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CASTELNUOVO-MUMFORD REGULARITY AND A-INVARIANT OF GRADED RINGS

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Abstract

This is a survey of recent results on the Castelnuovo-Mumford regularity and a-invariant of graded rings.

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Preliminaries

Let $S = \oplus_{n\geq 0} S_n$ be a finitely generated graded standard algebra over a local ring S_0. For convenience we assume that the residue field of S_0 is infinite.

Let $Q \subset S$ be any graded ideal. We will denote by $H^i_Q(S)$ the i-th local cohomology module of S with respect to Q. Put

$$a^i_Q = \sup\{n|\ H^i_Q(S)_n \neq 0\}.$$

Let $S_+ := \oplus_{n>0} S_n$. If $Q \supseteq S_+$, $H^i_Q(S)_n = 0$ for all n large enough, hence $a^i_Q < \infty$.

Definition. [EG] [O1] The number

$$\operatorname{reg}(S) := \max\{a^i_{S_+} - i|\ i \geq 0\}$$

is called the *Castelnuovo-Mumford regularity* of S.

Remark. Let $\mathfrak{m}$ denote the maximal ideal of S_0. The number

$$\ell(S_+) := \dim \oplus_{n\geq 0} S_n/\mathfrak{m}S_n$$

is called the *analytic spread* of S_+. It is well known that $H^i_{S_+}(S) = 0$ for $i > \ell(S_+)$.

The Castelnuovo-Mumford regularity carries a lot of information on the presentation of S. Let $S = S_0[T]/\mathcal{J}$ be a presentation of S as a factor ring of a polynomial ring $S_0[T]$. Then we denote by $\operatorname{reltype}(S)$ the maximal degree of the generators of $\mathcal{J}$. This number is independent of the presentation, and we call it the *relation type* of S.

Proposition. [T1] $\operatorname{reltype}(S) \leq \operatorname{reg}(S) + 1$.

If S_0 is a field k, this bound for $\operatorname{reltype}(S)$ is only a consequence of a more general fact. Let

$$0 \to F_s \to \cdots \to F_1 \to k[X] \to S$$

be a minimal free resolution of R as a $k[X]$-module. Let a_i be the maximal degree of the generators of F_i. Then

$$\operatorname{reg}(S) = \max\{a_i - i|\ i = 1, \dots, s\}.$$

This fact has often been used as the definition for $\operatorname{reg}(S)$ when S_0 is a field.

Let M denote the maximal graded ideal of S and $d = \dim S$. It is well known that $H^i_M(S) = 0$ for $i > d$. The number

$$a(S) := a^d_M$$

is an important invariant of S [GW]. For instance, if S has a canonical module K, then

$$a(S) = \min\{n|\ K_n \neq 0\}.$$

In order to estimate $a(S)$ we have studied in [T2] the number

$$a^*(S) := \max\{a^i_M|\ i \geq 0\}.$$

Recently, Hyry [Hy] has shown that

$$a^*(S) = \max\{a^i_{S_+} \mid i \geq 0\}.$$

This invariant has its own interest as shown by the following fact.

Remark. Let S_0 be of finite length. Let $H_S(n) := \text{length}(S_n)$ be the Hilbert function of S. Then there is a polynomial $P_S(n)$ such that $H_S(n) = P_S(n)$ for all n large enough. The number

$$p(S) := \min\{t \mid H_S(n) = P_S(n) \text{ for } n \geq t\}$$

is called the *postulation number* (or *Hilbert regularity*) of S. By a result of Serre we know that

$$p(S) \leq a^*(S) + 1.$$

As local cohomology modules are complicated objects, we want to find effective characterizations for $\text{reg}(S)$ and $a^*(S)$.

2. reg(S) and $a^*(S)$ via filter-regular sequences

We start with some observations on an arbitrary sequence $\mathbf{z} := z_1, \ldots, z_v$ of homogeneous elements of degree 1 in S. Put

$$Q_i := (z_1, \ldots, z_i).$$

Definition. [T1] The sequence $\mathbf{z}$ is called a *filter-regular* (with respect to S_+) if $z_i \notin P$ for all associated primes P of Q_{i-1}, $P \not\supseteq S_+$, $i = 1, \ldots, v$.

For any graded S-module E we set

$$\delta(E) := \inf\{n \mid E_n \neq 0\}.$$

Theorem. [T3] [T4] *Let* $\mathbf{z}$ *be a filter-regular sequence. Then*

$$\max\{a^i_{S_+} + i \mid i = 0, \ldots, v\} = \max\{\delta(Q_i : S_+/S_+) \mid i = 0, \ldots, v\},$$
$$\max\{a^i_{S_+} \mid i = 0, \ldots, v\} = \max\{\delta(Q_i : S_+/Q_i) - i \mid i = 0, \ldots, v\}.$$

It is surprising that the above formulas do not depend on the choice of the sequence $\mathbf{z}$. Moreover, we have

$$\delta(Q_i : S_+/Q_i) = \delta(Q_i : z_{i+1}/Q_i),\ i = 0, \ldots, v-1.$$

Thus, it makes sense to introduce the following invariants:

$$a(\mathbf{z}) := \min\{\delta(Q_{i-1} : z_i/Q_{i-1}) \mid i = 1, \ldots, v\},$$
$$s(\mathbf{z}) := \min\{\delta(Q_{i-1} : z_i/Q_{i-1}) - i + 1 \mid i = 1, \ldots, v\}.$$

They are called the *regularity* resp. *sliding regularity* of the sequence $\mathbf{z}$ [AHT]. It is not hard to see that $a(\mathbf{z}) < \infty$ or $s(\mathbf{z}) < \infty$ if and only if $\mathbf{z}$ is a filter-regular sequence.

Let Q be a reduction of S_+, i.e. a graded ideal in S_+ with $S_n = Q_n$ for n large enough. The number

$$r_Q(S_+) := \min\{n |\ S_{n+1} = Q_{n+1}\}$$

is called the *reduction number* of S_+ with respect to Q.

We will consider only the case when Q is generated by a sequence $\mathbf{z} = z_1, \dots, z_v$ of homogeneous elements of degree 1. Since the residue field of S_0 is infinite, we may always choose (and therefore assume) $\mathbf{z}$ to be a filter-regular sequence.

It is easy to check that

$$\delta(Q_v : S_+/Q_v) = r_Q(S_+).$$

Since $v \geq \ell(S_+)$, the above theorem yields the following result:

Theorem. [T3] [T4] *Let Q and $\mathbf{z}$ be as above. Then*

$$\begin{aligned} \operatorname{reg}(S) &= \max\{a(\mathbf{z}),\ r_Q(S_+)\},\\ a^*(S) &= \max\{s(\mathbf{z}),\ r_Q(S_+) - v\}. \end{aligned}$$

Note that these formulas are independent of the choice of $\mathbf{z}$ and Q. If Q is a minimal reduction of S_+, the formula for $a^*(S)$ was also found by Herrmann et al [HHK]. From the above formulas we can deduce that

$$\begin{aligned} \operatorname{reg}(S) &\geq 0,\\ a^*(S) &\geq -\operatorname{grade}(S_+). \end{aligned}$$

Moreover, there is the following relationship between $\operatorname{reg}(S)$ and $a^*(S)$:

Corollary. $a^*(S) + \operatorname{grade}(S_+) \leq \operatorname{reg}(S) \leq a^*(S) + \ell(S_+)$.

3. Applications to the associated graded ring

Let A be a local ring with infinite residue field and I an ideal of A. Let

$$G := \oplus_{n\geq 0} I^n/I^{n+1}$$

denote the associated graded ring of I. We will apply the above results to compute $\operatorname{reg}(G)$ and $a^*(G)$ by means of elements of I.

Our starting point is a reduction J of I, i.e. an ideal $J \subseteq I$ such that there is an integer n for which

$$I^{n+1} = JI^n.$$

The least number n with this property is called the *reduction number* of I with respect to J, and we will denote it by $r_J(I)$.

Let $x_1, \dots, x_v$ be a generating set of J. Let z_i be the image of x_i in I/I^2. Since the residue field of A is infinite, we may always choose (and therefore assume) that $z_1, \dots, z_v$ is a filter-regular sequence of G. Put

$$J_i = (x_1, \dots, x_i).$$

Theorem. [T3] reg(G) *is the least number* $r \geq r_J(I)$ *such that*

$$(J_{i-1} : x_i) \cap I^{r+1} = J_{i-1}I^r,\ i = 1, \dots, v.$$

The above characterization of reg(G) is independent of the choice of J. In particular, the case $J = I$ yields the following interesting result.

Corollary. reg$(G) = r$ *if and only if* I *is generated by a sequence* $x_1, \dots, x_v$ *such that*

$$(J_{i-1} : x_i) \cap I^{r+1} = J_{i-1}I^r,\ i = 1, \dots, v.$$

For $r = 0$ the above formula is one of the defining conditions for $x_1, \dots, x_v$ to be a d-sequence [Hu2].

Corollary. reg$(G) = 0$ *if and only if* I *is generated by a* d*-sequence.*

Remark. Let $R := \oplus_{n \geq 0} I^n t^n$ denote the Rees algebra of I. Ooishi [O2] has proven that we always have

$$\text{reg}(R) = \text{reg}(G).$$

One of the most important properties of an ideal generated by a d-sequence is that it is of linear type, i.e. reltype$(R) = 1$ [Hu1], [V]. Since reltype$(R) \leq$ reg$(R) + 1$, this is only a consequence of the fact reg$(R) = 0$. Thus, the above corollary clearly sets apart ideals of linear type from those generated by d-sequences.

There is a similar characterization for $a^*(G)$ in terms of any reduction J of I.

Theorem. [T4] $a^*(G)$ *is the least number* $s \geq r_J(I) - v$ *such that*

$$(J_{i-1} : x_i) \cap I^{s+i} = J_{i-1}I^{s+i-1},\ i = 1, \dots, v.$$

Corollary. *If* I *is generated by a* d*-sequence, then*

$$a^*(G) = -\operatorname{grade}(I).$$

Example. (1) I is a generic codimension 2 Cohen-Macaulay ideal: $a(G) = -2$.
(2) I is a generic codimension 3 Gorenstein ideal: $a(G) = -3$.

We conclude this survey with a less trivial application to the class of analytic deviation 1 ideals recently studied by Huckaba and Huneke [HH].

Corollary. *Let* A *be a regular local ring. Let* I *be a prime ideal with* $ht(I) \geq 1$, $ad(I) = 1$. *Assume that* I_P *is a complete intersection for all primes* $P \supset I$ *with* $ht(P/I) = 1$. *Then* reg$(G) \leq 1$ *and* $a^*(G) = -ht(I)$.

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