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YANG-BAXTER EQUATION ON TWO-DIMENSIONAL LATTICE AND SOME INFINITE DIMENSIONAL ALGEBRAS

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Abstract

We show that the Yang-Baxter equation is equivalent to the associativity of the algebra generated by non-commuting link operators. Starting from these link operators we build out the (FFZ) algebras, the $s\ell q(2)$ is derived by considering a special combination of the generators of (FFZ) algebra.

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1 Introduction

The Yang-Baxter equation (YBE) is a unifying basis of several studies in two-dimensional integrable systems described by the quantum inverse scattering method [1]. A particular solution of this equation has lead to the definition of quantum groups [2]. The latter are mathematical objects which arose in the solution of some models of statistical mechanics [3] and in the study of factorized scattering of solitons and strings [4].

One of the earlier discoveries in particle physics was the realization of the existence of two different types of particles: bosons and fermions. In the algebraic context they are distinguished by the fact that the bosonic (fermionic) operators generally satisfy simple commutation (anti-commutation) relations. In the context of two-dimensional quantum field theory, it is natural to accept that one would encounter more exotic objects than just bosons and fermions. In fact, anyons which are two-dimensional particles with arbitrary statistics, interpolate between bosons and fermions (for a review see [5]-[7] and references therein). In the last few years they have attracted a spectacular interest, especially in the interpretation of certain condensed matter phenomena, most notably the fractional quantum Hall effect [8] and high T_c-superconductivity [9].

Quantum groups present themselves as natural mathematical objects allowing the description of the fractional statistics. Indeed, it has been proved in several works that the connection between quantum deformations and intermediate statistics holds [10-19].

The aim of this paper consists in obtaining the YBE by introducing some non-commuting operators denoted by L_p. These ones link between different sites of a given arbitrary two-dimensional lattice, we show that the associativity of the algebra generated by L_p's is equivalent to the YBE. Starting from these link operators, we consider the anyonic algebra which coincides exactly with the one obtained in the work [10]. We realize in a pure mathematical context the Fairke-Fletcher-Zachos (FFZ) algebra, that can be seen as a quantum W_∞ algebra [20]. We also derive the $s\ell q(2)$ from the (FFZ) algebra.

The article is organized as follows. In the second section we obtain the YBE by requiring the non-commutativity of the link operators introduced on an arbitrary two-dimensional lattice. In the third section we establish a correspondence between the phase breaking the commutativity of the link operators and the angle function on the two-dimensional lattice. This construction leads to the anyonic algebra.

The fourth section is devoted to the introduction of the translation operators on a lattice, we show that they are nothing but a generator of the (FFZ) algebra. We derive the $s\ell q(2)$ algebra. The final section consists on giving some concluding remarks.

2 The Yang-Baxter Equation (or two-dimensional lattice)

This section is devoted to obtaining the Yang-Baxter equation starting from the definition of some operator denoted by L_p. These ones allow the transition from one site to an arbitrary other one on a given two-dimensional lattice. We begin with the definition of the above operator and we show that the Yang-Baxter equation is nothing but an equation which is equivalent to the associativity of the algebra generated by the non-commuting elements L_ℓ.

We denote by Ω a two-dimensional lattice and we define the link operators Lp as follows:

Definition1

$$\phi_i^\ell(n) \equiv L_p \phi_i(n) \tag{1}$$

where $\phi_i(n)/i = 1, \ldots, d$ is a d-dimensional vector function on Ω, n is defined to be a couple of two integer $n \equiv (n_1, n_2)$ (n_1 and n_2 are respectively the horizontal and longitudinal coordinates of a given site n of Ω).

In the equation (1) $\ell \equiv \pm 1, \pm 2$ are the four possible orientations on Ω (Fig.1), so the index "ℓ" indicates *a* moving along the direction ℓ on Ω;

Definition 2

$$\phi_i^2(n) \equiv \phi_i(n_1+1, n_2), \quad \phi_i^2(n) \equiv \phi_i(n_1, n_2+1)$$

$$\phi_i^{-1}(n) \equiv \phi_i(n_1-1, n_2), \quad \phi_i^{-2}(n) \equiv \phi_i(n_1, n_2-1) \tag{2}$$

We note that in these definitions, the element L_ℓ is regarded as an operator linking two neighbouring lattice sites, so we call it the link operator. In the simple case where these operators commute, one can write:

$$L_\ell^{(\ell')} \cdot L_{\ell'} = L_{\ell'}^{(\ell)} \cdot L_\ell \tag{3}$$

we point out that in this relation, we adopt the notation $L_\ell^{(\ell')}$ for which we suppose that the link operator $L_\ell^{(\ell')}$ act on a given vector function $\phi_i^\ell(n)$ by keeping invariant the direction described by ℓ'.

The product "$\cdot$" in Eq.(3) is simply the composition of the operators L_ℓ's, this composition occurs from the definition (1). Now, by introducing a matrix denoted by $R_{\ell\ell'}$, we break the commutativity of the product (Eq.(3) as follows:

$$R_{\ell\ell'} : V \otimes V \to V \otimes V$$

and

$$(R_{\ell\ell'})_{mn}^{ij} \left(L_\ell^{(\ell')}\right)_{m'}^{m} \cdot (L_{\ell'})_{n'}^{n} = \left(L_{\ell'}^{(\ell)}\right)_{n'}^{j} \cdot (L_\ell)_{m'}^{i} \tag{4}$$

where V is a d-dimensional vector space in which lives the multi-component functions "ϕ_i". The indices ij, m' and n' take the values $1, 2, \ldots, d$.

The equation (4) can be rewritten in a compact form as:

$$(R_{\ell\ell'})_{12}\left((L_{\ell}^{(\ell')})_1 \otimes (L_{\ell'})_2\right) = \left(L_{\ell'}^{(\ell)}\right)_2 \otimes (L_{\ell})_1) \tag{5}$$

owing to this equality, one can prove, by a direct computation that the relation:

$$(R_{\ell\ell'})_{12}(R_{\ell'\ell})_{21} = 1\!\!1 \otimes 1\!\!1 \ . \tag{6}$$

where $1\!\!1$ is the unit $d \times d$ matrix acting on V.

Starting from the above tools, we derive the following result:

Proposition 1 By requiring the product in Eq.(3) to be associative, we obtain the well known Yang-Baxter equation on $(R_{\ell\ell'})$:

$$(R_{\ell\ell'})_{12}(R_{\ell\ell''}^{(\ell')})_{13}(R_{\ell'\ell''})_{23} = (R_{\ell'\ell''}^{(\ell)})_{23}(R_{\ell\ell''})_{13}(R_{\ell\ell'}^{\ell''})_{12} \tag{7}$$

We recall that in the literature this equation is seen as a representation of the braid group. The latter plays for the intermediate quantum statistics the same role played by the permutation group for bosonic and fermionic statistics. In mathematical sense, it suffices to multiply the matrix R in (7) by a permutation one P as:

$$B = P \cdot R$$

and the YBE becomes:

$$B_{12}B_{23}B_{12} = B_{23}B_{12}B_{23} \tag{8}$$

We notice that in the classical limit $R = 1\!\!1 \otimes 1\!\!1$, this equation becomes trivial.

The equality (8), known as the Braid relation appears in the study of intermediate statistics (especially the anyonic ones). So, we construct, using the above mathematical tools, an algebra, interpolating between the bosonic and fermionic algebras. This matter constitutes the purpose of the next section.

3 Anyonic Algebra

Firstly, we point out that the approach leading to the anyon algebra is of purely mathematical aspect. Moreover, it is treated in a way which is different from those usually used in the literature.

We start by considering a plaquette (Fig. 1). The operators L_ℓ along the P_n are written as:

Definition 3

$$L_1 \equiv \ell^{iA_1((n_1,n_2)/n_1+1,n_2))}$$

$$L_2 \equiv e^{iA_2((n_1+1,n_2),(n_1+1,n_2+1))} \tag{9}$$

$$L_{-1} \equiv e^{iA_1((n_1+1,n_2+1),(n_1,n_2+1))}$$

$$L_{-2} \equiv e^{iA_2((n_1,n_1+1),(n_1,n_2))} \; .$$

The functions $A_\ell(n,p)$ are subject to the following constraint:

$$\partial_\ell f_{\bar{\ell}}(n) = A_{\bar{\ell}}(n, n+\ell), \quad n \in \Omega \tag{10}$$

In the relation (10), the derivative on Ω is defined by:

$$\partial_\ell f_{\bar{\ell}}(n) = f(n+\ell) - f(n) \tag{11}$$

$$\partial_{-\ell} = -\partial_\ell$$

ℓ means the direction on the lattice and $\bar{\ell}$ is his absolute value.

The motivation for the choice of these link operators will be made clear when we give the construction of the anyonic algebra.

The phases in (10) may be rewritten as:

$$\begin{aligned}
A_1((n_1,n_2),(n_1+1,n_2)) &= f_1(n_1+1,n_2) - f(n_1,n_2) \\
&= \partial_1 f_1(n_1,n_2) \\
&\equiv f^1(n_1,n_2) \\
A_2((n_1+1,n_2),(n_1+1,n_2+1)) & f_2(n_1+1,n_2+1) - f_2(n_1+1,n_2) \\
&= \partial_2 f_2(n_1,n_2) \\
&\equiv f_2(n_1+1,n_2) \\
A_1((n_1+1,n_2+1)),(n_1,n_2+1)) &= f_1(n_1,n_2+1) - f_1(n_s+1,n_2+1) \\
&= -\partial_1 f_1(n_1,n_2) \\
&\equiv -f^1(n_1,n_2+1) \\
A_2((n_1.n_2+1),(n_1,n_2)) &= f_2(n_1,n_2) - f_2(n_1,n_2+1) \\
&= -\partial_2 f_1(n_1,n_2) \\
&= -f^2(n_1,n_2)
\end{aligned} \tag{13}$$

In this construction the $f^\alpha(n)$ can be interpreted as the angles between the sites n_i and n_{i+1}. These angles are given via a point $n^* \in \Omega^*$ (we denote by Ω^* the dual lattice of Ω and $n^* \in \Omega^*$, $n^* \equiv (n_1 + \frac{a}{2}, n_2 + \frac{a}{2})$, a is the lattice spacing between two neighbourhood sites).

It is natural, owing to Eqs.(13) to see that the sum of the four phases over a plaquette P_n in Ω is given by the relation:

$$f^2(n_1+1,n_2) - f^2(n_1,n_2) - f^1(n_1,n_2+1) + f^1(n_1,n_2) = 2\pi \tag{14}$$

So this expression can be rewritten for one tower as:

$$\oint_{P_n} f(n) = 2\pi \tag{15}$$

And thus for several turns one obtains:

$$\oint_{P_n} f(n) = 2\pi\kappa \tag{16}$$

where κ is the winding number of the closed loop P_n. Thus in general, for any curve Γ_n on the lattice we can define the function $\Theta(n)$ as:

$$\Theta_{\Gamma_n(n)} \equiv \int_{\Gamma_n} f(n), \quad n \in \Omega \tag{17}$$

Such that Γ_n is the curve from a point p at the infinity of the x-axis to the point n on the lattice Ω (Fig.2). We call $\Theta_{\Gamma_n(n)}$ the lattice angle function under which the point n may be regarded by another site m on the lattice. The relevance of angle function appearing in the intermediate statistics has been introduced firstly in the work [5]. According to this work, the angle function is measured from another point $m^* \in \Omega$ instead of $n^* \in \Omega$ (Fig.2). We take therefore $\Theta_{\Gamma_n(n,m)}$ and $f(n,m)$ instead of $\Theta_{\Gamma_n(n)}$ and $f(n)$.

In the same way and accordingly of the above definitions one defines.

Definition 4

$$\Theta_{F_n}(n,m) - \Theta_{\Gamma'_n}(n,m) \equiv \oint_{\Gamma_n\Gamma_n^{-1}} f(n,m) = 2\pi\kappa \tag{18}$$

where κ is the winding number of the loop $\Gamma_n\Gamma_n^{-1}$ around the dual point m^* (Fig.2).

Basing on the fact, there are two kinds of D angle function, when one considers the origin point "p" at infinity) of the positive x. axes or at the infinity of the negative x-axes. One can prove that these functions are subject to the following conditions:

$$\Theta_{\pm\Gamma_n}(n,m) - \Theta_{\pm Gamma_n}(m,n) = \begin{cases} \pm\prod \mathrm{sgn}(n_2 - m_2), & n_2 \neq m_2 \\ \pm\prod \mathrm{sgn}(n_1 - m_1), & n_2 = m_2 \end{cases}$$

$$\Theta_{-\Gamma_n}(n,m) - \Theta_{\pm Gamma_n}(n,m) = \begin{cases} -\prod \mathrm{sgn}(n_2 - m_2), & n_2 \neq m_2 \\ -\prod \mathrm{sgn}(n_1 - m_1), & n_2 = m_2 \end{cases} \tag{19}$$

with "$+\Gamma_n$" the curve following the anti-clockwise sense and "$-\Gamma_n$" is the clockwise one.

Now, we are in a position to construct the anyonic operator. They are given starting from the introducing of fermionic field on a two-dimensional lattice. Let us give the two-component fermionic spinor field by:

$$\psi(n) = \begin{pmatrix} \psi_1(n) \\ \psi_2(n) \end{pmatrix} \tag{20}$$

The quantized components in relation (20) obey to the following equation:

$$\{\psi_\alpha(n),\ \psi_\beta(m)\} = 0$$

$$\{\psi_\alpha(n),\ \psi_\beta^+(m)\} = \delta_{nm}\delta_{\alpha\beta} \tag{21}$$

$$\{\psi_\alpha^+(n),\ \psi_\beta^+(m)\} = 0$$

where $\alpha, \beta = 1, 2$ and $\{\varphi, \eta\} \equiv \varphi\eta + \eta\varphi$.

Let us recall also that the Fock vacuum state $|0\rangle$ is defined by:

$$\psi_\alpha(n)|0\rangle = 0 \tag{22}$$

starting from this vacuum, one can generate all the other states describing the fermions on Ω.

Returning to our purpose; the construction of the anyonic algebra. To start let us at first introduce the elements:

$$\Delta_\alpha(n_\pm) = \sum_m \psi_\alpha^+(m)\Theta_{\pm\Gamma_n}(n, m)\ \psi_\alpha(m) \tag{23}$$

By straightforward calculation, one obtains

$$\begin{aligned}
[\Delta_\alpha(n_\pm),\ \psi_\beta(m)] &= -\delta_{\alpha\beta}\Theta_{\pm\Gamma_n}(n, m)\ \psi_\alpha(m) \\
[\Delta_\alpha(n_\pm),\ \psi_\beta^+(m)] &= \delta_{\alpha\beta}\Theta_{\pm\Gamma_n}(n, m)\psi_\alpha^+(m) \\
[\Delta_\alpha(n_\pm),\ \Delta_\beta(m_\pm)] &= 0
\end{aligned} \tag{24}$$

Now we can give the expression of the anyonic operators. Indeed one can prove the following proposition:

Proposition 2 By defining the operators $\varphi_\alpha(n_\pm)$ and $\varphi_\alpha^+(n_\pm)$ as:

$$\begin{aligned}
\varphi_\alpha(n_\pm) &= e^{i\nu\Delta_\alpha(n_\pm)}\psi_\alpha(n) \\
\varphi_\alpha^+(n_\pm) &= \psi_\alpha^+(n)e^{-i\nu\Delta_\alpha(n_\pm)}
\end{aligned} \tag{25}$$

We obtain the algebra described by the following algebraic relations:

$$\begin{aligned}
&\{\varphi_\alpha(n_\pm),\ \varphi_\alpha^+(n_\pm)\} = 1 \\
&\{\varphi_\alpha(n_\pm),\ \varphi_\alpha(m_\mp)\}_{\Lambda^\mp} = 0 \qquad n > m \\
&\{\varphi_\alpha(n_\pm),\ \varphi_\alpha^+(m_\pm)\}_{\Lambda^\mp} = 0 \qquad n > m \\
&\{\varphi_\alpha^+(n_\pm),\ \varphi_\alpha(m_\pm)\}_{\Lambda^\mp} = 0 \qquad n > m \\
&\{\varphi_\alpha^+(n_\pm),\ \varphi_\alpha^+(m_\mp)\}_{\Lambda^\mp} = 0 \qquad n > m \\
&\{\varphi_\alpha(n_\pm),\ \varphi_\beta(m_\pm)\} = 0 \qquad \alpha \neq \beta \\
&\{\varphi_\alpha^+(n_\pm),\ \varphi_\beta(m_\pm)\} = 0 \qquad \alpha \neq \beta \\
&\{\varphi_\alpha^+(n_\pm),\ \varphi_\beta^+(m_\pm)\} = 0 \qquad \alpha \neq \beta
\end{aligned} \tag{26}$$

where $\Lambda^\pm = e^{\pm i\nu\pi}$ and $n > m \Leftrightarrow \left\{ \begin{array}{l} n_+ > m_+ \Leftrightarrow \left\{ \begin{array}{l} n_2 > m_2 \\ n_1 > m_1, n_2 - n_1 \end{array} \right. \\ n_- < m_- \Leftrightarrow \left\{ \begin{array}{l} n_2 < m_2 \\ n_1 < m_1 \\ n_2 = m_2 \end{array} \right. \end{array} \right.$ and $\{x, y\}_\Lambda = xy + \Lambda yx$.

We have also

$$[\varphi_\alpha(n_\pm)]^2 = [\varphi_\alpha^+(n_\pm)]^2 = 0 \tag{27}$$

This constraint seems to describe an important property of anyonic systems. Indeed it appears when one discusses the statistics corresponding to anyons, the relation Eq.(27) is viewed as a hard core condition; at the same point of a two-dimensional lattice, cannot live more than one particle. For this reason, many of the authors in the literature consider the anyons as a fermion but defined on a given two-dimensional lattice. We add also to this remark that, owing to the above condition, anyons obey the Pauli exclusion principle.

The parameter ν in the equalities (26) is seen as a statistical parameter. The obtained algebra interpolates between the bosonic algebra ($\nu = 1$ mod 2) and the fermionic one ($\nu = 0$ mod 2).

Consequently, we have realized the anyonic algebra starting from one special definition of the link operators on the two-dimensional lattice Ω. We showed also the correspondance between them and the angle function discussed in [5]. In the latter, the authors show that the Schwinger realization of SU(2) which is bosonic or fermionic, can be generalized to anyons of intermediate statistics. In this case, the Schwinger construction does not lead to ordinary group SU(2), but rather to its q-analogue, the $U_q(2)$ ($q = \exp i\pi\nu$).

In order to investigate some other quantum symmetries appearing when the studying of these exotic statistics, we will show that it is possible to obtain the FFZ algebra from which we derive the quantum group algebra $s\ell q(2)$.

4 (FFZ) and $s\ell q(2)$ algebras

Starting from the above link operators (Eq.(9)), we define the generators T_n as follows:

Definition 5

$$T_n = T_{(n_1,n_2)} = R_{ij}^{\frac{\bar{n}_1\bar{n}_2}{2}} L_{\pm i}^{\bar{n}_i} L_{\pm j}^{\bar{n}_j} \tag{28}$$

where $n_1, n_2 \in \mathbb{Z}$, $i, j = 1, 2$ and $i \neq j$. $\bar{n}$ is the absolute value of $x \in \mathbb{Z}$, the indices are omitted in the notation of $T_{(n_\pm,n_\alpha)}$.

As already seen (Fig.1) the operator L_ℓ allows the transition from the site X_i to the site $X_{i+\ell}$ on Ω. This is described in Fig.3, $\ell = \pm1, \pm2$. The generators T_n are regarded as translations from the point O (the origin of referential) to a point $n \in \Omega$ (Fig.4).

At first we require that the product of these operators are given by formula:

$$T_n T_m = e^{i\alpha_2(n,m)} T_{n+m} \tag{29}$$

The function $\alpha(n, m)$ depending on two sites of Ω is introduced to be anti-symmetric and is defined as:

$$\alpha_2 : \Omega \times \Omega \to \mathbb{R}$$

the motivation of this choice of the product between these two translation operators is due to the fact that the link operators do not commute and thus the composition of two translations must lead naturally to a translation.

Before giving the complete description of the (FFZ), we notice that in the literature this algebra has been poorly realized in a mathematical way. So, one of the main results of this work is to construct the FFZ algebra on the lattice Ω. To do this we are lead, due to a pure mathematical reason, to divide Ω onto four subsets given by:

$$
\begin{aligned}
&i) \quad \theta^{++};\ n \in D^{++} \Leftrightarrow (n_2 > 0,\ n_1 > 0) \\
&ii) \quad D^{+-};\ n \in D^{+-} \Leftrightarrow (n_2 > 0, n_1 < 0) \\
&iii) \quad D^{-+};\ n \in D^{-+} \Leftrightarrow (n_2 < 0, n_1 > 0) \\
&iv) \quad D^{--};\ n \in D^{--} \Leftrightarrow (n_2 < 0,\ n_1 < 0)\ .
\end{aligned}
\tag{30}
$$

and

$$\Omega = D^{++} \oplus D^{+-} \oplus D^{-+} \oplus D^{--}\ .$$

Now, by requiring that the translation operators T_n do not commute, but this non-commutativity property is described by the introducing of some matrix R as follows:

$$T_n T_m = R^{ab}_{nm} T_a T_b \tag{30a}$$

we find that the matrix obey, owing to the assumption (Eq.(29)), the following relation:

Proposition 3

$$R^{ab}_{nm} = \delta^{a+b}_{n+m} e^{-i(\alpha_2(a,b)-\alpha_2(m,n))} \tag{30b}$$

one can check that this matrix satisfies the Yang-Baxter Equation, because the product composing the translation operators in the expression (Eq.(30b)) is associative.

At this stage we are able to give the expression of the generators of the (FFZ) algebra on every subset of the lattice Ω.

Definition 6

i) for D^{++}:

$$
\begin{aligned}
T^{++}_n &\equiv Q_{2,1}^{\frac{\bar{n}_2 \bar{n}_1}{2}} L_2^{\bar{n}_2} L_1^{\bar{n}_1} \\
S^{++}_n &\equiv Q_{1,2}^{\frac{\bar{n}_2 \bar{n}_1}{2}} L_1^{\bar{n}_1} L_2^{\bar{n}_2}
\end{aligned}
\tag{31a}
$$

ii) for D^{+-}:

$$
\begin{aligned}
T^{+-}_n &\equiv Q_{1,2}^{\bar{n}_2 \bar{n}_1} L_{-1}^{\bar{n}-1} L_2^{\bar{n}-2} \\
S^{+-}_n &\equiv Q_{2,1}^{\frac{\bar{n}_2 \bar{n}_1}{2}} L_2^{\bar{n}_1} L_{-1}^{\bar{n}_2}
\end{aligned}
\tag{31b}
$$

iii) for D^{-+}:

$$T_n^{-+} \equiv Q_{1,2}^{\frac{\bar{n}_2\bar{n}_1}{2}} L_1^{\bar{n}_1} L_{-2}^{\bar{n}_2}$$

$$S^{-+} \equiv Q_{2,1}^{\frac{\bar{n}_2\bar{n}_1}{2}} L_{-2}^{\bar{n}_2} L_1^{\bar{n}_1} \tag{31c}$$

iv) for D^{--}:

$$T_n^{--} \equiv Q^{\frac{\bar{n}_2\bar{n}_1}{2}} L_{-2}^{\bar{n}_2} L_{-1}^{\bar{n}_1}$$

$$S_n^{--} \equiv Q_{1,2}^{\frac{\bar{n}_2\bar{n}_1}{2}} L_{-1}^{\bar{n}_1} L_{-2}^{\bar{n}_2} \tag{31d}$$

Following the assumption (Eq.(30)), we prove by a direct calculation that the parameters $Q_{1,2}$ appearing in the relations (31) have the expression:

$$Q_{1,2} = e^{\frac{i\alpha(n,n')}{\bar{n}'\times\bar{n}}}$$

Consequently, the elements (T's) are nothing but the generators of (FFZ) algebra for both subset in the lattice Ω, we have then:

Proposition 4

$$\begin{aligned}
&\text{for } D^{++}; \quad [T_n^{++}, T_{n'}^{++}] = 2i\sin\alpha(n,n')T_{n+n'}^{++} \\
&\text{for } D^{+-}; \quad [T_n^{+-}, T_{n'}^{+-}] = 2i\sin\alpha(n,n')T_{n+n'}^{+-} \\
&\text{for } D^{-+}; \quad [T_n^{-+}, T_{n'}^{-+}] = 2i\sin\alpha(n,n')T_{n'+n}^{-+} \\
&\text{for } D^{--}; \quad [T_n^{--}, T_{n'}^{++}] = 2i\sin\alpha(n,n')T_{n+n'}^{--}
\end{aligned} \tag{32}$$

Using these particular realizations of (FFZ) algebra, we can derive the $s\ell q(2)$ starting from the generators defined by:

Definition 7

$$\begin{aligned}
J_+ &\equiv \frac{1}{(q-q^{-1})}(T_{(1,1)}^{++} - T_{(-1,1)}^{+-}) \\
J_- &\equiv \frac{1}{(q-q^{-1})}(T_{(-1,-1)}^{--} - T_{(1,-1)}^{-+}) \\
q^{2J_3} &\equiv T_{(2,0)}^{++} \\
q^{-2J_3} &\equiv T_{(-2,0)}^{--}
\end{aligned} \tag{33}$$

Basing on these definitions, we obtain:

$$[J_+, J_-] = [2J_3]_q$$

$$q^{J_3} J_\pm q^{-J_3} = q^{\pm 1} J_\pm \tag{34}$$

where q is taken to equal $Q_{1,2} = Q_{2,1}$ in the above equations and

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \ .$$

We point out that surprisingly enough, we have constructed the $s\ell q(2)$ algebra starting from the introduction of the α in k operators on a lattice. Our realization is different than the one given in the work [5] where the authors obtain the same quantum symmetry by using the Schwinger construction.

5 Concluding Remarks

In some mathematical point of view, the notion of discretization of the two-dimensional manifold has been seen in the literature as the main and essential conception allowing the connection between the intermediate statistics and quantum algebras. In this context, starting from the definition of the link operators on a given two-dimensional lattice Ω, we have constructed an algebra coinciding exactly with the anyonic algebra. We have realized, in a mathematical way the (FFZ) algebra build out from the introduced link operators and Ω. The $s\ell q(2)$ is thus obtained in an original way.

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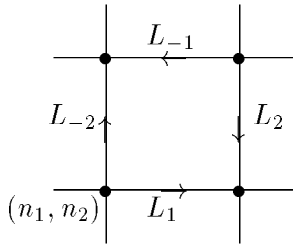

Fig.1 – Elementary plaquette A_n, $n = (n_1, n_2) \in \Omega$, on which we translate from n to n.

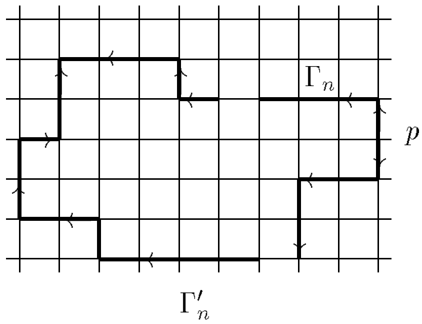

Fig.2 - $\Theta_{\Gamma_n(n,m)}$ is the angle under which the point n is seen by m^* in the positive direction $(+)$ and $\Theta_{\Gamma'_n(n,m)}$ in the opposite direction$(-)$

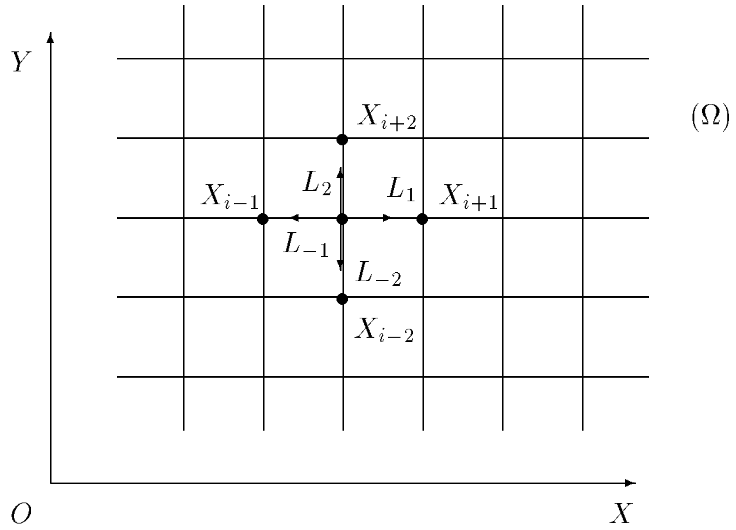

Fig.3 - Lattice Ω and its reference XOY

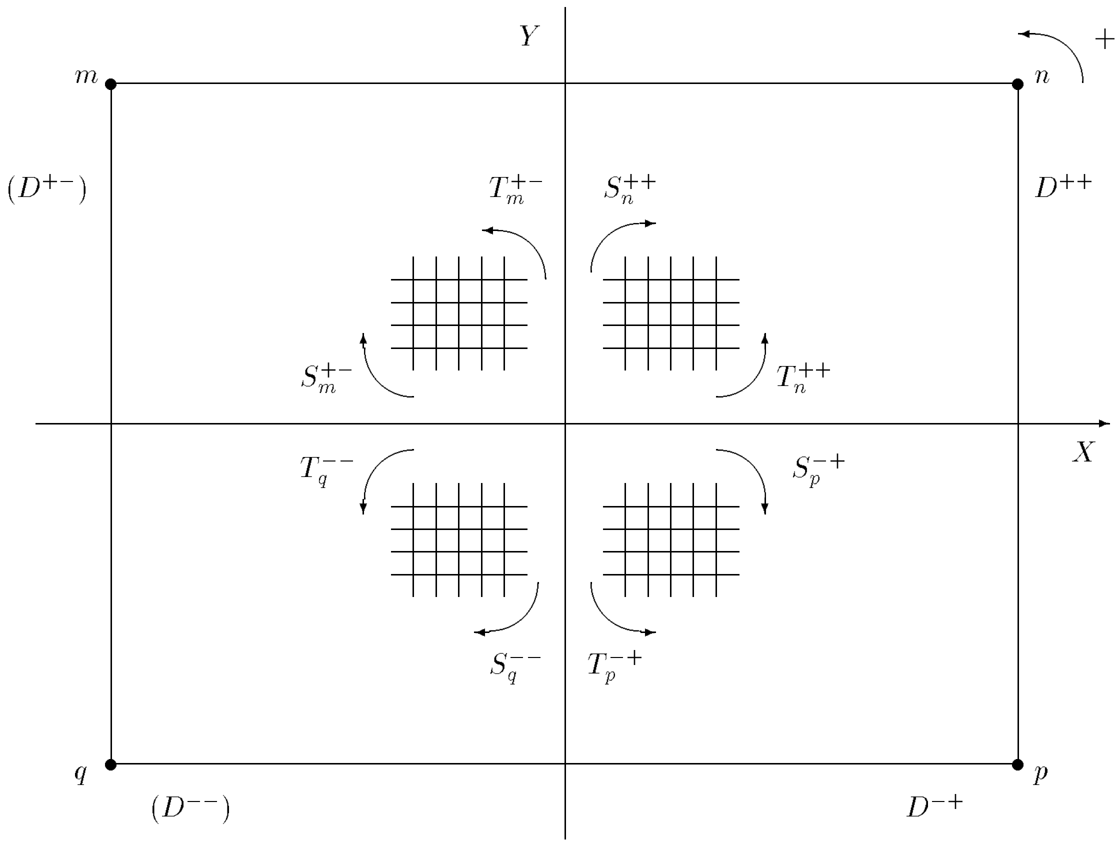

Fig.4 - Lattice Ω is divided in four parts: D^{++}, D^{+-}, D^{-+} and D^{--}