phi')([x, y]) &= [\phi(x), \phi(y)] - [\phi'(x), \phi'(y)] \\ &= [\phi(x) - \phi'(x), \phi(y)] + [\phi'(x), \phi(y) - \phi'(y)]. \end{aligned}$$

Since the map  $\phi - \phi'$  takes value in  $\ker(p)$  which is central in  $\mathcal{C}$ , it follows that  $\phi - \phi'$  is trivial on  $[\mathcal{U}, \mathcal{U}] = \mathcal{U}$ . From whence follows the Lemma.  $\square$

For the existence of such a morphism, consider the product  $\mathcal{C} \times \mathcal{U}$  equipped with the bracket given by  $[(c, u), (c', u')] := ([c, c'], [u, u'])$ . Denote by  $\mathcal{C} \times_{\mathcal{L}} \mathcal{U}$  the subalgebra

$$\mathcal{C} \times_{\mathcal{L}} \mathcal{U} := \{(c, u) \in \mathcal{C} \times \mathcal{U} \mid p(c) = \alpha(u)\}.$$

The second projection  $\rho_2 : \mathcal{C} \times_{\mathcal{L}} \mathcal{U} \rightarrow \mathcal{U}$  is a surjective morphism of algebras whose kernel is obviously central in  $\mathcal{C} \times_{\mathcal{L}} \mathcal{U}$ . Therefore there exists a morphism  $s : \mathcal{U} \rightarrow \mathcal{C} \times_{\mathcal{L}} \mathcal{U}$  such that  $\rho_2 s = \text{id}_{\mathcal{U}}$ . Consider the morphism  $\phi := \rho_1 s : \mathcal{U} \rightarrow \mathcal{C}$  where  $\rho_1 : \mathcal{C} \times_{\mathcal{L}} \mathcal{U} \rightarrow \mathcal{C}$  is the first projection. By definition of  $\rho_1, \rho_2, \phi$  and  $s$ , one has  $s(u) = (\rho_1 s(u), \rho_2 s(u)) = (\phi(u), u)$ . But  $(\phi(u), u) \in \mathcal{C} \times_{\mathcal{L}} \mathcal{U}$  means that  $p\phi(u) = \alpha(u)$ ; from whence  $p\phi = \alpha$ , and the Theorem is proved.  $\square$

**2.3. Remark.** These properties of the universal central extension are homologically characterized by the equalities

$$\text{HL}_1(\mathcal{L}) = \text{HL}_1(\mathcal{U}) = \text{HL}_2(\mathcal{U}) = 0.$$

**2.4. Existence criterion.** Now we characterize Leibniz algebras which admit a universal central extension.

**Theorem 2.5.** *A Leibniz algebra  $\mathcal{L}$  (free as a  $\mathbb{K}$ -module) admits a universal central extension if, and only if, it is perfect. Moreover, the kernel of the universal central extension is canonically isomorphic to  $\text{HL}_2(\mathcal{L})$ .*

**Proof.** The condition is necessary because a universal central extension is perfect and the surjective image of a perfect Leibniz algebra is also perfect.

Conversely, suppose that the Leibniz algebra  $\mathcal{L}$  is perfect. Let  $\text{Im}(d_3)$  be the image of the Leibniz boundary  $d_3$  i.e., the submodule of  $\mathcal{L}^{\otimes 2}$  generated by the elements

$$d_3(x \otimes y \otimes z) = [x, y] \otimes z - [x, z] \otimes y - x \otimes [y, z], \quad \forall x, y, z \in \mathcal{L}.$$

Consider the quotient module  $M := \mathcal{L}^{\otimes 2} / \text{Im}(d_3)$ , equipped with the trivial action of  $\mathcal{L}$ . The canonical projection  $\pi : \mathcal{L}^{\otimes 2} \rightarrow M$  is obviously a 2-cocycle of  $\text{HL}^2(\mathcal{L}, M)$ . Then it determines a central extension  $\mathcal{L}_{\pi}$ ; recall that  $\mathcal{L}_{\pi}$  is the  $\mathbb{K}$ -module  $\mathcal{L}_{\pi} = \mathcal{L} \oplus M$  equipped with the bracket given by  $[(x, m), (x', m')] := ([x, x'], \pi(x, x'))$  (see [10]). Let us show that the subalgebra  $\mathcal{L}'_{\pi} := [\mathcal{L}_{\pi}, \mathcal{L}_{\pi}]$  is perfect. First, remark that we have  $\mathcal{L}_{\pi} = \mathcal{L}'_{\pi} + M$ . In fact, since  $\mathcal{L}$  is perfect, any element of  $\mathcal{L}_{\pi}$  takes the form  $(\sum [x_i, x'_i], m)$ ; by definition of the bracket on  $\mathcal{L}_{\pi}$ , one has

$$\left(\sum [x_i, x'_i], m\right) = \sum [(x_i, 0), (x'_i, 0)] + (0, m - \sum \pi(x_i, x'_i)) \in \mathcal{L}'_{\pi} + M,$$

which proves the equality  $\mathcal{L}_{\pi} = \mathcal{L}'_{\pi} + M$ . Since  $M$  is central in  $\mathcal{L}_{\pi}$ , one gets

$$\mathcal{L}'_{\pi} = [\mathcal{L}_{\pi}, \mathcal{L}_{\pi}] = [\mathcal{L}'_{\pi} + M, \mathcal{L}'_{\pi} + M] = [\mathcal{L}'_{\pi}, \mathcal{L}'_{\pi}].$$

From whence follows the perfectness of the algebra  $\mathcal{L}'_{\pi}$ .

The first projection  $\rho_1 : \mathcal{L}'_{\pi} \rightarrow \mathcal{L}$  is a central extension whose kernel is generated by the elements of the form  $(0, \sum \pi(x_i, x'_i))$  such that  $\sum [x_i, x'_i] = 0$ .

Since the  $\mathbb{K}$ -module  $\mathcal{L}$  is free, any central (hence abelian) extension of  $\mathcal{L}$  is of the form  $p : \mathcal{L}_f \rightarrow \mathcal{L}$  for a well-determined 2-cocycle  $f$  of  $\mathrm{HL}^2(\mathcal{L}, V)$ . Then the map

$$\phi : \mathcal{L}'_{\pi} \rightarrow \mathcal{L}_f, \quad (x, \pi(y, z)) \mapsto (x, f(y, z))$$

is an algebra morphism satisfying  $p\phi = \rho_1$ . The uniqueness follows from the Lemma 2.4 ( $\mathcal{L}'_{\pi}$  is perfect).

Now let us show that  $\ker(\rho_1)$  is canonically isomorphic to  $\mathrm{HL}_2(\mathcal{L})$ . Since the Leibniz boundary  $d_2 : \mathcal{L}^{\otimes 2} \rightarrow \mathcal{L}$  acts by  $x \otimes y \mapsto [x, y]$ , the  $\mathbb{K}$ -linear map

$$\xi : \ker(\rho_1) \rightarrow \mathrm{HL}_2(\mathcal{L}), \quad (0, \sum \pi(x_i, x'_i)) \mapsto \sum x_i \otimes x'_i$$

is well-defined and surjective. Suppose that  $\xi(0, \sum \pi(x_i, x'_i))$  is a boundary *i.e.*,

$$\xi(0, \sum \pi(x_i, x'_i)) = \sum x_i \otimes x'_i = \sum d_3(y_j \otimes y'_j \otimes y''_j).$$

Then, by the definition of  $M = \mathcal{L}^{\otimes 2} / \mathrm{Im}(d_3)$ , one has

$$\sum \pi(x_i, x'_i) = \sum \pi d_3(y_j \otimes y'_j \otimes y''_j) = 0.$$

Thus the map  $\xi$  is also injective; which proves the isomorphism  $\ker(\rho_1) \cong \mathrm{HL}_2(\mathcal{L})$ .  $\square$

**2.5. Remark.** The universal central extension can also be characterized by the following. Consider  $\mathrm{HL}_2(\mathcal{L})$  as a trivial representation of  $\mathcal{L}$ . By the universal coefficient theorem, there is an isomorphism  $\mathrm{HL}^2(\mathcal{L}, \mathrm{HL}_2(\mathcal{L})) \cong \mathrm{Hom}(\mathrm{HL}_2(\mathcal{L}), \mathrm{HL}_2(\mathcal{L}))$ . But one knows that there is a natural bijection  $\mathrm{HL}^2(\mathcal{L}, \mathrm{HL}_2(\mathcal{L})) \cong \mathrm{Ext}(\mathcal{L}, \mathrm{HL}_2(\mathcal{L}))$ , where  $\mathrm{Ext}(\mathcal{L}, \mathrm{HL}_2(\mathcal{L}))$  denotes the set of isomorphism classes of *abelian* extensions of  $\mathcal{L}$  by  $\mathrm{HL}_2(\mathcal{L})$ . The universal central extension corresponds to the element

$$\mathrm{id}_{\mathrm{HL}_2(\mathcal{L})} \in \mathrm{Hom}(\mathrm{HL}_2(\mathcal{L}), \mathrm{HL}_2(\mathcal{L})).$$

**2.6. Case of Lie algebras.** Let  $\mathfrak{g}$  be a perfect Lie algebra which is free as a  $\mathbb{K}$ -module. Then there exist a universal central extension  $\alpha : \mathfrak{U} \rightarrow \mathfrak{g}$  in the category (**Leib**) of Leibniz algebras, and a universal central extension  $\alpha' : \mathfrak{u} \rightarrow \mathfrak{g}$  in the category (**Lie**) of Lie algebras.

**Proposition 2.6.** *If the  $\mathbb{K}$ -module  $\mathfrak{u}$  is free (e.g. when  $\mathbb{K}$  is a field), then the Leibniz algebra  $\mathfrak{U}$  is the universal central extension of the perfect Lie algebra  $\mathfrak{u}$  in the category (**Leib**). Moreover there is an isomorphism of Lie algebras  $\mathfrak{u} \xrightarrow{\sim} \mathfrak{U}_{\mathrm{Lie}}$ .*

**Proof.** In fact, the extension  $\alpha' : \mathfrak{u} \rightarrow \mathfrak{g}$  is also central in the category (**Leib**). Therefore there exists a morphism of Leibniz algebras  $\phi : \mathfrak{U} \rightarrow \mathfrak{u}$  such that  $\alpha'\phi = \alpha$ . Since  $\mathfrak{U}$  is perfect and any central extension  $\mathfrak{U}$  splits, it suffices to show that the map  $\phi : \mathfrak{U} \rightarrow \mathfrak{u}$  is a central extension. Since  $\ker(\phi) \subset \ker(\alpha)$ , it is clear that  $\ker(\phi)$  is central in  $\mathfrak{U}$ . Let us show that  $\phi$  is surjective. First, remark that one has  $\mathfrak{u} = \mathrm{Im}(\phi) + \ker(\alpha')$ . In fact, let  $z' \in \mathfrak{u}$ ; since  $\alpha$  is surjective and  $\alpha'\phi = \alpha$ , there exists an element  $z \in \mathfrak{U}$  such that  $\alpha'(z') = \alpha(z) = \alpha'\phi(z)$  *i.e.*,  $z' - \phi(z) \in \ker(\alpha')$ . From whence the equality  $\mathfrak{u} = \mathrm{Im}(\phi) + \ker(\alpha')$ . Since the algebra  $\mathfrak{u}$  is perfect and  $\ker(\alpha')$  is central in  $\mathfrak{u}$ , we have

$$\mathfrak{u} = [\mathfrak{u}, \mathfrak{u}] = [\mathrm{Im}(\phi) + \ker(\alpha'), \mathrm{Im}(\phi) + \ker(\alpha')] = [\mathrm{Im}(\phi), \mathrm{Im}(\phi)] \subset \mathrm{Im}(\phi).$$

Therefore the morphism  $\phi$  is surjective.

Furthermore, denote by  $\bar{\alpha}$  (resp.  $\bar{\phi}$ ) the morphism induced by  $\alpha$  (resp.  $\phi$ ) on  $\mathfrak{U}_{Lie}$ . It is clear that the extension  $\bar{\alpha} : \mathfrak{U}_{Lie} \rightarrow \mathfrak{g}$  is central in the category **(Lie)** and that  $\alpha' \bar{\phi} = \bar{\alpha}$ . Therefore there exists a Lie algebra morphism  $\psi : \mathfrak{u} \rightarrow \mathfrak{U}_{Lie}$  such that  $\bar{\alpha} \psi = \alpha'$ . By composition, we get

$$\bar{\alpha}(\psi \bar{\phi}) = \alpha' \bar{\phi} = \bar{\alpha} \quad \text{and} \quad \alpha'(\bar{\phi} \psi) = \bar{\alpha} \psi = \alpha'.$$

Since the Lie algebras  $\mathfrak{u}$  and  $\mathfrak{U}_{Lie}$  are perfect, we deduce from the Lemma 2.4 that  $\bar{\phi} \psi = \text{id}$  on  $\mathfrak{u}$  and  $\psi \bar{\phi} = \text{id}$  on  $\mathfrak{U}$ ; from whence the isomorphism  $\mathfrak{u} \xrightarrow{\sim} \mathfrak{U}_{Lie}$ .  $\square$

As consequence, we obtain a commutative diagram with exact columns and rows

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \ker(\text{can}_2) & \longrightarrow & \text{HL}_2(\mathfrak{u}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{HL}_2(\mathfrak{g}) & \longrightarrow & \mathfrak{U} & \longrightarrow & \mathfrak{g} \longrightarrow 0 \quad (\mathbf{Leib}) \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \text{H}_2(\mathfrak{g}) & \longrightarrow & \mathfrak{u} \cong \mathfrak{U}_{Lie} & \longrightarrow & \mathfrak{g} \longrightarrow 0 \quad (\mathbf{Lie}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0 \\
 & & & & (\mathbf{Leib}) & & ,
 \end{array}$$

which yields the following characterization

**Corollary 2.7.** *With the above notation, one has the isomorphisms*

$$\ker(\mathfrak{U} \rightarrow \mathfrak{U}_{Lie}) \cong \text{HL}_2(\mathfrak{U}_{Lie}) \cong \ker(\text{can}_2 : \text{HL}_2(\mathfrak{g}) \rightarrow \text{H}_2(\mathfrak{g})). \quad \square$$

**2.7. Examples: the Steinberg and Virasoro algebras.** Let  $A$  be an associative algebra and let  $\mathfrak{sl}_n(A)$  be the Lie algebra of  $(n \times n)$ -matrices with entries in  $A$  and with zero trace in the *abelianised*  $A/[A, A]$  (here  $[A, A]$  stands for the submodule of  $A$  generated by the commutators  $[a, b] = ab - ba$ , for any  $a, b \in A$ ). Recall that the Lie algebra  $\mathfrak{sl}_n(A)$  is perfect for  $n \geq 3$ .

Studying the universal central extension  $\mathfrak{stl}_n(A)$  of  $\mathfrak{sl}_n(A)$  in the category **(Lie)**, Ch. Kassel and J.-L. Loday (see [6]) generalize a result by S. Bloch (see [1]) and obtain the isomorphism

$$\text{H}_2(\mathfrak{sl}_n(A)) \cong \text{HC}_1(A), \quad \forall n \geq 5$$

where  $\text{HC}_1(A)$  denotes the cyclic homology of  $A$  (see [7]).

J.-L. Loday and T. Pirashvili (see [10]) construct the universal central extension  $\mathfrak{stl}_n(A)$  of  $\mathfrak{sl}_n(A)$  in the category **(Leib)** and obtain the isomorphism

$$\text{HL}_2(\mathfrak{stl}_n(A)) \cong \text{HH}_1(A), \quad \forall n \geq 5$$



where  $\mathrm{HH}_1(A)$  denotes the Hochschild homology of  $A$ . Corollary 2.7 implies that

$$\mathfrak{st}_n(A) \cong \mathfrak{st}_n(A)_{\mathrm{Lie}}.$$

They also show that there is an isomorphism

$$\mathrm{HL}_2(\mathrm{Der}(\mathbb{C}[z, z^{-1}])) \xrightarrow{\sim} \mathrm{H}_2(\mathrm{Der}(\mathbb{C}[z, z^{-1}]))$$

which proves that the two universal central extensions of the Lie algebra  $\mathrm{Der}(\mathbb{C}[z, z^{-1}])$  coincide. In fact, this universal central extension is the Virasoro algebra.

### 3. Third homology groups

From now on we assume that  $\mathbb{K}$  is a field.

**3.1. Helpful background.** While making explicit computations, one often needs the following characterization of spectral sequences, whose cohomological version can be found in [11, section 2.2.2].

Let  $(F_*^p)_{p \geq 0}$  be a filtration of a complex  $(C_*, d)$  and let  $(E^r)_{r \geq 1}$  be the spectral sequence derived from this filtration. Define

$$Z_{p,q}^r := F_{p+q}^p \cap d^{-1}(F_{p+q-1}^{p-r}) \quad \text{and} \quad B_{p,q}^r := F_{p+q}^p \cap d(F_{p+q+1}^{p+r}).$$

Then one has

$$E_{p,q}^r \cong Z_{p,q}^r / (Z_{p-1,q+1}^{r-1} + B_{p,q}^{r-1}).$$

The differential maps are given by the commutative diagram

$$\begin{array}{ccc} Z_{p,q}^r & \xrightarrow{d} & Z_{p-r,q+r-1}^r \\ \eta_{p,q}^r \downarrow & & \downarrow \eta_{p-r,q+r-1}^r \\ E_{p,q}^r & \xrightarrow{d_{p,q}^r} & E_{p-r,q+r-1}^r \end{array}$$

where  $\eta_{p,q}^r : Z_{p,q}^r \rightarrow E_{p,q}^r$  is the canonical projection with  $\ker(\eta_{p,q}^r) = Z_{p-1,q+1}^{r-1} + B_{p,q}^{r-1}$ . Moreover we have

$$\begin{aligned} d(Z_{p,q}^r) &= F_{p+q-1}^{p-r} \cap d(F_{p+q}^p) = B_{p-r,q+r-1}^r, \\ \mathrm{Im}(d_{p,q}^r) &= B_{p-r,q+r-1}^r / (Z_{p-r-1,q+r}^{r-1} + B_{p-r,q+r-1}^{r-1}), \\ E_{p,q}^r &\cong \ker(d_{p,q}^r) / \mathrm{Im}(d_{p+r,q-r+1}^r). \end{aligned}$$

**3.2. Hochschild-Serre spectral sequence.** Let  $\mathfrak{g}$  be a perfect Lie algebra and let  $\mathfrak{u}$  be its universal central extension in the category  $(\mathbf{Lie})$  with kernel  $\mathfrak{h} \cong \mathrm{H}_2(\mathfrak{g})$ . Since  $\mathfrak{h}$  is an abelian Lie algebra on which  $\mathfrak{g}$  acts trivially, the  $E^2$ -terms of the Hochschild-Serre spectral sequence are given by

$$E_{p,q}^2 \cong \mathrm{H}_p(\mathfrak{g}, \mathrm{H}_q(\mathfrak{h})) \cong \Lambda^q(\mathfrak{h}) \otimes \mathrm{H}_p(\mathfrak{g}) \implies \mathrm{H}_{p+q}(\mathfrak{u}).$$

**Theorem 3.1.** *Let  $\mathfrak{u}$  be the universal central extension of a perfect Lie algebra  $\mathfrak{g}$ . Then one has an exact sequence*

$$0 \rightarrow S^2(\mathbb{H}_2(\mathfrak{g})) \rightarrow \mathbb{H}_3(\mathfrak{u}) \rightarrow \mathbb{H}_3(\mathfrak{g}) \rightarrow 0,$$

from whence the (non-natural) isomorphism

$$\mathbb{H}_3(\mathfrak{u}) \cong S^2(\mathbb{H}_2(\mathfrak{g})) \oplus \mathbb{H}_3(\mathfrak{g})$$

where  $S^2$  denotes the symmetric functor.

**Proof.** We have to compute separately the four terms  $E_{p,q}^\infty$  with  $p+q=3$ . Firstly recall that the Hochschild-Serre spectral sequence is derived from the filtration  $(F_*^p)_{p \geq 0}$  where  $F_n^p$  is the subspace of  $\Lambda^n(\mathfrak{u})$  generated by the elements  $x_1 \wedge \cdots \wedge x_n$  with at least  $(n-p)$  factors in  $\mathfrak{h}$ .

**i) The term  $E_{0,3}^\infty$ .** One has  $E_{0,3}^2 \cong \Lambda^3(\mathfrak{h})$ . Let us determine the image  $\text{Im}(d_{2,2}^2) = B_{0,3}^2/B_{0,3}^1$ . Since  $\mathfrak{u}$  is perfect, one has

$$B_{0,3}^2 = F_3^0 \cap d(F_4^2) = \Lambda^3(\mathfrak{h}) \cap d(F_4^2) = \Lambda^3(\mathfrak{h}).$$

On the other hand, we have

$$B_{0,3}^1 = F_3^0 \cap d(F_4^1) = F_3^0 \cap \{0\} = 0.$$

Therefore  $\text{Im}(d_{2,2}^2) = \Lambda^3(\mathfrak{h})$ ; from whence we deduce

$$E_{0,3}^\infty \cong \cdots \cong E_{0,3}^3 = 0.$$

**ii) The term  $E_{1,2}^\infty$ .** Since the Lie algebra  $\mathfrak{g}$  is perfect, it is clear that

$$E_{1,2}^\infty \cong \cdots \cong E_{1,2}^2 = 0.$$

**iii) The term  $E_{2,1}^\infty$ .** One has  $E_{2,1}^2 \cong \mathfrak{h} \otimes \mathbb{H}_2(\mathfrak{g}) \cong \mathfrak{h}^{\otimes 2}$ . Firstly let us determine the image  $\text{Im}(d_{2,1}^2) = B_{0,2}^2/B_{0,2}^1$ . Since  $\mathfrak{u}$  is perfect, we have

$$B_{0,2}^2 = F_2^0 \cap d(F_3^2) = \Lambda^2(\mathfrak{h}) \cap d(F_3^2) = \Lambda^2(\mathfrak{h}).$$

On the other hand, we have

$$B_{0,2}^1 = F_2^0 \cap d(F_3^1) = \Lambda^2(\mathfrak{h}) \cap \{0\} = 0.$$

Therefore  $\text{Im}(d_{2,1}^2) = \Lambda^2(\mathfrak{h})$ ; from whence  $\ker(d_{2,1}^2) \cong S^2(\mathfrak{h})$ .

Now we have to calculate  $\text{Im}(d_{4,0}^2) = B_{2,1}^2/(Z_{1,2}^1 + B_{2,1}^1)$  with

$$\begin{aligned} B_{2,1}^2 &= F_3^2 \cap d(F_4^4), \\ Z_{1,2}^1 &= F_3^2 \cap d^{-1}(F_2^0) = F_3^2 \cap d^{-1}(\Lambda^2(\mathfrak{h})), \\ B_{2,1}^1 &= F_3^2 \cap d(F_4^3). \end{aligned}$$

As vector space we have

$$F_4^4 = \Lambda^4(\mathfrak{u}) = F_4^3 + \Lambda^4(\mathfrak{g})$$

which yields

$$d(F_4^4) = d(F_4^3) + d(\Lambda^4(\mathfrak{g})).$$

Thus it is clear, that

$$B_{2,1}^2 = (B_{2,1}^1 + Z_{1,2}^1).$$

From whence we deduce that  $\text{Im}(d_{4,0}^2) = 0$ . And then we have

$$E_{2,1}^\infty \cong \cdots \cong E_{2,1}^2 = S^2(\mathfrak{h}).$$

**iv) The term  $E_{3,0}^\infty$ .** Since  $E_{1,1}^2 = 0$  (perfectness of  $\mathfrak{g}$ ), we have

$$E_{3,0}^\infty \cong \cdots \cong E_{3,0}^2 = H_3(\mathfrak{g}).$$

**v) Conclusion.** Since  $\mathbb{K}$  is a field, the exact sequence (hence the isomorphism) of Proposition 3.1 is clear taking into account the above computations.  $\square$

**3.3. Leibniz spectral sequence.** Recall that for any Leibniz algebra  $\mathcal{G}$ , any two-sided ideal  $\mathcal{H}$  in  $\mathcal{G}$  and any semi-representation  $M$  of  $\mathcal{G}$ , the filtration  $(F_*^p)_{p \geq 0}$  defined by

$$F_n^p := \begin{cases} M \otimes \mathcal{H}^{\otimes(n-p)} \otimes \mathcal{G}^{\otimes p}, & \text{if } n \geq p \\ M \otimes \mathcal{G}^{\otimes n}, & \text{if } n \leq p, \end{cases}$$

gives rise to a spectral sequence  $(E^r(\mathcal{H}, M))_{r \geq 1}$  which converges to the Leibniz homology  $\text{HL}_*(\mathcal{G}, M)$  (see [4]). Moreover, if the adjoint diagonal action of  $\mathcal{G}/\mathcal{H}$  on  $\text{HL}_q(\mathcal{H}, M)$  is trivial, then one has the isomorphisms

$$\begin{aligned} E_{0,q}^2(\mathcal{H}, M) &\cong \text{HL}_q(\mathcal{H}, M), \\ E_{1,q}^2(\mathcal{H}, M) &\cong \text{HL}_q(\mathcal{H}, M) \otimes \text{HL}_1(\mathcal{G}/\mathcal{H}), \\ E_{p,q}^2(\mathcal{H}, M) &\cong \text{HL}_q(\mathcal{H}, M) \otimes \text{HL}_{p-1}(\mathcal{G}, \mathcal{G}/\mathcal{H}), \quad p \geq 2. \end{aligned}$$

Here we let  $\mathcal{U}(=\mathcal{G})$  be the universal central extension of a perfect Leibniz algebra  $\mathcal{L}$ . Denote by  $\mathcal{H} := \text{HL}_2(\mathcal{L})$  the kernel of this universal central extension. Since the Leibniz algebra  $\mathcal{L} \cong \mathcal{U}/\mathcal{H}$  acts trivially on  $\mathcal{H}$ , it also acts trivially on  $\text{HL}_q(\mathcal{H}) = \mathcal{H}^{\otimes q}$  for any integer  $q \geq 0$ . Therefore, by the perfectness of  $\mathcal{L}$  (i.e.,  $\text{HL}_1(\mathcal{L}) = 0$ ), the spectral sequence  $(E^r(\mathcal{H}, \mathbb{K}))_{r \geq 1}$  is characterized by

$$\begin{aligned} E_{0,q}^2(\mathcal{H}, \mathbb{K}) &\cong \mathcal{H}^{\otimes q}, \quad E_{1,q}^2(\mathcal{H}, \mathbb{K}) = 0, \\ E_{p,q}^2(\mathcal{H}, \mathbb{K}) &\cong \mathcal{H}^{\otimes q} \otimes \text{HL}_{p-1}(\mathcal{U}, \mathcal{L}), \quad p \geq 2. \end{aligned}$$

**Theorem 3.2.** *Let  $\mathcal{U}$  be the universal central extension of a perfect Leibniz algebra  $\mathcal{L}$ . Then one has an exact sequence*

$$0 \rightarrow \text{HL}_2(\mathcal{L})^{\otimes 2} \rightarrow \text{HL}_3(\mathcal{U}) \rightarrow \text{HL}_3(\mathcal{L}) \rightarrow 0,$$

from whence the isomorphism

$$\text{HL}_3(\mathcal{U}) \cong \text{HL}_2(\mathcal{L})^{\otimes 2} \oplus \text{HL}_3(\mathcal{L}).$$

**Proof.** In order to obtain  $\text{HL}_3(\mathcal{U})$ , we are led to compute  $\text{HL}_1(\mathcal{U}, \mathcal{L})$  and  $\text{HL}_2(\mathcal{U}, \mathcal{L})$ . To this end, we will consider the spectral sequence  $(E^r(\mathcal{H}, \mathcal{L}))_{r \geq 1}$ .

**Lemma 3.3.** *One has an isomorphism*

$$\mathrm{HL}_1(\mathcal{U}, \mathcal{L}) \cong \mathrm{HL}_2(\mathcal{L})$$

and an exact sequence

$$0 \rightarrow \mathrm{HL}_2(\mathcal{L})^{\otimes 2} \rightarrow \mathrm{HL}_3(\mathcal{U}) \rightarrow \mathrm{HL}_3(\mathcal{L}) \rightarrow 0.$$

**i)** From the perfectness of  $\mathcal{U}$ , one easily checks that

$$B_{0,3}^2 = \mathcal{H}^{\otimes 3} \cap (\mathcal{H}^{\otimes 2} \otimes d(\mathcal{U}^{\otimes 2})) = \mathcal{H}^{\otimes 3} \cap (\mathcal{H}^{\otimes 2} \otimes \mathcal{U}) = \mathcal{H}^{\otimes 3}.$$

And since

$$Z_{0,3}^3 = \mathcal{H}^{\otimes 3} \cap d^{-1}(0) = \mathcal{H}^{\otimes 3},$$

we have  $E_{0,3}^3(\mathcal{H}, \mathbb{K}) = 0$ .

**ii)** We already know that  $E_{1,2}^2(\mathcal{H}, \mathbb{K}) = 0$  (perfectness of  $\mathcal{L}$ ).

**iii)** From the perfectness of  $\mathcal{U}$ , we also get

$$B_{0,2}^2 = \mathcal{H}^{\otimes 2} \cap d(\mathcal{H} \otimes \mathcal{U}^{\otimes 2}) = \mathcal{H}^{\otimes 2} \cap (\mathcal{H} \otimes \mathcal{U}) = \mathcal{H}^{\otimes 2}.$$

And since

$$B_{0,2}^1 = \mathcal{H}^{\otimes 2} \cap d(\mathcal{H}^{\otimes 2} \otimes \mathcal{U}) = 0,$$

we deduce that

$$\mathrm{Im}(d_{2,1}^2) \cong B_{0,2}^2 / B_{0,2}^1 \cong \mathcal{H}^{\otimes 2}.$$

But we know that

$$E_{2,1}^2(\mathcal{H}, \mathbb{K}) \cong \mathcal{H} \otimes \mathrm{HL}_1(\mathcal{U}, \mathcal{L}) \cong \mathcal{H}^{\otimes 2};$$

from whence we have  $E_{2,1}^3(\mathcal{H}, \mathbb{K}) = 0$ .

**iv)** Since  $E_{1,1}^2(\mathcal{H}, \mathbb{K}) = 0$  (perfectness of  $\mathcal{L}$ ), one easily checks that

$$E_{3,0}^\infty(\mathcal{H}, \mathbb{K}) \cong \cdots \cong E_{3,0}^2(\mathcal{H}, \mathbb{K}) \cong \mathrm{HL}_2(\mathcal{U}, \mathcal{L}).$$

**v) Conclusion.** Therefore there exists an exact sequence

$$0 \rightarrow \mathrm{HL}_2(\mathcal{L})^{\otimes 2} \rightarrow \mathrm{HL}_3(\mathcal{U}) \rightarrow \mathrm{HL}_3(\mathcal{L}) \rightarrow 0,$$

from whence the isomorphism

$$\mathrm{HL}_3(\mathcal{U}) \cong \mathrm{HL}_2(\mathcal{L})^{\otimes 2} \oplus \mathrm{HL}_3(\mathcal{L}). \quad \square$$

**Proof of the Lemma.** Recall that we are using the spectral sequence  $(E^r(\mathcal{H}, \mathcal{L}))_{r \geq 1}$ .

**i) Computation of  $\mathrm{HL}_1(\mathcal{U}, \mathcal{L})$ .** One easily checks that

$$\begin{aligned} E_{0,1}^2(\mathcal{H}, \mathcal{L}) &\cong \mathrm{HL}_0(\mathcal{L}, \mathrm{HL}_1(\mathcal{H}, \mathcal{L})) \cong \mathrm{HL}_0(\mathcal{L}, \mathcal{L} \otimes \mathrm{HL}_1(\mathcal{H})) \\ &\cong \mathcal{L} \otimes \mathcal{H} / [\mathcal{L} \otimes \mathcal{H}, \mathcal{L}] \cong (\mathcal{L} / [\mathcal{L}, \mathcal{L}]) \otimes \mathcal{H} = 0, \\ E_{1,0}^2(\mathcal{H}, \mathcal{L}) &\cong \mathrm{HL}_1(\mathcal{L}, \mathrm{HL}_0(\mathcal{H}, \mathcal{L})) \cong \mathrm{HL}_1(\mathcal{L}, \mathcal{L}) \cong \mathrm{HL}_2(\mathcal{L}). \end{aligned}$$

Therefore we have

$$\mathrm{HL}_1(\mathcal{U}, \mathcal{L}) \cong \mathrm{HL}_2(\mathcal{L}) \cong \mathcal{H}.$$

**ii) Computation of  $\mathrm{HL}_2(\mathcal{U}, \mathcal{L})$ .** As before we have

$$\begin{aligned} E_{0,2}^2(\mathcal{H}, \mathcal{L}) &\cong \mathrm{HL}_0(\mathcal{L}, \mathrm{HL}_2(\mathcal{H}, \mathcal{L})) \cong \mathrm{HL}_0(\mathcal{L}, \mathcal{L} \otimes \mathrm{HL}_2(\mathcal{H})) \\ &\cong \mathcal{L} \otimes \mathcal{H}^{\otimes 2} / [\mathcal{L} \otimes \mathcal{H}^{\otimes 2}, \mathcal{L}] \cong (\mathcal{L} / [\mathcal{L}, \mathcal{L}]) \otimes \mathcal{H}^{\otimes 2} = 0, \\ E_{1,1}^2(\mathcal{H}, \mathcal{L}) &\cong \mathrm{HL}_1(\mathcal{L}, \mathrm{HL}_1(\mathcal{H}, \mathcal{L})) \cong \mathrm{HL}_1(\mathcal{L}, \mathcal{L} \otimes \mathrm{HL}_1(\mathcal{H})) \\ &\cong \mathrm{HL}_1(\mathcal{H}) \otimes \mathrm{HL}_1(\mathcal{L}, \mathcal{L}) \cong \mathcal{H} \otimes \mathrm{HL}_2(\mathcal{L}) \cong \mathcal{H}^{\otimes 2}. \end{aligned}$$

Here one also needs to return to the characterization of

$$E_{1,1}^3(\mathcal{H}, \mathcal{L}) = Z_{1,1}^3 / (Z_{0,2}^2 + B_{1,1}^2).$$

We have

$$\begin{aligned} Z_{1,1}^3 &= F_2^1 \cap d^{-1}(F_1^{-2}) = (\mathcal{L} \otimes \mathcal{H} \otimes \mathcal{U}) \cap d^{-1}(0), \\ Z_{0,2}^2 &= F_2^0 \cap d^{-1}(F_1^{-2}) = (\mathcal{L} \otimes \mathcal{H}^{\otimes 2}) \cap d^{-1}(0) = \mathcal{L} \otimes \mathcal{H}^{\otimes 2}, \\ B_{1,1}^2 &= F_2^1 \cap d(F_3^3) = (\mathcal{L} \otimes \mathcal{H} \otimes \mathcal{U}) \cap d(\mathcal{L} \otimes \mathcal{U}^{\otimes 3}). \end{aligned}$$

Therefore we obtain

$$E_{1,1}^3(\mathcal{H}, \mathcal{L}) \cong \mathcal{H} \otimes \mathrm{HL}_1(\mathcal{L}, \mathcal{L}) \cong \mathcal{H} \otimes \mathrm{HL}_2(\mathcal{L}) \cong \mathcal{H}^{\otimes 2}.$$

And it is clear that

$$E_{1,1}^\infty(\mathcal{H}, \mathcal{L}) \cong \dots \cong E_{1,1}^3(\mathcal{H}, \mathcal{L}) \cong \mathcal{H}^{\otimes 2}.$$

In order to determine  $E_{2,0}^2$ , we use the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{L}^{\otimes 2} \otimes \mathcal{H} \otimes \mathcal{U} & \longrightarrow & E_{3,0}^1 \cong \mathcal{L}^{\otimes 2} \otimes \mathcal{U}^{\otimes 2} & \longrightarrow & \mathcal{L}^{\otimes 3} \otimes \mathcal{U} \longrightarrow 0 \\ & & \downarrow d'_3 & & \downarrow d_{3,0}^1 & & \downarrow d''_3 \\ 0 & \longrightarrow & \mathcal{L}^{\otimes 2} \otimes \mathcal{H} & \longrightarrow & E_{2,0}^1 \cong \mathcal{L}^{\otimes 2} \otimes \mathcal{U} & \longrightarrow & \mathcal{L}^{\otimes 3} \longrightarrow 0 \\ & & \downarrow d'_2 & & \downarrow d_{2,0}^1 & & \downarrow d''_2 \\ 0 & \longrightarrow & 0 & \longrightarrow & E_{1,0}^1 \cong \mathcal{L}^{\otimes 2} & \longrightarrow & \mathcal{L}^{\otimes 2} \longrightarrow 0. \end{array}$$

Denoting by  $\bar{a}$  the image in  $\mathcal{L} \cong \mathcal{U}/\mathcal{H}$  of an element  $a \in \mathcal{U}$ , one easily checks that

$$\begin{aligned} d'_3(\bar{y}_1 \otimes \bar{y}_2 \otimes x \otimes z) &= d(\bar{y}_1 \otimes \bar{y}_2 \otimes x \otimes \bar{z}), \quad d'_2 \equiv 0, \\ d''_3(\bar{y}_1 \otimes \bar{y}_2 \otimes \bar{y}_3 \otimes z) &= d(\bar{y}_1 \otimes \bar{y}_2 \otimes \bar{y}_3 \otimes \bar{z}) \\ d''_2(\bar{y}_1 \otimes \bar{y}_2 \otimes \bar{y}_3) &= d(\bar{y}_1 \otimes \bar{y}_2 \otimes \bar{y}_3). \end{aligned}$$

Therefore, the long exact sequence in homology yields the exact sequence

$$(\mathcal{L}^{\otimes 2} / [\mathcal{L}^{\otimes 2}, \mathcal{L}]) \otimes \mathcal{H} \xrightarrow{i} E_{2,0}^2(\mathcal{H}, \mathcal{L}) \xrightarrow{\pi} \mathrm{HL}_3(\mathcal{L}) \rightarrow 0.$$

On the other hand, ones knows that

$$E_{2,0}^2(\mathcal{H}, \mathcal{L}) \cong Z_{2,0}^2 / (Z_{1,1}^1 + B_{2,0}^1).$$

But here we have

$$Z_{1,1}^1 = F_2^1 \cap d^{-1}(F_1^0) = (\mathcal{L}^{\otimes 2} \otimes \mathcal{U}) \cap d^{-1}(\mathcal{L} \otimes \mathcal{H}).$$

Thus we have  $\text{Im}(i) \subset Z_{1,1}^1$ ; from whence we deduce that

$$E_{2,0}^\infty(\mathcal{H}, \mathcal{L}) \cong \cdots \cong E_{2,0}^2(\mathcal{H}, \mathcal{L}) \cong \text{HL}_3(\mathcal{L}).$$

Therefore the Lemma is clear.  $\square$

**3.4. Interpretation of the natural maps.** From now on, let  $\mathfrak{g}$  be a perfect Lie algebra and let  $\mathfrak{U}$  (resp.  $\mathfrak{u}$ ) be its universal central extension in the category **(Leib)** (resp. **(Lie)**). Since the Leibniz algebra  $\mathfrak{U}$  is also the universal central extension of  $\mathfrak{u}$ , the Proposition 3.2 yields the isomorphisms

$$\text{HL}_3(\mathfrak{U}) \cong \text{HL}_2(\mathfrak{g})^{\otimes 2} \oplus \text{HL}_3(\mathfrak{g}) \cong \text{HL}_2(\mathfrak{u})^{\otimes 2} \oplus \text{HL}_3(\mathfrak{u}),$$

and we know that

$$\text{HL}_2(\mathfrak{u}) \cong \ker(\text{HL}_2(\mathfrak{g}) \rightarrow \text{H}_2(\mathfrak{g})) \cong \ker(\mathfrak{U} \rightarrow \mathfrak{u} \cong \mathfrak{U}_{Lie}).$$

Therefore we can state the characterization of the natural maps  $\text{HL}_3(\mathfrak{U}) \rightarrow \text{HL}_3(\mathfrak{u})$  and  $\text{HL}_3(\mathfrak{u}) \rightarrow \text{H}_3(\mathfrak{u})$ .

**Theorem 3.4.** *Let  $\mathfrak{U}$  (resp.  $\mathfrak{u}$ ) be the universal central extension in the category **(Leib)** (resp. **(Lie)**) of a perfect Lie algebra  $\mathfrak{g}$ . Then we have a (non-natural) commutative diagram*

$$\begin{array}{ccccc} \text{HL}_3(\mathfrak{U}) & \xrightarrow{\cong} & \text{HL}_2(\mathfrak{g})^{\otimes 2} & \oplus & \text{HL}_3(\mathfrak{g}) \\ \downarrow & & \downarrow & & \parallel \\ \text{HL}_3(\mathfrak{u}) & \xrightarrow{\cong} & \text{HL}_2(\mathfrak{g})^{\otimes 2} / K^{\otimes 2} & \oplus & \text{HL}_3(\mathfrak{g}) \\ \downarrow & & \downarrow & & \downarrow \\ \text{H}_3(\mathfrak{u}) & \xrightarrow{\cong} & \text{S}^2(\text{H}_2(\mathfrak{g})) & \oplus & \text{H}_3(\mathfrak{g}) \end{array}$$

where

$$K := \ker(\text{HL}_2(\mathfrak{g}) \rightarrow \text{H}_2(\mathfrak{g})) \cong \ker(\mathfrak{U} \rightarrow \mathfrak{u} \cong \mathfrak{U}_{Lie}) \cong \text{HL}_2(\mathfrak{u}).$$

**Proof.** Recall that the natural map  $\text{HL}_*(\mathfrak{U}) \rightarrow \text{HL}_*(\mathfrak{u})$  (resp.  $\text{HL}_*(\mathfrak{u}) \rightarrow \text{H}_*(\mathfrak{u})$ ) is induced by the canonical projection

$$\mathfrak{U}^{\otimes *} \longrightarrow (\mathfrak{U}_{Lie})^{\otimes *} \quad (\text{resp. } \mathfrak{u}^{\otimes *} \longrightarrow \Lambda^*(\mathfrak{u})).$$

Given a Lie algebra  $\mathfrak{l}$ , one has a commutative diagram of «little» complexes

$$\begin{array}{ccccccc} \longrightarrow & ?? & \longrightarrow & \text{S}^2(\mathfrak{l}) & \longrightarrow & 0 \\ & \downarrow & & \downarrow & & \\ \longrightarrow & \mathfrak{l}^{\otimes 3} & \longrightarrow & \mathfrak{l}^{\otimes 2} & \longrightarrow & \mathfrak{l} \\ & \downarrow & & \downarrow & & \parallel \\ \longrightarrow & \Lambda^3(\mathfrak{l}) & \longrightarrow & \Lambda^2(\mathfrak{l}) & \longrightarrow & \mathfrak{l}, \end{array}$$

which yields in homology the exact sequence

$$(†) \quad \mathrm{HL}_3(\mathfrak{l}) \rightarrow \mathrm{H}_3(\mathfrak{l}) \rightarrow \mathrm{S}^2(\mathfrak{l})/\sim \rightarrow \mathrm{HL}_2(\mathfrak{l}) \rightarrow \mathrm{H}_2(\mathfrak{l}) \rightarrow 0.$$

Here “ $\sim$ ” stands for the relations generated in  $\mathrm{S}^2(\mathfrak{l})$  by the elements

$$([x, y], z) - ([x, z], y) - (x, [y, z]), \quad \forall x, y, z \in \mathfrak{l}.$$

Applied to the Lie algebras  $\mathfrak{u}$  and  $\mathfrak{g}$ , we obtain the diagram

$$\begin{array}{ccccccccc} \mathrm{HL}_3(\mathfrak{u}) & \longrightarrow & \mathrm{H}_3(\mathfrak{u}) & \longrightarrow & \mathrm{S}^2(\mathfrak{u})/\sim & \longrightarrow & K & \longrightarrow & 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\ \mathrm{HL}_3(\mathfrak{g}) & \longrightarrow & \mathrm{H}_3(\mathfrak{g}) & \longrightarrow & \mathrm{S}^2(\mathfrak{g})/\sim & \longrightarrow & \mathrm{HL}_2(\mathfrak{g}) & \longrightarrow & \mathrm{H}_2(\mathfrak{g}) & \longrightarrow & 0. \end{array}$$

Therefore it is clear that  $\mathrm{HL}_3(\mathfrak{g}) \cap K^{\otimes 2} = 0$ ; from whence the computation of  $\mathrm{HL}_3(\mathfrak{u})$ . The commutative diagram is now obvious taking into account the previous calculations and the last diagram.  $\square$

**3.5. Remark.** T. Pirashvili studies in general the kernel of the canonical map

$$\mathfrak{l}^{\otimes * } \longrightarrow \Lambda^*(\mathfrak{l}).$$

This gives rise to a notion of *relative homology* which enables him to construct a long exact sequence generalizing the sequence (†) (see [13]).

#### 4. A finer Leibniz spectral sequence

There exists another filtration  $(FL_*^p)_{p \geq 0}$  of the Leibniz complex, more efficient than the above one. The vector space  $FL_n^p$  is the submodule of  $M \otimes \mathcal{G}^{\otimes n}$  generated by the elements  $m \otimes x_1 \otimes \cdots \otimes x_n$  such that at least  $(n - p)$  factors  $x_i$  are in  $\mathcal{H}$ . Obviously the filtration  $(F_*^p)_{p \geq 0}$  is a subfiltration of  $(FL_*^p)_{p \geq 0}$ . Nevertheless the  $EL^2$ -terms of the spectral sequence derived from this finer filtration are slightly more complicated to determine. It is expected that if  $M = \mathbb{K}$  and if the adjoint diagonal action of  $\mathcal{G}/\mathcal{H}$  on  $\mathrm{HL}_*(\mathcal{H})$  is trivial, then the  $EL^2$ -terms are given by

$$EL_{p,q}^2 \cong (\mathrm{HL}_*(\mathcal{G}/\mathcal{H}) \star \mathrm{HL}_*(\mathcal{H}))_{p+q}.$$

Here  $\star$  stands for the *free product* of graded modules that is, the non-commutative analogue of the classical tensor product  $\otimes$ , which arises in the isomorphism

$$\mathrm{T}(V \oplus V') \cong \mathrm{T}(V) \star \mathrm{T}(V').$$

To be more precise,  $EL_{p,q}^2$  is the direct sum of the components

$$X_{i_1} \otimes Y_{i_2} \otimes X_{i_3} \otimes Y_{i_4} \otimes \cdots, \quad i_1 + i_2 + i_3 + i_4 + \cdots = p + q$$

such that either

$$X_* = \mathrm{HL}_*(\mathcal{G}/\mathcal{H}) \quad \text{and} \quad Y_* = \mathrm{HL}_*(\mathcal{H})$$

or

$$X_* = \mathrm{HL}_*(\mathcal{H}) \quad \text{and} \quad Y_* = \mathrm{HL}_*(\mathcal{G}/\mathcal{H}).$$

In particular, this will give an idle way of getting the Künneth-style formula for the Leibniz homology (see [9], [12]).

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