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THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

AN ALGORITHM FOR GENERAL QUADRATIC PROGRAMMING

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Abstract

The method to be described here is an extension to the Dantzig-Wolfe method for convex QP problems. Our method can successfully locate a KT point for a general QP problem. It even solves concave QP problems without any additional effort. The main difference from the Dantzig-Wolfe method is that it allows for the decreasing of the multipliers during non-complementary iterations. The main effort of this work is devoted to proving results that lead to the conclusion of successful termination with the assumption of boundedness and non-degeneracy.

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1. Introduction

The model problem to be solved is

$$Minimize\ f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\ G\mathbf{x} + \mathbf{g}^T\mathbf{x}$$

$$\textit{subject to} \tag{1.1}$$

$$A^T\mathbf{x} \geq \mathbf{b},$$

where G is an $n \times n$ symmetric matrix and A is $n \times m$. The ith column of A is denoted by $\mathbf{a}_i$, and carries the coefficients of the ith constraint. The vector $\mathbf{x}$ is the unknown vector to be found.

Historically, the name of quadratic programming was restricted to the specific problem of minimizing a convex quadratic function subject to linear constraints (see Wolfe[16] and Dantzig[7]). Later on, the definition was extended to include the problem of finding a local minimum of any quadratic function subject to linear constraints (see Beale[1]).

When the function to be minimized is convex, the problem is well understood both theoretically and computationally. Many active set methods have been designed. Best[2] has pointed out that under certain assumptions various methods designed to solve convex problems are equivalent. Fletcher[8] has shown that the active set method is equivalent to the Dantzig-Wolfe method. In solving the general problem when the function is nonconvex some of the convex programming solvers can be modified to terminate successfully , as was done by Gill and Murray [9]. Gill and Murray did modify , in a stable way , the active set method. Murray[13] also made an algorithm for the indefinite case.

There are also other classes of methods designed to solve the general problem. Among those are the Ritter cutting plane methods (Cottle and Mylander [6]). Of more interest to us, here, are the complementarity pivoting methods. Our method could be considered to be one of them. This class of methods has been applied to solve convex programming problems.(See Cottle[3] , Cottle and Dantzig[5], and recently Cottle[4].) Complementarity pivoting methods can work for a wider range of quadratic programming problems (even nonconex ones.) However, Hashim[10] has shown the failure of such methods by giving an example. There are also other methods with the same idea of pivoting. Among these is the algorithm designed by Keller[11], a method by Van de Panne and Whinston[15], and also Lemke[12].

Our method is capable to solve concave quadratic programming. It practically searches the solution at the vertices. (For more about concave minimization see Pardalose and Rosen [14].)

In section(3) we introduce our method and prove theorems to show its successful termination at a KT point under the practical assumption of boundedness. Also cycling is not supposed to happen as a result of degeneracy. In the next section we introduce and give preliminaries that pave the way to the description of the method. The next section will end up with a general description of the Dantzig-Wolfe method to which our method is an extension.

2. Preliminaries

The KT conditions for (1.1) are

$$G\mathbf{x} - A\lambda + \mathbf{g} = \mathbf{0},$$

$$-A^T\mathbf{x} + \mathbf{v} + \mathbf{b} = \mathbf{0}, \tag{2.1}$$

$$\mathbf{v}, \lambda \geq \mathbf{0} \textit{ and } \lambda^T\mathbf{v} = 0,$$

where λ are the multipliers corresponding to $A^T\mathbf{x} \geq \mathbf{b}$ and $\mathbf{v}$ are slack variables. The condition $\lambda^T\mathbf{v} = 0$ is known as the complementarity condition. In that case we say that the tableau is complementary. (2.1) could be rewritten as

$$M\mathbf{t} = \mathbf{q}, \tag{2.2}$$

where

$$M = \begin{bmatrix} G & -A & O \\ -A^T & O & I \end{bmatrix}, \mathbf{t} = \begin{bmatrix} \mathbf{x} \\ \lambda \\ \mathbf{v} \end{bmatrix}, \textit{ and } \mathbf{q} = \begin{bmatrix} -\mathbf{g} \\ -\mathbf{b} \end{bmatrix}.$$

Now , rearrange M in the following partitioned form

$$M = [M_B : M_N],$$

where M_B is an $(n+m) \times (n+m)$ non-singular matrix (called the basis matrix) , and M_N is $(n+m) \times m$. Correspondingly $\mathbf{t}$ and $\mathbf{q}$ are rearranged and partitioned to

$$\mathbf{t} = \begin{bmatrix} \mathbf{t}_B \\ \mathbf{t}_N \end{bmatrix}, \textit{ and } \mathbf{q} = \begin{bmatrix} \mathbf{q}_B \\ \mathbf{q}_N \end{bmatrix}.$$

The components of $\mathbf{t}_B$ are called basic variables and those of $\mathbf{t}_N$ are called nonbasic. The nonbasic variables are always kept at zero. Notice that the components of $\mathbf{x}$ are always among the basic variables.

The basis matrix will initially take the form

$$M_B = \begin{bmatrix} G & -A_1 & O \\ -A_1^T & O & O \\ -A_2^T & O & I \end{bmatrix}.$$

(This form always appears when the tableau is complementary , i.e. when $\lambda^T\mathbf{v} = 0$.) Here A_1 contains those (l say) columns of A corresponding to the (basic) Lagrange multipliers, λ_1 say. A_2 carrys the remaining $m - l$ columns. The basic vector $\mathbf{t}_B$ takes the form

$$\mathbf{t}_B = \begin{bmatrix} \mathbf{x} \\ \lambda_1 \\ \mathbf{v}_2 \end{bmatrix},$$

where $\mathbf{v}_2$ is the $(m-l) - \textit{vector}$ carrying the basic slack variables. Respectively we obtain the resulting forms for $M_N, \mathbf{t}_N, \mathbf{q}_B$, and $\mathbf{q}_N$ as

$$\begin{bmatrix} -A_2 & O \\ O & I \\ O & O \end{bmatrix}, \begin{bmatrix} \lambda_2 \\ \mathbf{v}_1 \end{bmatrix}, \begin{bmatrix} -\mathbf{g} \\ -\mathbf{b}_1 \end{bmatrix}, \textit{ and } \mathbf{b}_2.$$

We now move to give a representation of f in terms of $\mathbf{x}, \lambda$ and $\mathbf{v}$. This representation has been used by Keller[11]. Pre-multiplying the first equation of (2.1) by $\frac{1}{2}\mathbf{x}^T$ and the second by $\frac{1}{2}\mathbf{x}\lambda^T$ we get, respectively

$$\frac{1}{2}\mathbf{x}^T G\mathbf{x} + \frac{1}{2}\mathbf{x}^T A\lambda + \frac{1}{2}\mathbf{g}^T\mathbf{x} = \mathbf{0}$$

and

$$0.5\lambda^T A^T\mathbf{x} + \frac{1}{2}\lambda^T\mathbf{v} + \frac{1}{2}\mathbf{b}^T\lambda = \mathbf{0}.$$

On subtraction of the above two equations we get

$$f(\mathbf{x}) = \frac{1}{2}(\mathbf{g}^T\mathbf{x} + \mathbf{b}^T\lambda + \mathbf{v}^T\lambda) \tag{2.3}$$

Before ending up this section we give a general description of the Dantzig-Wolfe method to solve (2.1). The method is iterative. It starts with a complementary tableau with $\mathbf{v}_2 \geq \mathbf{0}$, $\lambda_2 = \mathbf{0}$, $\lambda_1 \geq \mathbf{0}$, and $\mathbf{v}_1 = \mathbf{0}$. During the remaining iterations the basic slack variables are to be kept non-negative.

Let us assume that at the kth iteration the tableau is complementary. Using superscripts for the relevant matrices , vectors , and scalars we can write $M^{(k)}$ for the basis matrix , $A_1^{(k)}$ for the partition of A corresponding to the basic multipliers $\lambda_1^{(k)}$, and so on. Also define at the kth iteration the set $I^{(k)}$of active constraints. That is

$$I^{(k)} = \{i : v_i^{(k)} \ is \ nonbasic\}.$$

At the kth iteration of the Dantzig-Wolfe method if $\lambda_1^{(k)} \geq \mathbf{0}$ then $\mathbf{x}^{(k)}$ is a KT point and the method will terminate. Otherwise, $q \in I^{(k)}$ is chosen for which $\lambda_q^{(k)} < 0$. It has been a general agreement , although not always the best computationally, to choose q that solves

$$\min_{i \in I^{(k)}} \lambda_i^{(k)}. \tag{2.4}$$

In the next step the complementary variable v_q is chosen to be increased. The effect on the basic variables is then observed. As long as all the basic slack variables stay non-negative, v_q is increased until $\lambda_q \uparrow 0$. The resulting tableau is again complementary and $I^{(k)}$ is updated to $I^{(k+1)}$ by removing q. It may be , however , that when v_q is increased a basic slack variable $v_{p_1}(p_1 \notin I^{(k)})$ decreases to zero. In this case an interchange takes place by removing v_{p_1} from the basic variables and adding v_q to the basic variables. As a result $I^{(k)}$ is updated by removing q and adding p_1. In the next iteration the complement of v_{p_1}, λ_{p_1} , is then increased. If $\lambda_q \uparrow 0$ then complementarity is restored, and the process is repeated again as above. In general, the process might add $p_1, p_2, \ldots , p_r$ to $I^{(k)}$ before complementarity is restored. This will not go on indefinitely (assuming non-degeneracy) since $r + l_1^{(k)} \leq min(m, n)$.

It has been proved [7] that the values of the function keep on decreasing from iteration to iteration when G is positive definite. This ensures termination if the problem is bounded.

3. The Method

In this section we start by giving a general description of the method. The description is given parallel to that of the Dantzig-Wolfe method, so that the slight difference becomes obvious. The main difference occurs when proving the successful termination , since we are dealing with a general objective function.

If the kth iteration is complementary (i.e. when the tableau is complementary) and $\lambda^{(k)} \geq \mathbf{0}$ is not satisfied, the next iteration is the same as in the Dantzig-Wolfe method, that is , the slack variable corresponding to the most negative Lagrange multiplier is to be increased. The increase might be blocked by v_{p_1} decreasing to zero before complementarity is restored. The next iteration in our method is slightly different from that of the Dantzig-Wolfe method. In the latter λ_{p_1} is to be increased , while in ours it might be increased or decreased. The choice between increasing or decreasing λ_{p_1} is made to ensure that the objective function f decreases. The case when G is positive definite the increase of λ_{p_1} guarantees the decrease. However , in the general case , when G is indefinite , the increase might not decrease f. So at each iteration a decision has to be made (based on a simple condition) on whether to increase or decrease λ_{p_1}. Thus it is better to say "λ_{p_1} is changed" to mean either increased or decreased. Now if the change of λ_{p_1} is blocked again by v_{p_2} the next iteration will be to change "increase or decrease" λ_{p_2} , and so on , until complementarity is restored. The process is thus repeated again.

Before getting into proving results that guarantee a successful termination , we show out the shape of some basis matrices and the corresponding expression of the objective function. If at the kth iteration the tableau is complementary, the basis matrix will have the form

$$M_B^{(k)} = \begin{bmatrix} G & -A_1^{(k)} & O \\ -A_1^{(k)^T} & O & O \\ -A_2^{(k)^T} & O & I \end{bmatrix}, \tag{3.1}$$

The general shape of the basis matrix after r successive non-complementary tableaux ($l^{(k)} + r \leq min(m, n)$), with suitable rearrangements, is

$$M_B^{(k+r)} = \begin{bmatrix} G & -A_1^{(k)} & -W & \mathbf{0} & O \\ -A_1^{(k)^T} & O & O & \mathbf{e}_q & O \\ -W^T & O & O & \mathbf{0} & O \\ -\mathbf{a}_{p_r}^T & \mathbf{0}^T & \mathbf{0}^T & 0 & \mathbf{0}^T \\ -A_2^{(k+r)^T} & O & O & \mathbf{0} & I \end{bmatrix}, \tag{3.2}$$

where $W = [\mathbf{a}_{p_1}, \; ... \; , \mathbf{a}_{p_{r-1}}]$ and $A_2^{(k+r))}$ is the matrix that results when removing $\mathbf{a}_{p_1}, \; ... \; , \mathbf{a}_{p_r}$ from $A_2^{(k)}$. For example when $r = 1$ we get

$$M_B^{(k+1)} = \begin{bmatrix} G & -A_1^{(k)} & \mathbf{0} & O \\ -A_1^{(k)^T} & O & \mathbf{e}_q & O \\ -\mathbf{a}_{p_r}^T & \mathbf{o}^T & 0 & \mathbf{0}^T \\ -A_2^{(k+1)^T} & O & \mathbf{0} & I \end{bmatrix}. \tag{3.3}$$

Corresponding to (3.1) and using (2.3) the function value $f^{(k)}(\equiv f(\mathbf{x}^{(k)}))$ will be given by

$$f^{(k)} = \frac{1}{2}(\mathbf{g}^T\mathbf{x}^{(k)} + \mathbf{b}_1^{(k)^T}\lambda_1^{(k)}). \tag{3.4}$$

When $v_q, q \in I^{(k)}$, is increased from zero f changes according to

$$f = \frac{1}{2}(\mathbf{g}^T\mathbf{x} + \mathbf{b}_1^{(k)^T}\lambda_1 + \lambda_q v_q), \tag{3.5}$$

where $\mathbf{x}$, λ, and λ_q are the corresponding changes in these basic variables. In the next iteration if λ_{p_1} is to be changed (as a result of v_q being blocked by v_{p_1}), the corresponding expression of f is given by

$$f = \frac{1}{2}(\mathbf{g}^T\mathbf{x} + \mathbf{b}_1^{(k)^T}\lambda_1 + b_{p_1}\lambda_{p_1} + \lambda_q v_q). \tag{3.6}$$

If after $r-1$ successive non-complementary tableaux $p_1, p_2, \ldots, p_{r-1}$ are added to $I^{(k)}$, and in the $(k+r)th$ iteration it is decided to change λ_{p_r} (as a result of $\lambda_{p_{r-1}}$ being blocked by v_{p_r}) , the function f changes according to

$$f = \frac{1}{2}(\mathbf{g}^T\mathbf{x} + \mathbf{b}_1^{(k)^T}\lambda_1 + \mathbf{b}_W^T\lambda_W + b_{p_r}\lambda_{p_r} + \lambda_q v_q), \tag{3.7}$$

where $\mathbf{b}_w^T = [b_{p_1}, \ldots, b_{p_{r-1}}]$ and $\lambda_W^T = [\lambda_{p_1}, \ldots, \lambda_{p_{r-1}}]$.

In the proof of the following results we will be using the matrices H, T, and U which are defined by

$$\begin{bmatrix} G & -A_1^{(k)} \\ -A_1^{(k)^T} & O \end{bmatrix}^{-1} = \begin{bmatrix} H & -T \\ -T^T & U \end{bmatrix}, \tag{3.8}$$

(see [8]). It is obvious that both H and U are symmetric. We now start with the following theorem.

Theorem 3.1. *Suppose $\lambda_q^{(k)} < 0$ at the kth iteration when the tableau is complementary. Let the choice of the next move be to increase v_q. Suppose , before complementarity is restored, the following r iterations added $p_1, \ldots, p_r$ to $I^{(k)}$ $(r + l^{(k)} \leq min(n,m))$. Then , in the coming iteration , we will have the following changes , in terms λ_{p_r}, in f, λ_q, and v_q respectively*

$$f = f^{(k+r)} + \alpha(\lambda_q^{(k+r)}\lambda_{p_r} + \frac{1}{2}\beta\lambda_{p_r}^2), \tag{3.9}$$

$$\lambda_q = \lambda_q^{k+r} + \beta\lambda_{p_r}, \tag{3.10}$$

$$v_q = v_q^{k+r} + \alpha\lambda_{p_r}, \tag{3.11}$$

where α and β are related, respectively, to v_q and λ_q in the $(k+r)th$ tableau.

Before proving this theorem we have to show that the equations from (3.9) to (3.11) imply that the function value will not increase by any change of λ_{p_r}. This is shown in the following theorem.

Theorem 3.2. *The change in λ_{p_r} in theorem 3.1 will not increase the function value.*

Proof: There are three cases to consider :

The first case is when $\alpha > 0$. In this case there are two cases. We consider first the case when $\beta \leq 0$. Since $\lambda_q^{(k+r)} < 0$ (3.9) implies that the increase of λ_{p_r} will decrease f until it is blocked by $v_{p_{r+1}}$ ($p_{r+1} \notin I^{(k+1)}$ since from (3.11) v_q increases with the increase of λ_{p_r}). In the second case when $\beta > 0$ the increase of λ_{p_r} causes f to decrease until λ_{p_r} reaches $-\frac{\lambda_q^{(k+r)}}{\beta}$. This is the value at which $\lambda_q \uparrow 0$ (from (3.10)). However , it might happen that a slack basic variable different from v_q might block this increase before λ_{p_r} reaches $-\frac{\lambda_q^{(k+r)}}{\beta}$.

The second case is when $\alpha < 0$. We consider two cases here also. When $\beta \geq 0$ the decrease of λ_{p_r} will decrease f until it is blocked by $v_{p_{r+1}}$, ($p_{r+1} \notin I^{(k+1)}$ for the same reason as above). When $\beta < 0$ the decrease of λ_{p_r} causes f to decrease until λ_{p_r} reaches $-\frac{\lambda_q^{(k+r)}}{\beta}$. As above, this decrease will either be blocked by a basic slack variable different from v_q or restore complementarity by increasing λ_q to zero.

The third case is when $\alpha = 0$. In this case f stays fixed at $f^{(k+r)}$. The proof is thus complete.

Note that, the third case in the above proof leaves a question to be asked. If both α and β are zero, what happens in the next iteration if no blockage to the change in λ_{p_r} takes place. This blockage is only guaranteed when f is decreasing. Fortunately , when $\alpha = 0$, β cannot be zero. This will be shown later in this section. Now we return to prove theorem 3.1.

Proof of Theorem 3.1 The values of the basic variables $\mathbf{x}^{(k+r)}$, $\lambda_1^{(k+r)}$, $\lambda_W^{(k+r)}$, and $v_q^{(k+r)}$ are given by

$$\text{(3.12)} \qquad \begin{bmatrix} G & -A_1^{(k)} & -W & \mathbf{0} \\ -A_1^{(k)T} & O & O & \mathbf{e}_q \\ -W^T & O & O & \mathbf{0} \\ -\mathbf{a}_{p_r}^T & \mathbf{0}^T & \mathbf{0}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}^{(k+r)} \\ \lambda_1^{(k+r)} \\ \lambda_W^{(k+r)} \\ v_q^{(k+r)} \end{bmatrix} = \begin{bmatrix} -\mathbf{g} \\ -\mathbf{b}_1^{(k)} \\ -\mathbf{b}_W \\ -b_{p_r} \end{bmatrix}.$$

Pre-multiplying (3.12) by

$$\begin{bmatrix} I & O & O & \mathbf{0} \\ O & I & O & \mathbf{0} \\ W^T & O & I & \mathbf{0} \\ \mathbf{a}_{p_r}^T & \mathbf{0}^T & \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} H & -T & O & \mathbf{0} \\ -T^T & U & O & \mathbf{0} \\ O & O & I & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & 1 \end{bmatrix},$$

we get

$$\text{(3.13)} \qquad \begin{bmatrix} I & O & -HW & -T\mathbf{e}_q \\ O & I & T^TW & U\mathbf{e}_q \\ O & O & -W^THW & -W^TT\mathbf{e}_q \\ \mathbf{0}^T & \mathbf{0}^T & -\mathbf{a}_{p_r}^THW & -\mathbf{a}_{p_r}^TT\mathbf{e}_q \end{bmatrix} \begin{bmatrix} \mathbf{x}^{(k+r} \\ \lambda_1^{(k+r)} \\ \lambda_W^{(k+r)} \\ v_q^{(k+r)} \end{bmatrix} =$$

$$\begin{bmatrix} -H\mathbf{g} + T\mathbf{b}_1^{(k)} \\ T^T\mathbf{g} - U\mathbf{b}_1^{(k)} \\ -W^TH\mathbf{g} + W^TT\mathbf{b}_1^{(k)} - \mathbf{b}_W \\ -\mathbf{a}_{p_r}^TH\mathbf{g} + \mathbf{a}_{p_r}^TT\mathbf{b}_1^{(k)} - b_{p_r} \end{bmatrix}.$$

Now , suppose $\lambda_W^{(k+r)}$ and $v_q^{(k+r)}$ are obtained from

$$\begin{bmatrix} W^T HW & W^T T\mathbf{e}_q \\ \mathbf{a}_{p_r}^T HW & \mathbf{a}_{p_r}^T T\mathbf{e}_q \end{bmatrix} \begin{bmatrix} \lambda_W^{(k+r)} \\ v_q^{(k+r)} \end{bmatrix} = \begin{bmatrix} W^T H\mathbf{g} - W^T T\mathbf{b}_1^{(k)} + \mathbf{b}_W \\ \mathbf{a}_{p_r}^T H\mathbf{g} - \mathbf{a}_{p_r}^T T\mathbf{b}_1^{(k)} + b_{p_r} \end{bmatrix}. \tag{3.14}$$

Then $\mathbf{x}^{(k+r)}$ and $\lambda_1^{(k+r)}$ can be found , in terms of $\lambda_W^{(k+r)}$ and $v_q^{(k+r)}$, by

$$\mathbf{x}^{(k+r)} = HW\lambda_W^{(k+r)} + T\mathbf{e}_q v_q^{(k+r)} - H\mathbf{g} + T\mathbf{b}_1^{(k)}, \tag{3.15}$$

$$\lambda_1^{(k+r)} = -T^T W\lambda_W^{(k+r)} - U\mathbf{e}_q v_q^{(k+r)} + T^T\mathbf{g} - U\mathbf{b}_1^{(k)}. \tag{3.16}$$

From (3.16) $\lambda_q^{(k+r)}$ is given by

$$\lambda_q^{(k+r)} = -\mathbf{e}_q^T T^T W\lambda_W^{(k+r)} - \mathbf{e}_q^T U\mathbf{e}_q v_q^{(k+r)} + \mathbf{e}_q^T T^T\mathbf{g} - \mathbf{e}_q^T U\mathbf{b}_1^{(k)}. \tag{3.17}$$

Now the change in λ_{P_r} causes these basic variables to change according to

$$\mathbf{x} = \mathbf{x}^{(k+r)} - \mathbf{d}_x^{(k+r)}\lambda_{p_r}, \tag{3.18}$$

$$\lambda_1 = \lambda_1^{(k+r)} - \mathbf{d}_\lambda^{(k+r)}\lambda_{p_r}, \tag{3.19}$$

$$\lambda_W = \lambda_W^{(k+r)} - \mathbf{d}_W^{(k+r)}\lambda_{p_r}, \tag{3.20}$$

$$v_q = v_q^{(k+r)} - d_{v_q}^{(k+r)}\lambda_{p_r}, \tag{3.21}$$

and , writing $d_{\lambda_q}^{(k+r)} = \mathbf{e}_q^T\mathbf{d}_\lambda^{(k+r)}$, (3.19) gives

$$\lambda_q = \lambda_q^{(k+r)} - d_{\lambda_q}^{(k+r)}\lambda_{p_r}, \tag{3.22}$$

where $\mathbf{d}_x^{(k+r)}$, $\mathbf{d}_\lambda^{(k+r)}$, $\mathbf{d}_W^{(k+r)}$, and $d_{v_q}^{(k+r)}$ are given by

$$\begin{bmatrix} G & -A_1^{(k)} & -W & \mathbf{0} \\ -A_1^{(k)T} & O & O & \mathbf{e}_q \\ -W^T & O & O & \mathbf{0} \\ -\mathbf{a}_{p_r}^T & \mathbf{0}^T & \mathbf{0}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{d}_x^{(k+r)} \\ \mathbf{d}_\lambda^{(k+r)} \\ \mathbf{d}_W^{(k+r)} \\ d_{v_q}^{(k+r)} \end{bmatrix} = \begin{bmatrix} -\mathbf{a}_{p_r} \\ \mathbf{0} \\ \mathbf{0} \\ 0 \end{bmatrix}. \tag{3.23}$$

(3.23) can be transformed , as we did for (3.12) , to

$$\begin{bmatrix} I & O & -HW & -T\mathbf{e}_q \\ O & I & T^T W & U\mathbf{e}_q \\ O & O & -W^T HW & -W^T T\mathbf{e}_q \\ \mathbf{0}^T & \mathbf{0}^T & -\mathbf{a}_{p_r}^T HW & -\mathbf{a}_{p_r}^T T\mathbf{e}_q \end{bmatrix} \begin{bmatrix} \mathbf{x}^{(k+r} \\ \lambda_1^{(k+r)} \\ \lambda_W^{(k+r)} \\ v_q^{(k+r)} \end{bmatrix} = \begin{bmatrix} -H\mathbf{a}_{p_r} \\ T^T\mathbf{a}_{p_r} \\ -W^T H\mathbf{a}_{p_r} \\ -\mathbf{a}_{p_r}^T H\mathbf{a}_{p_r} \end{bmatrix}. \tag{3.24}$$

Now , suppose $\mathbf{d}_W^{(k+r)}$ and $d_{v_q}^{(k+r)}$ are obtained from

$$\begin{bmatrix} W^T H W & W^T T \mathbf{e}_q \\ \mathbf{a}_{p_r}^T H W & \mathbf{a}_{p_r}^T T \mathbf{e}_q \end{bmatrix} \begin{bmatrix} \mathbf{d}_W^{(k+r)} \\ d_{v_q}^{(k+r)} \end{bmatrix} = \begin{bmatrix} W^T H \mathbf{a}_{p_r} \\ \mathbf{a}_{p_r}^T H \mathbf{a}_{p_r} \end{bmatrix}. \tag{3.25}$$

Then $\mathbf{d}_x^{(k+r)}$ and $\mathbf{d}_\lambda^{(k+r)}$ are recovered , in terms of $\mathbf{d}_W^{(k+r)}$ and $d_{v_q}^{(k+r)}$, by

$$\mathbf{d}_x^{(k+r)} = HW\mathbf{d}_W^{(k+r)} + T\mathbf{e}_q d_{v_q}^{(k+r)} - H\mathbf{a}_{p_r}, \tag{3.26}$$

$$\mathbf{d}_\lambda^{(k+r)} = -T^T W\mathbf{d}_W^{(k+r)} - U\mathbf{e}_q d_{v_q}^{(k+r)} + T^T\mathbf{a}_{p_r}. \tag{3.27}$$

Using (3.27),

$$d_{\lambda_q}^{(k+r)} = -\mathbf{e}_q^T T^T W\mathbf{d}_W^{(k+r)} - \mathbf{e}_q^T U\mathbf{e}_q d_{v_q}^{(k+r)} + \mathbf{e}_q^T T^T\mathbf{a}_{p_r}. \tag{3.28}$$

Now , using equations from (3.18) to (3.22) in (3.7) , f changes with λ_{p_r} according to

$$f = f^{(k+r)} + \frac{1}{2}[(b_{p_r} - \mathbf{g}^T\mathbf{d}_x^{(k+r)} - \mathbf{b}_1^{(k)^T}\mathbf{d}_\lambda^{(k+r)} - \mathbf{b}_W^T\mathbf{d}_W^{(k+r)} \\ -v_q^{(k+r)}d_{\lambda_q}^{(k+r)} - d_{v_q}^{(k+r)}\lambda_q^{(k+r)})\lambda_{p_r} + d_{\lambda_q}^{(k+r)}d_{v_q}^{(k+r)}\lambda_{p_r}^2]. \tag{3.29}$$

We now move to simplify the coefficient of λ_{p_r} in the expression for f in the above equation. Using (3.26) and (3.27) we get

$$\mathbf{g}^T\mathbf{d}_x^{(k+r)} = \mathbf{g}^T HW\mathbf{d}_W^{(k+r)} + \mathbf{g}^T T\mathbf{e}_q d_{v_q}^{(k+r)} - \mathbf{g}^T H\mathbf{a}_{p_r}, \tag{3.30}$$

$$\mathbf{b}_1^{(k)^T}\mathbf{d}_\lambda^{(k+r)} = -\mathbf{b}_1^{(k)^T} T^T W\mathbf{d}_W^{(k+r)} - \mathbf{b}_1^{(k)^T} U\mathbf{e}_q d_{v_q}^{(k+r)}. \\ +\mathbf{b}_1^{(k)^T} T^T\mathbf{a}_{p_r} \tag{3.31}$$

From (3.12) we have $\mathbf{b}_W = W^T\mathbf{x}^{(k+r)}$ and $b_{p_r} = \mathbf{a}_{p_r}^T\mathbf{x}^{(k+r)}$, and so , with (3.15) we get

$$\mathbf{b}_W^T\mathbf{d}_W^{(k+r)} = [W^T\mathbf{x}^{(k+r)}]^T\mathbf{d}_W^{(k+r)} = \mathbf{x}^{(k+r)^T}W\mathbf{d}_W^{k+r)} \\ -\mathbf{g}^T HW\mathbf{d}_W^{(k+r)} + \mathbf{b}_1^{(k)^T} T^T W\mathbf{d}_W^{(k+r)} + \lambda_W^{(k+r)^T} W^T HW\mathbf{d}_W^{(k+r)} \\ +\mathbf{e}_q^T T^T W\mathbf{d}_W^{(k+r)} v_q^{(k+r)}, \tag{3.32}$$

and

$$b_{p_r} = \mathbf{a}_{p_r}^T\mathbf{x}^{(k+r)} = -\mathbf{a}_{p_r}^T H\mathbf{g} + \mathbf{a}_{p_r}^T T\mathbf{b}_1^{(k)} \\ +\mathbf{a}_{p_r}^T HW\lambda_W^{(k+r)} + \mathbf{a}_{p_r}^T T\mathbf{e}_q v_q^{(k+r)}. \tag{3.33}$$

Using (3.28) we get

$$d_{\lambda_q}^{(k+r)}v_q^{(k+r)} = \mathbf{e}_q^T T^T\mathbf{a}_{p_r} v_q^{(k+r)} - \mathbf{e}_q^T T^T W\mathbf{d}_W^{(k+r)} v_q^{(k+r)} \\ -\mathbf{e}_q^T U\mathbf{e}_q d_{v_q}^{(k+r)} v_q^{(k+r)}. \tag{3.34}$$

Now combining the equations from (3.30) to (3.34) we get

$$b_{p_r} - \mathbf{g}^T\mathbf{d}_x^{(k+r)} - \mathbf{b}_1^{(k)^T}\mathbf{d}_\lambda^{(k+r)} - \mathbf{b}_W^T\mathbf{d}_W^{(k+r)} - v_q^{(k+r)}d_{\lambda_q}^{(k+r)} = \\ -\lambda_W^{(k+r)^T)} W^T HW\mathbf{d}_W^{(k+r)} + \mathbf{a}_{p_r}^T HW\lambda_W^{(k+r)} \\ +\mathbf{e}_q^T U\mathbf{e}_q d_{v_q}^{(k+r)} v_q^{(k+r)} - \mathbf{g}^T T\mathbf{e}_q d_{v_q}^{(k+r)} + \mathbf{b}_1^{(k)^T} U\mathbf{e}_q d_{v_q}^{(k+r)} \tag{3.35}$$

From (3.25) we have

$$W^T HW\mathbf{d}_W^{(k+r)} + W^T T\mathbf{e}_q d_{v_q}^{(k+r)} = W^T H\mathbf{a}_{p_r}. \tag{3.36}$$

Post-multiply the transpose of (3.36) by $\lambda_W^{(k+r)}$, and with suitable rearrangement , to get

$$-\mathbf{d}_W^{(k+r)^T} W^T HW\lambda_W^{(k+r)} + \mathbf{a}_{p_r}^T HW\lambda_W^{(k+r)} = \mathbf{e}_q^T T^T W\lambda_W^{(k+r)} d_{v_q}^{(k+r)}. \tag{3.37}$$

Using (3.37) in (3.35) results in

$$\begin{aligned} b_{p_r} - \mathbf{g}^T\mathbf{d}_x^{(k+r)} - \mathbf{b}_1^{(k)^T}\mathbf{d}_\lambda^{(k+r)} - \mathbf{b}_W^{(T}\mathbf{d}_W^{(k+r)} - v_q^{(k+r)} d_{\lambda_q}^{(k+r)} &= \\ -(-\mathbf{e}_q^T T^T W\lambda_W^{(k+r)} - \mathbf{e}_q^T U\mathbf{e}_q v_q^{(k+r)} + \mathbf{e}_q^T T^T\mathbf{g} - \mathbf{e}_q^T U\mathbf{b}_1^{(k)}) d_{v_q}^{(k+r)} & \\ = -\lambda_q^{(k+r)} d_{v_q}^{(k+r)}, & \end{aligned} \tag{3.38}$$

from (3.17). Finally , substituting (3.38) into (3.29) reduces it to

$$f = f^{(k+r)} + (-d_{v_q}^{(k+r)})\ (\lambda_q^{k+r}\lambda_{p_r} + \frac{1}{2}(-d_{\lambda_q}^{(k+r)})\ \lambda_{p_r}^2).$$

Letting $\alpha = -d_{v_q}^{(k+r)}$ and $\beta = -d_{\lambda_q}^{(k+r)}$ the proof is complete.

We note that the case when $r = 1$ the matrix W does not appear. However , following the same steps (see [10]) we can prove the same result. So this case is considered as a special case.

The first move from a complementary tableau will definitely reduce the function value (assuming non-degeneracy). This is quite clear from the fact that moving away from a constraint corresponding to a negative Lagrange multiplier reduces f locally. It can be proved that , following the same steps of the proof of theorem 3.1 , f changes with v_q according to the relation

$$f = f^{(k)} + (\lambda_q^{(k)} v_q + \frac{1}{2}(-d_{\lambda_q}^{(k)}) v_q^2)$$

It becomes obvious now, that the expression for f which appeared in theorem 3.1 is more general between the non-complementary moves. So we can write $\alpha^{(k+r)}$ and $\beta^{(k+r)}$ for $r = 1, 2, 3, ...$ in the expressions for f. We now have the following theorem.

Theorem 3.3. *If $\alpha^{(k+r)} \neq 0$ and $\alpha^{(k+r+1)} = 0$ then $\beta^{(k+r+1)} \neq 0$*

Proof: The basic variable $v_{p_{r+1}}$ changes with λ_{p_r} according to

$$v_{p_{r+1}} = v_{p_{r+1}}^{(k+r)} - d_{v_{p_{r+1}}}^{(k+r)}\lambda_{p_r}.$$

When the change in λ_{p_r} is blocked by $v_{p_{r+1}}$, the value of λ_{p_r} in the next iteration will be $\lambda_{p_r}^{(k+r+1)} = \frac{v_{p_{r+1}}^{(k+r)}}{d_{v_{p_{r+1}}}^{(k+r)}}$. This happens when $d_{v_{p_{r+1}}} \neq 0$. We want to show that $d_{\lambda_q}^{(k+r+1)} d_{v_q}^{(k+r)} = d_{v_{p_{r+1}}}^{(k+r)}$. This will prove the theorem.

Using (3.26) , $d_{v_{p_{r+1}}}$ is given by

$$d_{v_{p_{r+1}}}^{(k+r)} = \mathbf{a}_{p_{r+1}}^T HW\mathbf{d}_W^{(k+r)} + \mathbf{a}_{p_{r+1}}^T T\mathbf{e}_q d_{v_q}^{(k+r)} - \mathbf{a}_{p_{r+1}}^T H\mathbf{a}_{p_r}. \tag{3.39}$$

The system corresponding to (3.25) in the $(k + r + 1)th$ iteration is

$$\begin{bmatrix} W^T HW & W^T H\mathbf{a}_{p_r} & W^T T\mathbf{e}_q \\ \mathbf{a}_{p_r}^T HW & \mathbf{a}_{p_r}^T H\mathbf{a}_{p_r} & \mathbf{a}_{p_r}^T T\mathbf{e}_q \\ \mathbf{a}_{p_{r+1}}^T HW & \mathbf{a}_{p_{r+1}}^T H\mathbf{a}_{p_r} & \mathbf{a}_{p_{r+1}}^T T\mathbf{e}_q \end{bmatrix} \begin{bmatrix} \mathbf{d}_W^{(k+r+1)} \\ d_{\lambda_{p_r}}^{(k+r+1)} \\ d_{v_q}^{(k+r+1)} \end{bmatrix} = \tag{3.40}$$

$$\begin{bmatrix} W^T H\mathbf{a}_{p_{r+1}} \\ \mathbf{a}_{p_r}^T H\mathbf{a}_{p_{r+1}} \\ \mathbf{a}_{p_{r+1}}^T H\mathbf{a}_{p_{r+1}} \end{bmatrix}.$$

Thus, in a way similar to (3.27) we get, in terms of $\mathbf{d}_W^{(k+r+1)}$, $d_{v_q}^{(k+r+1)}$, and $d_{\lambda_{pr}}^{(k+r+1)}$,

$$\mathbf{d}_\lambda^{(k+r+1)} = -T^T W\mathbf{d}_W^{(k+r+1)} - U\mathbf{e}_q d_{v_q}^{(k+r+1)} - T^T\mathbf{a}_{p_r} d_{\lambda_{pr}}^{(k+r+1)} + T^T\mathbf{a}_{p_{r+1}},$$

and it is reduced to

$$\mathbf{d}_\lambda^{(k+r+1)} = -T^T W\mathbf{d}_W^{(k+r+1)} - T^T\mathbf{a}_{p_r} d_{\lambda_{pr}}^{(k+r+1)} + T^T\mathbf{a}_{p_{r+1}}, \tag{3.41}$$

since $\alpha^{(k+r+1)} = -d_{v_q}^{(k+r+1)} = 0$. Now , (3.41) gives

$$d_{\lambda_q}^{(k+r+1)} = -\mathbf{e}_q^T T^T W\mathbf{d}_W^{(k+r+1)} - \mathbf{e}_q^T T^T\mathbf{a}_{p_r} d_{\lambda_{pr}}^{(k+r+1)} + \mathbf{e}_q^T T^T\mathbf{a}_{p_{r+1}}. \tag{3.42}$$

Now , multiply both sides of (3.42) by $d_{v_q}^{(k+r)}$ to get

$$\begin{aligned} d_{\lambda_q}^{(k+r+1)} d_{v_q}^{(k+r)} &= -\mathbf{e}_q^T T^T W\mathbf{d}_W^{(k+r+1)} d_{v_q}^{(k+r)} \\ &-\mathbf{e}_q^T T^T\mathbf{a}_{p_r} d_{\lambda_{pr}}^{(k+r+1)} d_{v_q}^{(k+r)} + \mathbf{e}_q^T T^T\mathbf{a}_{p_{r+1}} d_{v_q}^{(k+r)}. \end{aligned} \tag{3.43}$$

We move on to simplify the expression on the R.H.S. of (3.43). We have

$$-\mathbf{e}_q^T T^T W\mathbf{d}_W^{(k+r+1)} d_{v_q}^{(k+r)} = -\mathbf{d}_W^{(k+r+1)^T}[\ W^T T\mathbf{e}_q d_{v_q}^{(k+r)}\ ] =$$
$$-\mathbf{d}_W^{(k+r+1)^T}[\ W^T H\mathbf{a}_{p_r} - W^T HW\mathbf{d}_W^{(k+r)}\ ],$$

using (3.25). Rewrite the above equation as

$$\begin{aligned} -\mathbf{e}_q^T T^T W\mathbf{d}_W^{(k+r+1)} d_{v_q}^{(k+r)} &= -\mathbf{a}_{p_r}^T HW\mathbf{d}_W^{(k+r+1)} + \\ &+\mathbf{d}_W^{(k+r)^T} W^T HW\mathbf{d}_W^{(k+r+1)}. \end{aligned} \tag{3.44}$$

We also have

$$\begin{aligned} -\mathbf{e}_q^T T^T\mathbf{a}_{p_r} d_{\lambda_{pr}}^{(k+r+1)} d_{v_q}^{(k+r)} &= -[\mathbf{a}_{p_r}^T T\mathbf{e}_q d_{v_q}^{(k+r)}] d_{\lambda_{pr}}^{(k+r+1)} \\ &= -[\mathbf{a}_{p_r}^T H\mathbf{a}_{p_r} - \mathbf{a}_{p_r}^T HW\mathbf{d}_W^{(k+r)}] d_{\lambda_{pr}}^{(k+r+1)}, \end{aligned}$$

using (3.25). Rewrite the above equation as

$$\begin{aligned} -\mathbf{e}_q^T T^T\mathbf{a}_{p_r} d_{\lambda_{pr}}^{(k+r+1)} d_{v_q}^{(k+r)} &= -\mathbf{a}_{p_r}^T H\mathbf{a}_{p_r} d_{\lambda_{pr}}^{(k+r+1)} \\ &+\mathbf{d}_W^{(k+r)^T} W^T H\mathbf{a}_{p_r} d_{\lambda_{pr}}^{(k+r+1)}. \end{aligned} \tag{3.45}$$

Add (3.44) and (3.45) to get

$$\begin{aligned} &-\mathbf{e}_q^T T^T W\mathbf{d}_W^{(k+r+1)} d_{v_q}^{(k+r)} - \mathbf{e}_q^T T^T\mathbf{a}_{p_r} d_{\lambda_{pr}}^{(k+r+1)} d_{v_q}^{(k+r)} \\ &= -[\mathbf{a}_{p_r}^T HW\mathbf{d}_W^{(k+r+1)} + \mathbf{a}_{p_r}^T H\mathbf{a}_{p_r} d_{\lambda_{pr}}^{(k+r+1)}] \\ &+\mathbf{d}_W^{(k+r)^T}[W^T HW\mathbf{d}_W^{(k+r+1)} + W^T H\mathbf{a}_{p_r} d_{\lambda_{pr}}^{(k+r+1)}] \\ &= -\mathbf{a}_{p_r}^T H\mathbf{a}_{p_{r+1}} + \mathbf{d}_W^{(k+r)^T} W^T H\mathbf{a}_{p_{r+1}}, \end{aligned}$$

using (3.40) with $d_{v_q}^{(k+r+1)} = 0$. Thus

$$\begin{aligned} &-\mathbf{e}_q^T T^T W\mathbf{d}_W^{(k+r+1)} d_{v_q}^{(k+r)} - \mathbf{e}_q^T T^T\mathbf{a}_{p_r} d_{\lambda_{pr}}^{(k+r+1)} d_{v_q}^{(k+r)} \\ &= -\mathbf{a}_{p_{r+1}}^T H\mathbf{a}_{p_r} + \mathbf{a}_{p_{r+1}}^T HW\mathbf{d}_W^{(k+r)}. \end{aligned} \tag{3.46}$$

Now, substitute (3.46) in (3.43) to get

$$d_{\lambda_q}^{(k+r+1)} d_{v_q}^{(k+r)} = -\mathbf{a}_{p_{r+1}}^T H \mathbf{a}_{p_r} + \mathbf{a}_{p_{r+1}}^T HW \mathbf{d}_W^{(k+r)} + \mathbf{a}_{p_{r+1}}^T T \mathbf{e}_q d_{v_q}^{(k+r)} = d_{v_{p_{r+1}}}^{(k+r)} \tag{3.47}$$

using (3.39). Hence $\beta^{(k+r+1)} = -d_{\lambda_q}^{(k+r+1)} = \frac{d_{v_{p+r}}^{(k+r)}}{-\alpha^{(k+r)}} \neq 0$ as required.

The above theorem is not applied when $\alpha^{(k+1)} = 0$. The following theorem caters for that separately.

Theorem 3.4. *Suppose that $\lambda_q^{(k)} < 0$ at the kth iteration when the tableau is complementary. Let the increase of v_q be blocked by v_{p_1}. In the coming iteration if $\alpha^{(k+1)} = 0$ then $\beta^{(k+r)} > 0$*

Proof: The basic slack variable v_{p_1}changes with v_q according to

$$v_{p_1} = v_{p_1}^{(k)} - d_{v_{p_1}}^{(k)} v_q.$$

Since the increase of v_q is blocked by v_{p_1} , $v_q^{(k)}$ becomes

$$v_q^{(k)} = \frac{v_{p_1}^{(k)}}{d_{v_{p_1}}^{(k)}},$$

and this happens when $d_{v_{p_1}}^{(k)} > 0$.

It can be shown that , by solving the systems in the kth and $(k+1)th$ iterations (similar to (3.23)) ,

$$\mathbf{d}_x^{(k)} = -T\mathbf{e}_q \tag{3.48}$$

$$\mathbf{d}_\lambda^{(k+1)} = T^T \mathbf{a}_{p_1} - U\mathbf{e}_q d_{v_q}^{(k+1)}. \tag{3.49}$$

Using (3.48) $d_{v_{p_1}}^{(k)} = \mathbf{a}_{p_1}^T \mathbf{d}_x^{(k)} = -\mathbf{a}_{p_1}^T T\mathbf{e}_q$. Since $d_{v_q}^{(k+1)} = 0$ (3.49) gives

$$d_{\lambda_q}^{(k+1)} = \mathbf{e}_q^T T^T \mathbf{a}_{p_1} = \mathbf{a}_{p_1}^T T \mathbf{e}_q = -d_{v_{p_1}}^{(k)}.$$

Thus $\beta^{(k+1)} = -d_{\lambda_q}^{(k+1)} = d_{v_{p_1}}^{(k)} > 0$. Hence the proof.

4. The Algorithm

As our method is a feasible-point method , we have to use a method to obtain a starting feasible point. We are not going to get into the details of this since the literature is full of methods of this kind (see [9]). The description of the algorithm is given in two parts. **Part 1** will describe the steps to be carried out after a complementary tableau. **Part 2** shows the moves to be followed when we are at a non-complementary tableau.

Part 1 : Suppose the kth iteration is complementary. The first step is to find $q \in I^{(k)}$ which solves

$$\min_{i \in I^{(k)}} \lambda_i^{(k)}. \tag{4.1}$$

If $\lambda_q^{(k)} \geq 0$, then the algorithm terminates at $\mathbf{x}^{(k)}$. If not then the system

$$M^{(k)} \begin{bmatrix} \mathbf{d}_x^{(k)} \\ \mathbf{d}_\lambda^{(k)} \\ \mathbf{d}_v^{(k)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_q \\ \mathbf{0} \end{bmatrix}, \tag{4.2}$$

is solved to obtain $\mathbf{d}_x^{(k)}$, $\mathbf{d}_\lambda^{(k)}$, and $\mathbf{d}_v^{(k)}$. Then p_1 is chosen to solve

$$\min_{p \notin I^{(k)} \& d_{v_p}^{(k)} > 0} \quad \frac{v_p^{(k)}}{d_{v_p}^{(k)}}. \tag{4.3}$$

If

$$d_{\lambda_q}^{(k)} < 0 \; and \; \frac{\lambda_q^{(k)}}{d_{\lambda_q}^{(k)}} \leq \frac{v_{p_1}^{(k)}}{d_{v_{p_1}}^{(k)}}, \tag{4.4}$$

then the next tableau is complementary, and $I^{(k)}$ is updated by removing q , and also the basic variables are updated. The next iteration is repeated as above. If (4.4) is not satisfied then $I^{(k)}$ is updated by adding p_1 and removing q . The basic variables will also be updated. The next iteration will the first step in **Part 2**.

Part 2 : In this part we assume that $p_1, p_2, \ldots, p_r$ were successively added to $I^{(k)}$ before complementarity is restored. The next step is to solve

$$M^{(k+r)} \begin{bmatrix} \mathbf{d}_x^{(k+r)} \\ \mathbf{d}_\lambda^{(k+r)} \\ \mathbf{d}_W^{(k+r)} \\ d_{v_q}^{(k+r)} \\ \mathbf{d}_v^{(k)} \end{bmatrix} = \begin{bmatrix} -\mathbf{a}_{p_r} \\ \mathbf{0} \\ \mathbf{0} \\ 0 \\ \mathbf{0} \end{bmatrix}, \tag{4.5}$$

for $\mathbf{d}_x^{(k+r)}$, $\mathbf{d}_\lambda^{(k+r)}$, $\mathbf{d}_W^{(k+r)}$, $d_{v_q}^{(k+r)}$, and $\mathbf{d}_v^{(k)}$. Now define σ by

$$\sigma = \begin{cases} 1 & if \quad (\; (d_{v_q}^{(k+r)} < 0) \; or \; (d_{v_q}^{(k+r)} = 0 \; and \; d_{\lambda_q}^{(k+r)} < 0) \;) \\ -1 & if \quad (\; (d_{v_q}^{(k+r)} > 0) \; or \; (d_{v_q}^{(k+r)} = 0 \; and \; d_{\lambda_q}^{(k+r)} > 0) \;) \end{cases}$$

The next step will be to obtain p_{r+1} which solves

$$\min_{p \notin I^{(k)} \bigcup \{q\} \& \sigma d_{v_p}^{(k+r)} > 0} \quad \sigma \frac{v_p^{(k+r)}}{d_{v_p}^{(k+r)}}. \tag{4.6}$$

If

$$\sigma d_{\lambda_q}^{(k+r)} < 0 \; and \; \frac{\lambda_q^{(k+r)}}{\sigma d_{\lambda_q}^{(k+r)}} \leq \frac{v_{p_1}^{(k+r)}}{\sigma d_{v_{p_1}}^{(k+r)}}, \tag{4.7}$$

the next tableau will be complementary and in the coming step we move to **Part 1**, after updating the basic variables. Otherwise, if (4.7) is not satisfied, the corresponding tableau will continue to be non-complementary. Thus we continue as above after updating the basic variables and updating $I^{(k+r)}$ to $I^{(k+r+1)}$ by adding p_{r+1}.

We now conclude by giving a compact outline of the algorithm.

(a) Given $\mathbf{x}^{(1)}$, $\lambda_1^{(1)}$, *and* $\mathbf{v}_2^{(1)}$, *set* $k = 1$

(b) Solve (4.1) for q.

(c) If $\lambda_q^{(k)} \geq 0$ *terminate with* $\mathbf{x}^{(k)}$ *as a solution.*

Otherwise solve (4.3) for p_1*.*

(d) If (4.4) is satisfied remove q *from* $I^{(k)}$*,*

update the basic variables,

set $k = k + 1$ *, and go to (b).*

Otherwise , remove q *from* $I^{(k)}$*,*

update the basic variables,

set $r = 1$ *and* $s = k$*.*

(e) Set $k = s + r$ *, and add* p_r *to* $I^{(k)}$*.*

(f) Solve (4.6) for p_{r+1}*.*

(g) If (4.7) is satisfied , update the basic variables,

set $k = k + 1$ *, and go to (b).*

Otherwise , update the basic variables,

set r=r+1 , and go to (e).

Very detailed work is required to apply this algorithm. It concerns updating factors of the matrices, such as $Z^T G Z$, where Z is an $n \times (n - l^{(k)})$ matrix satisfying $Z^T A_1^{(k)} = O$. This matrix is involved in using H, T, and U in the computation (see [8] and [9]). In [10] a practical implementation of this algorithm is made. Fletcher in his book[8] has pointed out that complementarity pivoting-like methods could be promising if efficient and stable factorizations are made. This was the base for this work.

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