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HIGHER DIMENSIONAL COMPLEX KLEINIAN GROUPS

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0. Introduction

Kleinian groups were introduced by H. Poincaré [Po] in the 1880's as the monodromy groups of certain 2^{nd} order differential equations on the complex plane $\mathbb{C}$, and they have played a major role in many parts of mathematics throughout this century, as for example in Riemann surfaces and Teichmüller theory, automorphic forms, holomorphic dynamics, conformal and hyperbolic geometry, 3-manifolds theory, etc. Many interesting results about the dynamics of rational maps on $P^1_{\mathbb{C}}$ in the last two decades have been motivated by the dynamics of Kleinian groups (see, for instance, [Su1,5-7, Mc1,2]). These are, by definition, discrete groups of holomorphic automorphisms of the complex projective line $P^1_{\mathbb{C}}$, whose limit set is not the whole $P^1_{\mathbb{C}}$. Equivalently, these can be regarded as groups of isometries of the hyperbolic 3-space, or as groups of conformal automorphisms of the sphere S^2. Much of the theory of Kleinian groups has been generalised to conformal Kleinian groups in higher dimensions (also called *Möbius* or *hyperbolic* Kleinian groups), *i.e.,* to discrete groups of conformal automorphisms of the sphere S^n whose limit set is not the whole sphere (see, for instance, [A2, Ku1,2, Ma1, Su1,5-7]). This theory has also been generalised to automorphisms of $P^2_{\mathbb{C}}$, and recently many results are being obtained about the dynamics of automorphisms and rational endomorphisms of $P^n_{\mathbb{C}}$ in general (see, for instance, the surveys [FS, SB, BS]).

The purpose of this work is to study what we call *higher dimensional complex Kleinian groups.* By this we mean (infinite) discrete subgroups of $PSL(n{+}1,\mathbb{C})$, the group of *holomorphic* automorphisms of $P^n_{\mathbb{C}}$, $n > 1$, whose limit set is not all of $P^n_{\mathbb{C}}$.

The theory of holomorphic Kleinian actions on higher dimensional projective spaces goes back to Myrberg [My], Schubart [Schb] and others, generalising part of the theory of automorphic functions to several complex variables. Kleinian actions on topological spaces, in general, were studied in [Ku1]; M. Nori [No] generalised the classical Schottky groups (see [Ma2]) to higher dimensional projective spaces (compare with section 5 below); Deligne and Mostow [DM] studied discrete subgroups of $PU(1,n)$, the projectivization of $U(1,n)$, which are uniform, non-arithmetic and they have a non-empty region of discontinuity on the ball in $P^n_{\mathbb{C}}$ whose homogeneous coordinates satisfy $-||z_0||^2 + \cdots + ||z_n||^2 < 0$; these can be regarded as complex Kleinian groups that leave invariant a ball, so they are analogous to the Fuchsian groups [Ma1]. Studying groups of automorphisms which act properly discontinuously on an open set in $P^n_{\mathbb{C}}$ provides an interesting way of extending the theory of automorphic functions to several complex variables (see [Bor] for a survey until 1952). This also provides an important method for constructing compact complex manifolds (and orbifolds) which carry a natural projective structure. As pointed out by Gunning [Gu], this is a rich geometric structure that can be used to study these manifolds (and orbifolds) in a uniform way. This is already a classical subject in dimension 1, i.e., for Riemann surfaces (see, for instance, [Be, Fa, KM, Gol, Gu]). In dimension 2, Kobayashi and Ochiai [KO] classified the compact, complex surfaces which admit a holomorphic projective structure; their results show that all these manifolds are quotients of an open set in $P^2_{\mathbb{C}}$ divided by a complex Kleinian group; it would be interesting to extend their classification to orbifolds. In dimension 3, interesting results have been obtained by M. Kato [Ka1-4]. Our constructions below provide families of manifolds which are "Pretzel twistor spaces" in the sense of Penrose [Pe3]; these are related with generalisations of string theory and quantum field theory to complex dimension 3, see [Si]. However, the main motivation for this work is to study complex Kleinian groups from the dynamical/ergodic point of view. To our knowledge, little is known in this direction and we aim to provide interesting insights into discrete holomorphic dynamics in several complex variables, a beautiful subject which is bound to grow enormously in the near future (c.f., [BLS, FS, SB, BS]). Just as the dynamics of classical Kleinian groups go side-to-side with the dynamics of rational maps on the Riemann sphere (see, for instance, [Su1,5-7, Mc1,2]), the dynamics of complex Kleinian groups in higher dimensions ought to go side-to-side with the dynamics of rational self-maps on projective spaces. This is yet to be explored.

This work can be regarded as belonging to the geometric framework of discrete subgroups of Lie groups (as in [Bor, Da, DM, Ma, Rag, Rat]), and it is of course related to the analytic framework of [A1-3, Be, BLS, FS, Fur, Kr, SB, Tu] and others. Though, our main motivation actually comes from hyperbolic geometry and conformal dynamics (as for instance [Bo, Ku1,2, Ma1,2, Mc1,2, Su1-7, Th1,2]). Yet, from a different viewpoint, some of the basic ideas of this article are indeed part of the "Penrose twistor program": the interplay between conformal geometry on Riemannian manifolds and the holomorphic structure of their twistor spaces (see [Pe1-3, AHS, Hi, BR, EL, Sa]). Here we look at the interplay between the conformal dynamics of Kleinian actions on even dimensional spheres and the holomorphic dynamics on their twistor spaces, which are complex projective manifolds. Thus we obtain, for instance, discrete subgroups of $PSL(4,\mathbb{C})$ acting minimally on all of $P^3_{\mathbb{C}}$, and the action is ergodic with respect to a natural lifting to $P^3_{\mathbb{C}}$ of the Patterson-Sullivan measure for discrete subgroups of $Iso_+(\mathbb{H}^5)$.

This article consists of two main parts. The first of these, sections 2 to 4 below, is concerned with twistor theory. We show that the (conformal) dynamics of Kleinian groups on spheres embeds in the holomorphic dynamics of complex Kleinian groups on projective spaces. In the second part, sections 5 to 7, we drop the "reality condition" imposed in the first part, to study complex Kleinian groups that generalise the classical Schottky groups.

We first give a method, in section 1, that we call the *conical* (or *suspension*) construction, which allows us to define a complex Kleinian group in dimension $n+1$ out of a Kleinian group in dimension n. The Kleinian actions that we get in this way are "suspensions" of Kleinian actions in dimension n. This leads to a more general construction, that we call the *join* of two Kleinian actions, that will be explored in a future article. Given two Kleinian actions on $P^n_{\mathbb{C}}$ and $P^m_{\mathbb{C}}$, their join is a Kleinian group on $P^{n+m+1}_{\mathbb{C}}$, whose discontinuity and limit sets are the projective join of the corresponding discontinuity and limit sets of the previous actions. This construction can be iterated to any number of Kleinian actions.

In section 2 we use the twistor fibration $P^3_{\mathbb{C}} \to S^4$, with fibre $P^1_{\mathbb{C}} = SO(4)/U(2)$, to show that the dynamics of conformal Kleinian groups on S^4 embeds in the holomorphic dynamics of complex Kleinian groups on $P^3_{\mathbb{C}}$. If Γ is a conformal Kleinian group on S^4 with limit set Λ, then Γ lifts canonically to a complex Kleinian group $\widetilde{\Gamma} \subset PSL(4,\mathbb{C})$, whose limit set $\widetilde{\Gamma}$ is $\Lambda \times P^1_{\mathbb{C}}$. In section 3 we generalise this result to arbitrary Riemannian manifolds of dimension $2n$, $n>1$, and their twistor spaces. If N^{2n} is such a manifold, its twistor space is the total space of the bundle $p: \mathfrak{Z}(N) \to N$, whose fibre at $x \in N$ is the symmetric space $SO(2n)/U(n)$, of all complex structures on T_xN which are compatible with the metric and the orientation on N. $\mathfrak{Z}(N)$ is an almost Hermitian manifold, whose complex structure is integrable whenever N is (locally) conformally flat, by [AHS, DV, OR]; $\mathfrak{Z}(N)$ is actually projective if N is a sphere S^{2n} and $\mathfrak{Z}(S^4)$ is $P^3_{\mathbb{C}}$. We show that if Γ is a discrete group of orientation preserving conformal diffeomorphisms of N, then Γ has a canonical lifting to a group $\widetilde{\Gamma}$ of diffeomorphisms of $\mathfrak{Z}(N)$ that preserve the almost complex structure and carry fibres isometrically into fibres. Moreover, if the almost complex structure on $\mathfrak{Z}(N)$ is integrable, then the elements of $\widetilde{\Gamma}$ are indeed holomorphic transformations. The limit set of $\widetilde{\Gamma}$ is $p^{-1}(\Lambda(\Gamma))$, the inverse image of the limit set of Γ. We show (theorems 3.6 and 4.3) that an interesting feature occurs, which does not happen for conformal Kleinian groups: the action of $\widetilde{\Gamma}$ on its limit set $\Lambda(\widetilde{\Gamma})$ may or may not be minimal and ergodic, with respect to the densities $\widetilde{\mu}_y$ defined from the Patterson-Sullivan densities for Γ. More precisely, if Γ is any (geometrically-finite) discrete subgroup of $Iso_+(\mathbb{H}^{2n+1}) \cong Conf_+(S^{2n}) \cong SO_0(2n+1,1)$ which is Zariski-dense, then the action of $\widetilde{\Gamma}$ is minimal and ergodic on its limit set $\widetilde{\Lambda}$; the same statements hold for subgroups of $Iso_+(\mathbb{H}^{2n})$, regarded as subgroups of $Iso_+(\mathbb{H}^{2n+1})$. However, if Γ is the fundamental group of a hyperbolic orbifold of dimension $m < 2n$, regarded as a subgroup of $Conf(S^{2n})$ via the inclusion $Iso(\mathbb{H}^m) \hookrightarrow Iso(\mathbb{H}^{2n+1})$, then the action of $\widetilde{\Gamma}$ on its limit set is neither minimal nor ergodic. The manifold $\mathfrak{Z}(S^{2n}) = SO(2n+1)/U(n)$ has a canonical projective embedding in $P_{\mathbb{C}}^{2^n-1}$, given via the spin representation [In]. This is the projective embedding of $\mathfrak{Z}(S^{2n})$ of smallest codimension, and the group of holomorphic automorphisms of $\mathfrak{Z}(S^{2n})$ extends uniquely to a subgroup of $PSL(2^n, \mathbb{C})$, by [Bl]. Therefore, every conformal Kleinian group on S^{2n} is canonically a complex Kleinian group on $P_{\mathbb{C}}^{2^n-1}$, for all $n \geq 1$.

The next step is to forget the "reality condition" of sections 2 to 4, and consider arbitrary lines in $P_{\mathbb{C}}^3$, not necessarily twistor lines. More generally, we consider an arbitrary configuration $\{(L_1, M_1), ..., (L_r, M_r)\}$ of pairs of projective n-spaces in $P_{\mathbb{C}}^{2n+1}$, which are, all of them, pairwise disjoint. Given arbitrary neighbourhoods $U_1, ..., U_r$ of the $L_i{}'s$, pairwise disjoint, we show that there exists, for each $i = 1, ..., r$, projective transformations T_i of $P_{\mathbb{C}}^{2n+1}$, which interchange the interior with the exterior of a compact tubular neighbourhood N_i of L_i contained in U_i, leaving invariant the boundary $E_i = \partial(N_i)$. The $E_i{}'s$ are *mirrors*, they play the same role in $P_{\mathbb{C}}^{2n+1}$ as circles play in S^2 to define the classical Schottky groups. Each mirror E_i is a $(2n+1)$-sphere bundle over $P_{\mathbb{C}}^n$. The group of automorphisms of $P_{\mathbb{C}}^{2n+1}$ generated by the $T_i{}'s$ is a complex Kleinian group Γ. The region of discontinuity $\Omega(\Gamma)$ is a fibre bundle over $P_{\mathbb{C}}^n$ with fibre S^{2n+2} minus a Cantor set $\mathcal{C}$. The limit set Λ is the complement of $\Omega(\Gamma)$ in $P_{\mathbb{C}}^{2n+1}$; it is the closure of the Γ-orbit of the $L_i's$, and it is a product $\mathcal{C} \times P_{\mathbb{C}}^n$. The action of Γ on this set of projective lines is minimal in the sense that the Γ-orbit of every point x_o in $P_{\mathbb{C}}^{2n+1}$ accumulates to (at least a point in) each one of the projective lines in Λ. This set is transversally *projectively self-similar*, i.e., Λ corresponds to a Cantor set in the Grassmannian $G_{2n+1,n}$, which is dynamically-defined. Hence Λ is a *solenoid* (or *lamination*) by projective spaces, which is transversally Cantor and projectively self-similar. Each of these groups Γ contains a subgroup $\check{\Gamma}$ of index two, which is a free group of rank $r-1$ and acts *freely* on $\Omega(\Gamma)$. The quotient $\Omega(\Gamma)/\check{\Gamma}$ is a compact complex manifold, which is a fibre bundle over $P_{\mathbb{C}}^n$ with fibre the connected sum of $(r-1)$ copies of $S^{2n+1} \times S^1$. As mentioned above, these manifolds have a canonical projective structure, *i.e.,* they have an atlas $\{(\mathcal{U}_i, \phi_i)\}$ whose changes of coordinates are restrictions of complex projective transformations. However, these manifolds are never Kähler, due to cohomological reasons. When $n = 1$ and the configuration $\{(L_1, M_1), ..., (L_r, M_r)\}$ consists of twistor lines of the fibration $p: P_{\mathbb{C}}^3 \to S^4$, then Γ and $\check{\Gamma}$ descend to conformal Schottky groups on S^4. In this case $\Omega(\Gamma)/\check{\Gamma}$ is the twistor space of the conformally flat manifold $S^4/p(\check{\Gamma})$, which is a Schottky manifold [Ku2]; $\Omega(\Gamma)/\check{\Gamma}$ is a *flat twistor space* [Si]. We also generalise (in section 6) our construction of Schottky groups to $P_{\mathbb{C}}^{\infty}$, the projectivization of a separable complex infinite dimensional Hilbert space.

In section 7 we compare the deformations of our Schottky groups with the deformations of the complex manifolds that one gets as quotients of the action of the group on its region of discontinuity. For this we estimate an upper bound for the Hausdorff dimension of the limit set of the complex Schottky groups. We use this to show that, with the appropriate conditions for the Schottky group $\check{\Gamma}$, the Kuranishi space $\mathfrak{K}$ of versal deformations of the complex manifold $M_{\check{\Gamma}} := \Omega(\check{\Gamma})/\check{\Gamma}$, is smooth near the reference point determined by $M_{\check{\Gamma}}$. Furthermore, we estimate the dimension of $\mathfrak{K}$ and we prove that every infinitesimal deformation of $M_{\check{\Gamma}}$ actually corresponds to an infinitesimal deformation of the group $\check{\Gamma}$ in the projective group $PSL(2n+2, \mathbb{C})$, in analogy with the classical Teichmüller and moduli theory for Riemann surfaces.

Since this article touches several branches of mathematics which might not be familiar to the reader, we have included an extense bibliography, which is not at all exhaustive, but can be useful.

1. Complex Kleinian groups

Let $SL(2,\mathbb{C})$ be the group of 2×2 complex matrices with determinant 1. This group acts linearly on $\mathbb{C}^2$, so it acts on the complex projective line $P^1_{\mathbb{C}}$. Moreover, if we identify $P^1_{\mathbb{C}}$ with the Riemann sphere $S^2 \cong \mathbb{C}\cup\infty := \widehat{\mathbb{C}}$, then the action of $SL(2,\mathbb{C})$ on $P^1_{\mathbb{C}}$ is via the Möbius transformations:

$$z \mapsto \frac{az+b}{cz+d}\,.$$

Two such matrices define the same Möbius transformation iff they differ by multiplication by ± 1, so the group $PSL(2,\mathbb{C}) \cong SL(2,\mathbb{C})/\{\pm I\}$ can be identified with the group of all Möbius transformations acting on $P^1_{\mathbb{C}}$. This coincides with the group of holomorphic automorphisms of the Riemann sphere; it also coincides with $Conf_+(S^2)$, the group of orientation preserving conformal automorphisms on S^2.

We recall that if Γ is a discrete subgroup of $PSL(2,\mathbb{C})$, then its *limit set* $\Lambda(\Gamma)\subset P^1_{\mathbb{C}}$ is the set of accumulation points of some (any) Γ-orbit; by definition, $\Lambda(\Gamma)$ is closed, hence compact. The complement $\Omega(\Gamma) = P^1_{\mathbb{C}} - \Lambda(\Gamma)$ is the *region of discontinuity* of Γ. A *(classical)* Kleinian group is a discrete subgroup Γ of $PSL(2,\mathbb{C})$, whose limit set is not all of $P^1_{\mathbb{C}}$, *i.e.,* its region of discontinuity is not empty.

Let us now discuss generalisations of these groups to higher dimensions. We consider first the conformal case. Let $Conf(S^n)$ be the group of all conformal automorphisms of the n-sphere, and let $Conf_+(S^n)$ be the index-two subgroup of orientation preserving maps. $Conf(S^n)$ is generated by inversions on all possible $(n-1)$-spheres in the unit sphere $S^n\subset\mathbb{R}^{n+1}$; $Conf_+(S^n)$ consists of the elements obtained by an even number of inversions. S^n bounds the unit ball D^{n+1} and given a $(n-1)$-sphere $S^{n-1}_\delta\subset S^n$, there exists a unique n-sphere $S^n_\delta\subset\mathbb{R}^{n+1}$ intersecting S^n orthogonally at S^{n-1}_δ. The inversion in $\mathbb{R}^{n+1}$ on this S^n_δ, takes the unit ball D^{n+1} into itself, preserving the hyperbolic metric on D^{n+1}. This identifies $Conf(S^n)\cong Iso(\mathbb{H}^{n+1})$, the group of isometries of the hyperbolic $(n+1)$-space. Hence $Conf_+(S^n)$ acts transitively on $\mathbb{H}^{n+1}$ and the isotropy group is $SO(n+1)$, the group of orientation preserving maps in $\mathbb{R}^{n+1}$ generated by reflexions on hyperplanes through the origin. Therefore $Conf_+(S^n)$ is diffeomorphic to $SO(n+1)\times\mathbb{H}^{n+1}$.

We now focus our attention on the case $n=4$. We already know $Conf_+(S^4)\cong Iso_+(\mathbb{H}^5)$. As a manifold, $Conf_+(S^4)$ is diffeomorphic to $SO(5)\times\mathbb{H}^5$, so it has dimension 15. Let us give a different description of this group, which is appropriate for this article. We recall that S^4 can be thought of as being the *projective quaternionic line* $P^1_{\mathcal{H}}\cong S^4$. This is the space of *right* quaternionic lines in $\mathcal{H}^2$, *i.e.,* subspaces of the form

$$L_q := \{q\lambda\,:\,\lambda\in\mathcal{H}\}\ ,\ \ q\in\mathcal{H}^2 - \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\},$$

where $\mathcal{H}$ is the space of quaternions and $\mathcal{H}^2 := \left\{\begin{pmatrix}q_0\\q_1\end{pmatrix}\ :\ q_0,q_1\in\mathcal{H}\right\}$. Identify S^4 with $\mathcal{H}\cup\{\infty\} := \widehat{\mathcal{H}}$ via the stereographic projection. Let $GL(2,\mathcal{H}) := Gl_l(2,\mathcal{H})$ be the group of all invertible 2×2 quaternionic matrices $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ acting on $\mathcal{H}^2$ by the left:

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}q_0\\q_1\end{pmatrix} = \begin{pmatrix}aq_0+bq_1\\ cq_0+dq_1\end{pmatrix}.$$

$\mathcal{H}^2$ is a right module over $\mathcal{H}$ and the action of $GL(2,\mathcal{H})$ on $\mathcal{H}^2$ commutes with multiplication by the right: for every $\lambda\in\mathcal{H}$ and $A\in GL(2,\mathcal{H})$ one has,

$$A\circ R_\lambda = R_\lambda\circ A \tag{1.1}$$

where R_λ is multiplication on the right by λ. Thus $GL(2,\mathcal{H})$ carries right quaternionic lines into right quaternionic lines, so it defines an action of $GL(2,\mathcal{H})$ on $P^1_{\mathcal{H}} = S^4$. Now consider the map

$$\psi : \mathcal{H}^2 - \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\} \to S^4 \tag{1.2}$$

given by: $\psi\begin{pmatrix}q_0\\q_1\end{pmatrix} = q_0 q_1^{-1}$. For each $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in GL(2,\mathcal{H})$ and a point (q_0,q_1) in $\mathcal{H}^2 - \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$, one has: $T\circ\psi(q_0,q_1) = \psi(A\begin{pmatrix}q_0\\q_1\end{pmatrix})$, where T is the Möbius transformation $T(q) = (aq+b)(cq+d)^{-1} \in S^4$. Let us denote by $M\ddot{o}b(2,\mathcal{H})$ the set of all the quaternionic Möbius transformations in $\widehat{\mathcal{H}} := \mathcal{H}\cup\infty = S^4$ of the form

$$T(q) = (aq+b)(cq+d)^{-1}\,,\; q\in\widehat{\mathcal{H}}\,,$$

where a,b,c,d are quaternions and the matrix $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is in $GL(2,\mathcal{H})$. We make the usual conventions about the point at infinity.

We refer to [A1,2, At, Ku2, Gi] for the proof of the following theorem.

1.3 Theorem. *$M\ddot{o}b(2,\mathcal{H})$ is a group and one has the group isomorphisms:*

$$M\ddot{o}b(2,\mathcal{H}) \cong PSL(2,\mathcal{H}) \cong Conf_+(S^4) \cong SO_0(5,1)\,,$$

where $PSL(2,\mathcal{H}) = (SL(8,\mathbb{R})\cap GL(2,\mathcal{H}))/\{\pm I\}$ and $SO_0(5,1)$ is the connected component of the identity of $SO(5,1)$.

We now consider the holomorphic case. That is, we think of the classical Kleinian groups as consisting of holomorphic automorphisms of $P^1_{\mathbb{C}}$, and we want to extend this notion to higher dimensions. For this we need to define first the appropriate concept of the limit set for an action of a discrete group. The limit set of a discrete subgroup of $Conf_+(S^n)$ is the set of accumulation points of any single orbit. This definition is fine because the action of such a group on its limit set is always minimal, i.e., the orbits are dense. In the cases relevant for this article, this might not be the case, so we need the following notions and definitions, that we take from Kulkarni's paper [Ku1].

1.4 Definition. *Let X be a locally compact Hausdorff space with a countable base for its topology. Let G be a group acting on X and let $\Omega\subset X$ be a G-invariant subset. The action on Ω is properly discontinuous if for every pair of compact subsets C and D of Ω, the cardinality of the set $\{\gamma\in G \mid \gamma(C)\cap D\neq\emptyset\}$, is finite.*

1.5 Remark. If G and Ω are as above and G acts properly discontinuously on Ω, then the following three conditions hold:

a) The stabiliser of each point is finite.
b) Every $y\in\Omega$ has a neighbourhood $\mathcal{V}$, such that if $g(\mathcal{V})\cap\mathcal{V}\neq\varnothing$, then $g(y)=y$.
c) For each pair of points $x,y\in\Omega$ in different G-orbits, there exist neighbourhoods $\mathcal{V}$ and $\mathcal{U}$, of x and y respectively, such that $\mathcal{V}\cap g(\mathcal{U})=\varnothing$ for all $g\in G$.

Let $\{A_\beta\}$ be a family of subsets of X where β runs over some infinite indexing set B. A point $x\in X$ is a *cluster point* of $\{A_\beta\}$ if every neighbourhood of x intersects A_β for infinitely many $\beta\in B$. Let $L_0(G)$ be the closure of the set of points in X with infinite isotropy group. Let $L_1(G)$ be the closure of the set of cluster points of the family $\{\gamma(x)\}_{\gamma\in G}$, where x runs over $X - L_0(G)$. Finally, let $L_2(G)$ be the closure of the set of cluster points of $\{\gamma(K)\}_{\gamma\in G}$, where K runs over all the compact subsets of $X - \{L_0(G)\cup L_1(G)\}$. We have:

1.6 Definitions-propositions. *(Compare* [Ku1]*.)*

i) Let X be as above and let G be a group of homeomorphisms of X. The **limit set** *of G in X is the set $\Lambda(G) := L_0(G) \cup L_1(G) \cup L_2(G)$. This set is closed in X and it is G-invariant (it can be empty).*

ii) The **region of discontinuity** *of G is $\Omega(G) \subset X := X - \Lambda(G)$. This set (which can be empty) is open, G-invariant, and G acts properly discontinuously on $\Omega(G)$.*

iii) A **Möbius** *(or* **hyperbolic**, *or* **conformal***)* **Kleinian group** *is a discrete subgroup $\Gamma \subset Iso_+(\mathbb{H}^{n+1})$ whose limit set is not the whole sphere at infinity. In other words, the region of discontinuity for its action on S^n is not empty.*

iv) A **complex Kleinian group** *is a discrete subgroup Γ of $PSL(n+1, \mathbb{C})$, the group of automorphisms of the projective n-space $P^n_{\mathbb{C}}$, whose limit set is not all of $P^n_{\mathbb{C}}$.*

This definition is not standard; in fact questions regarding domains of discontinuity and limit sets are quite subtle, c.f., [Ku1]. With this definition, if G acts properly discontinuously on Ω, then the orbit space Ω/G, with the quotient topology, is Hausdorff and the canonical map $\pi : \Omega \to \Omega/G$ is open and continuous. If Ω is a manifold and the action of G is differentiable, then Ω/G is an orbifold; if in addition G acts freely on Ω, then Ω/G is a manifold and π is a covering map.

1.7 Remark. *In this paper we will consider only non-elementary Möbius groups, i.e., groups whose limit sets are infinite.*

The classical definitions of the *limit set* and the *discontinuity set* coincide with the above definitions when G is a Möbius (or conformal) group. Higher dimensional conformal Kleinian groups have been widely studied by many authors, see for instance [A1,2, Kr, Ku1,2, Ni, Ra, Su1-4, Th1,2]. For example, every isometry of the hyperbolic space $\mathbb{H}^n$ extends canonically to an isometry of $\mathbb{H}^{n+1}$, hence every hyperbolic Kleinian group in $\mathbb{H}^n$ determines a hyperbolic Kleinian group in $\mathbb{H}^{n+1}$, whose limit set is contained in the equator of the sphere at infinity. There is a similar construction for complex Kleinian groups. We call this the *cone construction*, or *suspension*, and it uses the *Schur multipliers*, see [Sch] and [Ki], pp 220. Let $\Gamma \subset PSL(n, \mathbb{C})$ be a complex Kleinian group, so Γ acts on $P^{n-1}_{\mathbb{C}}$. We note that $P^{n-1}_{\mathbb{C}}$ is the space of lines in $\mathbb{C}^n$ and $P^n_{\mathbb{C}}$ can be thought of as being $\mathbb{C}^n$ union the hyperplane at infinity. If we extend the action of Γ on $P^{n-1}_{\mathbb{C}}$ to a linear (unimodular) action on $\mathbb{C}^n$, *i.e.,* to an action of a subgroup $\widehat{\Gamma} \subset SL(n, \mathbb{C})$ on $\mathbb{C}^n$, then one has an action of $\widehat{\Gamma}$ on $P^n_{\mathbb{C}}$. The limit set of $\widehat{\Gamma}$ is the complex cone, with vertex at 0, over the limit set of Γ in the $P^{n-1}_{\mathbb{C}}$ at infinity. One has an exact sequence:

$$0 \to \mathbb{Z}_n \to SL(n, \mathbb{C}) \xrightarrow{\mathcal{P}} PSL(n, \mathbb{C}) \to 1,$$

where $\mathcal{P}$ denotes projectivization of linear maps. One can always lift Γ to $SL(n, \mathbb{C})$ by taking its pull back under $\mathcal{P}$, so that Γ can always be extended to a Kleinian group $\widehat{\Gamma}$ on $P^n_{\mathbb{C}}$, whose restriction to the $P^{n-1}_{\mathbb{C}}$ at infinity is Γ. If we can actually lift Γ to a subgroup $\widetilde{\Gamma} \subset SL(n, \mathbb{C})$ that intersects the kernel of $\mathcal{P}$ only at the identity, then Γ itself can be considered as a Kleinian group in $P^n_{\mathbb{C}}$. The obstruction to lift $\Gamma \subset PSL(n, \mathbb{C})$ to an isomorphic group in $SL(n, \mathbb{C})$ is an element in $H^2(\Gamma, \mathbb{Z}_n)$. If this obstruction vanishes, then Γ can be regarded as a Kleinian group on $P^n_{\mathbb{C}}$. This happens, for instance, if $H^2(\Gamma, \mathbb{Z}_n) \cong 0$.

If Γ is the fundamental group of a complete (non-necessarily compact), hyperbolic 3-manifold, so that $\Gamma \subset PSL(2, \mathbb{C})$, then the obstruction in question can be identified with the second Stiefel-Whitney class ω_2 of the 3-manifold, as pointed out by Thurston, see [Kr]. This class is always 0, because every oriented 3-manifold is parallelizable. Hence Γ can always be lifted isomorphically to $SL(2, \mathbb{C})$. Thus the fundamental group Γ of a hyperbolic 3-manifold acting on $\mathbb{H}^3$, whose action on the sphere at infinity is Kleinian, can be considered as a complex Kleinian group $\widetilde{\Gamma}$ acting on $P^2_{\mathbb{C}}$ and leaving the line at infinity invariant. The limit set of $\widetilde{\Gamma}$ is the cone, with vertex at 0, over the limit set of Γ on $P^1_{\mathbb{C}}$. By Ahlfors' Finiteness Theorem [A3], the quotient $\Omega(\Gamma)/\Gamma$ is a Riemann surface of finite type. Hence the quotient $\Omega(\Gamma)/\widehat{\Gamma}$, is a complex line bundle over a Riemann surface of finite type; $\Omega(\Gamma)/\widehat{\Gamma}$ is homotopically equivalent to $\mathbb{H}^3/\Gamma$.

It is worth noting that one has a canonical embedding $SL(n,\mathbb{C}) \to SL(n+k,\mathbb{C})$, for all $n > 1, k > 0$, given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & I_{k\times k}\end{pmatrix}$. Therefore, if $\Gamma \subset PSL(n,\mathbb{C})$ is a Kleinian group on $P^{n-1}_{\mathbb{C}}$ that can be suspended to a linear group in $SL(n,\mathbb{C})$, then Γ can be automatically suspended to a linear group in $SL(n+k,\mathbb{C})$; thus, via the cone construction, Γ can be regarded as a Kleinian group on $P^{n+k}_{\mathbb{C}}$. If Ω is the region of discontinuity of Γ in $P^{n-1}_{\mathbb{C}}$, then the region of discontinuity Ω^k of Γ in $P^{n+k}_{\mathbb{C}}$ is a $(k+1)$-dimensional complex bundle over $P^{n-1}_{\mathbb{C}}$. Hence, in particular, *the fundamental group of every complete, connected, open, 3-dimensional hyperbolic manifold can be regarded as a Kleinian group on* $P^n_{\mathbb{C}}$*, for all* $n > 0$. We note, however, that in all these cases the action on $P^n_{\mathbb{C}}$ is defined from an action on the previous $P^{n-1}_{\mathbb{C}}$, and it leaves $P^{n-1}_{\mathbb{C}}$ invariant.

A related construction is the *join* of two Kleinian actions: let $\Gamma_1 \subset GL(n+1,\mathbb{C})$ be a discrete group acting on $\mathbb{C}^{n+1}$ which induces a Kleinian action on $P^n_{\mathbb{C}}$; let $\Gamma_2 \subset GL(m+1,\mathbb{C})$ be a discrete group acting on $\mathbb{C}^{m+1}$, which induces a Kleinian action on $P^m_{\mathbb{C}}$. Then $\Gamma_1 \times \Gamma_2$ acts on $\mathbb{C}^{n+m+2}$ and induces a Kleinian action on $P^{n+m+1}_{\mathbb{C}}$. The limit set and discontinuity set of this action are the corresponding projective joins of the limit sets and discontinuity sets in $P^n_{\mathbb{C}}$ and $P^m_{\mathbb{C}}$.

It is clear that in this case $P^n_{\mathbb{C}}$ and $P^m_{\mathbb{C}}$ are both invariant subsets for the action on $P^{n+m+1}_{\mathbb{C}}$. This motivates the following definitions:

1.8 Definition. *Let* Γ *be a complex Kleinian group on* $P^n_{\mathbb{C}}$*,* $n > 1$*.*

i) The action of Γ *on* $P^n_{\mathbb{C}}$ *is* **reducible** *if it is obtained either by suspending a complex Kleinian action on* $P^{n-1}_{\mathbb{C}}$*, or as the join of two Kleinian actions on* $P^{n_1}_{\mathbb{C}}$ *and* $P^{n_2}_{\mathbb{C}}$*,* $n = n_1 + n_2 + 1$*. Otherwise we say that the action of* Γ *is* **irreducible**.

ii) The action of Γ *on* $P^n_{\mathbb{C}}$ *is* **(complex) algebraically-mixing** *if there are no proper complex compact submanifolds of* $P^n_{\mathbb{C}}$ *which are* Γ*-invariant.*

It is clear that algebraically-mixing implies irreducible.

2. Complex Kleinian groups on $P^3_{\mathbb{C}}$

Let M be a closed, oriented 4-manifold endowed with a conformal metric. The Calabi-Penrose twistor space over M (see, for instance, [At, AHS, BR, C1,2, Pe1,2, Sa]), is the total space of the twistor fibration, i.e., the 2-sphere bundle,

$$p : \mathfrak{Z}(M) \to M\,,$$

whose fibre $E_x \cong P^1_{\mathbb{C}}$ at each point $x \in M$, the *twistor line* at x, is the set of all complex structures on T_xM which are compatible with the metric and orientation. When $M = S^4$ with its canonical metric, the twistor space $\mathfrak{Z} := \mathfrak{Z}(S^4)$ is the complex projective space $P^3_{\mathbb{C}}$. Let us give an alternative (well known) construction of the twistor space $\mathfrak{Z}$. We know that S^4 is the space of right quaternionic lines in $\mathcal{H}^2$. Multiplication on the right by i determines a complex structure on $\mathcal{H}^2 \cong \mathbb{R}^8$. In this way, each *right* quaternionic line L_q in $\mathcal{H}^2$ becomes a 2-dimensional complex space in $\mathbb{C}^4 \cong \mathcal{H}^2$. Moreover, given any $\alpha \in \mathcal{H}$, multiplication by α by the right preserves $L_q := \{q\lambda : \lambda \in \mathcal{H}\,,\, q \in \mathcal{H}^2 - \{(0,0)\}\}$, so each line L_q is covered by the complex lines $l_{q\alpha} := \{q\alpha\lambda : \lambda \in \mathbb{C}\,,\}\subset L_q$. If we identify each complex line $l_{q\alpha}$ to a point we obtain $P^3_{\mathbb{C}} = \mathfrak{Z}$, and if we identify each quaternionic line L_q to a point we obtain $P^1_{\mathcal{H}} = S^4$. This gives the 2-sphere bundle, $p\colon P^3_{\mathbb{C}} \to S^4$, where each fibre is a projective line $P^1_{\mathbb{C}}$.

It is well known that every $h \in \mathit{Conf}_+(S^4)$ has a canonical lifting to a holomorphic map $\widetilde{h}\colon \mathfrak{Z} = P^3_{\mathbb{C}} \to P^3_{\mathbb{C}}$. This is given through the identification in (1.3):

$$\mathit{Conf}_+(S^4) \cong \mathit{M\ddot{o}b}(2,\mathcal{H}) \cong PSL(2,\mathcal{H}) \subset PSL(4,\mathbb{C})\,.$$

This also says that $\mathit{Conf}_+(S^4)$ actually lifts to $PSL(4,\mathbb{C})$ as a group, and it carries twistor lines into twistor lines.

We recall that $P^1_{\mathbb{C}}$ has the Fubini-Study metric [We], which coincides with the standard metric on S^2. This metric is essentially the angle between the complex lines in $\mathbb{C}^2$. For each line L_q, with the above complex structure, we consider the standard Hermitian metric. Then a transformation $A \in PSL(2,\mathcal{H})$ sends the right line L_q isometrically into the right line $L_{q'}$, because the vectors $\{q, qi, qj, qk\}$ form a real orthonormal basis in L_q, and their image in $L_{q'}$ is the basis $\{q', q'i, q'j, q'k\}$; therefore A preserves the angle between complex lines contained in the *same* right quaternionic line, so it preserves the Fubini-Study metric on the fibres of the twistor fibration. Since every biholomorphism of $P^3_{\mathbb{C}}$ is a projective linear transformation, we arrive to the following theorem:

2.1 Theorem. *Let $h \in \mathit{M\ddot{o}b}(2,\mathcal{H})$. Then h lifts canonically to an automorphism $\widetilde{h}$ of $P^3_{\mathbb{C}}$. This lifting preserves the Calabi-Penrose fibration and it is an isometry on each fibre $P^1_{\mathbb{C}}$, with respect to the Fubini-Study metric. Furthermore, the map $\phi\colon h \mapsto \widetilde{h}$ injects $\mathit{M\ddot{o}b}(2,\mathcal{H})$ into the complex projective group $PSL(4,\mathbb{C})$.*

Now consider a discrete subgroup Γ of $\mathit{Conf}_+(S^4)$; Γ is said to be *Fuchsian* if it leaves invariant a 3-dimensional round sphere S^3_δ in S^4 where by *round sphere* we mean, a sphere at infinity which is the boundary of a complete totally geodesic subspace of hyperbolic space $\mathbb{H}^n$. Every such group is automatically a Kleinian group in S^4, because its limit set is contained in S^3_δ. The fundamental group of every complete hyperbolic n-manifold with $n < 5$, is a Fuchsian group in S^4, hence also Kleinian, because the canonical inclusion $\mathit{Iso}(\mathbb{H}^n) \hookrightarrow \mathit{Iso}(\mathbb{H}^5)$ leaves invariant a hyperplane in $\mathbb{H}^5$.

2.2 Theorem.

i) Every conformal (discrete, Kleinian) group Γ in S^4 with limit set $\Lambda(\Gamma)$, is canonically a complex (discrete, Kleinian) group $\widetilde{\Gamma} = \phi(\Gamma)$ in $P^3_{\mathbb{C}}$. The limit set $\widetilde{\Lambda} := \Lambda(\widetilde{\Gamma})$ is $p^{-1}(\Lambda) \cong \Lambda \times P^1_{\mathbb{C}}$, where p is the Calabi-Penrose fibration $p\colon P^3_{\mathbb{C}} \to S^4$.

ii) Let Γ be a conformal Kleinian group in S^4, which is the fundamental group of a hyperbolic n-orbifold with $n < 4$, via the inclusion $\mathit{Iso}(\mathbb{H}^n) \hookrightarrow \mathit{Iso}(\mathbb{H}^5)$. Then the action of $\widetilde{\Gamma}$ on $P^3_{\mathbb{C}}$ leaves invariant a proper submanifold of $P^3_{\mathbb{C}}$ and it is not minimal on the limit set.

iii) If Γ is the fundamental group of a hyperbolic orbifold of dimension $n = 4, 5$ and $\Lambda(\Gamma)$ is the whole $S^{n-1} \subset S^4$, then the action of $\widetilde{\Gamma}$ is minimal on its limit set $\widetilde{\Lambda} \cong S^{n-1} \times P^1_{\mathbb{C}}$. Hence the action of $\widetilde{\Gamma}$ on $P^3_{\mathbb{C}}$ is algebraically-mixing, i.e., there is no proper complex submanifold (nor sub-variety) of $P^3_{\mathbb{C}}$ which is $\widetilde{\Gamma}$-invariant.

Statement (ii) is proved by showing that if Γ is a conformal Kleinian group in S^4 that leaves invariant a maximal round sphere $S^2 \subset S^4$, then S^2 lifts to a holomorphic Legendrian curve in $P^3_{\mathbb{C}}$, which is $\widetilde{\Gamma}$-invariant and it is transversal to all the twistor lines that this line meets. We recall that the action of a group acting on a topological space is said to be *minimal* if each orbit is dense.

To prove statement iii) in Theorem 2.2 we need the following theorem, which is of independent interest:

2.3 Theorem. *Let Γ be a discrete subgroup of $Conf_+(S^4)$. Let $\widetilde{\Gamma}$ be the canonical lift of Γ to $P^3_{\mathbb{C}}$ and let $H \subset \Lambda(\widetilde{\Gamma})$ be a non-empty minimal subset for the action of $\widetilde{\Gamma}$. Then:*

i) The restriction of $p : P^3_{\mathbb{C}} \to S^4$ to H is a locally trivial continuous fibre bundle over all of $\Lambda(\Gamma)$.

ii) If $H \neq \Lambda(\widetilde{\Gamma})$, then each fibre H_x of $p|_H$ is either a point or a copy of the round circle S^1, and there exists a $\widetilde{\Gamma}$-invariant continuous section of the bundle $p : P^3_{\mathbb{C}} \to S^4$ over the points in $\Lambda(\Gamma) \subset S^4$.

Proof of 2.3. We first note that, because H is compact, nonempty and the action of Γ on $\Lambda(\Gamma)$ is minimal, H intersects every twistor line over $\Lambda(\Gamma)$. Let $x \in \Lambda(\Gamma)$ and $H_x = H \cap p^{-1}(\{x\})$. Then $\widetilde{\gamma}(H_x) = H_{\gamma(x)}$ for every $\widetilde{\gamma} \in \widetilde{\Gamma}$, because $\widetilde{\Gamma}$ acts minimally on H and it carries twistor lines onto twistor lines . Moreover, the action on the twistor lines is by isometries. Thus, for every $x, y \in \Lambda(\Gamma)$, H_x is isometric to H_y. Also by minimality, if for a sequence $\{x_i\}$ of $\Lambda(\Gamma)$ one had

$$\lim_{i \to \infty} x_i = x, \ \ but \ \lim_{x_i \to x} H_{x_i} \neq H_x,$$

where H_{x_i} converges to F_x in the Hausdorff metric, then $F_x \cup H_x$ would be isometric to H_x, which is not possible. Thus H_x depends continuously on x in the Hausdorff metric of compact subsets of $P^3_{\mathbb{C}}$. Hence, for each $x \in \Lambda(\Gamma)$ there exists an open neighbourhood $U_x \subset \Lambda(\Gamma)$ and a continuous map $\psi : U_x \to SO(3)$, such that if we consider a trivialization of the Calabi-Penrose fibration $p^{-1}(U_x) \cong U_x \times S^2$, then $(y, H_y) = (y, \psi(y)(H_x))$. Thus, we can trivialise $p|_H$ in U_x by the function $(y, w) \mapsto (y, \psi(y)(w))$, $w \in H_x \subset S^2$, from $U_x \times H_x \subset U_x \times S^2$ to ${p|_H}^{-1}(U_x)$. This proves statement i).

Suppose that $H \neq p^{-1}(\Lambda(\Gamma))$, then we also have a fibration $p_1 : P^3_{\mathbb{C}} - H \to \Lambda(\Gamma)$, where the fibres are $p_1^{-1}(\{x\}) = p^{-1}(\{x\}) - H_x := \Sigma_x$, and Σ_x is an open subset of the sphere $H_{\pi^{-1}(\{x\})}$. Thus Σ_x is isometric to Σ_y for all $x \,, y \in \Lambda(\Gamma)$ and $\widetilde{\gamma}$ sends Σ_x isometrically onto $\Sigma_{\gamma(x)}$. Suppose that for a fixed $x \in \Lambda(\Gamma)$ the function $y \mapsto d(y, H_x)$, from Σ_x to $\mathbb{R}$, attains its maximum at a *unique* point z_x, where d denotes the spherical distance in $p^{-1}(\{x\})$. Then, by minimality, the closure of the orbit of z_x under $\widetilde{\Gamma}$ meets every fibre of p_1, and it cannot meet the fibre in more than one point because z_x is the unique point at maximal distance to H_x. Hence the closure of the $\widetilde{\Gamma}$-orbit of z_x is the graph of a continuous section of p_1. The image of this section is a closed set, it is $\widetilde{\Gamma}$-invariant, with a minimal action of $\widetilde{\Gamma}$. Let us now show that, for each $x \in \Lambda(\Gamma)$, H_x is homogeneous. Let $w_1 \,, w_2 \in H_x$. Then there exists a sequence $\{\widetilde{\gamma}_i\}$ in $\widetilde{\Gamma}$ such that $\widetilde{\gamma}_i(w_1)$ converges to w_2, by minimality, and we can obtain a subsequence $\widetilde{\gamma}_{i_j}$ such that the restriction $\widetilde{\gamma}_{i_j}|_{H_x}$ is convergent, because $SO(3)$ is compact. Hence the subgroup of $SO(3)$ that leaves invariant H_x is compact and it acts transitively on H_x. Then the connected component of this group is either trivial and H_x is a section of $p|_{\Lambda(\widetilde{\Gamma})}$, or else it is $SO(2)$ or $SO(3)$. If it is $SO(2)$, then H_x is a round circle and we can apply the previous argument to obtain an invariant section (for instance, we could take the set of points which are centers of one of the discs in which the circle divides the fibre). If this group is $SO(3)$, then $H = \Lambda(\widetilde{\Gamma})$, which is a contradiction. This proves statement ii). $\square$

The proof of (2.3) can easily be adapted to prove the following more general theorem:

Theorem 2.3.1. *Let X and Y be compact metric spaces, and let G be a compact group which acts minimally on X. Let $\pi : E \to X$ be a locally trivial fibre bundle with fibre Y, such that G acts on E as a skew-product, $g(x, y) = (g(x), F_{(g,x)}(y))$, where $F_{(g,x)} : Y \to Y$ is an isometry for* *each $(g,x) \in G \times X$. Let H be a minimal subset of E for the action of G. Then the restriction of π to H, $\pi|_H : H \to X$, is a locally trivial fibre bundle, whose fibres are homogeneous spaces on which G acts transitively.*

Proof of 2.2. We observe first that if $\widetilde{x} \in P^3_{\mathbb{C}}$ is not in $p^{-1}(\Lambda(\Gamma))$, then there is a neighbourhood $\mathcal{U}$ of $x = p(\widetilde{x})$ totally contained in $\Omega(\Gamma)$ which is a wandering domain for the action of Γ on S^4. Hence $\widetilde{\mathcal{U}} := p^{-1}(\mathcal{U})$ is a wandering domain for the action of $\widetilde{\Gamma}$ on $P^3_{\mathbb{C}}$. Therefore $p^{-1}(\Lambda(\Gamma))$ contains $\Lambda(\widetilde{\Gamma})$. To prove that $\Lambda(\widetilde{\Gamma})$ contains $p^{-1}(\Lambda(\Gamma))$, consider a trivialization of $p^{-1}(\Lambda(\Gamma)) \cong \Lambda(\Gamma) \times P^1_{\mathbb{C}}$, and let $\widetilde{z} = (x,w) \in p^{-1}(\Lambda)$ with $x \in \Lambda(\Gamma)$ and $w \in E_x$. Then there exists a sequence of different points of the form $\gamma_i(y)\,, y \in S^4$, which converges to x, and $\widetilde{\gamma}_i(y,w) = (\gamma_i(y), F_{(\gamma_i,y)}(w))$, where $F_{(\gamma_i,y)} \in SO(3)$. Since $SO(3)$ is compact, we can assume that the sequence $F_{(\gamma_i,y)}$ converges in $SO(3)$. Thus the whole fibre E_x is contained in $\Lambda(\widetilde{\Gamma})$. Hence the limit set in $P^3_{\mathbb{C}}$ is as stated and the group $\widetilde{\Gamma}$ is complex Kleinian, proving statement i).

Let us now prove statement ii). We first recall that the bundle normal to the twistor lines in $P^3_{\mathbb{C}}$, with respect to the Fubini-Study metric, is a complex two dimensional (holomorphic) sub-bundle of the tangent bundle of $P^3_{\mathbb{C}}$. This gives a holomorphic contact structure to $P^3_{\mathbb{C}}$ ([Ar or La, p. 204]). We recall that a complex structure on R^4 can be thought of as being a choice of an oriented 2-plane $P \subset R^4$: the orientation determines a complex structure on P, and also an orientation and a complex structure on the orthogonal complement of P. Hence, if $\Sigma \looparrowright S^4$ is an immersed oriented surface in S^4, then Σ can be lifted canonically to $P^3_{\mathbb{C}}$, and by [C1,2] this is a Legendrian (or *horizontal*) surface $\widehat{\Sigma}$ in $P^3_{\mathbb{C}}$, *i.e.,* it is tangent to the contact structure. Moreover, if Σ is the Riemann sphere, then every minimal immersion $\Sigma \looparrowright S^4$ lifts to a holomorphic curve $\widehat{\Sigma}$ in $P^3_{\mathbb{C}}$ (see [Br, p. 466, also C1,2, La]).

Let us now consider a maximal round sphere S^2 in S^4 and consider a conformal Kleinian group Γ on S^4 that leaves invariant this S^2. Then S^2 lifts to a holomorphic curve L in $P^3_{\mathbb{C}}$, which is horizontal. The action of $\widetilde{\Gamma}$ on $P^3_{\mathbb{C}}$ preserves L and it also preserves all lines in the Calabi-Penrose fibration that intersect L. Hence L is a proper complex submanifold of $P^3_{\mathbb{C}}$ which is $\widetilde{\Gamma}$-invariant and the action on $\Lambda(\widetilde{\Gamma})$ is not minimal, because the action on the fibers is by isometries, so the points in $p^{-1}(S^2) - L$ can never accumulate towards L. This proves statement ii).

To prove statement iii) we first observe the standard fact that Zorn's lemma implies that there exists a subset $H \subset \Lambda(\widetilde{\Gamma}) \subset P^3_{\mathbb{C}}$ where the action of $\widetilde{\Gamma}$ is minimal. We claim that one must have $H = \Lambda(\widetilde{\Gamma})$. Suppose $H \neq \Lambda(\widetilde{\Gamma})$ and $n = 4$, so that the limit set $\Lambda(\Gamma)$ is a round 3-sphere $S^3 \subset S^4$. By 2.3.ii, there exists a continuous family of almost complex structures $J_x : T_xS^4 \to T_xS^4$ for all $x \in S^3 \subset S^4$, which is compatible with the metric and the orientation of S^4, and which is Γ-invariant. Consider the associated 2-plane field $\Pi := \{\Pi_x := T_xS^3 \cap J_x(T_xS^3) \subset T_xS^4\,, x \in S^3\}$. This plane field is Γ-invariant. Let $\mathcal{L}$ be the line field tangent to S^3 which is orthogonal to Π, then $\mathcal{L}$ is also Γ-invariant by the conformality of the action, and this is not possible. In fact, following the idea of the proof of Mostow's Rigidity Theorem [Mo], if α is a geodesic whose endpoints are in S^3, then we can use parallel transport along α to transport the line at one end point of α at infinity, to a line at the other end point at infinity. The angle of these two lines is a continuous Γ-invariant function in $S^3 \times S^3$ which must be a constant because, under the hypothesis, Γ acts ergodically on pairs of points in S^3. This is impossible by Theorem 5.9.10 in [Th2], in which Thurston gives a proof of Mostow's Rigidity Theorem [Mo] using the non existence of Γ-invariant measurable line fields. We can also use the following argument: Let $\mathfrak{H}$ be the family of *all* horocycles, of dimension 1, which are contained in $\mathbb{H}^4 \subset \mathbb{H}^5$ and which are tangent at infinity to the line field $\mathcal{L}$. Since $\Gamma = \pi_1(M^4)$, this family $\mathfrak{H}$ determines a proper, closed and invariant subset for a unipotent one-parameter subgroup of $SO(4,1)$ on the unit tangent bundle of M. But this is not possible because every such action is minimal, hence every orbit is dense (see [Da]). Therefore $H = \Lambda(\widetilde{\Gamma})$ and $\widetilde{\Gamma}$ acts minimally on $\Lambda(\widetilde{\Gamma}) = S^3 \times P^1_{\mathbb{C}}$. Hence the action of $\widetilde{\Gamma}$ on $P^3_{\mathbb{C}}$ is algebraically mixing, since any invariant algebraic sub-variety of $P^3_{\mathbb{C}}$ must contain $\Lambda(\widetilde{\Gamma})$, which has real dimension 5, so it must have complex dimension 3. This proves statement iii) when $n = 4$. If $n = 5$ and the action of $\widetilde{\Gamma}$ on $\Lambda(\widetilde{\Gamma})$ were not minimal then, by theorem 2.3, there would exist a section of the Calabi-Penrose fibration over all of S^4. This is impossible since the sphere S^4 does not have an almost complex structure. $\square$

We notice that all the previous arguments could be applied if the limit set of Γ were a smooth compact 3-manifold, but this is useless because of the following:

Remark. *If the limit set of a non-elementary conformal group acting on S^n is a compact smooth k-manifold N, for some $0 < k \leq n$, then N is a round sphere S^k.* The proof, by Livio Flaminio, is a direct consequence, via stereographic projection of S^n into the tangent plane of S^n at a hyperbolic fixed point of the group, of the following fact: if $M \subset \mathbb{R}^n$ is a closed k-submanifold of $\mathbb{R}^n$ which is invariant under a homothetic transformation, then M is a k-dimensional subspace of $\mathbb{R}^n$.

Theorem 2.2 iv) implies the following corollary:

Corollary 2.4. *There exist* **discrete** *subgroups of the projective group $PSL(4,\mathbb{C})$ which act minimally on $P^3_{\mathbb{C}}$. More precisely, let Γ be a discrete subgroup of $Iso_+(\mathbb{H}^5)$ such that $\mathbb{H}^5/\Gamma$ has finite volume. Let $\widetilde{\Gamma}$ be its canonical lifting to $PSL(4,\mathbb{C})$. Then $\widetilde{\Gamma}$ acts minimally on $P^3_{\mathbb{C}}$.*

In fact we will prove in section 4 that these groups also act ergodically on $P^3_{\mathbb{C}}$ with respect to the geometric measure.

2.5 Definition. *A discrete subgroup Γ of $Iso_+(\mathbb{H}^n)$ is geometrically-finite if it has a finite-sided polyhedron as a fundamental domain in H^n (see* [Ra, Ni, FSp]*). Γ is said to be Zariski-dense in $Iso_+(\mathbb{H}^n)$ if its Zariski closure is $Iso_+(\mathbb{H}^n)$ (see* [FSp]*).*

The previous theorem 2.2.iii can be generalised as follows:

2.6 Theorem. *Let Γ be a geometrically-finite discrete subgroup of $Iso_+(\mathbb{H}^m)$, $m = 4, 5$, which is Zariski-dense. Let Λ be its limit set in S^4. Let $\widetilde{\Gamma}$ be the lifting of Γ to $P^3_{\mathbb{C}}$. Then, $\widetilde{\Gamma}$ acts minimally on its limit set $\widetilde{\Lambda} = \Lambda \times P^1_{\mathbb{C}} \subset P^3_{\mathbb{C}}$.*

Theorem 2.6 implies that if we consider a hyperbolic Schottky group acting on $\mathbb{H}^5$, whose Cantor limit set is not contained in any round sphere of dimension less than four, then its twistorial lifting acts minimally on its limit set. This is a question that C. Series asked us, motivating Theorem 2.6 and the equivalent statement in (4.3) below. To prove (2.6) we use the following theorem by L. Flaminio and R. Spatzier (Theorem 1.3 in [FSp]). For this we recall [Pa, Su2, FSp] that a finite measure μ on $\Lambda := \Lambda(\Gamma)$ is called *geometric*, or *the Patterson-Sullivan measure*, if for all $\gamma \in \Gamma$, $\gamma^*(\mu) = |\gamma\prime|^\delta \mu$, where δ is the Hausdorff dimension of Λ and $\gamma^*(\mu)(A) = \mu(\gamma(A))$. The support of μ is Λ. If Γ is geometrically finite then μ is unique up to scaling. In particular, if Λ is the whole sphere at infinity, then μ is the Lebesgue measure on the sphere, up to scaling.

2.7 Theorem [FSp]. *Let Γ be a geometrically-finite discrete subgroup of $Iso_+(\mathbb{H}^n)$ which is Zariski-dense. Let μ be the Patterson-Sullivan measure on the limit set Λ of Γ. Then, every Γ-invariant measurable distribution on $\Lambda \subset S^{n-1}$ by subspaces of dimension d is μ-almost everywhere trivial, i.e., either $d = n-1$ or $d = 0$.*

Proof of 2.6. Suppose the action is not minimal. Then, by (2.3), there exists a continuous invariant section of the Calabi-Penrose fibration restricted to Λ. This section is therefore a Γ-invariant continuous family of almost complex structures $\{J_x\}_{x\in\Lambda}$. If $m = 5$, let E_x be the subspace of dimension 2 of T_xS^4 which is the eigenspace corresponding to the eigenvalue i of J_x. Then, the family $\{E_x\}_{x\in S^4}$ is a 2-dimensional Γ-invariant distribution. This contradicts theorem 2.7. If $m = 4$, for each $x \in \Lambda \subset S^3 \subset S^4$, let E_x be the 2-plane $E_x = T_x(S^3) \cap J_x(T_x(S^3))$. Then, the family $\{E_x\}_{x\in S^3}$ is a 2-dimensional Γ-invariant distribution. This contradicts theorem 2.7. $\square$

The results of M. Kapovich and L. Potyagailo [KP] show that Ahlfors' Finiteness Theorem and Sullivan's Finite Number of Cusps Theorem fail for conformal groups in S^3. More precisely, there exist finitely generated conformal Kleinian groups Γ on all S^n, $n > 2$, whose region of discontinuity $\Omega(\Gamma)$ contains infinitely many connected components which are not Γ-equivalent. Also, there exist finitely generated conformal Kleinian groups in all spheres S^n, $n > 2$, having infinitely many non-equivalent cusps. Hence, by (2.2.i) and the cone construction, we obtain:

2.8 Corollary. *Ahlfors' Finiteness Theorem fails for complex Kleinian groups acting on* $P^n_{\mathbb{C}}$, *for all* $n > 1$.

Similarly, Sullivan's Finite Number of Cusps Theorem also fails for complex Kleinian groups in $P^n_{\mathbb{C}}$, but one has to make precise what a "cusp" means in this context.

Proof. Everything has been shown except for the fact that one can find an example $\Gamma \subset M\ddot{o}b(2,\mathcal{H})$ like the ones of Kapovich and Potyagailo in which Γ lifts to $SL(4,\mathbb{C})$. To show this we recall that, by (1.3), there exists an exact sequence

$$1 \to \mathbb{Z}_2 \to SL(2,\mathcal{H}) \xrightarrow{\mathcal{P}} M\ddot{o}b(2,\mathcal{H}) \to 1.$$

Now let $\Gamma \subset M\ddot{o}b(2,\mathcal{H})$ be a conformal Kleinian group for which either Ahlfors' Finiteness Theorem (or Sullivan's Finite Number of Cusps) fails. Such examples can be chosen to be the fundamental group of a hyperbolic 4-manifold M for which $H^1(M,\mathbb{Z}) \neq 0$ (Misha Kapovich, personal communication). Therefore M admits finite coverings of every order. The obstruction to lift Γ to $SL(2,\mathcal{H}) \subset SL(4,\mathbb{C})$ is an element $\omega \in H^2(\Gamma,\mathbb{Z}_2)$. Hence this obstruction vanishes if we take a double covering $\widetilde{M}$, of M. The corresponding subgroup $\widehat{\Gamma}$, of index-two in Γ, lifts to $SL(4,\mathbb{C})$, so that Ahlfors' Finiteness Theorem (or Sullivan's Finite Number of Ends) fail for $\widehat{\Gamma}$. The same argument shows, after passing to finite coverings, that we can iterate the process to find examples in every dimension, since M admits finite coverings of every order and the obstruction lies in $H^2(\widetilde{\Gamma}',\mathbb{Z}_n)$, where $\widetilde{\Gamma}'$ is a lifting of $\widetilde{\Gamma}$ to $SL(n-1,\mathbb{C})$, $n > 5$. $\square$

2.9 Remark. The construction in this section is a special case of a more general construction: let $P^n_{\mathcal{H}}$ denote the quaternionic projective n-space consisting of *right* quaternionic lines in $\mathcal{H}^{n+1}$. Then one has the classical fibration $p : P^{2n+1}_{\mathbb{C}} \to P^n_{\mathcal{H}}$ with fibre $P^1_{\mathbb{C}}$. If $SL(n+1,\mathcal{H})$ denotes the group of $n \times n$ matrices with coefficients in $\mathcal{H}$ acting on the *left* on $\mathcal{H}^{n+1}$, then this action can be projectivized to obtain an action on $P^n_{\mathcal{H}}$. Then, $PSL(n+1,\mathcal{H}) = SL(n+1,\mathcal{H})/\{\pm I\}$ is the group of *quaternionic* projective transformations. It sends fibres of p into fibres of p.

3. Kleinian groups and twistor spaces

We consider the unitary group $U(n) := \{A \in GL(n,\mathbb{C}) : A^{-1} = \overline{A^t}\}$; the columns of each such matrix define linearly independent vectors in $\mathbb{C}^n$, so $U(n)$ can be regarded as being the set of all unitary n-frames in $\mathbb{C}^n$. We also consider the special orthogonal group $SO(n)$, which consists of all orthonormal, oriented n-frames in $\mathbb{R}^n$. These are both compact Lie groups of dimensions

n^2 and $n(n-1)/2$, respectively. There are natural inclusions $U(n) \hookrightarrow SO(2n) \hookrightarrow SO(2n+1)$ and $U(n) \hookrightarrow U(n+1)$. This gives a map between symmetric spaces,

$$SO(2n+1)/U(n) \longrightarrow SO(2n+2)/U(n+1) \ , \ n > 0 \,,$$

which is easily seen to be a diffeomorphism. One also has a natural action of $SO(2n+1)$ on the sphere S^{2n}, with isotropy $SO(2n)$, so that $S^{2n} \cong SO(2n+1)/SO(2n)$. This induces a projection map

$$p \colon \mathfrak{Z}(S^{2n}) := SO(2n+1)/U(n) \ \to \ S^{2n} \,,$$

which is a submersion with fibre $\mathfrak{L}_x^{(2n)} := p^{-1}(x) \cong SO(2n)/U(n)$, the set of all complex structures on the tangent space $T_x S^{2n}$ which are compatible with the metric and orientation. For $n > 1$, the fibre $\mathfrak{L}_x^{(2n)}$ coincides with the space $\mathfrak{Z}(S^{2n-2})$.

3.1 Definition. $\mathfrak{Z}(S^{2n})$ is the *twistor space* of S^{2n}, and the map $p \colon \mathfrak{Z}(S^{2n}) \to S^{2n}$ is the *twistor fibration.* The fibres of p are the *twistor fibres.*

This generalises the twistor fibration studied in section 2. This was used by Calabi in [C1,2] to study minimal immersions of spheres (see also [EL, BR, Sa]), and more generally, for every even-dimensional Riemannian manifold, by Penrose [Pe1-3], Atiyah-Hitchin-Singer [AHS, Hi], Dubois-Violette [DV], O'Brien-Rawnsley [OR] and others. We refer to [BR,Sa] for clear accounts on the subject.

3.2 Definition. Let N be an oriented, Riemannian $2n$-manifold, $n > 0$. The *twistor space* of N is the total space of *the twistor fibration,* $p \colon \mathfrak{Z}(N) \to N$, whose fibre at $x \in N$ is the twistor space $SO(2n)/U(n)$ of S^{2n-2}, i.e., the set of all complex structures on $T_x N$ compatible with the metric and the orientation on N.

$\mathfrak{Z} := \mathfrak{Z}(N)$ is always an almost complex manifold. In fact, the Levi-Civita connection ∇ on N gives rise to a splitting $T(\mathfrak{Z}) = H \oplus V$ of the tangent bundle $T(\mathfrak{Z})$, into the horizontal and vertical components, where V is the bundle tangent to the fibres of p and H is isomorphic to the pull back $p^*(TN)$ of TN. Each $\widetilde{x} \in \mathfrak{Z}(N)$ represents a point $x := p(\widetilde{x})$ in N, together with a complex structure on $T_x N$; since at each $\widetilde{x} \in \mathfrak{Z}(N)$ one has $T_{\widetilde{x}}(\mathfrak{Z}) = H_{\widetilde{x}} \oplus V_{\widetilde{x}}$, and $H_{\widetilde{x}}$ is naturally isomorphic to $T_x N$, one has a tautological complex structure on $H_{\widetilde{x}}$. Hence an almost complex structure on the fibre $\mathfrak{Z}(S^{2n-2}) = SO(2n)/U(n)$, determines an almost complex structure $\widetilde{J}$ on $\mathfrak{Z}(N)$, and an almost complex structure on the fibre is easily defined by induction: $\mathfrak{Z}(S^{2n-2})$ fibres over S^{2n-2} with fibre $\mathfrak{Z}(S^{2n-4})$, and so on; at each step $T(\mathfrak{Z}(S^{2i}))$ decomposes as above, into a horizontal and a vertical component, with the horizontal component having a tautological almost complex structure. Hence, the complex structure on $P^1_{\mathbb{C}} = \mathfrak{Z}(S^2)$ determines an almost complex structure on $\mathfrak{Z}(S^4) = P^3_{\mathbb{C}}$ and so on, till we get an almost complex structure $\widetilde{J}$ on $\mathfrak{Z}(N)$. The question of the integrability of $\widetilde{J}$ is very subtle: it is integrable if N is (locally) conformally flat (by [AHS] for $n = 2$, by [DV, OR] for $n > 2$). In fact, this condition is also necessary for $n > 2$, see [Sa, Th.3.3]. Hence, in particular, $\mathfrak{Z}(S^{2n})$ is always a complex manifold with the almost complex structure $\widetilde{J}$. It has complex dimension $n(n+1)/2$.

We summarise the previous discussion in the following well known theorem.

3.3 Theorem. *Let N be a closed, oriented, Riemannian $2n$-manifold, $n > 1$, and let $p \colon \mathfrak{Z}(N) \to N$ be the twistor fibration of N. Then $\mathfrak{Z}(N)$ has a (preferred) almost complex structure $\widetilde{J}$, which is integrable whenever N is conformally flat. The twistor fibration $p \colon \mathfrak{Z}(N) \to N$, is a locally trivial fibre bundle with fibre $\mathfrak{Z}(S^{2n-2}) = SO(2n)/U(n)$. In particular, $\mathfrak{Z}(S^{2n}) \cong SO(2n+1)/U(n) \cong SO(2n+2)/U(n+1)$.*

The Lie group $SO(2n+1)$, being compact, has a canonical bi-invariant metric: the distance between two frames. This descends to a metric on $\mathfrak{Z}(S^{2n})$, which is invariant under the left action of $SO(2n+1)$ and restricts to the corresponding metric on each twistor fibre $\mathfrak{L}_x^{(2n)}$. With this, the projection $p: \mathfrak{Z}(S^{2n}) \to S^{2n}$ becomes a Riemannian submersion. Furthermore, each element $\mathcal{F} \in SO(2n+1)$ can be regarded as being of the form $(x, \mathcal{F}_x^{2n})$, where x is a point in S^{2n} and $\mathcal{F}_x^{2n} = (v_1(x), ..., v_{2n}(x))$ is an orthonormal basis of the tangent space $T_x S^{2n}$. If $\gamma \in Conf_+(S^{2n})$ is a conformal diffeomorphism, then the derivative of $d\gamma$ carries $\mathcal{F}_x^{2n}$ into a basis $\mathcal{F}_{\gamma(x)}^{2n}$ of $T_{\gamma(x)} S^{2n}$, which is orthonormal up to a scalar multiple. Thus γ lifts canonically to a diffeomorphism $\widetilde{\gamma}$ of the $\mathfrak{Z}(S^{2n})$. There is another way of defining this lifting of γ to $\mathfrak{Z}(S^{2n})$: at each point $x \in S^{2n}$, the basis $\mathcal{F}_x^{2n}$ provides an identification $T_x(S^{2n}) \cong \mathbb{C}^n$, so it endows $T_x(S^{2n})$ with a complex structure J_x^1; then $d\gamma$ determines the basis $d(\gamma)_x(\mathcal{F}_x^{2n})$ of $T_{\gamma(x)}(S^{2n})$, hence an isomorphism $T_{\gamma(x)}(S^{2n}) \cong \mathbb{C}^n$ and a complex structure $J_{\gamma(x)}^1$ on $T_{\gamma(x)}(S^{2n})$. This gives the lifting $\widetilde{\gamma}$ of γ. More generally, if N is a closed, oriented Riemannian $2n$-manifold, then its twistor space $\mathfrak{Z}(N)$ has a natural metric g, which turns it into an almost Hermitian manifold (following the notation in [Sa]): this metric is defined locally on $T_{\widetilde{x}}(\mathfrak{Z}(N)) = H_{\widetilde{x}} \oplus V_{\widetilde{x}}$ as the product of the metric on the horizontal subspace $H_{\widetilde{x}}$ and the above metric on the fibre $\mathfrak{Z}(S^{2n-2})$. It is clear that the constructions above can be extended to this more general setting. Hence, whenever we have an orientation preserving conformal automorphism γ of N, we have a canonical lifting of γ to an automorphism $\widetilde{\gamma}$ of $\mathfrak{Z}(N)$ that carries twistor fibres isometrically into twistor fibres. Moreover, it is clear that one has: $(d\widetilde{\gamma})_{\widetilde{x}}\, \widetilde{J}_{\widetilde{x}} = \widetilde{J}_{\widetilde{\gamma}(\widetilde{x})}\, (d\widetilde{\gamma})_{\widetilde{x}}$ for every $\widetilde{x} \in \mathfrak{Z}(N)$, so that $\widetilde{\gamma}$ is, in fact, an "almost-holomorphic" automorphism of $\mathfrak{Z}(N)$, i.e., an automorphism that preserves the almost complex structure. (These maps are called holomorphic in [Sa].) If the almost complex structure on $\mathfrak{Z}(N)$ is integrable, then $\widetilde{\gamma}$ is actually holomorphic.

When $n = 2$ and $N = S^4$ we are in the situation envisaged in section 2. The twistor space is $P^3_{\mathbb{C}}$ with its Fubini-Study metric.

3.4 Theorem. *Let N be as above, a closed, oriented, Riemannian $2n$-manifold, let $p: \mathfrak{Z}(N) \to N$ be its twistor fibration and endow $\mathfrak{Z}(N)$ with the metric g as above, i.e., it is locally the product of the metric on N, lifted to the horizontal distribution given by the Levi-Civita connection on N, by the metric on the fibre induced by the bi-invariant metric on $SO(2n)$. Then:*

i) The group $Conf_+(N)$, of conformal diffeomorphisms of N that preserve the orientation, lifts canonically to a subgroup $\widetilde{Conf_+}(N) \subset Aut_{hol}(\mathfrak{Z}(N))$ of almost-holomorphic transformations of $\mathfrak{Z}(N)$. Moreover, if the almost-complex structure $\widetilde{J}$ on $\mathfrak{Z}(N)$ is integrable, then these transformations are indeed holomorphic.

ii) Each element in $\widetilde{Conf_+}(N)$ carries twistor fibres isometrically into twistor fibres.

iii) If $\Gamma \subset Conf_+(N)$ is a discrete subgroup acting on N with limit set Λ, then its canonical lifting $\widetilde{Conf_+}(N)$ acts on $\mathfrak{Z}(N)$ with limit set $\widetilde{\Lambda} = p^{-1}(\Lambda)$, so $\widetilde{\Lambda}$ is a fibre bundle over Λ with fibre $\mathfrak{Z}(S^{2n-2})$.

The limit set in (3.4) is defined as in (1.6) above.

Proof. We already know that every $\gamma \in Conf_+(N)$ lifts canonically to an element $\widetilde{\gamma} \in Aut_{hol}(\mathfrak{Z}(N))$. So the only thing to prove for statement (i) is that $Conf_+(N)$ lifts to $Aut_{hol}(\mathfrak{Z}(N))$ as a group, i.e., that given any $\gamma_1, \gamma_2 \in Conf_+(N)$, one has $\widetilde{\gamma}_1 \circ \widetilde{\gamma}_2 = \widetilde{\gamma_1 \circ \gamma_2}$, but this is evident because the derivative satisfies the chain rule. We next recall that at each $x \in N$, the derivative $d\gamma_x : T_x N \to T_{\gamma(x)} N$ takes orthonormal framings into orthogonal framings. In other words, dividing $d_{\gamma(x)}$ by some positive real number, we obtain an orthogonal automorphism $T_x N \to T_{\gamma(x)} N$. Hence, given the splitting $T\mathfrak{Z}(N) = V \oplus H$, into the vertical and horizontal components, one has that, for each $\widetilde{x} \in \mathfrak{Z}(N)$, the induced action of the derivative, $d\widetilde{\gamma}(\widetilde{x})|_{V_{\widetilde{x}}} : V_{\widetilde{x}} \to V_{\widetilde{\gamma}(\widetilde{x})}$, is by orthogonal transformations. Therefore statement (ii) in Theorem 3.4 follows from the fact that the metric on the fibres comes from the bi-invariant metric on $SO(2n)$. The proof of (iii) is the same as that of (2.2.i), since $\widetilde{\Gamma}$ acts by isometries on the twistor fibres in $\mathfrak{Z}$, which are copies of the compact manifold $SO(2n)/U(n)$, and the group of isometries of every compact Riemannian manifold is compact (see [Ko], Th. II.1.2.) $\square$

Let us restrict now our attention to the case $N = S^{2n}$.

3.5 Definition. By a *twistorial Kleinian group* we mean a discrete subgroup $\Gamma \subset Aut_{hol}(\mathfrak{Z}(S^{2n}))$ of holomorphic automorphisms, which acts on $\mathfrak{Z}(S^{2n})$ with non-empty region of discontinuity $\Omega(\Gamma)$, where the latter is defined as in (1.6) above.

It follows from (3.4.iii) that if $\Gamma \subset Conf_+(S^{2n})$ is Kleinian, then its lifting $\widetilde{\Gamma}$ to $Aut_{hol}(\mathfrak{Z}(S^{2n}))$ is also Kleinian. We have the following generalisation of (2.2):

3.6 Theorem. *Let $\Gamma \subset Conf_+(S^{2n})$, $n > 1$, be a conformal Kleinian group. We set $\mathfrak{Z} := \mathfrak{Z}(S^{2n})$ and let $\widetilde{\Gamma}$ be the canonical lifting of Γ to $Aut_{hol}(\mathfrak{Z})$, whose limit set we denote by $\widetilde{\Lambda}$. Then one has:*

i) If Γ leaves invariant an m-sphere $S^m \subset S^{2n}$, $m < 2n-1$, then the action of $\widetilde{\Gamma}$ on $\mathfrak{Z}$ leaves invariant a copy of each twistor space $\mathfrak{Z}(S^{2r}) \subset \mathfrak{Z}$ for all $r \geq m/2$, which are all complex (algebraic) submanifolds of $\mathfrak{Z}$. Hence the action of $\widetilde{\Gamma}$ on $\widetilde{\Lambda} \subset \mathfrak{Z}$ is not minimal.

ii) If Γ is a geometrically-finite discrete subgroup of $Iso_+(\mathbb{H}^m)$, $m = 2n$, $2n+1$, which is Zariski-dense, then $\widetilde{\Gamma}$ acts minimally on $\widetilde{\Lambda}$. Hence, there are no proper complex submanifolds (nor subvarieties) of $\mathfrak{Z}$ which are $\widetilde{\Gamma}$-invariant, i.e., the action of $\widetilde{\Gamma}$ on $\mathfrak{Z}$ is algebraically-mixing.

iii) Let Γ be a geometrically-finite discrete subgroup of $Iso_+(\mathbb{H}^{2m+1})$, $m < n-1$, which is Zariski-dense (so $\widetilde{\Gamma}$ leaves invariant $\mathfrak{Z}(S^{2m})$). Then the action of $\widetilde{\Gamma}$ on $\mathfrak{Z}(S^{2m}) \subset \mathfrak{Z}$ has no invariant complex submanifolds, the restriction of the projection $p : \mathfrak{Z} \to S^{2n}$ to $\widetilde{\Lambda}_{2m} := \widetilde{\Lambda} \cap \mathfrak{Z}(S^{2m})$ is a fibre bundle over $\Lambda(\Gamma)$, with fibre $\mathfrak{Z}(S^{2m-2})$, and the action of $\widetilde{\Gamma}$ on $\widetilde{\Lambda}_{2m}$ is minimal.

Proof. If Γ leaves invariant an m-sphere S^m, then it leaves invariant, via the inclusion, a sequence of spheres $S^m \subset S^{m+1} \subset \cdots \subset S^{2n}$. Hence, for every sphere S^{2r} in this sequence, Γ takes almost complex structures on S^{2r} into almost complex structures on S^{2r}, so $\widetilde{\Gamma}$ preserves $\mathfrak{Z}(S^{2r}) \subset \mathfrak{Z}$. Since $\widetilde{\Gamma}$ takes twistor fibres isometrically into themselves, preserving $\mathfrak{Z}(S^{2r})$, one has that the action of $\widetilde{\Gamma}$ on $\widetilde{\Lambda}$ is not minimal, because the orbits cannot get too close to $\widetilde{\Lambda} \cap \mathfrak{Z}(S^{2r})$. This proves statement i).

To prove statement ii) we need the following generalization of Theorem 1.3 in [FSp].

3.7 Theorem. *Let Γ be a geometrically-finite and Zariski-dense discrete subgroup of $Iso(\mathbb{H}^{n+1})$. Let μ be the Patterson-Sullivan measure on the limit set Λ. Let $0 < p < n$. Let $G_{\Lambda,p}$ be the restriction to Λ of the Grassmannian fibre bundle, $G_{n,p}(S^n)$, of p-dimensional subspaces of TS^n . Then Γ acts, via the differential, minimally on $G_{\Lambda,p}$. Furthermore, Γ acts ergodically on $G_{\Lambda,p}$ with respect to the measure $\widetilde{\mu}$ which is (locally) the product of μ with the homogeneous measure on $G_{n,p} = SO(n)/(SO(p) \times SO(n-p))$.*

Proof. We prove first the ergodicity of the action. Notice that locally one has a product structure: given any point $x \in \Lambda$, there is an open neighbourhood U_x of x in S^n such that, $G_{\Lambda,p}|_{U_x} \cong (\Lambda \cap U_x) \times G_{n,p} \cong (\Lambda \cap U_x) \times (SO(n)/(SO(p) \times SO(n-p)))$. Furthermore, the action of Γ on $G_{\Lambda,p}$ sends a fibre isometrically onto a fibre, since Γ preserves angles. The action of Γ on $G_{\Lambda,p}$ is a factor of the action of Γ on $\mathcal{F}_{\Lambda,n}$, the restriction to Λ of the bundle of orthonormal n-frames on S^n, which locally is $U_x \times SO(n)$. This action is ergodic with respect to $\widetilde{\mu}$ by [FSp] (see appendix, section 9). Hence the action on $G_{\Lambda,p}$ is also ergodic.

Let us now show minimality. By theorem 2.3.1, if the action on $G_{\Lambda,p}$ is not minimal, then there exists a minimal set $F \subset G_{\Lambda,p}$ which is a fiber bundle over Λ and whose fibre F_x, at $x \in \Lambda$, is a proper submanifold of the fibre $\{x\} \times G_{n,p}$. All submanifolds F_x are isometric. Consider a small tubular neighbourhood $\mathcal{U}_x$ of F_x in the fibre $\{x\} \times G_{n,p}$ consisting of points at a distance less than $\epsilon > 0$ of F_x. Then, for ϵ small, the union $\mathcal{U} := \cup_{x\in\Lambda}\mathcal{U}_x$ is a measurable set of positive measure which is Γ-invariant and whose measure varies with ϵ. This contradicts ergodicity, so the action is minimal. $\square$

Let us now prove the first statement in (3.6.ii). As in section 2, there exists necessarily a compact set $H \subset \widetilde{\Lambda} \subset \mathfrak{Z}$, where $\widetilde{\Gamma}$ acts minimally. If $H = \widetilde{\Lambda}$, then there is nothing to prove. Assume $H \neq \widetilde{\Lambda}$, then by (2.3.1) we know that $p|_H : H \to \Lambda$ is a fibre bundle. The set H is a closed subset of the set of pairs (x, J), where $x \in S^{m-1} \subset S^{2n}$ and J is an almost complex structure on T_xS^{2n} compatible with the metric and orientation. If $m = 2n$, then H determines a closed family $\mathfrak{F}$ of hyperplanes of dimension $2n-2$ tangent to S^{2n-1}, in the same way as in (2.2.iii): $\mathfrak{F} := T_xS^{2n-1} \cap J(T_xS^{2n-1})$, $(x, J) \in H$. This contradicts Theorem 3.7, so the action is minimal on its limit set when $m = 2n$. If $m = 2n+1$, we consider the families of n-planes: $\Pi^{\pm i} := \cup_{x\in\Lambda}\Pi_x^{\pm i}$, where $\Pi_x^{\pm i}$ is the eigenspace in $T_x(S^{2n})$ corresponding to the multiplication by $\pm i$ given by the corresponding complex structure. These are disjoint, Γ-invariant families of n-planes over Λ, contradicting (3.7). Hence the action of $\widetilde{\Gamma}$ is minimal on its limit set. The second statement in (ii) now follows easily: the minimality of the action implies that any invariant complex submanifold (or subvariety) of $\mathfrak{Z}$ must have the same dimension as $\mathfrak{Z}$, so it must be all of $\mathfrak{Z}$. Statement (iii) is an easy combination of Theorem 2.3.1 with statements (i) and (ii), so we leave the proof to the reader. $\square$

We know (see for instance [BR]) that the twistor space embeds in a projective space $P_{\mathbb{C}}^N$, for some N. This can be proved in the usual way: showing that there exists a holomorphic line bundle $\mathcal{L}$ over $\mathfrak{Z}$ with "enough" sections, which provide a projective embedding of $\mathfrak{Z}$. However, in order to state our next result, we shall give a more precise description of such an embedding, following [In]. For this we first recall some facts about the spin representation. We refer to [ABS, LM or Gi, Ch.3] for details. If V is a real vector space of dimension m, with the usual quadratic form q, then its Clifford Algebra $\mathcal{C}(V)$ is the quotient $\mathcal{C}(V) := \bigotimes^r T^*(V)/I$, of the complete tensor algebra of V by the ideal generated by elements of the form $(e * e + q(e)\cdot 1)$. As a vector space, $\mathcal{C}(V)$ has dimension 2^m and it is isomorphic to the exterior algebra of V (see [LM] for a nice description of this isomorphism). For $m = 2n$ even, the group $Spin(2n)$ is defined to be the multiplicative subgroup of $\mathcal{C}(V)$ consisting of all the elements that can be expressed in the form $v_1 * ... * v_{2r}$, where each v_i is a vector in V of unit length. $Spin(2n)$ acts orthogonally on V, so there is a canonical surjective homomorphism $Spin(2n) \to SO(2n)$, whose kernel is the centre of $Spin(2n)$, which consists of ± 1. Hence, for all $n > 1$, $Spin(2n)$ is simply connected and it is the universal cover of $SO(2n)$. This group acts linearly on $\mathcal{C}(V)$, so it also acts on the complexification $\mathcal{C}_{\mathbb{C}}(V) = \mathcal{C}(V) \otimes \mathbb{C}$, which is a complex representation space for the spin group, of complex dimension 2^{2n}. As a left module, $\mathcal{C}_{\mathbb{C}}(V)$ splits as the direct sum of 2^n copies of a left module Δ of dimension 2^n, which is the *the spin representation* of $Spin(2n)$, by definition. This is, in fact, a complex representation space for the whole Clifford algebra $\mathcal{C}(V)$, and it is its unique irreducible complex representation, up to equivalence. However, as a representation space of the spin group, this is still reducible: Δ splits as the direct sum of two irreducible, non-equivalent representations $\Delta^{\pm}$ of dimension 2^{n-1}, called the *(positive and negative) half-spin representations.* Let $P_{\mathbb{C}}(\Delta^+) \cong P_{\mathbb{C}}^{2^{n-1}-1}$ be the projectivization of the positive half-spin representation space Δ^+. Then $Spin(2n)$ acts on $P_{\mathbb{C}}(\Delta^+)$ inducing an action of $SO(2n)$, whose isotropy group at a preferred point $\theta_{\varnothing}$ is $U(n)$. This gives an $SO(2n)$-equivariant embedding of $\mathfrak{Z}(S^{2n-2}) = SO(2n)/U(n)$ in the projective space $P_{\mathbb{C}}(\Delta^+) \cong P_{\mathbb{C}}^{2^{n-1}-1}$, see [In], pages 108 and 114. Furthermore, from [In], Th. 3.7, we know that this is the projective embedding of $\mathfrak{Z}(S^{2n-2})$ of smallest codimension.

It is clear that the first Betti number of $\mathfrak{Z}(S^{2n})$ is 0 and $H^2(\mathfrak{Z}(S^{2n}); \mathbb{Z}) \cong \mathbb{Z}$. Hence, by [Bl] or [Ko], Ch. III-9, given the above embedding $\mathfrak{Z}(S^{2n}) \hookrightarrow P_{\mathbb{C}}^{2^n-1}$, every holomorphic automorphism of $\mathfrak{Z}(S^{2n})$ extends to this projective space and, moreover, the group $Aut_{hol}(\mathfrak{Z}(S^{2n}))$ can be identified uniquely with the group of holomorphic automorphisms of $P_{\mathbb{C}}^{2^n-1}$ that preserve $\mathfrak{Z}(S^{2n})$. Thus we arrive to the following theorem:

3.8 Theorem. *Let Γ be a conformal Kleinian group on S^{2n}. Then Γ is a complex Kleinian group in $P_{\mathbb{C}}^{2^n-1}$. More precisely, Γ lifts canonically to a Kleinian group $\widetilde{\Gamma}$ on the twistor space $\mathfrak{Z}(S^{2n})$ and, given the natural embedding $\mathfrak{Z}(S^{2n}) \hookrightarrow P_{\mathbb{C}}^{2^n-1}$ via the spin representation, $\widetilde{\Gamma}$ extends uniquely to a complex Kleinian group in $P_{\mathbb{C}}^{2^n-1}$.*

We remark that the only Riemannian manifold which is not a sphere and whose twistor space is Kähler is $P_{\mathbb{C}}^2$, by [Hi, Sl], whose twistor space is the manifold $F_3(\mathbb{C})$ of flags in $\mathbb{C}^3$. The above discussion applies also in this case; however, the group $Conf_+(P_{\mathbb{C}}^2)$ is $PU(3)$, which is compact, hence every discrete subgroup of this group is finite.

4. Patterson-Sullivan measures on twistor spaces

Each Riemannian metric g on S^{2n} defines canonically a Riemannian metric $\widetilde{g}$ on $\mathfrak{Z}(S^{2n})$ via the twistor fibration $p : \mathfrak{Z}(S^{2n}) \to S^{2n}$: $\widetilde{g}$ is the unique metric for which the differential of p, (dp), is an isometry in each horizontal plane H_x and which coincides with the metric on each twistor fibre inherited from the bi-invariant metric on $SO(2n)$. Two conformally equivalent Riemannian metrics on S^{2n} lift to two Riemannian metrics on $\mathfrak{Z}(S^{2n})$ which are *horizontally conformal, i.e.,* they coincide in the twistor fibres and differ by a conformal factor in the horizontal distribution. A similar remark holds for other $2n$-dimensional Riemannian manifolds, and also for measures.

Let Γ be a *geometrically-finite* Kleinian group on $\mathbb{H}^{2n+1}$, $n > 1$. Let $y \in \mathbb{H}^{2n+1}$ and let μ_y be the Patterson-Sullivan measure on the sphere at infinity S^{2n} obtained from the orbit of y ([Pa, Su2,3, Ni]). Let $\widetilde{\Gamma}$ be the lifted group acting on $\mathfrak{Z}(S^{2n})$. For each $y \in \mathbb{H}^{2n+1}$, let $\widetilde{\mu}_y$ be the measure on $\mathfrak{Z}(S^{2n})$, supported in $\Lambda(\widetilde{\Gamma})$, which is the product of μ_y on S^{2n} and the measure on the twistor fibres determined by the metric. This is well defined and the family $\{\widetilde{\mu}_y\}$ is a *horizontally conformal density* in $\Lambda(\widetilde{\Gamma})$ of exponent δ, where $\delta := \delta(\Lambda(\Gamma))$ is the Hausdorff dimension of $\Lambda(\Gamma)$ (see [Ni] for the definition of conformal densities). These measures are all proportional for $y \in \mathbb{H}^{2n+1}$. Moreover, since the limit set of $\widetilde{\Gamma}$ is the cartesian product of $\Lambda(\Gamma)$ and a manifold of dimension $(n^2 - n)$, the Theorem 2 of [BM] (see also [Mr]) says that the Hausdorff dimension of $\Lambda(\widetilde{\Gamma})$ is $\delta(\Lambda(\widetilde{\Gamma})) = \delta(\Lambda(\Gamma)) + n^2 - n$. Thus one can apply known results of discrete hyperbolic groups to obtain results about the Hausdorff dimension of $\widetilde{\Gamma}$. In particular, by Theorem D in [Tu] (c.f., [Su3]) one has the following theorem, which will be used in section 7:

4.1 Theorem. *Let Γ be a geometrically-finite conformal Kleinian group on $\mathbb{H}^{2n+1}$, with $n > 1$. Let $\widetilde{\Gamma}$ be its lifting to $\mathfrak{Z}(S^{2n})$. Then $\delta(\Lambda(\widetilde{\Gamma})) = \delta(\Lambda(\Gamma)) + (n^2 - n) < n^2 + n$.*

Also, by results of R. Bowen [Bo] and D. Ruelle [Ru], we obtain:

4.2 Theorem. *Let $\{\Gamma_t\}$ be an analytic family of conformal Kleinian groups acting on $\mathbb{H}^{2n+1}$, which are geometrically-finite and without parabolic elements. Let $\widetilde{\Gamma}_t$ be their liftings to $\mathfrak{Z}(S^{2n})$. Then $\delta(t) := \delta(\Lambda(\widetilde{\Gamma}_t))$ is a real analytic function of t.*

It would be interesting to find conditions under which this theorem holds for general complex Kleinian groups on $P_{\mathbb{C}}^N$. We now recall: **i)** if Γ is a subgroup of $Iso(\mathbb{H}^m)$, $m \leq 2n+1$, then Γ is a subgroup of $Iso(\mathbb{H}^{2n+1})$ via the inclusion $Iso(\mathbb{H}^m) \hookrightarrow Iso(\mathbb{H}^{2n+1})$. **ii)** The Patterson-Sullivan density $\{\mu_y\}$, $y \in \mathbb{H}^{2n+1}$, associated to $\Gamma \subset Iso_+(\mathbb{H}^{2n+1})$ is *ergodic* if for any Γ-invariant Borel subset A, either $\mu_y(A) = 0$ or $\mu_y(-A) = 0$, where $-A := \Lambda(\Gamma) - A$. **iii)** If a discrete subgroup $\Gamma \subset Iso(\mathbb{H}^{2n+1})$ is geometrically-finite, then the densities $\{\mu_y\}$, $y \in \mathbb{H}^{2n+1}$, are all proportional, and so are their liftings $\{\widetilde{\mu}_y\}$ to $\mathfrak{Z}(S^{2n})$. Hence, ergodicity for a fixed μ_y implies ergodicity for all μ_y, $y \in \mathbb{H}^{2n+1}$.

4.3 Theorem. *Let $\widetilde{\Gamma}$ be a group of holomorphic transformations of $\mathfrak{Z} := \mathfrak{Z}(S^{2n})$ which is the lifting of a geometrically-finite discrete subgroup $\Gamma \subset Conf_+(S^{2n})$, $n > 1$. Assume Γ is indeed contained in $Iso_+(\mathbb{H}^{m+1}) = Conf_+(S^m) \subset Conf_+(S^{2n})$, for some $m \leq 2n$. Let $\widetilde{\Lambda} := \Lambda(\widetilde{\Gamma})$ be the limit set of $\widetilde{\Gamma}$.*

i) If $m < 2n-1$, then the action of $\widetilde{\Gamma}$ on $\widetilde{\Lambda}$ is not ergodic with respect to the measures $\widetilde{\mu}_y$, $y \in \mathbb{H}^{m+1}$.

ii) If m is either $2n-1$ or $2n$ and $\Lambda(\Gamma) = S^{m-1}$, then the action of $\widetilde{\Gamma}$ on $\widetilde{\Lambda}$ is ergodic with respect to the measures $\widetilde{\mu}_y$, $y \in \mathbb{H}^{m+1}$.

iii) In fact, if m is either $2n-1$ or $2n$ and Γ is Zariski-dense in $Iso_+(\mathbb{H}^{m+1})$, then $\widetilde{\mu}$ is ergodic, $y \in \mathbb{H}^{m+1}$.

iv) Let $m = 2r < 2n-1$, so that (by 3.6) one has a $\widetilde{\Gamma}$-invariant twistor space $\mathfrak{Z}(S^{2r})$ in $\mathfrak{Z}(S^{2n})$, whose intersection with $\widetilde{\Lambda}$ is a fibre bundle over Λ with fibre $\mathfrak{Z}(S^{2r-2})$. If Γ is Zariski-dense in $Iso_+(\mathbb{H}^{m+1})$, then the action on $\widetilde{\Lambda} \cap \mathfrak{Z}(S^{2r})$ is ergodic for $\widetilde{\mu}_y$, $y \in \mathbb{H}^{m+1}$.

Notice that statement iii) implies statement ii), so we only prove statement iii). We also notice that, by [Su2,3], the action of $\Gamma \subset Conf_+(S^m)$ on its limit set $\Lambda \subset S^m$ is ergodic with respect to the Patterson-Sullivan densities. If $\Lambda = S^m$, these measures are constant multiples of the Lebesgue measure on S^m.

Proof. Assume $m < 2n-1$ and suppose $m = 2r$ is even. Then, by (3.6), one has a $\widetilde{\Gamma}$-invariant twistor space $\mathfrak{Z}(S^{2r})$ in $\mathfrak{Z}(S^{2n})$, whose intersection with $\widetilde{\Lambda}$ is a fibre bundle over Λ with fibre $\mathfrak{Z}(S^{2r-2})$. Furthermore, by (3.4), $\widetilde{\Gamma}$ takes twistor fibres isometrically into twistor fibres. This implies that for every $\varepsilon > 0$, the ε-neighbourhood of $\mathfrak{Z}(S^{2r}) \cap \widetilde{\Gamma}$ in $\widetilde{\Gamma}$, is an invariant set of positive μ_y-measure, whose complement in $\widetilde{\Gamma}$ has also positive measure if ε is small. Hence these measures are not ergodic, proving i) when m is even. If $m < 2n-1$ is odd, then $m+1$ is even and $m+1 < 2n-1$, so we can apply the above arguments taking the inclusion $Conf_+(S^m) \hookrightarrow Conf_+(S^{m+1})$, thus proving i).

Now let $m = 2n-1$. Suppose there exists $A \subset \Lambda(\widetilde{\Gamma}) \subset \mathfrak{Z}(S^{2n})$ which is a $\widetilde{\Gamma}$-invariant Borel subset such that $\widetilde{\mu}_y(A) \neq 0 \neq \widetilde{\mu}_y(-A)$, where $-A$ is the complement of A in $\Lambda(\widetilde{\Gamma})$. The set $p(A) \subset \Lambda(\Gamma) \subset S^{2n-1} \subset S^{2n}$ is Γ-invariant. Since the measure μ_y is ergodic, by [Su2], then either $\mu_y(p(A)) = 0$ or $\mu_y(-p(A)) = 0$, where $-p(A) := S^{2n-1} - p(A)$. We can assume $\mu_y(-p(A)) = 0$, so that $p(A)$ has full measure, $\mu_y(p(A)) = 1$. Then, by Fubini's theorem applied to the fibration p, the set $p^{-1}(p(A))$ has full measure in $\Lambda(\widetilde{\Gamma})$. The set $p^{-1}(p(A))$ consists of A and $B = p^{-1}(p(A)) \cap (-A)$, which are disjoint sets of, necessarily, positive measure.

The limit set $\widetilde{\Lambda} \cong p^{-1}(\Lambda)$ is the set of all almost complex structures compatible with the orientation and the canonical metric of T_xS^{2n} for $x \in \Lambda \subset S^{2n-1}$. An almost complex structure J_x of T_xS^{2n}, at a point $x \in \Lambda \subset S^{2n-1}$, determines the oriented $(2n-2)$-plane $\mathcal{P}_x := T_xS^{2n-1} \cap J_x(T_xS^{2n-1})$, tangent to S^{2n-1} at $x \in \Lambda$. Let $\mathcal{L}_x$ be the line in T_xS^{2n-1} orthogonal to $\mathcal{P}_x$; this line determines the family $\mathcal{H}_x$, consisting of all horocycles in $\mathbb{H}^{2n}$ which are tangent to $\mathcal{L}_x$.

Let $\mathfrak{H}$ be the space of *all* one dimensional horocycles in $\mathbb{H}^{2n-1}$. It is clear that the group $Iso_+(\mathbb{H}^{2n}) \cong Conf_+(S^{2n-1})$ acts transitively on $\mathfrak{H}$. So $\mathfrak{H}$ is a homogeneous space with a unique invariant measure class, which is clearly ergodic, because the action of $Iso_+(\mathbb{H}^{2n})$ is transitive. Therefore the restriction of this action to Γ also acts ergodically on $\mathfrak{H}$, by Moore's Ergodicity Theorem [Zi, Th. 2.2.6].

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be the subsets of $\mathfrak{H}$ consisting of *all* horocycles in $\mathbb{H}^{2n} \subset \mathbb{H}^{2n+1}$ which are tangent to the lines determined by the points of $x \in \Lambda$ which are in A and B, respectively. Then $\mathcal{H}_A$ and $\mathcal{H}_B$ are two disjoint Borel subsets of $\mathfrak{H}$ which are Γ-invariant and of positive measure, because $p(A) = p(B)$ has full measure in S^{2n-1}. This is a contradiction and the statement ii) is proven when $m = 2n - 1$. Notice that this would also contradict Theorem 3.7 above.

Let now $m = 2n$. Then the limit set of Γ is contained in the sphere S^{2n}. Suppose that the action of $\widetilde{\Gamma}$ is not ergodic. Then, as before, there exists a $\widetilde{\Gamma}$-invariant open set $A \subset \mathfrak{Z}$ such that both A and $-A$ have positive Lebesgue measure. If $z \in A$, then z corresponds to an almost complex structure J_z at the tangent space, $T_x(S^{2n})$, of the point $x := p(z)$. The tangent space decomposes as the direct sum: $T_x(S^{2n}) = E_z^1 \oplus E_z^2$, where E_z^1 and E_z^2 are the eigenspaces which correspond to the eigenvalues i and $-i$, respectively. Now we use the same argument as before: the set of horocycles which are tangent to the family $\{E_z^1\}_{z \in \mathcal{U}}$ is a $\widetilde{\Gamma}$-invariant set in the space of all horocycles which has positive measure and whose complement has also positive measure. This contradicts both, Moore's ergodicity theorem and Theorem 3.7 above. This completes the proof of statement iii).

The proof of statement iv) is immediate from the above discussion. $\square$

5. Complex Schottky groups

Now we want to generalise, to the holomorphic case, the construction of conformal Schottky groups. We recall that in the conformal case, the Schottky groups are obtained by considering pairwise disjoint $(n-1)$-spheres $\mathbb{S}_1, ..., \mathbb{S}_r$ in S^n. Each sphere $\mathbb{S}_i$ plays the role of a mirror: it divides S^n in two diffeomorphic components, and one has an involution T_i of S^n interchanging these components, the inversion on $\mathbb{S}_i$. The Schottky group is defined to be the group of conformal transformations generated by these involutions. We are going to make a similar construction on $P_{\mathbb{C}}^{2n+1}$, $n > 0$. (For $n = 0$, if we take $P_{\mathbb{C}}^0$ to be a point, this construction gives the classical Schottky groups.)

Consider the subspaces of $\mathbb{C}^{2n+2} = \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ defined by $\widehat{L_0} := \{(a, 0) \in \mathbb{C}^{2n+2}\}$ and $\widehat{M_0} := \{(0, b) \in \mathbb{C}^{2n+2}\}$. Let $\widehat{S}$ be the involution of $\mathbb{C}^{2n+2}$ defined by $\widehat{S}(a, b) = (b, a)$. This interchanges $\widehat{L_0}$ and $\widehat{M_0}$.

5.1 Lemma. *Let $\Phi\colon \mathbb{C}^{2n+2} \to \mathbb{R}$ be given by $\Phi(a, b) = |a|^2 - |b|^2$. Then:*

i) $\widehat{E}_{\widehat{S}} := \Phi^{-1}(0)$ is a real algebraic hypersurface in $\mathbb{C}^{2n+2}$ with an isolated singularity at the origin 0. It is embedded in $\mathbb{C}^{2n+2}$ as a (real) cone over $S^{2n+1} \times S^{2n+1}$, with vertex at $0 \in \mathbb{C}^{2n+2}$.

ii) $\widehat{E}_{\widehat{S}}$ is invariant under multiplication by $\lambda \in \mathbb{C}$, so it is in fact a complex cone. $\widehat{E}_{\widehat{S}}$ separates $\mathbb{C}^{2n+2} - \{(0, 0)\}$ in two diffeomorphic connected components U and V, which contain respectively $\widehat{L_0} - \{(0, 0)\}$ and $\widehat{M_0} - \{(0, 0)\}$. These two components are interchanged by the involution $\widehat{S}$, for which $\widehat{E}_{\widehat{S}}$ is an invariant set.

iii) Every linear subspace $\widehat{K}$ of $\mathbb{C}^{2n+2}$ of dimension $n+2$ containing $\widehat{L_0}$ meets transversally $\widehat{E}_{\widehat{S}}$ and $\widehat{M_0}$. Therefore a tubular neighbourhood V of $\widehat{M_0} - \{(0, 0)\}$ in $P_{\mathbb{C}}^{2n+1}$ is obtained, whose normal disc fibres are of the form $\widehat{K} \cap V$, with $\widehat{K}$ as above.

Proof. The first statement is clear because Φ is a quadratic form with $0 \in \mathbb{C}^{2n+2}$ as unique critical point. Clearly $\widehat{E}_{\widehat{S}}$ is invariant under multiplication by complex numbers, so it is a complex cone. That $\widehat{E}_{\widehat{S}} \cap S^{4n+3} = S^{2n+1} \times S^{2n+1} \subset \mathbb{C}^{2n+2}$, is because this intersection consists of all pairs (x, y) so that $|x| = |y| = \frac{1}{\sqrt{2}}$. That $\widehat{S}$ leaves $\widehat{E}_{\widehat{S}}$ invariant is obvious, and so is that $\widehat{S}$ interchanges the two components of $\mathbb{C}^{2n+2} - \{(0, 0)\}$ determined by $\widehat{E}_{\widehat{S}}$, which must be diffeomorphic because $\widehat{S}$ is an automorphism. Finally, if $\widehat{K}$ is a subspace as in the statement (iii), then $\widehat{K}$ meets transversally $\widehat{E}_{\widehat{S}}$, because through every point in $\widehat{E}_{\widehat{S}}$ there exists an affine line in $\widehat{K}$ which is transverse to $\widehat{E}_{\widehat{S}}$. □

Let S be the linear projective involution of $P_{\mathbb{C}}^{2n+1}$ defined by $\widehat{S}$. Since $\widehat{E}_{\widehat{S}}$ is a complex cone, it projects to a codimension 1 real submanifold of $P_{\mathbb{C}}^{2n+1}$, that we denote by E_S .

Definition. *We call* E_S **the canonical mirror** *and* S **the canonical involution**.

5.2 Corollary.
i) E_S *is an invariant set of* S.
ii) E_S *is a* S^{2n+1}*-bundle over* $P_{\mathbb{C}}^n$*, in fact* E_S *is the sphere bundle associated to the holomorphic bundle* $(n+1)\mathcal{O}_{P_{\mathbb{C}}^n}$*, which is the normal bundle of* $P_{\mathbb{C}}^n$ *in* $P_{\mathbb{C}}^{2n+1}$.
iii) E_S *separates* $P_{\mathbb{C}}^{2n+1}$ *in two connected components which are interchanged by* S *and each one is diffeomorphic to a tubular neighbourhood of the canonical* $P_{\mathbb{C}}^n$ *in* $P_{\mathbb{C}}^{2n+1}$.

It is an exercise to show that (5.1) holds in the following more generally setting. Of course one has the equivalent of (5.2) too.

5.3 Lemma. *Let* λ *be a positive real number and consider the involution*

$$\widehat{S}_\lambda : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} ,$$

given by $\widehat{S}_\lambda(a,b) = (\lambda b, \lambda^{-1} a)$*. Then* $\widehat{S}_\lambda$ *also interchanges* $\widehat{L}_0$ *and* $\widehat{M}_0$*, and the set*

$$\widehat{E}_\lambda = \{(a,b) : |a|^2 = \lambda^2 |b|^2\}$$

satisfies, with respect to $\widehat{S}_\lambda$*, the analogous properties* $(i) - (iii)$ *of (5.1) above.*

We notice that as λ tends to 0, the manifold E_λ gets thinner and approaches the L_0-axes. Consider now two arbitrary disjoint projective subspaces L and M of dimension n in $P_{\mathbb{C}}^{2n+1}$, and the corresponding linear subspaces $\widehat{L}$, $\widehat{M}$ of $\mathbb{C}^{2n+2}$. It is clear that $\mathbb{C}^{2n+2} = \widehat{L} \oplus \widehat{M}$ and there is a linear automorphism $\widehat{H}$ of $\mathbb{C}^{2n+2}$ taking $\widehat{L}$ to $\widehat{L}_0$ and $\widehat{M}$ to $\widehat{M}_0$. For every $\lambda \in \mathbb{R}_+$, the automorphism $\widehat{H}^{-1} \circ \widehat{S}_\lambda \circ \widehat{H}$, is an involution that descends to an involution $H^{-1} \circ S_\lambda \circ H$ of $P_{\mathbb{C}}^{2n+1}$ that interchanges L and M. It is clear that one has results analogous to (5.1) and to (5.2). One also has:

5.4 Lemma. *Let* T *be a linear projective involution of* $P_{\mathbb{C}}^{2n+1}$ *that interchanges* L *and* M*. Then* T *is conjugate in* $PSL(2n+2, \mathbb{C})$ *to the canonical involution* S.

Proof. Let $\widehat{L}$ and $\widehat{M}$ be linear subspaces of $\mathbb{C}^{2n+2}$ as above. Let $\{l_1, ..., l_{n+1}\}$ be a basis of $\widehat{L}$. Then $\{l_1, \ldots, l_{n+1}, \widehat{T}(l_1), \ldots, \widehat{T}(l_{n+1})\}$ is a basis of $\mathbb{C}^{2n+2}$. The linear transformation that sends the canonical basis of $\mathbb{C}^{2n+2} = \mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1}$ to this basis induces a projective transformation which realizes the required conjugation. □

In this paper, **mirrors** in $P_{\mathbb{C}}^{2n+1}$ are, by definition, the images of E_S under the action of $PSL(2n+2, \mathbb{C})$. A mirror is the boundary of a tubular neighbourhood of a $P_{\mathbb{C}}^n$ in $P_{\mathbb{C}}^{2n+1}$, so it is an S^{2n+1}-bundle over $P_{\mathbb{C}}^n$.

We summarise the previous discussion in the following result.

5.5 Proposition. *Let* $L \cong M \cong P_{\mathbb{C}}^n$ *be disjoint projective subspaces of* $P_{\mathbb{C}}^{2n+1}$*. Then:*
i) There exist involutions of $P_{\mathbb{C}}^{2n+1}$ *that interchange* L *and* M.
ii) Each of these involutions has a mirror, i.e., an invariant set $E = E_T \subset P_{\mathbb{C}}^{2n+1}$ *which separates* $P_{\mathbb{C}}^{2n+1}$ *in two connected components which are interchanged by* T*. Each component is diffeomorphic to a tubular neighbourhood of the canonical* $P_{\mathbb{C}}^n \subset P_{\mathbb{C}}^{2n+1}$.

iii) Given an arbitrary tubular neighbourhood U of L, we can choose T so that the corresponding mirror E_T is contained in the interior of U.

In fact one can obviously make stronger the last statement of (5.5):

5.6 Lemma. *Let L and M be as above. Given an arbitrary constant λ, $0 < \lambda < 1$, we can find an involution T interchanging L and M, with a mirror E such that if U^* is the open component of $P^{2n+1}_{\mathbb{C}} - E$ which contains M and $x \in U^*$, then $d(T(x), L) < \lambda d(x, M)$, where the distance d is induced by the Fubini-Study metric.*

Proof. The involution $T_\lambda := H^{-1} \circ S_\lambda \circ H$, with H and S_λ as above, satisfies (5.6). $\square$

We notice that the parameter λ in (5.6) gives control upon the degree of expansion and contraction of the generators of the groups, so one can estimate bounds on the Hausdorff dimension of the limit set (see section 7 below).

The previous discussion can be summarized in the following theorem (Compare [No]):

5.7 Theorem. *Let $\mathcal{L} := \{(L_1, M_1), \dots, (L_r, M_r)\}$, $r > 1$, be a set of r pairs of projective subspaces of dimension n of $P^{2n+1}_{\mathbb{C}}$, all of them pairwise disjoint. Then:*

i) There exist involutions $T_1, ..., T_r$ of $P^{2n+1}_{\mathbb{C}}$, such that each T_i, $i = 1, ..., r$, interchanges L_i and M_i, and the corresponding mirrors E_{T_i} are all pairwise disjoint.

ii) If we choose the $T_i^!s$ in this way, then the subgroup of $PSL(2n+2, \mathbb{C})$ that they generate is complex Kleinian.

iii) Moreover, given a constant $C > 0$, we can choose the $T_i^!s$ so that if $T := T_{j_1} \cdots T_{j_k}$ is a reduced word of length $k > 0$ (i.e., $j_1 \neq j_2 \neq \cdots \neq j_{k-1} \neq j_k$), then $T(N_i)$ is a tubular neighbourhood of the projective subspace $T(L_i)$ which becomes very thin as k increases: $d(x, T(L_i)) < C\lambda^k$ for all $x \in T(N_i)$, where N_i is the connected component of $P^{2n+1}_{\mathbb{C}} - E_{T_i}$ that contains L_i, for all $i = 1, ...r$.

A Kleinian group constructed as above will be called a **complex Schottky group**.

5.8 Theorem. *Let Γ be a complex Schottky group in $P^{2n+1}_{\mathbb{C}}$, generated by involutions $\{T_1, .., T_r\}$, $n \geq 1$, $r > 1$, as in (5.7) above. Let $\Omega(\Gamma)$ be the region of discontinuity of Γ and let $\Lambda(\Gamma) = P^{2n+1}_{\mathbb{C}} - \Omega(\Gamma)$ be the limit set. Then, one has:*

i) Let $W = P^{2n+1}_{\mathbb{C}} - \cup_{i=1}^r \overset{\circ}{N}_i$, where $\overset{\circ}{N}_i$ is the interior of the tubular neighbourhood N_i as in (5.7). Then W is a compact fundamental domain for the action of Γ on $\Omega(\Gamma)$. One has: $\Omega(\Gamma) = \bigcup_{\gamma \in \Gamma} \gamma(W)$.

ii) $\Lambda(\Gamma)$ is an intersection of nested sets: $\Lambda(\Gamma) = \cap_{i=1}^\infty \gamma_i(N_{j(i)})$, where $\{\gamma_i\}_{i=1}^\infty$ is a sequence of distinct elements of Γ and $j \colon \mathbb{N} \to \{1, ..., r\}$ is a function such that $\gamma_{i+1}(N_{j(i+1)}) \subset \gamma_i(N_{j(i)})$. Hence $\Lambda(\Gamma)$ is the closure of the Γ-orbit of the union $L_1 \cup ... \cup L_r$.

iii) If $r = 2$, then $\Gamma \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, the infinite dihedral group, and $\Lambda(\Gamma)$ is the union of two disjoint projective subspaces L and M of dimension n. In this case we say that Γ is **elementary**, *in analogy with Kleinian groups acting on $P^1_{\mathbb{C}}$.*

iv) If $r > 2$, then $\Lambda(\Gamma)$ is a complex solenoid (lamination), homeomorphic to $P^n_{\mathbb{C}} \times \mathcal{C}$, where $\mathcal{C}$ is a Cantor set. Γ acts minimally on the set of projective subspaces in $\Lambda(\Gamma)$ considered as a closed subset of the Grassmannian $G_{2n+1,n}$.

v) If $r > 2$, let $\check{\Gamma} \subset \Gamma$ be the index 2 subgroup consisting of the elements which are reduced words of even length. Then $\check{\Gamma}$ is free of rank $r - 1$ and acts freely on $\Omega(\Gamma)$. The compact manifold with boundary $\check{W} = W \cup T_1(W)$ is a fundamental domain for the action of $\check{\Gamma}$ on $\Omega(\Gamma)$. We also call $\check{\Gamma}$ a **complex Schottky group**.

vi) Each element $\gamma \in \check{\Gamma}$ leaves invariant two copies, P_1 and P_2, of $P^n_{\mathbb{C}}$ in $\Lambda(\Gamma)$. For every $L \subset \Lambda(\Gamma)$, $\gamma^i(L)$ converges to P_1 (or to P_2) as $i \to \infty$ (or $i \to -\infty$).

In fact we prove that if $r > 2$, then Γ acts on a graph whose vertices have all valence either 2 or r. This graph is actually a tree, which can be compactified by adding its "ends". These form a Cantor set and the action of Γ can be extended to this compactification. The limit set $\Lambda(\Gamma)$ corresponds to the uncountable set of ends of this tree. We use this to prove statement $v)$ above.

Proof of i) Let ∂W be the boundary of $W = P^{2n+1}_{\mathbb{C}} - \cup \overset{\circ}{N}_i$, i.e., the union $E_1 \cup \cdots \cup E_r$ of the mirrors. Set $W_0 := W$. Now define $W_1 = \bigcup_{i=0}^{r} T_i(W)$, where T_0 is the identity, by definition. Then W_1 is a manifold whose boundary consists of $r(r-1)$ components $E_{ij} := T_i(E_j)$, $i \neq j\,; i, j = 1, \ldots, r$, each one being a mirror. Define, by induction on $k > 1$, $W_k = \bigcup_{i=o}^{r} T_i(W_{k-1})$. Then W_k is a manifold whose boundary consists of $r(r-1)^k$ components, $E_{j_1,\ldots,j_k} := T_{j_1} \cdots Tj_{k-1}(E_{j_k})$, where $j_1, j_2, \ldots, j_k \in \{1, \ldots, r\}$ and $j_1 \neq j_2, \ldots, j_{k-1} \neq j_k$. Thus W_k is contained in the interior of W_{k+1}: $W_k \subset \overset{\circ}{W}_{k+1}$.

Let $U = \bigcup_{k=0}^{\infty} W_k$, so U is Γ-invariant, since $T_j(W_k) \subset W_{k+1}$ for every $j \in \{1, \ldots, r\}$. It is clear that U is open, since any $x \in U$ is contained in the interior of some W_k. Let $\gamma = T_{j_1} \cdots T_{j_k}$ be any element of Γ represented as a reduced word of length $k > 1$. Then $\gamma(W) \subset W_k - \overset{\circ}{W}_{k-1}$. Thus, for any $\gamma \neq \beta$, $\gamma(\overset{\circ}{W}) \bigcap \beta(\overset{\circ}{W}) = \varnothing$. Since $U = \bigcup_{\gamma\in\Gamma} \gamma(W)$, then U is obtained from translates of W, glued along some boundary components. Thus U is open, connected, with a properly discontinuous action of Γ. Therefore $U \subset \Omega(\Gamma)$. To finish the proof of i) we must prove $P^{2n+1}_{\mathbb{C}} - U = \Lambda(\Gamma)$. For this we consider, for each $k \geqslant 0$, the set $F_k := P^{2n+1}_{\mathbb{C}} - \overset{\circ}{W}_k$. Then $F_{k+1} \subset F_k$, hence $\bigcap_{k=0}^{\infty} F_k = P^{2n+1}_{\mathbb{C}} - U$ is a nonempty closed invariant set. For each $k \geqslant 0$, F_k is a disjoint union of closed tubular neighbourhoods of projective subspaces of dimension n of $P^{2n+1}_{\mathbb{C}}$. These are of the form $\gamma(N_i) = T_{j_1} \cdots T_{j_k}(N_i)$, for a $\gamma \in \Gamma$ which is represented in terms of the generators as the reduced word $T_{j_1} \cdots T_{j_k}$. They are closed tubular neighbourhoods of the projective subspace $T_{j_1} \cdots T_{j_k}(L_i)$. For each sequence $\{\gamma_j\}_{j=1}^{\infty}$ in Γ, such that the length of γ_{j+1} is bigger than the length of γ_j and $\gamma_{j+1}(N_i) \subset \gamma_j(N_i)$, the tubular neighbourhood becomes thinner. By (5.7), the sequence $\{\gamma_j(L_i)\}_{j=1}^{\infty}$ converges, in the Hausdorff metric, to a linear subspace of dimension n. Hence, also by (5.7), $P^{2n+1}_{\mathbb{C}} - U$ is a nowhere dense closed subset of $P^{2n+1}_{\mathbb{C}}$, which is a disjoint union of projective subspaces of dimension n. Thus U is open and dense in $P^{2n+1}_{\mathbb{C}}$; since $U \subset \Omega(\Gamma)$, it follows that $\Omega(\Gamma)$ is also connected. We have that U/Γ is compact and it is obtained from the compact fundamental domain W after identifications in each component of its boundary. If $\Omega(\Gamma) \neq U$ we arrive to a contradiction, because Ω/Γ is connected and U/Γ is open, compact and properly contained in Ω/Γ. Therefore, $\Omega(\Gamma) = U$ and $\Lambda(\Gamma) = \bigcap_{i=0}^{\infty} F_i$. This proves i).

Proof of ii). If $x \in \Lambda(\Gamma)$ then, as shown above, $x \in \bigcap_{i=0}^{\infty} F_i$. To prove ii) it is sufficient to choose, for each i, the component of F_i which contains x. Such component is of the form $\gamma(N_j)$ for a *unique* $\gamma \in \Gamma$ (we set $\gamma = \gamma_i$) and a *unique* $j \in \{1, \ldots, r\}$. We set $j = j(i)$. This proves ii). This also shows that $\bigcap_{i=0}^{\infty} F_i$ is indeed the limit set according to Kulkarni's definition (1.6.i).

Proof of iii). We have two involutions, T and S, and two neighbourhoods, N_T and N_S, whose boundaries are the mirrors of T and S, respectively. The limit set is the disjoint union $A \cup B$, where $A := \bigcap_{\gamma\in\Gamma'} \gamma(N_S)$, $B := \bigcap_{\gamma\in\Gamma''} \gamma(N_T)$, Γ' is the set of elements in Γ which are words ending in T and Γ'' is the set of elements which are words ending in S. By (5.7), A and B are each the intersection of a nested sequence of tubular neighbourhoods of projective subspaces of dimension n, whose intersection is a projective subspace of dimension n. Hence A and B are both projective subspaces of dimension n, and they are disjoint. Two reduced words ending in

T and S, act differently on N_T (or N_S). Hence Γ is the free product of the groups generated T and S, proving iii).

Proof of iv). Let $L \subset P_{\mathbb{C}}^{2n+1}$ be a subspace of dimension n and let N be a closed tubular neighbourhood of L as above. Let D be a closed disc which is an intersection of the form $\widehat{L} \cap N$, where $\widehat{L}$ is a subspace of complex dimension $n+1$, transversal to L. If M is a subspace of dimension n contained in the interior of N, then M is transverse to D, otherwise the intersection of M with $\widehat{L}$ would contain a complex line and M would not be contained in N. From the proofs of i) and ii) we know that $\Lambda(\Gamma)$ is the disjoint union of uncountable subspaces of dimension n. Let $x \in \Lambda(\Gamma)$ and let $L \subset \Lambda(\Gamma)$ be a projective subspace with $x \in L$. Let N be a tubular neighbourhood of L and D a transverse disc as above. Then $\Lambda(\Gamma) \cap D$ is obtained as the intersection of families of discs of decreasing diameters, exactly as in the construction of Cantor sets. Therefore $\Lambda(\Gamma) \cap D$ is a Cantor set and $\Lambda(\Gamma)$ is a solenoid (or lamination) by projective subspaces which is transversally Cantor. It follows that $\Lambda(\Gamma)$ is a fibre bundle over $P_{\mathbb{C}}^n$, with fibre a Cantor set $\mathcal{C}$. Since $P_{\mathbb{C}}^n$ is simply connected and $\mathcal{C}$ is totally disconnected, this fibre bundle must be trivial, hence the limit set is a product $P_{\mathbb{C}}^n \times \mathcal{C}$, as stated.

There is another way to describe the above construction: Γ acts, via the differential, on the Grassmannian $G_{2n+1,n}$ of projective subspaces of dimension n of $P_{\mathbb{C}}^{2n+1}$. This action also has a region of discontinuity and contains a Cantor set which is invariant. This Cantor set corresponds to the closed family of disjoint projective subspaces in $\Lambda(\Gamma)$. It is clear that the action on the Grassmannian is minimal on this Cantor set.

Proof of v). Choose a point x_0 in the interior of W. Let Γ_{x_0} be the Γ-orbit of x_0. We construct a graph $\check{\mathcal{G}}$ as follows: to each $\gamma(x_0) \in \Gamma_{x_0}$ we assign a vertex v_γ. Two vertices v_γ , $v_{\gamma'}$ are joined by an edge if $\gamma(W)$ and $\gamma'(W)$ have a common boundary component, which corresponds to a mirror E_i. This means that γ' is γ followed by an involution T_i or vice-versa. This graph can be realized geometrically by joining the corresponding points $\gamma(x_0), \gamma'(x_0) \in \Omega(\Gamma)$ by an arc $\alpha_{\gamma,\gamma'}$ in $\Omega(\Gamma)$, which is chosen to be transversal to the corresponding boundary component of $\gamma(W)$; we also choose these arcs so that no two of them intersect but at the extreme points. Clearly $\check{\mathcal{G}}$ is a tree and each vertex has valence r. To construct a graph $\mathcal{G}$ with an appropriate Γ-action we introduce more vertices in $\check{\mathcal{G}}$: we put one vertex at the middle point of each edge in $\check{\mathcal{G}}$; these new vertices correspond to the points where the above arcs intersect the boundary components of $\gamma(W)$. Then we have an obvious simplicial action of Γ on $\mathcal{G}$. Let $\check{\Gamma}$ be the index-two subgroup of Γ consisting of elements which can be written as reduced words of even length in terms of $T_1, ..., T_r$. A fundamental domain for $\check{\Gamma}$ in $\Omega(\Gamma)$ is $\check{W} = W \cup T_1(W)$, so this group acts freely on the vertices of $\check{\mathcal{G}}$. Hence $\check{\Gamma}$ is a free group of rank $r-1$. The tree $\check{\mathcal{G}}$ can be compactified by its ends by adding a Cantor set on which $\check{\Gamma}$ acts minimally; this corresponds to the fact that Γ acts minimally on the set of projective subspaces which constitute $\Lambda(\Gamma)$.

Proof of vi). By (5.7), if $\gamma \in \check{\Gamma}$, then either $\gamma(N_1)$ is contained in N_1 or $\gamma^{-1}(N_1)$ is contained in N_1; say $\gamma(N_1)$ is contained in N_1. Thus $\{\gamma^i(N_1)\}$, $i > 0$, is a nested sequence of tubular neighbourhoods of projective subspaces whose intersection is a projective subspace P_1 of dimension n; $\{\gamma^i(N_1)\}$, $i < 0$, is also a nested sequence of tubular neighbourhoods of projective subspaces whose intersection is a projective subspace P_2 of dimension n. For every $L \subset \Lambda(\Gamma)$, $\gamma^i(L)$ converges to P_1 and P_2 as $i \to \infty$ or $i \to -\infty$, respectively, and both P_1 and P_2 are invariant under γ, as claimed. $\square$

5.9 Remarks. i) It is worth noting that one may let the mirrors overlap slightly and still obtain complex Kleinian groups, in the same way as in the case of subgroups of conformal transformations of $P_{\mathbb{C}}^1$ obtained by inversions on overlapping circles, c.f., [Bo]. Also, in theorem 5.8 we could take $r = \infty$, *i.e.,* an infinite countable set of disjoint mirrors (see 6.5 below).

ii) The action of $\check{\Gamma}$ in the Cantor set of projective subspaces is analogous to the action of a classical Fuchsian group of the second kind on its Cantor limit set. We also observe that, since each involution T_i is conjugate to the canonical involution defined in lemma 5.1, the laminations obtained in theorem 5.8 are transversally *projectively self-similar*. Hence one could try to apply results analogous to the results for (conformally) self-similar sets (for instance Bowen's formula [Bo]) to estimate the transverse Hausdorff dimension of the laminations obtained. Here by *transverse Hausdorff dimension* we mean the Hausdorff dimension of the Cantor set $\mathcal{C}$ of projective subspaces of $G_{2n+1,n}$ which conform the limit set. If $\widetilde{T}_i$, $i = 1, \ldots, r$, denote the maps induced in the Grassmannian $G_{2n+1,n}$ by the linear projective transformations T_i, then $\mathcal{C}$ is dynamically-defined by the group generated by the set $\{\widetilde{T}_i\}$.

iii) The construction of Kleinian groups given in 5.8 actually provides families of Kleinian groups, obtained by varying the size of the mirrors that bound tubular neighbourhoods around the L_i's. In Section 7 below we will look at these families .

iv) The above construction of complex Kleinian groups, using involutions and mirrors, can be adapted to produce discrete groups of automorphisms of quaternionic projective spaces of odd (quaternionic) dimension. Every "quaternionic Kleinian group" on $P_{\mathcal{H}}^{2n+1}$ lifts canonically to a complex Kleinian group on $P_{\mathbb{C}}^{4n+3}$, c.f., (2.5.ii) above.

6. Quotient Spaces of the region of discontinuity

We now discuss the nature of the quotients $\Omega(\Gamma)/\Gamma$ and $\Omega(\Gamma)/\check{\Gamma}$, for the groups of section 5. The proof of proposition (6.1) is straightforward and is left to the reader.

6.1 Proposition. *Let L be a copy of the projective space $P_{\mathbb{C}}^n$ in $P_{\mathbb{C}}^{2n+1}$ and let x be a point in $P_{\mathbb{C}}^{2n+1} - L$. Let $K_x \subset P_{\mathbb{C}}^{2n+1}$ be the unique copy of the projective space $P_{\mathbb{C}}^{n+1}$ in $P_{\mathbb{C}}^{2n+1}$ that contains L and x. Then K_x intersects transversally every other copy of $P_{\mathbb{C}}^n$ embedded in $P_{\mathbb{C}}^{2n+1} - L$, and this intersection consists of one single point. Thus, given two disjoint copies L and M of $P_{\mathbb{C}}^n$ in $P_{\mathbb{C}}^{2n+1}$, there is a canonical projection map*

$$\pi := \pi_L : P_{\mathbb{C}}^{2n+1} - L \to M ,$$

which is a (holomorphic) submersion. Each fibre $\pi^{-1}(x)$ is diffeomorphic to $\mathbb{R}^{2n+2}$.

6.2 Theorem. *Let Γ be a complex Schottky group as in theorem 5.8, with $r > 2$. Then:*

i) The fundamental domain W of Γ is (the total space of) a locally trivial differentiable fibre bundle over $P_{\mathbb{C}}^n$ with fibre $S^{2n+2} - \mathring{D}_1 \cup \cdots \cup \mathring{D}_r$, where each $\mathring{D}_i$ is the interior of a smooth closed $(2n+2)$-disc D_i in S^{2n+2} and the D_i's are pairwise disjoint.

ii) $\Omega(\Gamma)$ fibres differentiably over $P_{\mathbb{C}}^n$ with fibre S^{2n+2} minus a Cantor set.

iii) If $\check{\Gamma}$ is the subgroup of index two as in theorem 5.8, which acts freely on $\Omega(\Gamma)$, then $\Omega(\Gamma)/\check{\Gamma}$ is a compact complex manifold that fibres differentiably over $P_{\mathbb{C}}^n$ with fibre $(S^{2n+1}\times S^1)\# \cdots \#(S^{2n+1}\times S^1)$, the connected sum of $r-1$ copies of $S^{2n+1} \times S^1$.

Proof of i). Let $P_1, P_2 \subset \Lambda(\Gamma)$ be two disjoint projective subspaces of dimension n contained in $\Lambda(\Gamma) \subset P_{\mathbb{C}}^{2n+1}$. Since $\Omega(\Gamma)$ is open in P^{2n+1}, the restriction to $\Omega(\Gamma)$ of the map π given by 6.1, using P_1 as L and P_2 as M, is a holomorphic submersion. We know, by theorem 5.8.iv, that $\Lambda(\Gamma)$ is a compact set which is a *disjoint* union of projective subspaces of dimension n and which is a transversally Cantor lamination. By 6.1, for each $y \in P_2$, K_y meets transversally each of these projective subspaces (in other words, K_y is transverse to the lamination $\Lambda(\Gamma)$, outside P_1). Hence, by theorem 5.8, for each $y \in P_2$, K_y intersects $\Lambda(\Gamma) - P_1$ in a Cantor set minus one point (this point corresponds to P_1). The family of subspaces K_y of dimension $n+1$ are all transverse to P_2.

Let us now choose P_1 and P_2 as in 5.8.vi, so they are invariant sets for some $\gamma \in \check{\Gamma}$, and $\gamma^j(L)$ converges to P_2 as $j \to \infty$ for every projective n-subspace $L \subset \Lambda(\Gamma) - P_1$. We see that *every* mirror E_i, $i \in \{1, \ldots, r\}$ is transverse to all K_y. Hence the restriction

$$\pi_1 := \pi_{P_1}|_W : W \to P_2 \cong P^n_{\mathbb{C}} ,$$

of π to W, is a submersion which restricted to each component of the boundary is also a submersion. For each $y \in P_2$ one has $\pi_1^{-1}(\{y\}) = K_y \cap W$, so $\pi_1^{-1}(\{y\})$ is compact. Thus π_1 is the projection of a locally trivial fibre bundle with fibres $K_y \cap W$, $y \in P_2$, by Ehresmann's lemma [Eh]. On the other hand, for a fixed $y_0 \in P_2$, $K_{y_0} \cap W$ is a closed $(2n+2)$-disc with $r-1$ smooth closed $(2n+2)$-discs removed from its interior. This is true because P_1 is contained in exactly one of the $N_i^\prime s$, say N_1, the tubular neighbourhood of P_1, and K_{y_0} intersects each N_j, $j \neq 1$, in a smooth closed $(2n+2)$-disc. This proves i).

Proof of ii). The above arguments show that for each $\bar{\gamma} \in \Gamma$, the image $\bar{\gamma}(E_i)$ of a mirror E_i is transverse to K_y for all $y \in P_2$ and $i \in \{1, \ldots, r\}$. Hence the restriction $\pi_1^k := \pi_{P_1}|_{W_k}$, where W_k is as above, is a submersion whose restriction to each boundary component of W_k is also a submersion. Thus π_1^k is a locally trivial fibration. Since $\Omega(\Gamma) = \bigcup_{k \geq 0} W_k$, we finish the proof of the first part of ii) by applying the slight generalisation of Ehresmann fibration lemma [Eh]; we leave the proof to the reader.

Lemma. *Let $\mathcal{M} = \bigcup_{i=1}^{\infty} \mathcal{N}_i$ be a smooth manifold which is the union of compact manifolds with boundary $\mathcal{N}_i$, so that each $\mathcal{N}_i$ is contained in the interior of $\mathcal{N}_{i+1}$. Let $\mathcal{L}$ be a smooth manifold and $f : \mathcal{M} \to \mathcal{L}$ a submersion whose restriction to each boundary component of $\mathcal{N}_i$, for every i, is also a submersion. Then f is a locally trivial fibration.*

Thus $\pi_{P_1} : \Omega(\Gamma) \to P_2 \cong P^n_{\mathbb{C}}$ is a holomorphic submersion which is a locally trivial differentiable fibration. To finish the proof of ii) we only need to show that the fibres of π_{P_1} are S^{2n+2} minus a Cantor set. Just as above, one shows that $K_y \cap W_k$ is diffeomorphic to the sphere S^{2n+2} minus the interior of $r(r-1)^k$ disjoint $(2n+2)$-discs. Therefore the fibre of π_{P_1} at y, which is $K_y \cap \Omega(\Gamma)$, is the intersection of S^{2n+2} minus a nested union of discs, which gives a Cantor set as claimed in ii).

Proof of iii). We recall that by theorem 5.8.v, the fundamental domain of $\check{\Gamma}$ is the manifold $\check{W} = W \cup T_1(W)$. Then, as above, the restriction of π to $\check{W}$ is a submersion which is also a submersion in each connected component of the boundary:

$$\partial \check{W} = \left(\bigcup_{j \neq 1} T_1(E_j) \right) \bigcup_{j \neq 1} E_j ,$$

which is the disjoint union of the $r-1$ mirrors E_j, $j \neq 1$, together with the mirrors $E_{1j} := T_1(E_j)$, $j \neq 1$. The mirror E_j is identified with E_{1j}, $j \neq 1$, by T_1, and $\Omega(\Gamma)/\check{\Gamma}$ is obtained through these identifications. Let $\check{\pi} : \check{W} \to P_2$ be the restriction of π to $\check{W}$. By the proof of i), $\check{\pi}^{-1}(y) = K_y \cap \check{W}$, $y \in P_2$, is diffeomorphic to S^{2n+2} minus the interior of $2(r-1)$ disjoint $(2n+2)$-discs. The restriction of $\check{\pi}$ to each E_j and E_{1j} determines fibrations $\check{\pi}_j : E_j \to P_2$ and $\check{\pi}_{1j} : E_{1j} \to P_2$, respectively, whose fibres are S^{2n+1}. Set $\widehat{\pi}_j := \check{\pi}_{1j} \circ (T_1|_{E_j})$. If we had that $\widehat{\pi}_j = \check{\pi}_j$ for all $j = 2, ..., r$, then we would have a fibration from $\check{W}/\check{\Gamma}$ to P_2, because we would have compatibility of the projections on the boundary. In fact we only need that $\widehat{\pi}_j$ and $\check{\pi}_j$ be homotopic through a smooth family of fibrations $\pi_t : E_{1j} \to P_2$; $\pi_1 = \widehat{\pi}_j$, $\pi_0 = \check{\pi}_j$, $t \in [0,1]$. Actually, to be able to glue well the fibrations at the boundary we need that $\pi_t = \check{\pi}_j$ for t in a neighbourhood of 0 and $\pi_t = \widehat{\pi}_j$ for t in a neighbourhood of 1. But this is almost trivial: $\check{\pi}_j : E_{1j} \to P_2$ is the projection of E_{1j} onto P_2 from P_1 and $\widehat{\pi} - j$ is the projection of E_{1j} from $T(P_1)$ onto P_2. The n-dimensional subspaces P_1 and $T(P_1)$ are disjoint from P_2, so there exists a smooth family of n-dimensional subspaces P_t, $t \in [0,1]$, such that the family is disjoint from P_2 and $P_t = P_1$ for t in a neighbourhood of 0 and $P_t = T(P_1)$ for t in a neighbourhood of 1. We can choose the family so that for each $t \in [0,1]$, the set of $n+1$ dimensional subspaces which contain P_t meet transversally E_{1j}. To achieve this we only need to take an appropriate curve in the Grassmannian of projective n-planes in $P^{2n+1}_{\mathbb{C}}$, consisting of a family P_t which is transverse to all K_y; this is possible by (6.1) and the fact that the set of n-dimensional subspaces which are *not* transverse to the $K'_y s$, is a proper algebraic variety of $P^{2n+1}_{\mathbb{C}}$. In this way we obtain the desired homotopy. Hence $\check{W}$ fibres over $P_2 \cong P^n_{\mathbb{C}}$; the fibre is obtained from S^{2n+2} minus the interior of $2(r-1)$ disjoint $(2n+2)$-discs whose boundaries are diffeomorphic to S^{2n+1} and are identified by pairs by diffeomorphisms which are *isotopic* to the identity (using a fixed diffeomorphism to S^{2n+1}). Hence the fibre is *diffeomorphic* to $(S^{2n+1} \times S^1)\# \cdots \#(S^{2n+1} \times S^1)$, the connected sum of $r-1$ copies of $S^{2n+1} \times S^1$. This proves iii). $\square$

6.3 Theorem. *Let M_Γ be the compact complex orbifold $M_\Gamma := \Omega(\Gamma)/\Gamma$, which has complex dimension $(2n+1)$. Then:*

i) The singular set of M_Γ, $Sing(M_\Gamma)$, is the disjoint union of r submanifolds analytically equivalent to $P^n_{\mathbb{C}}$, one contained in (the image in M_Γ of) each mirror E_i of Γ.

ii) Each component of $Sing(M_\Gamma)$ has a neighbourhood homeomorphic to the normal bundle of $P^n_{\mathbb{C}}$ in $P^{2n+1}_{\mathbb{C}}$ modulo the involution $v \mapsto -v$, for v a normal vector.

iii) M_Γ fibres over $P^n_{\mathbb{C}}$ with fibre a real analytic orbifold with r singular points, each having a neighbourhood (in the fibre) homeomorphic to the cone over the real projective space $P^{2n+1}_{\mathbb{R}}$.

Proof. We notice that M_Γ is obtained from the fundamental domain W after an identification on the boundary E_j by the action of T_j. The singular set of M_Γ is the union of the images, under the canonical projection $p : \Omega(\Gamma) \to \Omega(\Gamma)/\Gamma$, of the fixed point sets of the r involutions T_j. Now, T_j is conjugate to the canonical involution S of (5.2). The lifting of S to $\mathbb{C}^{2n+2} = \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ has, as a fixed point set, the $(n+1)$-subspace $\{(a,a) : a \in \mathbb{C}^{n+1}\}$. This projectivizes to a n-dimensional projective subspace. Since we can assume, *for a fixed j*, that T_j is an isometry, we obtain the local structure of a neighbourhood of each component of the singular set. The same arguments as in theorem 6.2.iii prove that $\Omega(\Gamma)/\Gamma$ fibres over $P^n_{\mathbb{C}}$ and that the fibre has r singular points, corresponding to the r components of $Sing(M_\Gamma)$, and each of these r points has a neighbourhood (in the fibre) homeomorphic to the cone over $P^{2n+1}_{\mathbb{R}}$. $\square$

6.4 Remarks. i) The map π in (6.2.ii) is holomorphic, but the fibration is not holomorphically locally trivial, because the complex structure on the fibres may change.

ii) The Kleinian groups of 6.2 provide a method for constructing complex manifolds which is likely to produce interesting examples (c.f., [No, Ka1-4, Pe3, Si]). These are never Kähler, because the fibration $\pi : \Omega(\Gamma)/\check{\Gamma} \to P^n_{\mathbb{C}}$ has a section, by dimensional reasons, so there cannot exist a 2-cocycle with a power which is the fundamental class of $\Omega(\Gamma)/\check{\Gamma}$. The bundle $(n+1)\mathcal{O}_{P^n_{\mathbb{C}}}$ is nontrivial as a real bundle, because it has non-vanishing Pontryagin classes (except for $n=1$), hence π is a nontrivial fibration.

iii) The manifolds obtained by resolving the singularities of the orbifolds in (6.3) have very interesting topology. We recall that the orbifold M_Γ is singular along r disjoint copies of $P^n_{\mathbb{C}}$: $S_1, \ldots, S_r$. The resolution $\widetilde{M}_\Gamma$ of M_Γ is obtained by a monoidal transformation along each S_i, and it replaces each point $x \in S_i$, $1 \le i \le r$ by a projective space $P^n_{\mathbb{C}}$. Hence, if $\mathcal{P} : \widetilde{M} \to M$ denotes the resolution map, then $\mathcal{P}^{-1}(S_i)$ is a non-singular divisor in $\widetilde{M}$, which fibres holomorphically over $P^n_{\mathbb{C}}$ with fibre $P^n_{\mathbb{C}}$, $1 \leq i \leq r$.

6.5 Complex Schottky groups in infinite dimensional projective space $P^\infty_{\mathbb{C}}$. The constructions and theorems of sections 5 and 6 can be realized in the infinite dimensional complex projective space: let H be an infinite dimensional, separable, complex Hilbert space, and let $P^\infty_{\mathbb{C}}$ be the associated projective space. Let $\{e_i\}_{i\in\mathbb{Z}}$ be an orthonormal base of H, indexed by the integers. Then $H = H_+ \oplus H_-$, where H_+ is the closed subspace generated by $\{e_i\}, i \geq 0$, and H_- the subspace generated by $\{e_i\}, i < 0$. Let $z = (x, y)$ be the expression of an element in H with respect to this decomposition. Let $\widehat{T} : H \to H$ be the involution $\widehat{T}(e_i) = e_{1-i}$, *i.e.* $\widehat{T}(x, y) = (y, x)$. We call $\widehat{T}$ the *canonical* involution. The set $\widehat{E} := \{(x, y) \mid ||x||^2 = ||y||^2\}$, is a complex cone in H, whose projectivization is a Hilbert manifold E in $P^\infty_{\mathbb{C}}$. We call it *the canonical mirror.* Everything done in sections 5 and 6 goes through, but one has a new interesting feature: if we subtract from $P^\infty_{\mathbb{C}}$ a projective subspace L of *infinite* co-dimension, the resulting space has the same homotopy type as $P^\infty_{\mathbb{C}}$, by Theorem 2 of Eells and Kuiper in [EK]. Hence, by a theorem of Eells and Elworthy [EE] (see also [BK]), there exists a diffeomorphism from $P^\infty_{\mathbb{C}} - L$ onto $P^\infty_{\mathbb{C}}$. So we can construct complex Schottky groups Γ in $P^\infty_{\mathbb{C}}$ as above, but to define the region of discontinuity we must take into consideration the fact that $P^\infty_{\mathbb{C}}$ is not locally compact. Let $\{(\widehat{L}_i, \widehat{M}_i)\}_i$, $1 < i \leq r$ or $r = \infty$, be a disjoint collection of (at most) countable many pairs of subspaces of infinite dimension and codimension, such that $\widehat{L}_i \oplus \widehat{M}_i = H$; let $\widehat{T}_i$ be an involution which interchanges $\widehat{L}_i$ with $\widehat{M}_i$. Let T_i denote the projective transformation of $P^\infty_{\mathbb{C}}$ induced by $\widehat{T}_i$ and denote by (L_i, M_i) the projectivization of $(\widehat{L}_i, \widehat{M}_i)$. As before, we can choose the $T_i's$ to have pairwise disjoint mirrors. Let Γ be the group of holomorphic projective transformations of $P^\infty_{\mathbb{C}}$ generated by the T_i's. We define the limit set $\Lambda(\Gamma)$ to be the closure of the Γ-orbit of $\cup_1^r L_i$. The region of discontinuity is $\Omega(\Gamma) := P^\infty_{\mathbb{C}} - \Lambda(\Gamma)$. In this case $\Omega(\Gamma)$ fibres over $P^\infty_{\mathbb{C}}$ with fibre an infinite dimensional complex manifold modeled on a separable Hilbert space, hence diffeomorphic to this Hilbert space. Then, proceeding exactly as before, we have:

6.6 Theorem. *Let $\{T_i\}$, $1 < i \leq r$ or $r = \infty$, be a (finite or countable) family of involutions of $P^\infty_{\mathbb{C}}$, which interchange pairs of infinite dimensional projective subspaces of $P^\infty_{\mathbb{C}}$, with pairwise disjoint mirrors. Let Γ be the group of holomorphic transformations of $P^\infty_{\mathbb{C}}$ that they generate. Then:*

i) The region of discontinuity $\Omega(\Gamma)$ fibres differentiably over $P^\infty_{\mathbb{C}}$, with fibre a contractible complex Hilbert manifold (hence diffeomorphic to the Hilbert space, by [EK]*).*

ii) If $\check{\Gamma}$ is the subgroup of index 2 of Γ, consisting of all the reduced words of even length, then $\check{\Gamma}$ acts freely on $\Omega(\Gamma)$ and the quotient $\Omega(\Gamma)/\check{\Gamma}$ is a complex Hilbert manifold.

iii) $\Omega(\Gamma)/\check{\Gamma}$ fibres differentiably over $P^\infty_{\mathbb{C}}$, with fibre an infinite dimensional complex manifold, which is an Eilenberg-MacLane space $K(F_{r-1}, 1)$ for the free group of rank $r - 1$.

7. Hausdorff dimension and moduli spaces

Let $\mathcal{L} := \{(L_1, M_1), \ldots, (L_r, M_r)\}$ be a configuration of $P^n_{\mathbb{C}}$'s in $P^{2n+1}_{\mathbb{C}}$ as before, $r > 2$. Let Γ and Γ' be complex Schottky groups obtained from this same configuration, i.e., they are generated by sets $\{T_1, ..., T_r\}$ and $\{T'_1, ..., T'_r\}$ of holomorphic involutions of $P^{2n+1}_{\mathbb{C}}$ that interchange the L_i's with the M_i's. For each $i = 1, ..., r$, the composition $T'_i \circ T_i^{-1}$ preserves the subspaces L_i, M_i. It is an exercise to see that the subgroup of $PSL(n+2, \mathbb{C})$ of transformations that preserve these subspaces is the projectivization of a copy of $GL(n+1, \mathbb{C}) \times GL(n+1, \mathbb{C}) \subset GL(2n+2, \mathbb{C})$. Therefore, we can always find an analytic family $\{\Gamma_t\}$, $0 \leq t \leq 1$, of complex Schottky groups, with configuration $\mathcal{L}$, such that $\{\Gamma_0\} = \Gamma$ and $\{\Gamma_1\} = \Gamma'$. Furthermore, let $\mathcal{L} := \{(L_1, M_1), \ldots, (L_r, M_r)\}$ and $\mathcal{L}' := \{(L'_1, M'_1), \ldots, (L'_r, M'_r)\}$ be two configurations of $P^n_{\mathbb{C}}$'s in $P^{2n+1}_{\mathbb{C}}$ as before. Due to dimensional reasons, we can always move these configurations to obtain a differentiable family of pairs of disjoint n-dimensional subspaces $\{(L_{1,t}, M_{1,t}), \ldots, (L_{r,t}, M_{r,t})\}$, with $0 \le t \le 1$, providing an isotopy between $\mathcal{L}$ and $\mathcal{L}'$. Thus one has a differentiable family Γ_t of complex Kleinian groups, where $\Gamma_0 = \Gamma$ and $\Gamma_1 = \Gamma'$. The same statements hold if we replace Γ and Γ' by their subgroups $\check{\Gamma}$ and $\check{\Gamma}'$, consisting of words of even length. So one has a differentiable family $\check{\Gamma}_t$ of Kleinian groups, where $\check{\Gamma}_0 = \check{\Gamma}$ and $\check{\Gamma}_1 = \check{\Gamma}'$. Hence the manifolds $\Omega(\Gamma_t)/\check{\Gamma}_t$ are all diffeomorphic. By section 6, these manifolds are (in general non-trivial) fibre bundles over $P^n_{\mathbb{C}}$ with fibre $\#^{(r-1)}(S^{2n+1} \times S^1)$, a connected sum of $(r-1)$-copies of $S^{2n+1} \times S^1$. If $n = 1$, given any configuration of r pairwise disjoint lines in $P^3_{\mathbb{C}}$, there exists an isotopy of $P^3_{\mathbb{C}}$ which carries the configuration into a family of r twistor lines. Hence $P^3_{\mathbb{C}}$ minus this configuration is diffeomorphic to the Cartesian product of S^4 minus r points with $P^1_{\mathbb{C}}$. Moreover, the attaching functions that we use to glue the boundary components of W, the fundamental domain of Γ, are all isotopic to the identity, because they live in $PSL(4, \mathbb{C})$, which is connected. Thus, if $n = 1$, then $\Omega(\Gamma_t)/\check{\Gamma}_t$ is diffeomorphic to a product $P^1_{\mathbb{C}} \times \#^{(r-1)}(S^3 \times S^1)$. Hence we have:

7.1 Proposition. *The differentiable type of the compact (complex) manifold $\Omega(\Gamma_t)/\check{\Gamma}_t$ is independent of the choice of configuration. It is a manifold of real dimension $(4n+2)$, which is a fibre bundle over $P^n_{\mathbb{C}}$ with fibre $\#^{(r-1)}(S^{2n+1} \times S^1)$; moreover, this bundle is trivial if $n = 1$. We denote the corresponding manifold by M^n_r.*

The fact that the bundle is trivial when $n = 1$ is interesting because, as pointed out in the introduction, when the configuration $\mathcal{L}$ consists of twistor lines in $P^3_{\mathbb{C}}$, the quotient $\Omega(\Gamma)/\check{\Gamma}$ is the twistor space of the conformally flat manifold $p(\Omega(\Gamma))/p(\check{\Gamma})$, which is a connected sum of the form $\#^{(r-1)}(S^3 \times S^1)$. Hence, in this case the natural fibration goes the other way round, i.e., it is a fibre bundle over $\#^{(r-1)}(S^3 \times S^1)$ with fibre $P^1_{\mathbb{C}}$.

Given a configuration $\mathcal{L}$ as above, let us denote by $[\mathcal{L}]_G$ its orbit under the action of the group $G = PSL(2n+2, \mathbb{C})$. These orbits are equivalence classes of such configurations. Let us denote by $\mathcal{C}^n_r$ the set of equivalence classes of configurations consisting of r pairs of $P^n_{\mathbb{C}}$'s as above. Then $\mathcal{C}^n_r$ is a Zariski open set of the moduli space $\mathfrak{M}^n_r$, of configurations of r unordered couples of projective subspaces of dimension n in $P^{2n+1}_{\mathbb{C}}$, which is obtained as the Mumford quotient [MFK] of the action of G on such configurations. By [MFK], $\mathcal{C}^n_r$ is a complex algebraic variety: the *moduli space of configurations* of r pairs of n-planes $P^n_{\mathbb{C}}$ in $P^{2n+1}_{\mathbb{C}}$. Similarly, we denote by $\mathfrak{S}^n_r$ the equivalence classes, or moduli space, of the corresponding Schottky groups, where two such groups are equivalent if they are conjugated by an element in $PSL(n+2, \mathbb{C})$. Given $\mathcal{L} := \{(L_1, M_1), \ldots, (L_r, M_r)\}$, and r-tuples of involutions $(T_1, \ldots, T_r)$ and $(S_1, \ldots, S_r)$ as above, i.e., interchanging L_i with M_i for all $i = 1, .., r$ and having pairwise disjoint mirrors, we say that these r-tuples are *equivalent* if there exists $h \in G$ such that $hT_ih^{-1} = S_i$ for all i. Let $\mathfrak{T}_{\mathcal{L}}$ denote the set of equivalence classes of such r-tuples of involutions. It is clear that a conjugation h as above must leave $\mathcal{L}$ invariant. Hence, if r is big enough with respect to n, then h must be actually the identity, so the equivalence classes in fact consist of a single element.

7.2 Theorem. *There exists a holomorphic surjective map $\pi\colon \mathfrak{S}^n_r \to \mathcal{C}^n_r$ which is a C^∞ locally trivial fibration with fibre $\mathfrak{T}_{\mathcal{L}}$. Furthermore, let $\Gamma\,, \Gamma'$ be complex Schottky groups as above and let $\Omega(\Gamma)\,, \Omega(\Gamma')$ be their regions of discontinuity. Then the complex orbifolds $M_\Gamma := \Omega(\Gamma)/\Gamma$ and $M_{\Gamma'} := \Omega(\Gamma')/\Gamma'$ are biholomorphically equivalent if and only if Γ and Γ' are projectively conjugate, i.e., they represent the same element in $\mathfrak{S}^n_r$. Similarly, if $\check{\Gamma}\,, \check{\Gamma}'$ are the corresponding index 2 subgroups, consisting of the elements which are words of even length, then the manifolds $M_{\check{\Gamma}} := \Omega(\Gamma)/\check{\Gamma}$ and $M_{\check{\Gamma}'} := \Omega(\Gamma')/\check{\Gamma}'$, are biholomorphically equivalent if and only if $\check{\Gamma}$ and $\check{\Gamma}'$ are projectively conjugate.*

Proof. The first statement in (7.2) is obvious, i.e., that we have a holomorphic surjection $\pi: \mathfrak{G}_r^n \to \mathcal{C}_r^n$ with kernel $\mathfrak{T}_{\mathcal{L}}$. The other statements are immediate consequences of the following lemma (7.3), proved for us by Sergei Ivashkovich. Our proof below is a variation of Ivashkovich's proof.

7.3 Lemma. *Let U be a connected open set in $P_{\mathbb{C}}^{2n+1}$ that contains a subspace $L \subset P_{\mathbb{C}}^{2n+1}$ of dimension n, and let $h: U \to V$ be a biholomorphism onto an open set $V \subset P_{\mathbb{C}}^{2n+1}$. Suppose that V also contains a subspace M of dimension n. Then h extends uniquely to an element in $PSL(2n+2, \mathbb{C})$.*

Proof. Let $f : U \to P_{\mathbb{C}}^n$ be a holomorphic map. Then f is defined by n meromorphic functions $f_1, \ldots, f_n$ from U to $P_{\mathbb{C}}^1$ (see [Iva]), *i.e.,* holomorphic functions which are defined outside of an analytic subset of U (the indeterminacy set).

Consider the set of all subspaces of $P_{\mathbb{C}}^{2n+1}$ of dimension $n+1$ which contain L. Then, if N is such a subspace, one has a neighbourhood U_N of L in N which is the complement of a round ball in the affine part, $\mathbb{C}^{n+1}$, of N. Since the boundary of such a ball is a round sphere S_N and, hence, it is pseudo-convex, it follows from the E. Levi extension theorem, applied to each f_i, that the restriction, f_N, of f to $U \cap N$ extends to all of N as a meromorphic function. The union of all subspaces N is $P_{\mathbb{C}}^{2n+1}$ and they all meet in L. The functions f_N depend holomorphically on N as is shown in [Iva]. One direct way to prove this is by considering the Henkin-Ramirez reproducing kernel defined on each round sphere S_N, [He, Ram]. One can choose the spheres S_N in such a way that the kernel depends holomorphically on N by considereing a tubular neighbourhood of L in N whose radius is independent of N. Hence the extended functions to all $N's$ define a meromorphic function in all of $P_{\mathbb{C}}^{2n+1}$, which extends f. Now let h be as in the statement lemma 7.3 and let $\widetilde{h}$ be its meromorphic extension. Then, since by hypothesis h is a biholomorphism from the open set $U \subset P_{\mathbb{C}}^n$ onto the the open set $V := h(U) \subset P_{\mathbb{C}}^n$, one can apply the above arguments to $h^{-1} : V \to U$. Let $g : P_{\mathbb{C}}^n \to P_{\mathbb{C}}^n$ be the meromorphic extension of h^{-1}. Then, outside of their sets of indeterminacy, one has $\widetilde{h}g = g\widetilde{h} = Id$. Hence the indeterminacy sets are empty and both $\widetilde{h}$ and g are biholomorphisms of $P_{\mathbb{C}}^n$. In fact, in [Iva] it is shown that if f is as in the statement of lemma 7.3 and if f is required only to be locally injective, then f extends as a holomorphic function $\square$

Notice that if $n = 1$, then (7.3) becomes Lemma 3.2 in [Ka1].

7.4 Corollary. *For $r > 2$ sufficiently large, the manifold $\Omega(\Gamma)/\check{\Gamma}$ has non-trivial moduli.*

In fact, if the manifolds $\Omega(\Gamma)/\check{\Gamma}$ and $\Omega(\Gamma')/\check{\Gamma}'$ are complex analytically equivalent, then $\check{\Gamma}$ is conjugate to $\check{\Gamma}'$ in $PSL(2n+2, \mathbb{C})$, by (7.2), and the corresponding configurations $\mathcal{L}$ and $\mathcal{L}'$ are projectively equivalent. Now it is sufficient to choose r big enough to have two such configurations which are not projectively equivalent. This is possible because the action induced by the projective linear group G on the Grassmannian $G_{2n+1,n}$ is obtained from the projectivization of the action of $SL(2n+2, \mathbb{C})$ acting on the Grassmann algebra Λ^{n+1}, of $(n+1)$-vectors of $\mathbb{C}^{2n+2}$, restricted to the set of decomposable $(n+1)$-vectors $\mathcal{D}^{n+1}$. The set $\mathcal{D}^{n+1}$ generates the Grassmann algebra and $G_{2n+1,n} = (\mathcal{D}^{n+1} - \{0\})/\sim$, where $\sim$ is the equivalence relation of projectivization.

If r is small with respect to n, then $\mathcal{C}_r^n$ consists of one point, because any two such configurations are in the same $PSL(2n+2, \mathbb{C})$-orbit. Therefore, in this case $\mathfrak{T}_{\mathcal{L}}$ coincides with $\mathfrak{G}_r^n$. That is, to change the complex structure of M_r^n we need to change the corresponding involutions into a family of involutions, with the same configuration (up to conjugation), which is not conjugate to the given one.

The following result is a generalization of Theorem 5.2 in [Ka1]. This can be regarded as a restriction for a complex orbifold (or manifold) to be of the form $\Omega(\Gamma)/\Gamma$ (or $\Omega(\Gamma)/\check{\Gamma}$).

7.5 Proposition. *If $r > 2$, then the compact complex manifolds and orbifolds $\Omega(\Gamma)/\check{\Gamma}$ and $\Omega(\Gamma)/\Gamma$, obtained in theorem 6.2, have no non-constant meromorphic functions.*

Proof. Let f be a meromorphic function on one of these manifolds (or orbifolds). Then f lifts to a meromorphic function $\widetilde{f}$ on $\Omega(\Gamma) \subset P^{2n+1}_{\mathbb{C}}$, which is $\check{\Gamma}$-invariant. By lemma (7.6) below, f extends to a meromorphic function on all of $P^{2n+1}_{\mathbb{C}}$. Hence $\widetilde{f}$ must be constant, because $\check{\Gamma}$ is an infinite group. $\square$

7.6 Lemma [Iva]. *Let $U \subset P^{2n+1}_{\mathbb{C}}$, $n \geq 1$, be an open set that contains a projective subspace $P^n_{\mathbb{C}}$. Let $f : U \to P^1_{\mathbb{C}}$ be a meromorphic function. Then f can be extended to a meromorphic function $\widetilde{f} : U \to P^1_{\mathbb{C}}$.*

We refer to [Iva] for the proof of (7.6). In the following proposition we estimate an upper bound for the Hausdorff dimension of the limit set of some Schottky groups.

7.7 Proposition. *Let $r > 2$, $0 < \lambda < (r-1)^{-1}$ and let Γ and $\check{\Gamma}$ be as in (5.7). Then, for every $\delta > 0$, the Hausdorff dimension of $\Lambda(\Gamma) = \Lambda(\check{\Gamma})$ is less than $2n+1+\delta$, i.e., the* **transverse** *Hausdorff dimension of $\Lambda(\Gamma) = \Lambda(\check{\Gamma})$ is less than $1 + \delta$.*

Proof. We recall that $\Lambda(\Gamma) = \cap_{k=0}^{\infty} F_k$, by the proof of theorem 5.8.i), where F_k is the disjoint union of the $r(r-1)^k$ closed tubular neighbourhoods $\gamma(N_i)$, $i \in \{1, \ldots, r\}$, where $\gamma \in \Gamma$ is an element which can be represented as a reduced word of length k in terms of the generators. $\gamma(N_i)$ is a closed tubular neighbourhood of $\gamma(L_i)$, as in theorem 5.7, and the "width" of each $\gamma(N_i)$, $w_{(\gamma,i)} := d(\gamma(E_i), L_i)$, satisfies $w_{(\gamma,i)} \leq C\lambda^k$, as was shown in lemma 5.6 and corollary 5.7. Hence,

$$w(k) := \sum_{\substack{l(\gamma)=k \\ i\in\{1,\ldots,r\}}} w_{(\gamma,i)}^{1+\delta} \leq Cr(r-1)^k \lambda^{k(1+\delta)} < C\,r(r-1)^{-\delta k}.$$

Thus, $\lim_{k\to\infty} w(k) = 0$. Hence, just as in the proof of the theorem of Marstrand [Mr], the Hausdorff dimension of $\Lambda(\Gamma)$ cannot exceed $2n+1+\delta$. $\square$

Next we will apply the previous estimates to compute the versal deformations of manifolds obtained from complex Schottky groups as in (7.7), whose limit sets have small Hausdorff dimension.

We first recall [Kod] that given a compact complex manifold X, a deformation of X consists of a triple $(\mathcal{X}, \mathcal{B}, \omega)$, where $\mathcal{X}$ and $\mathcal{B}$ are complex analytic spaces and $\omega : \mathcal{X} \to \mathcal{B}$ is a surjective holomorphic map such that $\omega^{-1}(t)$ is a complex manifold for all $t \in \mathcal{B}$ and $\omega^{-1}(t_0) = X$ for some t_0, which is called the reference point. It is known [Kur] that given X, there is always a deformation $(\mathcal{X}, \mathfrak{K}_X, \omega)$ which is *universal*, in the sense that every other deformation is induced from it (see also [KNS, Kod]). The space $\mathfrak{K}_X$ is the Kuranishi space of *versal* deformations of X [Kur]. If we let $\Theta := \Theta_X$ be the sheaf of germs of local holomorphic vector fields on X, then every deformation of X determines, via differentiation, an element in $H^1(X, \Theta)$, so $H^1(X, \Theta)$ is called the *space of infinitesimal deformations* of X ([Kod], Ch. 4). Furthermore ([KNS] or [Kod, Th. 5.6]), if $H^2(X, \Theta) = 0$, then the Kuranishi space $\mathfrak{K}_X$ is smooth at the reference point t_0 and its tangent space at t_0 is canonically identified with $H^1(X, \Theta)$. In particular, in this case *every infinitesimal deformation* of X comes from an actual deformation, and vice-versa, every deformation of the complex structure on X, which is near the original complex structure, comes from an infinitesimal deformation.

The following lemma is an immediate application of (7.7), and Harvey's Theorem 1 in [Ha], which generalises the results of Scheja [Schj].

7.8 Lemma. *Let* $r > 2$, $0 < \lambda < (r-1)^{-1}$, *let* $\check{\Gamma}$ *be as in proposition 7.7 and let* $\Omega := \Omega(\Gamma) \subset P_{\mathbb{C}}^{2n+1}$ *be its region of discontinuity . Then one has:*

$$H^j(\Omega, i^*(\Theta_{P_{\mathbb{C}}^{2n+1}})) \cong H^j(P_{\mathbb{C}}^{2n+1}, \Theta_{P_{\mathbb{C}}^{2n+1}}) \,,\ for\ 0 \leq j < n \,,$$

where i *is the inclusion of* Ω *in* $P_{\mathbb{C}}^{2n+1}$. *Hence, if* $n > 1$, *then one has:*

$$H^0(\Omega, i^*(\Theta_{P_{\mathbb{C}}^{2n+1}})) \cong \mathfrak{sl}(2n+2, \mathbb{C}) \quad and \quad H^j(\Omega, i^*(\Theta_{P_{\mathbb{C}}^{2n+1}})) \cong 0 \,,$$

for all $0 < j < n$, *where* $\mathfrak{sl}(2n+2, \mathbb{C})$ *is the Lie algebra of* $PSL(2n+2, \mathbb{C})$, *and it is being considered throughout this section as an additive group.*

Proof. By (7.7) we have that the Hausdorff dimension d of the limit set $\Lambda(\check{\Gamma})$ satisfies $d < 2n+1+\delta$ for every $\delta > 0$. Therefore the Hausdorff measure of $\Lambda(\Gamma)$ of dimension s, $\mathcal{H}_s(\Lambda(\Gamma))$, is zero for every $s > 2n+1$. Hence the first isomorphism in (7.8) follows from Theorem 1.ii) in [Ha], because the sheaf Θ is locally free. The second statement in (7.8) is now immediate, because $H^0(P_{\mathbb{C}}^{2n+1}, \Theta_{P_{\mathbb{C}}^{2n+1}}) \cong \mathfrak{sl}(2n+2, \mathbb{C})$ and $H^j(P_{\mathbb{C}}^{2n+1}, \Theta_{P_{\mathbb{C}}^{2n+1}}) \cong 0$ for $j > 0$, a fact which follows immediately by applying the long exact sequence in cohomology derived from the short exact sequence:

$$0 \to \mathcal{O} \to [\mathcal{O}(1)]^{n+1} \to \Theta_{P_{\mathbb{C}}^{2n+1}} \to 0,$$

where $\mathcal{O}$ is the structural sheaf of $P_{\mathbb{C}}^{2n+1}$ and $[\mathcal{O}(1)]^{n+1}$ is the direct sum of $n+1$ copies of $\mathcal{O}_{P_{\mathbb{C}}^{2n+1}}(1)$, the sheaf of germs of holomorphic sections of the holomorphic line bundle over $P_{\mathbb{C}}^{2n+1}$ with Chern class 1. See Hartshorne [Ht], Example 8.20.1, page 182. $\square$

We let $M := \Omega/\check{\Gamma}$, where $\check{\Gamma}$ is as above. We notice that Ω is simply connected when $n > 0$, so that Ω is the universal covering $\widetilde{M}$ of M. Let $p : \widetilde{M} \to M$ be the covering projection; since $\check{\Gamma}$ acts freely on Ω, this projection is actually given by the group action. Let Θ_M be the sheaf of germs of local holomorphic vector fields on M and let $\widetilde{\Theta}$ be the pull-back of Θ to $\widetilde{M}$ under the covering p; $\widetilde{\Theta}$ is the sheaf $i^*(\Theta_{P_{\mathbb{C}}^{2n+1}})$ on $\widetilde{M} = \Omega$.

7.9 Lemma. *If* $n > 2$, *then for* $0 \leq j \leq 2$ *we have:*

$$H^j(M, \Theta_M) \cong H^j(\check{\Gamma}, \mathfrak{sl}(2n+2, \mathbb{C})) \,.$$

Proof. If $n > 2$, then (7.8) and Mumford's formula (c) in [Mu], pag 23, (see also Grothendieck [Gr], Chapter V) imply that there exists an isomorphism

$$\phi : H^j(\check{\Gamma}, H^0(\Omega, \widetilde{\Theta})) \to H^j(M, \Theta_M) \,,$$

for $0 \leq j \leq 2$, where $H^0(\Omega, \widetilde{\Theta})$ is the vector space of holomorphic vector fields on the universal covering $\widetilde{M} = \Omega \subset P_{\mathbb{C}}^{2n+1}$ of M.

Now, by [Ha], Theorem 1.i), every holomorphic vector field in $\Omega(\Gamma)$, extends to a holomorphic vector field defined in all of $P_{\mathbb{C}}^{2n+1}$. Therefore,

$$H^0(\Omega, \widetilde{\Theta}) = H^0(P_{\mathbb{C}}^{2n+1}, \Theta_{P_{\mathbb{C}}^{2n+1}}) = \mathfrak{sl}(2n+2, \mathbb{C}) \,. \quad \square$$

We recall that $H^1(\check{\Gamma}, \mathfrak{sl}(2n+2, \mathbb{C})) \cong Hom_{\mathbb{Z}}(\check{\Gamma}, \mathfrak{sl}(2n+2, \mathbb{C}))$; since $\mathfrak{sl}(2n+2, \mathbb{C})$ is being considered as commutative group, every homomorphism from Γ into this group factors through the commutator of Γ. Thus we have:

$$H^1(\check{\Gamma}, \mathfrak{sl}(2n+2, \mathbb{C})) \cong Hom_{\mathbb{Z}}(\check{\Gamma}/[\check{\Gamma}, \check{\Gamma}], \mathfrak{sl}(2n+2, \mathbb{C})) \,.$$

As a vector space, $\mathfrak{sl}(2n+2,\mathbb{C})$ is isomorphic to $\mathbb{C}^{(2n+2)^2-1}$. Therefore we have that $H^1(\check{\Gamma},\mathfrak{sl}(2n+2,\mathbb{C}))$ is a complex vector space of dimension $(r-1)((2n+2)^2-1)$, because $\check{\Gamma}$ is a free group of rank $r-1$, so $\check{\Gamma}/[\check{\Gamma},\check{\Gamma}]=\mathbb{Z}^{r-1}$. We also notice that, since $\check{\Gamma}$ is a free group, the Eilenberg-MacLane space $K(\check{\Gamma},1)$ is a bouquet of circles S^1, so its cohomological dimension is 1. By the Universal Coefficients Theorem one has

$$H^2(\check{\Gamma},\mathfrak{sl}(2n+2,\mathbb{C}))\cong H^2(\check{\Gamma},\mathbb{Z})\otimes_{\mathbb{Z}}\mathfrak{sl}(2n+2,\mathbb{C})\cong 0\,,$$

Hence, one obtains,

$$H^2(M,\Theta_M)\cong H^2(\check{\Gamma},\mathfrak{sl}(2n+2,\mathbb{C}))\cong 0\,.$$

Thus we arrive to the following theorem:

7.10 Theorem. *Let $n,r>2$ and let λ be an arbitrary scalar such that $0<\lambda<(r-1)^{-1}$. Let Γ be a Schottky group as in (5.7.iii), so that the (Fubini-Study) distance from $\gamma(x)$ to the limit set Λ decreases faster that $C\lambda^k$ for every point $x\in P^{2n+1}_{\mathbb{C}}$ and any $\gamma\in\Gamma$ of word-length k (where C is some positive constant). Let $\check{\Gamma}$ be the index-two subgroup of Γ consisting of words of even length. Let Ω be the region of discontinuity of Γ, $M:=\Omega/\check{\Gamma}$, and let $\mathfrak{K}^n_r$ denote the Kuranishi space of versal deformations of M, with reference point $t_0\in\mathfrak{K}^n_r$ corresponding to M. Then, we have:*

$$H^1(M,\Theta_M)\;\cong\;H^1(\check{\Gamma},\mathfrak{sl}(2n+2,\mathbb{C}))\;\cong\;\mathbb{C}^{(r-1)((2n+2)^2-1)}\,,$$

and

$$H^2(M,\Theta_M)=0\,.$$

Hence $\mathfrak{K}^n_r$ is non-singular at t_0, of complex dimension $(r-1)((2n+2)^2-1)$, and every infinitesimal deformation of M is obtained by an infinitesimal deformation of $\check{\Gamma}$ as a subgroup of $PSL(2n+2,\mathbb{C})$.

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