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THE SOLUTION OF BBGKY HIERARCHY OF KINETIC EQUATIONS THROUGH THE PARTICLE SOLUTION OF VLASOV EQUATION

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Abstract

The solution of BBGKY hierarchy of kinetic equations is defined through particle method solution of Vlasov equation.

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Suppose we are given a system of monoatomic molecules. Suppose that the molecules interact through a two-body potential ϕ. In the framework of classical statistical physics, we consider for the given system the problem of solving the hierarchy of BBGKY kinetic equations [N.N.Bogoluibov, 1970]:

$$\frac{\partial}{\partial t} f_n(t) = [H_n, f_n(t)] + \frac{1}{v} \int \sum_{1 \le i \le n} [\phi(q_i - q), f_{n+1}(t)]\, dx, \tag{1}$$

where f_n is the probability density of the gas ensemble at time $t \in I\!R_+$ at position $q_1 \in \wedge, q_2 \in \wedge, \cdots, q_n \in \wedge$ with the velocities $V_1 \in R^3 \cdots; V_n \in I\!R^3$ of particles. Therefore, $f : I\!R_+ \times F \to I\!R_+$ with the phase space $F = \left(\wedge \times I\!R^3\right)^n$.

Here,

$$H_n = \sum_{1 \le i \le n} T_i + \sum_{1 \le i < j \le n} \phi(q_i - q_j), \quad T_i = \frac{p_i}{2m},$$

$m = 1$ is the mass of a molecule, p the momentum of a molecule, $n \in N$, N is the number of molecules, V - the volume of the system; $N \to \infty, V \to \infty$, $v = \frac{V}{N} =$ const is volume per molecule, $[,]$ denotes the Poisson brackets.

Introducing the notation

$$\begin{aligned}
(\mathcal{H} f)_n &= [H_n, f_n]\,; \quad (\mathcal{D}_x f)_n(x_1, \cdots, x_n) = f_{n+1}, (x_1, \cdots x_n, x)\,; \\
(\mathcal{A}_x f)_n &= \tfrac{1}{v} \sum_{1 \le i \le n} [\phi(q_i - q), f_n]\,; \\
f(t) &= \{f_1(t_1 x_1), \cdots\cdots, f_n(t, x_1, \cdots, x_n), \cdots\}\,, n = 1, 2, \cdots
\end{aligned}$$

we can cast Eq.(1) in the form

$$\frac{\partial}{\partial t} f(t) = \mathcal{H} f(t) + \int \mathcal{A}_x \mathcal{D}_x f(t) dx. \tag{2}$$

Derivation of Hierarchy of Kinetic Equations for correlation functions.

Proposition 1. The hierarchy of kinetic equations for the correlation functions has the form

$$\frac{\partial}{\partial t}\varphi(t) = \mathcal{H}\varphi(t) + \frac{1}{2}\mathcal{W}(\varphi(t), \varphi(t)) + \int \mathcal{A}_x \mathcal{D}_x \varphi(t) dx + \int \mathcal{A}_x \varphi(t) \star \mathcal{D}_x \varphi(t) dx, \tag{3}$$

where [D.Ruelle, 1969],[M.Yu.Rasulova, 1980],[M.Yu.Rasulova, A.K.Vidibida, 1976]:

$$f(t) = \Gamma\varphi(t) = I + \varphi(t) + \frac{\varphi(t) \star \varphi(t)}{2!} + \cdots \frac{(\star\varphi(t))^n}{n!} + \cdots, \tag{4}$$

$$\varphi(t)=\{\varphi_1(t,x_1),\cdots,\varphi(t,x_1,\cdots,x_n),\cdots\};$$

$$(\varphi\star\varphi)(x)=\sum_{YCX}\varphi(Y)\varphi(X\setminus Y);\quad I\star\varphi=\varphi;\quad (\star\varphi)^n=\underbrace{\varphi\star\varphi\star\cdots\star\varphi}\text{ n times;}$$

$$X=(x_1,\cdots,x_n)=(x_{(n)});\quad Y=(x_{n'}),\quad n^{'}\in;\ n\cdot n^{'}=1,2,\cdots;$$

$$(\mathcal{U}\varphi_n)=\left[\sum_{1\le i<j\le n}\phi(q_i-q_j),\varphi_n\right],\mathcal{W}(\varphi,\varphi)=\sum_{YCX}\mathcal{U}(Y;X\setminus Y)\varphi(Y)\varphi(X\setminus Y).$$

Proof: To obtain (3), we substitute (4) in (2) :

$$\frac{\partial}{\partial t}\Gamma\varphi(t)=\mathcal{H}\Gamma\varphi(t)+\int\mathcal{A}_x\mathcal{D}_x\Gamma\varphi(t)dx. \tag{5}$$

We have

$$\mathcal{D}_x\Gamma\varphi(t)=\mathcal{D}_x\varphi(t)\star\Gamma\varphi(t), \tag{6}$$

$$\mathcal{A}_x\Gamma\varphi(t)=\mathcal{A}_x\varphi(t)\star\Gamma\varphi(t), \tag{7}$$

$$\mathcal{A}_x\mathcal{D}_x\Gamma\varphi(t)=\mathcal{A}_x\mathcal{D}_x\varphi(t)\star\Gamma\varphi(t)+\mathcal{A}_x\varphi(t)\star\mathcal{D}_x\varphi(t)\star\Gamma\varphi(t), \tag{8}$$

$$T\Gamma\varphi(t)=T\varphi(t)\star\Gamma\varphi(t), \tag{9}$$

$$\mathcal{U}\Gamma\varphi(t)=\mathcal{U}\varphi(t)\star\Gamma\varphi(t)+\frac{1}{2}\mathcal{W}\left(\varphi(t),\varphi(t)\star\Gamma\varphi(t)\right), \tag{10}$$

$$\frac{\partial}{\partial t}\Gamma\varphi(t)=\frac{\partial}{\partial t}\varphi(t)\star\Gamma\varphi(t). \tag{11}$$

substituting $(6)-(11)$ in (5), multiplying both sides by $\Gamma(-\varphi(t))$ we obtain (3). This proves the proposition.

To investigate our system on the basis of arguments similar to those in [1], we can choose as expansion parameter v, setting

$$\phi(q_i-q_j)=v\theta(q_i-q_j) \tag{12}$$

and making substitution [N.N.Bogoluibov, 1970], [M.Yu.Rasulova, 1980], [S.Ichimary, 1968], [R.L.Liboff, G.Perona, 1967], [A.I.Akhiezer (ed.), 1974]:

$$\varphi_n(t)=v^{n-1}\psi_n(t) \tag{13}$$

On the basis of $(12),(13)$ Eq.(3) for n molecules takes the form

$$\frac{\partial}{\partial t}\psi_n(t,X)=\left[\sum_{1\le i\le n}T_i,\psi_n(t,X)\right]+v\left(\mathcal{U}\psi(t)\right)_n(X)$$

$$+\frac{v}{2}\left(\mathcal{W}\psi(t),\psi(t)\right)_n(X)+v^2\int\left(\mathcal{A}_x\mathcal{D}_x\psi(t)\right)_n(X)dx \tag{14}$$

$$+v\int\left(\mathcal{A}_x\psi(t)\star\mathcal{D}_x\psi(t)\right)_n(X)dx$$

To solve Eq.(14), we apply perturbation theory, we shall seek a solution in the form of the series

$$\psi_n(t,X)=\sum_{\mu}v^{\mu}\psi_n^{\mu}(t,X),\, n=1,2,3,\cdots,\mu=0,1,2,\cdots \tag{15}$$

Substituting the series (15) in Eq.(14) and equating the coefficients of equal powers of v we obtain

$$\left(\frac{\partial}{\partial t}+\mathcal{L}_1\right)\psi_1^{o}(t)=0, \tag{16}$$

$$\left(\frac{\partial}{\partial t}+\mathcal{L}_1+\mathcal{L}_2\right)\psi_2^{o}(t)=S_2^{o}, \tag{17}$$

....

$$\left(\frac{\partial}{\partial t}+\sum_{i=1}\mathcal{L}_i\right)\psi_n^{\mu}(t)=S_n^{\mu}, \tag{18}$$

where we have introduced the notation

$$\mathcal{L}_1\psi_1^{o}(t)=v_1\frac{\partial}{\partial q_1}\psi_1^{o}(t,x_1)-\int\frac{\partial\theta(q_1-q)}{\partial q_1}\frac{\partial\psi_1^{o}(t,x)}{\partial p_1}\psi_1^{o}(t,x)dx,$$

$$\mathcal{L}_i\psi_n^{\mu}(t)=v_i\frac{\partial}{\partial q_1}\psi_n^{\mu}(t,X)-v\int\left(\mathcal{A}_x\psi_{(t)}\right)(x_i)\left(\mathcal{D}_x\psi^{\mu}\right)_{n-1}(t,X\setminus x_i)\,dx,$$

and

$$S_n^{\mu}=\left(\mathcal{U}\psi^{\mu-1}(t)\right)_n(X)+\frac{1}{2}\sum_{\delta_1+\delta_2=\mu-1}\left(\mathcal{W}(\psi^{\delta_1}(t),\psi^{\delta_2}(t)\right)(X) \tag{19}$$

$$+v\int\left(\mathcal{A}_x\mathcal{D}_x\psi^{\mu-1}(t)\right)_n(X)dx.$$

Thus, the solution of Eq. (14) reduces to the solution of the homogeneous (16) and inhomogenous (17), (18) Vlasov's [A.A.Vlasov, 1950] equations for $\psi_1^{o}(t)$ and $\psi_n^{\mu}(t)$, accordingly.

Proposition 2. The series (15), $\psi_n(t,X)=\sum_{\mu}v^{\mu}\psi_n^{\mu}(t,X)$, where ψ_1^{o} is defined in accordance with solution of Vlasov's equation and the remaining ψ_n^{μ} on the basis of the formula

$$\psi_n^{\mu}(t,X)=\int dx_1^{'}\cdots\int dx_n^{'}\int_{-\infty}^{t}dt'S_n^{\mu}\left(t,x_1^{'},\cdots,x_n^{'}\right)\bigcap_{1\le i\le n}G\left(t-t,x,x_i^{'}\right), \tag{20}$$

is a solution of Eq. (14).

Proof: We consider Eqs. (16) and (17) where (16) is the Vlasov equation. This system of coupled equations for the single-molecule and two-molecule perturbations can serve to determine the successive approximations $\psi_n^\mu(t)$. $\psi_1^o(t, X)$ is the solution of Vlasov's equation.

Substituting [M.Yu.Rasulova, 1980],[S.Ichimary, 1968]

$$\psi_2^o(t, x_1, x_2) = \int dx_1^{'} \int dx_2^{'} \int_{-\infty}^{t} dt^{'} S_2^o(t^{'}; x_1^{'}, x_2^{'}). \tag{21}$$

$$G(t - t^{'}; x_1, x_1^{'})G(t - t^{'}; x_2, x_2^{'})$$

in (17), we see that (21) is a solution of (17) if

$$S_2^o(t, x_1, x_2) = [\theta(q_1 - q_2), \psi_1^o(t; x_1)\psi_1^o(t, x_2)]$$

$$+\int_{1 \leq i \leq 2} [\theta(q_i - q), \psi_1^o(t; x_1)\psi_1^o(t; x)]\, dx$$

and if G satisfies equation

$$\left(\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial q_1}\right) G\left(t \cdot t_o; x_1, x_1^{'}\right) - \frac{\partial \psi(t, x_1}{\partial v_1}. \tag{22}$$

$$\int \frac{\partial \theta(q_1 - q)}{\partial q_1} G(t - t^{'}; x, x_1^{'})dx-$$

$$\int \frac{\partial \theta\left(q_1 - q\right)}{\partial q_1} \partial \frac{G\left(t - t^{'}; x_1, x_1^{'}\right)}{\partial v_1} \psi(t, x)dx = 0$$

with the initial condition

$$G(0; x_1, x_1^{'}) = \delta(x_1 - x_1^{'}) \tag{23}$$

The recursive system of Eq. (18) can, with allowance for the established structure of the solutions, serve to determine the successive approximations $\psi_n^\mu(t)$ and, therefore, formula (15). Indeed substituting again (20) directly in (18), we can see that (20) is a solution of (18) if S_n^μ is defined in accordance with (19) and if G satisfies Eq. (22) with the initial condition (23).

[H.Neunzert, 1978], [K.Steiner, 1995], [H.Neunzert and A.H.Siddiqi, 1997] by the particle method have proved the existence of unique solution of Vlasov equation

$$\partial_t \psi_1^o\left(t_1 x_1\right) = -v_1 \nabla_x \psi_1^o\left(t_1 x_1\right) + \frac{e_s}{m_s} \nabla_x \mathcal{A}^{k-1} \nabla_{V_1} \psi_1^o\left(t_1, x_1\right), \psi_1^o\left(T_k\right) = f_1^{k-1}\left(T_k\right) \tag{24}$$

$$-\triangle_x U^k = \frac{1}{\in_o} \sum_s \int_{\Gamma_s} e_1 f_1^k dS \qquad T = T_k \tag{25}$$

where $T_k = \frac{k}{n}T, k = 1, \cdots, n, n \in I\!N$ of size $\frac{1}{n}T, \boldsymbol{U}^0$ is solution (25) with $f^o(o, P) = f^o(P); \theta(\mid q_i - q_j \mid)$ is Coulomb potential; $\boldsymbol{U}$-potential by $E = -\nabla \boldsymbol{U}$ satisfies Poisson's equation. Inkn:Neinzert2, kn:Steiner it is shown that $\psi_1^o(t, x_1, v_1) = (\psi^o \Phi_{o,t})(x_1, v_1)$ is solution of the Vlasov equation. Here we assume that E is Lipschitz continuous, $\Phi_{t,\tau} : F \to F$ is a measure preserving group homomorphism [H.Neunzert, 1978] and ψ^o is continuous initial conditions.
A Numerical Scheme for the Vlasov equation is as follows [K.Steiner, 1995]:

For every time step $t_k = k\triangle t, k = 0, 1, \cdots$

$$\begin{aligned} v_i^N(t_{k+1}) &= v_i^N(t_k) + \triangle t E\left(q_i^N(t_k)\right) \\ q_i^N(t_{k+1}) &= q_i^N(t_k) + \triangle t v_i^N(t_{k+1}) \\ \alpha_i^N(t_{k+1}) &= \alpha_i^N(t_k). \end{aligned}$$

Solution (20) of two equations (16), (17) of hierarchy are in good agreement with results of [S.Ichimary, 1968] for plasma physics and this method is opening possibilities to calculate the solutions of the next complex kinetic equations of BBGKY hierarchy.

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