Available at: http://www.ictp.trieste.it/~pub_off IC/98/82

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency

THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

A REFINEMENT OF THE TORAL RANK CONJECTURE FOR 2-STEP NILPOTENT LIE ALGEBRAS

Paulo Tirao
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

It is known that the total (co)-homology of a 2-step nilpotent Lie algebra $\mathfrak{g}$ is $\geq 2^{|\mathfrak{z}|}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$. We improve this result by showing that actually 2^t, where $t = |\mathfrak{z}| + \left[\frac{|v|+1}{2}\right]$, v a complement of $\mathfrak{z}$ in $\mathfrak{g}$, is still a lower bound for the total cohomology.

MIRAMARE – TRIESTE

July 1998

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

§1. Introduction

An outstanding conjecture, known as the *Toral Rank Conjecture* (TRC) claims that for any nilpotent Lie algebra $\mathfrak{g}$ (over $\mathbf{R}$ or $\mathbf{C}$) the total (co)-homology, with trivial coefficients, satisfies the inequality $\left|\mathrm{H}_*(\mathfrak{g})\right| \geq 2^{|\mathfrak{z}|}$, where $\mathfrak{z} = \text{center}\,(\mathfrak{g})$.

The TRC is due to S. Halperin ([Ha], 1987). In 1988, in [DS], Deninger and Singhof proved it for 2-step nilpotent Lie algebras. Besides this class, only some particular cases have been added recently. It was shown in [CJ] that the TRC holds for $\mathfrak{g}$ if its center has dimension ≤ 5 or has codimension ≤ 7.

It turns out that, in general, $2^{|\mathfrak{z}|}$ is quite far from $|\mathrm{H}_*(\mathfrak{g})|$, specially when $|\mathfrak{z}|$ is comparatively small respect to $|\mathfrak{g}|$.

In this short note we give an alternative proof for the Deninger-Singhof result. Furthermore, we give a new bound that takes into account, not only the dimension of the center but also, the dimension of the Lie algebra itself. On the other hand, by using existing calculations, we show that actually it is the best possible general lower bound of the form 2^t.

Precisely, we give a direct proof to the following

Theorem. *Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra of finite dimension over $\mathbf{R}$ or $\mathbf{C}$. Let v be any direct complement, as vector spaces, of $\mathfrak{z} = center(\mathfrak{g})$. Then,*[1]

$$\left|\mathrm{H}_*(\mathfrak{g})\right| \geq 2^{|\mathfrak{z}|+\left[\frac{|v|+1}{2}\right]}.$$

§2. The proof of the Theorem

We will make use of the following combinatorial result.

Lemma 2.1. *Let n be a positive integer. Then,*

$$\left|\sum_{j=0}(-1)^j\binom{n}{2j}\right| + \left|\sum_{j=0}(-1)^j\binom{n}{2j+1}\right| \geq 2^{\frac{n}{2}}.$$

Proof. Consider the following integral representation for the binomial numbers:

$$\binom{n}{j} = \frac{1}{2\pi i}\oint \frac{(1+z)^n}{z^{j+1}}\,dz.$$

Hence,

$$\sum_{j=0}(-1)^j\binom{n}{2j} = \frac{1}{2\pi i}\oint \frac{(1+z)^n}{z}\left(\sum_{j=0}\left(\frac{-1}{z^2}\right)^j\right)dz$$

and for $|z|>1$ we get

$$\begin{aligned} &= \frac{1}{2\pi i}\oint \frac{(1+z)^n z}{1+z^2}\,dz \\ &= \mathrm{res}_{z=i} f(z) + \mathrm{res}_{z=-i} f(z), \end{aligned}$$

[1] The right brackets denote integral part.

where $f(z)=\frac{(1+z)^n z}{1+z^2}$. Being i and $-i$ simple poles of f it is easy to get

$$=\frac{(1+i)^n+(1-i)^n}{2}.$$

Writing $1+i=2^{1/2}e^{\pi i/4}$ and $1-i=2^{1/2}e^{-\pi i/4}$ is not difficult to conclude that

$$\left|\sum_{j=0}(-1)^j\binom{n}{2j}\right| = \left|\frac{(1+i)^n+(1-i)^n}{2}\right|$$
$$=\begin{cases} 2^{n/2}, & \text{if } n\equiv 0 \mod 4;\\ 2^{(n-1)/2}, & \text{if } n\equiv 1 \mod 4;\\ 0, & \text{if } n\equiv 2 \mod 4;\\ 2^{(n-1)/2}, & \text{if } n\equiv 3 \mod 4.\end{cases}$$

In an analogous way, one can show that

$$\left|\sum_{j=0}(-1)^j\binom{n}{2j+1}\right| = \begin{cases} 0, & \text{if } n\equiv 0 \mod 4;\\ 2^{(n-1)/2}, & \text{if } n\equiv 1 \mod 4;\\ 2^{n/2}, & \text{if } n\equiv 2 \mod 4;\\ 2^{(n-1)/2}, & \text{if } n\equiv 3 \mod 4.\end{cases}$$

Finally, summing up, the Lemma is proved. □

Proof of the Theorem. The homology of $\mathfrak{g}$ is the homology of the Koszul complex $(\bigwedge\mathfrak{g},\partial)$[2]

Put $\mathfrak{g}=v\oplus\mathfrak{z}$. Hence, we can write $\bigwedge\mathfrak{g}=\bigwedge v\otimes\bigwedge\mathfrak{z}$. It is straightforward to see that

$$\partial : \bigwedge^{p} v\otimes\bigwedge^{q}\mathfrak{z}\longrightarrow\bigwedge^{p-2} v\otimes\bigwedge^{q+1}\mathfrak{z}.$$

Therefore, it follows that the complex $(\bigwedge\mathfrak{g},\partial)$ is the direct sum of an *even* and an *odd* subcomplexes, precisely $(\bigwedge^{2p} v\otimes\bigwedge\mathfrak{z},\partial)$ and $(\bigwedge^{2p+1} v\otimes\mathfrak{z})$. Accordingly, the homology of the Koszul complex is the sum of the homologies of each of the even and odd subcomplexes.

It is well known that if $(\mathcal{C}=(C_i),\partial)$ is a finite complex of finite dimensional vector spaces, then

$$\left|\mathrm{H}_*(\mathcal{C})\right|\geq\left|\sum_i(-1)^i\dim(C_i)\right|.$$

By applying this to each of the even and odd subcomplexes we get that

$$\left|\mathrm{H}_*(\mathfrak{g})\right|\geq\left|\sum_{j=0}(-1)^j\dim\left(\bigwedge^{2j} v\otimes\bigwedge\mathfrak{z}\right)\right|+\left|\sum_{j=0}(-1)^j\dim\left(\bigwedge^{2j+1} v\otimes\bigwedge\mathfrak{z}\right)\right|$$
$$=2^{|\mathfrak{z}|}\left|\sum_{j=0}(-1)^j\binom{|v|}{2j}\right|+2^{|\mathfrak{z}|}\left|\sum_{j=0}(-1)^j\binom{|v|}{2j+1}\right|.$$

[2] $\partial(x_1\wedge\ldots\wedge x_p)=\sum_{i<j}(-1)^{i+j+1}[x_i,x_j]\wedge x_1\ldots\wedge\hat{x_i}\ldots\wedge\hat{x_j}\ldots\wedge x_p$.

Now the Theorem follows from Lemma 2.1. □

Remark 2.2. The bound obtained in the Theorem is the best possible, of the form 2^t, that can be given in general. In fact, among the few available computations there are examples that support this claim.

Example 2.3. Let $\mathfrak{g}_2$ denote the Lie algebra with basis $\{x_1, x_2, y_1, y_2, z\}$ and non-zero relations $[z, x_i] = y_i$ for each $1 \le i \le 2$.

It was shown in [ACJ] that

$$b_0 = 1, \quad b_1 = 3, \quad b_2 = 6, \quad b_3 = 6, \quad b_4 = 3, \quad b_5 = 1,$$

where $b_i = |\mathrm{H}^i(\mathfrak{g}_2)|$.

Therefore, the total cohomology of $\mathfrak{g}_2$ is equal to 20 and our bound is $2^{2+2} = 16$.

Example 2.4. Let $\mathfrak{g}$ denote the Lie algebra with basis $\{a, b, c, d, e, f, g\}$ and non-zero relations $[a, b] = e$, $[b, d] = g$, $[c, d] = e$ and $[a, c] = f$. This Lie algebra appears as $3, 7_D$ in Seeley's classification ([Se]).

In [CJP] one can explicitly find that

$$b_0 = 1, \quad b_1 = 4, \quad b_2 = 11, \quad b_3 = 14, \quad b_4 = 14, \quad b_5 = 11, \quad b_6 = 4, \quad b_7 = 1,$$

where $b_i = |\mathrm{H}^i(\mathfrak{g})|$. Hence $|\mathrm{H}^*(\mathfrak{g})| = 60$, while our bound is $2^{3+2} = 32$.

Example 2.5. We consider here a family of examples. For each $r \ge 2$ let E be an r-dimensional vector space, then $\mathfrak{g}_r = E \oplus \bigwedge^2 E$ is the rank r 2-step free nilpotent Lie algebra,

Their homology has been computed by Sigg in [Si]. Using his result one can compute, for each r, the total homology for the rank r algebra.

On the other hand, it is clear that our bound is $2^{\binom{r}{2}+[\frac{r+1}{2}]}$.

With a simple computer program written in Maple V we have gone up to $r = 11$.

r	$\lvert\mathfrak{g}_r\rvert$	Total cohomology	t	2^t
2	3	6	2	4
3	6	36	5	32
4	10	420	8	256
5	15	9800	13	8192
6	21	452760	18	262144
7	28	4.1835024×10^7	25	3.3554432×10^7
8	36	7.691667984×10^9	32	4.294967296×10^9
9	45	$2.828336198688 \times 10^{12}$	41	$2.199023255552 \times 10^{12}$
10	55	$2.073619375892064 \times 10^{15}$	50	$1.125899906842624 \times 10^{15}$
11	66	$3.040584296923128384 \times 10^{18}$	61	$2.305843009213693952 \times 10^{18}$

$\left(t = \binom{r}{2} + \left[\frac{r+1}{2}\right]\right)$

References

[ACJ] Armstrong G., Cairns G and Jessup B., *Explicit Betti numbers for a family of nilpotent Lie algebras*, Proc. AMS **125** (1997), 381–385.

[CJ] Cairns G. and Jessup B., *New bounds on the Betti numbers of nilpotent Lie algebras*, Communications in Algebra **25** (1997), 415–430.

[CJP] Cairns G., Jessup B., and Pitkethly J., *On the Betti numbers of nilpotent Lie algebras of small dimension*, Prog. in Math.: Integrable systems and foliations (Claude A. et al, eds.), vol. 145, Birkhaeuser, 1997, pp. 19–31.

[DS] Deninger Ch. and Singhof W., *On the cohomology of nilpotent Lie algebras*, Bull. Soc. Math. France **116** (1988), 3–14.

[Ha] Halperin S., *Le complexe de Koszul en algèbre et topologie*, Ann. l'Inst. Fourier **37** (1987), 77–97.

[Se] Seeley C, *7-dimensional nilpotent Lie algebras*, Transactions of the AMS **335** (1993), 479–496.

[Si] Sigg S., *Laplacian and homology of free 2-step nilpotent Lie algebras*, Jour. of Algbera **185** (1996), 144–161.