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SOLUTIONS OF q-DEFORMED EQUATIONS WITH QUANTUM CONFORMAL SYMMETRY AND NONZERO SPIN

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Abstract

We consider the construction of explicit solutions of a hierarchy of q-deformed equations which are (conditionally) quantum conformal invariant. We give two types of solutions - polynomial solutions and solutions in terms of q-deformations of the plane wave. We use two q-deformations of the plane wave as a formal power series in the noncommutative coordinates of q-Minkowski space-time and four-momenta. One q-plane wave was proposed earlier by the first named author and B.S. Kostadinov, the other is new. The difference between the two is that they are written in conjugated bases. These q-plane waves are used here for the construction of solutions of the massless Dirac equation - one is used for the neutrino, the other for the antineutrino. It is also interesting that the neutrino solutions are deformed only through the q-pane wave, while the prefactor is classical. Thus, we can speak of a definite left-right asymmetry of the quantum conformal deformation of the neutrino-antineutrino system.

1. Introduction

One of the purposes of quantum deformations is to provide an alternative of the regularization procedures of quantum field theory. Applied to Minkowski space-time the quantum deformations approach is also an alternative to Connes' noncommutative geometry [1]. The first step in such an approach is to construct a noncommutative quantum deformation of Minkowski space-time. There are several possible such deformations, cf. [2], [3], [4], [5], [6], [7], [8]. We shall follow the deformation of [5] which is different from the others, the most important aspect being that it is related to a deformation of the conformal group.

The first problem to tackle in a noncommutative deformed setting is to analyze the behavior of the wave equations analogues. Thus, we start here with the study of a hierarchy of deformed equations derived in [9] with the use of quantum conformal symmetry. The hierarchy is parametrized by a natural number a. In fact, the case $a = 1$ corresponds to the q-d'Alembert equation, while for each $a > 1$ there are two couples of equations involving fields of conjugated Lorentz representations of dimension a. For instance, the case $a = 2$ corresponds to the massless Dirac equation, one couple of equations describing the neutrino, the other couple of equations describing the antineutrino, while the case $a = 3$ corresponds to the Maxwell equations.

In [9] the solution spaces were given only via their transformation properties under $U_q(sl(4))$ and quantum conformal symmetry. However, no explicit solutions, which are important for the applications, were given. This was started in [10] with the construction of solutions of the q-d'Alembert equation. Two classes of solutions were given: polynomial ones and a deformation of the plane wave. It is a formal power series in the noncommutative coordinates of q-Minkowski space-time and four-momenta. This q-plane wave has analogous properties to the classical one. In particular, it has the properties of q-Lorentz covariance, and it satisfies the q-d'Alembert equation on the q-Lorentz covariant momentum cone. On the other hand, this q-plane wave is not an exponent or q-exponent, cf. [11]. Thus, it differs conceptually from the classical plane wave and may serve as a regularization of the latter. In the same sense it differs from the q-plane wave in the paper [12], which is not surprising, since there is used different q-Minkowski space-time (from [2], [3], [4] and different q-d'Alembert equation both based only on a (different) q-Lorentz algebra, and not on q-conformal (or $U_q(sl(4))$) symmetry as in our case.

In the present paper we give polynomial solutions for all equations of the hierarchy. We give also another q-deformation of the usual plane wave written in a conjugated basis with respect to the first q-plane wave derived in [10]. We give solutions of the massless Dirac equation involving the two conjugated q-plane waves - one for the neutrino, the other for the antineutrino. It is also interesting that the neutrino solutions are deformed only through the q-pane wave, while the prefactor is classical. Thus, we can speak of a definite left-right asymmetry of the quantum conformal deformation of the neutrino-antineutrino system.

2. Preliminaries

First we introduce new Minkowski variables:

$$x_\pm \ \equiv \ x_0 \pm x_3 \ , \qquad v \ \equiv \ x_1 - ix_2 \ , \qquad \bar{v} \ \equiv \ x_1 + ix_2 \ , \tag{1}$$

which, (unlike the x_μ), have definite group-theoretical interpretation as part of a coset of the conformal group $SU(2,2)$ (or of $SL(4)$ with the appropriate conjugation) [5]. The d'Alembert equation in terms of these variables is:

$$\Box \ \varphi \ = \ (\partial_- \ \partial_+ \ - \ \partial_v \ \partial_{\bar{v}}) \ \varphi \ = \ 0 \ , \tag{2}$$

while the Minkowski length is: $\mathcal{L} \ = \ x_- x_+ - v\bar{v} \ = \ x_0^2 - \vec{x}^2$.

In the q-deformed case we use the noncommutative q-Minkowski space-time of [5] which is given by the following commutation relations:

$$\begin{aligned} &x_\pm v \ = \ q^{\pm 1} v x_\pm \ , \quad x_\pm \bar{v} \ = \ q^{\pm 1} \bar{v} x_\pm \ , \\ &x_+ x_- - x_- x_+ \ = \ \lambda v \bar{v} \ , \quad \bar{v} v \ = \ v \bar{v} \ , \qquad (\lambda \equiv q - q^{-1}) \ , \end{aligned} \tag{3}$$

with the deformation parameter being a phase: $|q| = 1$. The q-Minkowski length is:

$$\mathcal{L}_q \ = \ x_- x_+ - q^{-1} v \bar{v} \ . \tag{4}$$

It commutes with the q-Minkowski coordinates and has the correct classical limit $\mathcal{L}_{q=1} \ = \ \mathcal{L}$. Relations (3) are preserved by the anti-linear anti-involution ω acting as :

$$\omega(x_\pm) \ = \ x_\pm \ , \quad \omega(v) \ = \ \bar{v} \ , \quad \omega(q) \ = \ \bar{q} \ = \ q^{-1} \ , \quad (\omega(\lambda) \ = \ -\lambda) \ , \tag{5}$$

from which follows also that $\omega(\mathcal{L}_q) \ = \ \mathcal{L}_q$.

The solution space (for q-d'Alambert) consists of formal power series in the q-Minkowski coordinates:

$$\varphi \ = \sum_{j,n,\ell,m \in \mathbb{Z}_+} \mu_{jn\ell m} \ \varphi_{jn\ell m} \ , \qquad \varphi_{jn\ell m} \ = \ \hat{\varphi}_{jn\ell m}, \ \tilde{\varphi}_{jn\ell m} \tag{6a}$$

$$\hat{\varphi}_{jn\ell m} \ = \ v^j \ x_-^n \ x_+^\ell \ \bar{v}^m \tag{6b}$$

$$\tilde{\varphi}_{jn\ell m} \ = \ \bar{v}^m \ x_+^\ell \ x_-^n \ v^j \ = \ \omega(\hat{\varphi}_{jn\ell m}) \tag{6c}$$

The solution spaces (6) are representation spaces of the quantum algebra $U_q(sl(4))$. The latter is defined, (cf. [13]), as the associative algebra over $\mathbb{C}$ with unit element $1_{\mathcal{U}}$, 'Chevalley' generators $k_i^\pm$, $X_i^\pm$, $i = 1,2,3$, and nontrivial relations $([x]_q \equiv (q^x - q^{-x})/\lambda)$:

$$k_i k_j \ = \ k_j k_i \ , \quad k_i k_i^{-1} \ = \ k_i^{-1} k_i \ = \ 1_{\mathcal{U}} \ , \quad k_i X_j^\pm \ = \ q^{\pm c_{ij}} X_j^\pm k_i \ , \tag{7a}$$

$$[X_i^+, X_j^-] \ = \ \delta_{ij} \left(k_i^2 - k_i^{-2}\right) / \lambda \ , \tag{7b}$$

$$\left(X_i^\pm\right)^2 X_j^\pm \ - \ [2]_q X_i^\pm X_j^\pm X_i^\pm \ + \ X_j^\pm \left(X_i^\pm\right)^2 \ = \ 0 \ , \quad |i-j| = 1 \ . \tag{7c}$$

Further we suppose that q is not a nontrivial root of unity. The action of $U_q(sl(4))$ on $\hat{\varphi}_{jn\ell m}$ was given in [14], (cf. also [10]). Using those formulae we can find also the action on $\tilde{\varphi}_{jn\ell m}$:

$$\begin{aligned}
\pi(k_1)\ \tilde{\varphi}_{jn\ell m} &= q^{(j-n+\ell-m)/2}\ \tilde{\varphi}_{jn\ell m}\ , && (8a)\\
\pi(k_3)\ \tilde{\varphi}_{jn\ell m} &= q^{(-j-n+\ell+m)/2}\ \tilde{\varphi}_{jn\ell m}\ , && (8b)\\
\pi(X_1^+)\ \tilde{\varphi}_{jn\ell m} &= q^{-2+(-j+n+\ell-m)/2}\ [n]_q\ \tilde{\varphi}_{j+1,n-1,\ell m} + \\
&\quad +\ q^{-2+(-j+n-\ell+m)/2}\ [m]_q\ \tilde{\varphi}_{jn,\ell+1,m-1}\ , && (9a)\\
\pi(X_3^+)\ \tilde{\varphi}_{jn\ell m} &= -\ q^{(-j-n+\ell+m)/2}\ [j]_q\ \tilde{\varphi}_{j-1,n,\ell+1,m} - \\
&\quad -\ q^{(-3j-n+3\ell+m)/2}\ [n]_q\ \tilde{\varphi}_{j,n-1,\ell,m+1}\ , && (9b)\\
\pi(X_1^-)\ \tilde{\varphi}_{jn\ell m} &= q^{1+(j-n+\ell-m)/2}\ [j]_q\ \tilde{\varphi}_{j-1,n+1,\ell m} + \\
&\quad +\ q^{1+(-j+n+\ell-m)/2}\ [\ell]_q\ \tilde{\varphi}_{jn,\ell-1,m+1}\ , && (10a)\\
\pi(X_3^-)\ \tilde{\varphi}_{jn\ell m} &= -\ q^{1+(j+3n-\ell-3m)/2}\ [\ell]_q\ \tilde{\varphi}_{j+1,n,\ell-1,m} - \\
&\quad -\ q^{1+(j+n-\ell-m)/2}\ [m]_q\ \tilde{\varphi}_{j,n+1,\ell,m-1} && (10b)
\end{aligned}$$

We have written only the action of $X_j^{\pm}, k_j$ for $j = 1,3$ since we shall use only those. (Note that the representation formulae in [14] are for general holomorphic representations of $U_q(sl(4))$ characterized by three integers (r_1, r_2, r_3), (which here are $(0,-1,0)$), and the functions depend in general on two additional variables $z, \bar{z}$.)

Next we need the q-d'Alembert equation:

$$\Box_q\ \varphi\ =\ \sum_{j,n,\ell,m\in\mathbb{Z}_+} \mu_{jn\ell m}\ \Box_q\ \varphi_{jn\ell m}\ =\ 0\ , \qquad (11a)$$

$$\begin{aligned}
\Box_q\ \hat{\varphi}_{jn\ell m} &= q^{1+n+2m+2j+\ell}[n]_q[\ell]_q\ \hat{\varphi}_{j,n-1,\ell-1,m} - \\
&\quad -\ q^{n+j+\ell+m}[j]_q[m]_q\ \hat{\varphi}_{j-1,n,\ell,m-1} && (11b)\\
\Box_q\ \tilde{\varphi}_{jn\ell m} &= q^{n+\ell}[n]_q[\ell]_q\ \tilde{\varphi}_{j,n-1,\ell-1,m} - \\
&\quad -\ q^{n+j+\ell+m+1}[j]_q[m]_q\ \tilde{\varphi}_{j-1,n,\ell,m-1} && (11c)
\end{aligned}$$

(for $(11b)$ cf. [9], [10]).

Remark: Equation (11) may be rewritten in a form closer to the $q = 1$ in (2) by introducing q-difference operators. For this we first define the operators:

$$\hat{M}_\kappa^{\pm}\ \varphi\ =\ \sum_{j,n,\ell,m\in\mathbb{Z}_+} \mu_{jn\ell m}\ \hat{M}_\kappa^{\pm}\ \varphi_{jn\ell m}\ , \qquad \kappa = \pm, v, \bar{v}\ , \qquad (12a)$$

$$T_\kappa^{\pm}\ \varphi\ =\ \sum_{j,n,\ell,m\in\mathbb{Z}_+} \mu_{jn\ell m}\ T_\kappa^{\pm}\ \varphi_{jn\ell m}\ , \qquad \kappa = \pm, v, \bar{v}\ , \qquad (12b)$$

and $\hat{M}_+^{\pm}$, $\hat{M}_-^{\pm}$, $\hat{M}_v^{\pm}$, $\hat{M}_{\bar{v}}^{\pm}$, resp., acts on $\varphi_{jn\ell m}$ by changing by ± 1 the value of j, n, ℓ, m, resp., while $T_+^{\pm}$, $T_-^{\pm}$, $T_v^{\pm}$, $T_{\bar{v}}^{\pm}$, resp., acts on $\varphi_{jn\ell m}$ by multiplication by $q^{\pm j}, q^{\pm n}, q^{\pm \ell}, q^{\pm m}$, resp. Using the above we define the q-difference operators as follows:

$$\hat{\mathcal{D}}_\kappa\ \varphi\ =\ \frac{1}{\lambda}\ \hat{M}_\kappa^{-1}\ \left(T_\kappa - T_\kappa^{-1}\right)\ \varphi\ . \qquad (13)$$

Using (12) and (13) then $(11b, c)$ may be rewritten as:

$$\left(q\ \hat{\mathcal{D}}_-\ \hat{\mathcal{D}}_+\ T_v\ T_{\bar{v}}\ -\ \hat{\mathcal{D}}_v\ \hat{\mathcal{D}}_{\bar{v}}\right)\ T_v\ T_-\ T_+\ T_{\bar{v}}\ \hat{\varphi}\ =\ 0 \tag{14a}$$

$$\left(\hat{\mathcal{D}}_-\ \hat{\mathcal{D}}_+\ -\ \hat{\mathcal{D}}_v\ \hat{\mathcal{D}}_{\bar{v}}\ T_v\ T_{\bar{v}}\right)\ T_-\ T_+\ \tilde{\varphi}\ =\ 0 \tag{14b}$$

Note that the operators in (12), (13), (14) for different variables commute, i.e., using these one is technically passing to commuting variables. Note that keeping the normal ordering it is straightforward to interchange commuting and noncommuting variables. $\diamondsuit$

3. q-Plane Waves

We want to q-deform the plane wave. Clearly, the most general q-deformation is:

$$\left(\exp(k\cdot x)\right)_q\ =\ \sum_{s=0}^{\infty}\ \frac{1}{[s]_q!}\ f_s\ , \tag{15}$$

where f_s is a homogeneous polynomial of degree s in both sets of variables, i.e., q-momenta $(k_v, k_-, k_+, k_{\bar{v}})$ and q-Minkowski coordinates $(v, x_-, x_+, \bar{v})$, such that $(f_s)|_{q=1}\ =\ (k\cdot x)^s$. Thus, we set $f_0\ =\ 1$. One may expect that f_s for $s > 1$ would be equal or at least proportional to $(f_1)^s$, but the outcome would be that this is not the case. In order to proceed systematically we have to impose the conditions of q-Lorentz covariance and the q-d'Alembert equation.

The complexification of the q-Lorentz subalgebra of the q-conformal algebra is generated by $k_j^{\pm}$, $X_j^{\pm}$, $j = 1, 3$. Using $(9a, b)$, $(10a, b)$ it is easy to check that:

$$\pi(X_j^{\pm})\ \mathcal{L}_q\ =\ 0\ ,\quad \Longrightarrow\quad \pi(X_j^{\pm})\ (\mathcal{L}_q)^s\ =\ 0\ ,\qquad j\ =\ 1,3. \tag{16}$$

Since $(k\cdot x)^s$ is a scalar as $(\mathcal{L}_q)^s$, then also the q-deformations f_s should be scalars, and thus also should obey (16). In order to implement this we suppose that the momentum components are also non-commutative obeying the same rules (3) as the q-Minkowski coordinates, and that they commute with the coordinates. Also the ordering of the momentum basis will be the same for the coordinates. Taking all this into account one can see that a natural expression for f_s is:

$$\begin{aligned} f_s\ =\ &\sum_{a,b,n\ \in\mathbb{Z}_+} \beta^s_{a,b,n}\ \frac{(-1)^{s-a-b}}{\Gamma_q(a-n+1)\ \Gamma_q(b-n+1)\ \Gamma_q(s-a-b+n+1)\ [n]_q!}\ \times \\ &\times\ k_v^{s-a-b+n}\ k_-^{b-n}\ k_+^{a-n}\ k_{\bar{v}}^n\ v^n\ x_-^{a-n}\ x_+^{b-n}\ \bar{v}^{s-a-b+n}\ , \end{aligned} \tag{17}$$

where some factors are introduced that are obvious from the correspondence with the case $q = 1$. (The expression in (17) does not involve terms that would vanish for $q = 1$. Actually, one may see that such expressions would lead to noncovariant momenta light-cone.) In order to implement q-Lorentz covariance we impose the conditions:

$$\pi(X_j^{\pm})\ f_s\ =\ 0,\qquad j\ =\ 1,3\ . \tag{18}$$

For this calculation we suppose that the q-Lorentz action on the noncommutative momenta is given by $(8a,c)$, $(9a,c)$, $(10a,c)$. This was done in [10] and the following result for β was found:

$$\beta^s_{a,b,n} \ = \ q^{n(s-2a-2b+2n) \ + \ a(s-a-1) \ + \ b(-s+a+b+1)} \ \beta^s_{0,0,0} \ , \tag{19}$$

where

$$\left(\beta^s_{0,0,0}\right)^{-1} \ = \ \sum_{p=0}^{s} \frac{q^{(s-p)(p-1)+p}}{[p]_q! \ [s-p]_q!} \ . \tag{20}$$

The functions f_s obey the q-d'Alambert equation if the momenta are on the q-light cone:

$$\mathcal{L}^k_q \ = \ k_- k_+ \ - \ q^{-1} \ k_v k_{\bar{v}} \ = \ 0 \ . \tag{21}$$

We turn now to the conjugated case. The q-plane wave now is:

$$\begin{aligned}
\widetilde{\exp}_q(k\cdot x) \ &= \ \sum_{s=0}^{\infty} \frac{1}{[s]_q!} \ \tilde{f}_s \\
\tilde{f}_s \ &= \ \sum_{a,b,n} \tilde{\beta}^s_{a,b,n} \ \frac{(-1)^{s-a-b}}{\Gamma_q(a-n+1) \ \Gamma_q(b-n+1) \ \Gamma_q(s-a-b+n+1) \ [n]_q!} \ \times \\
&\times \ k^n_{\bar{v}} k^{a-n}_+ k^{b-n}_- k^{s-a-b+n}_v \ \bar{v}^{s-a-b+n} x^{b-n}_+ x^{a-n}_- v^n
\end{aligned} \tag{22}$$

For q-Lorentz covariance we impose the conditions:

$$\pi(X^\pm_j) \ \tilde{f}_s \ = \ 0, \qquad j \ = \ 1,3 \ . \tag{23}$$

We use the commutation relations for the momenta components as for as the q-Minkowski coordinates, and that momenta commute with the coordinates. We also have to use the twisted derivation rule [14] :

$$\begin{aligned}
\pi(X^\pm_j) \ \psi\cdot\psi' \ &= \ \pi(X^\pm_j) \ \psi \ \cdot \ \pi(k^{-1}_j) \ \psi' \ + \ \pi(k_j) \ \psi \ \cdot \ \pi(X^\pm_j) \ \psi' \ , \\
\psi \ &= \ k^n_{\bar{v}} k^{a-n}_+ k^{b-n}_- k^{s-a-b+n}_v \quad , \qquad \psi' \ = \ \bar{v}^{s-a-b+n} x^{b-n}_+ x^{a-n}_- v^n \ .
\end{aligned} \tag{24}$$

The four conditions (23) bring eight relations between the coefficients $\tilde{\beta}$, however only three are independent, namely, the relations:

$$\tilde{\beta}^s_{a,b,n} \ = \ q^{2n+2a-b-2-s} \ \tilde{\beta}^s_{a-1,b,n} \ , \tag{25a}$$
$$\tilde{\beta}^s_{a,b,n} \ = \ q^{2n-a-2b+2+s} \ \tilde{\beta}^s_{a,b-1,n} \ , \tag{25b}$$
$$\tilde{\beta}^s_{a,b,n} \ = \ q^{2a+2b-4n+2-s} \ \tilde{\beta}^s_{a,b,n-1} \ , \tag{25c}$$

solving which we find the following solution:

$$\tilde{\beta}^s_{a,b,n} \ = \ q^{n(2a+2b-2n-s) \ + \ a(a-s-1) \ + \ b(s-a-b+1)} \ \tilde{\beta}^s_{0,0,0} \tag{26}$$

i.e., for each $s \geq 1$ only one constant remains to be fixed. Next we impose the q-d'Alembert equation:

$$\Box_q \, \tilde{f}_s \; = \; 0 \; , \tag{27}$$

which holds trivially for $s = 0, 1$. For $s \geq 2$ we substitute (17) to obtain that (27)holds iff the momentum operators are on the q-Lorentz covariant q-light cone (cf. (4)):

$$\mathcal{L}_q^k \;\; = \;\; k_- k_+ \; - \; q^{-1} \; k_v k_{\bar{v}} \;\; = \;\; 0 \; . \tag{28}$$

Now it remains only to fix the coefficient $\beta^s_{0,0,0}$. We note that for q=1 it holds:

$$(k \cdot x)|_{k \,\to\, x} \;\; = \;\; (x \cdot x) \;\; = \;\; \mathcal{L} \; , \tag{29}$$

and thus we shall impose the conditions:

$$(\tilde{f}_s)|_{k \,\to\, x} \quad = \quad (\mathcal{L}_q)^s \; . \tag{30}$$

A tedious calculation shows that:

$$\left(\tilde{\beta}^s_{0,0,0}\right)^{-1} \;\; = \;\; \sum_{p=0}^{s} \; \frac{q^{(p-s)(p-1)+p}}{[p]_q! \; [s-p]_q!} \; . \tag{31}$$

Note that $\left(\tilde{\beta}^s_{0,0,0}\right)^{-1} |_{q=1} \;\; = \;\; 2^s/s!$, as expected.

4. Polynomial solutions for nonzero spin

As we mentioned for $a \in \mathbb{N} + 1$ there are two couples of equations involving fields of conjugated Lorentz representations of dimension a. One of the couples is [9]:

$$\{q^a [a - 1 - N_z]_q \hat{\mathcal{D}}_v T_- - \hat{\mathcal{D}}_- \hat{\mathcal{D}}_z\} \; T_z T_v T_+ \; \hat{\varphi} = 0 \tag{32}$$

$$\{q^a [a - 1 - N_z]_q \hat{\mathcal{D}}_+ T_- T_{\bar{v}} T_v - \hat{\mathcal{D}}_{\bar{v}} \hat{\mathcal{D}}_z\} \; T_z T_v T_+ \; \hat{\varphi} = 0 \tag{33}$$

The solutions are polynomials in z of degree $a - 1$ and formal power series in the q- Minkowski coordinates:

$$\hat{\varphi} \;\; = \sum_{i,j,n,l,m \in \mathbb{Z}_+} p_{ijnlm} \; z^i v^j x_-^n x_+^l \bar{v}^m \tag{34}$$

Substituting (34) in (32) and (33) we obtain two recurrence relations:

$$p_{i+1,j,n+1,lm} \;\; = \;\; q^{a+n} \frac{[a-i-1]_q [j+1]_q}{[i+1]_q [n+1]_q} \; p_{i,j+1,nlm} \tag{35}$$

$$p_{i+1,jnl,m+1} \;\; = \;\; q^{a+j+n+m} \frac{[a-i-1]_q [l+1]_q}{[i+1]_q [m+1]_q} \; p_{ijn,l+1,m} \tag{36}$$

Solving (35) and (36) we obtain, respectively:

$$p_{ijnlm} \;\; = \;\; q^{i(a+n-1) - \frac{i(i-1)}{2}} \frac{[a-1]_q! [j+i]_q! [n-i]_q!}{[a-i-1]_q! [i]_q! [j]_q! [n]_q!} \; p_{0,j+i,n-i,lm} \tag{37}$$

$$p_{ijnlm} \ = \ q^{i(a+j+n+m-1)} \frac{[a-1]_q![l+i]_q![m-i]_q!}{[a-i-1]_q![i]_q![l]_q![m]_q!} \ p_{0jn,l+i,m-i} \tag{38}$$

Combining (37) and (38) we obtain:

$$\begin{aligned} p_{nmdbc} \ &= q^{n(a+d+c+\frac{n}{2}-\frac{1}{2})+m(c+m+n)} \frac{[a-1]_q![b]_q![d]_q![c]_q!}{[a-n-1]_q![n]_q![m]_q![d-m]_q!} \times \\ &\times \frac{1}{[b-n-m]_q![c+n+m]_q!} \ p_{00dbc} \end{aligned} \tag{39}$$

Accordingly, the solution of (32) and (33) is given using (34) as follows (since p_{00dbc} are constants):

$$\begin{aligned} \hat{\varphi}_{dbc} \ &= \ \sum_{m=0}^{min(d,b)} \sum_{n=0}^{a-1} q^{n(a+d+c+\frac{n}{2}-\frac{1}{2})+m(c+m+n)} \frac{[a-1]_q![b]_q![d]_q![c]_q!}{[a-n-1]_q![n]_q![m]_q![d-m]_q!} \times \\ &\times \frac{1}{[b-n-m]_q![c+n+m]_q!} \ z^n v^m x_-^{d-m} x_+^{b-n-m} \bar{v}^{c+n+m} \end{aligned} \tag{40}$$

Formula (40) is valid also for noncommutative coordinates. For commuting coordinates it may be written in more compact form as follows:

$$\hat{\varphi}_{dbc} \ = \ x_-^d x_+^b \bar{v}^c \ F_D^q \left(-b, 1-a, -d, c+1; q^{a+d+c-\frac{1}{2}} \frac{z\bar{v}}{x_+}, q^c \frac{v\bar{v}}{x_+x_-} \right) \tag{41}$$

where F_D^q is a q-deformation of a double hypergeometric function:

$$F_D^q[a, b_1, b_2, c, z_1, z_2] = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q^{\frac{n^2}{2}+m^2+nm} (a)_{n+m}^q (b_1)_n^q (b_2)_m^q}{(c)_{n+m}^q [n]_q! [m]_q!} \ z_1^n z_2^m \tag{42}$$

which for $q = 1$ is given by [15], (f-la 5.7.1.6). For $a = 1$ these solutions coincide with the results of [10].

We pass now to the other couple of equations [9]:

$$\left([a - N_{\bar{z}} - 1]_q \hat{\mathcal{D}}_+ T_{\bar{v}} - q^a \hat{\mathcal{D}}_v \hat{\mathcal{D}}_{\bar{z}} T_- \right) \ T_- T_{\bar{v}} \ \hat{\varphi}' \ = \ 0 \tag{43}$$

$$\left([a - N_{\bar{z}} - 1]_q \hat{\mathcal{D}}_{\bar{v}} - q^a \hat{\mathcal{D}}_- \hat{\mathcal{D}}_{\bar{z}} T_v^2 T_- \right) \ T_{\bar{v}} \ \hat{\varphi}' \ = \ 0 \tag{44}$$

The solutions are polynomials in $\bar{z}$ of degree $a-1$ and formal power series in the q-Minkowski coordinates:

$$\hat{\varphi}' \ = \ \sum_{i,j,n,l,m \in \mathbb{Z}_+} p_{ijnlm} \ v^m x_-^n x_+^l \bar{v}^j \bar{z}^i \tag{45}$$

Substituting (45) in (43) and (44) we obtain two recurrence relations:

$$p_{i+1,jnl,m+1} \ = \ q^{-a+j-n-m} \frac{[a-i-1]_q[l+1]_q}{[i+1]_q[m+1]_q} \ p_{ijn,l+1,m} \tag{46}$$

$$p_{i+1,j,n+1,l,m} = q^{-a-n-2m}\frac{[a-i-1]_q[j+1]_q}{[i+1]_q[n+1]_q}\ p_{i,j+1,nlm} \tag{47}$$

The solution of (46) and (47) is:

$$\begin{aligned} p_{nmd'b'c'} &= q^{-n(a+d'+c'+\frac{n}{2}-\frac{1}{2})+m(c'+m+n)}\frac{[a-1]_q![b']_q![d']_q![c']_q!}{[a-n-1]_q![n]_q![m]_q![d'-m]_q!}\times \\ &\times \frac{1}{[b'-n-m]_q![c'+n+m]_q!}\ p_{00d'b'c'} \end{aligned} \tag{48}$$

Accordingly, the solution of (43) and (44) is given by:

$$\begin{aligned} \hat{\varphi}'_{d'b'c'} &= \sum_{m=0}^{min(d',b')}\sum_{n=0}^{a-1} q^{-n(a+d'+c'+\frac{n}{2}-\frac{1}{2})+m(c'+m+n)}\frac{[a-1]_q![b']_q![d']_q![c']_q!}{[a-n-1]_q![n]_q![m]_q![d'-m]_q!}\times \\ &\times \frac{1}{[b'-n-m]_q![c'+n+m]_q!}\ v^{c'+m+n}x_-^{d'-m}x_+^{b'-n-m}\bar{v}^m\bar{z}^n = \\ &= x_-^{d'}x_+^{b'}\bar{v}^{c'}\ F'^q_D\left(-b', 1-a, -d', c'+1; q^{-(a+d+c-\frac{1}{2})}\frac{z\bar{v}}{x_+}, q^{c'}\frac{v\bar{v}}{x_+x_-}\right) \end{aligned} \tag{49}$$

where another deformation of the same double hypergeometric function as above is used:

$$F'^q_D[a', b'_1, b'_2, c', z_1, z_2] = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{\frac{-n^2}{2}+m^2+nm}(a')^q_{n+m}(b'_1)^q_n(b'_2)^q_m}{(c')^q_{n+m}[n]_q![m]_q!}\ z_1^n z_2^m \tag{50}$$

The first formula in (49) is valid also for noncommutative coordinates. For $a=1$ these solutions coincide with (40), (41), and all reduce to the result of [10].

5. Solutions of the massless q-Dirac equation in terms of q-plane waves

As it was shown in [9] if a function satisfies (32) and (33) or (43) and (44) then it satisfies also the q-d'Alambert equation. Thus, it is justified to look for solutions in terms of q-plane waves.

Here we shall restrict to the case $a=2$. In this case the relevant equations are:

$$\{q^2[1-N_z]_q\hat{\mathcal{D}}_v T_- - \hat{\mathcal{D}}_-\hat{\mathcal{D}}_z\}\ T_z T_v T_+\ \hat{\varphi} = 0 \tag{51}$$

$$\{q^2[1-N_z]_q\hat{\mathcal{D}}_+ T_- T_{\bar{v}} T_v - \hat{\mathcal{D}}_{\bar{v}}\hat{\mathcal{D}}_z\}\ T_z T_v T_+\ \hat{\varphi} = 0 \tag{52}$$

$$([1-N_{\bar{z}}]_q\hat{\mathcal{D}}_+ T_{\bar{v}} T_v^{-1} - q^2\hat{\mathcal{D}}_v\hat{\mathcal{D}}_{\bar{z}} T_-)\ T_- T_{\bar{v}}\ \hat{\varphi}' = 0 \tag{53}$$

$$([1-N_{\bar{z}}]_q\hat{\mathcal{D}}_{\bar{v}} - q^2\hat{\mathcal{D}}_-\hat{\mathcal{D}}_{\bar{z}} T_v^2)\ T_{\bar{v}}\ \hat{\varphi}' = 0 \tag{54}$$

obtained by setting $a=2$ in equations (32), (33), (43), (44). The functions $\hat{\varphi}$, $\hat{\varphi}'$ are polynomials of first degree in z, $\bar{z}$, respectively, and can be written as:

$$\hat{\varphi} = \hat{\varphi}_0 + z\hat{\varphi}_1\ , \qquad \hat{\varphi}' = \hat{\varphi}'_0 + \bar{z}\hat{\varphi}'_1 \tag{55}$$

We note that the above equations are a q-deformation of the massless Dirac equation. Indeed, for $q = 1$ they can be rewritten in the two-component form of the massless Dirac equation. It is well known that the latter splits into independent equations for the neutrino $\Phi_{(-)}$ and the antineutrino $\Phi_{(+)}$:

$$(\partial_{x_0} \pm (\sigma_1 \partial_{x_1} + \sigma_2 \partial_{x_2} + \sigma_3 \partial_{x_3}))\ \Phi_{(\pm)}(x) \ = \ 0 \tag{56}$$

where σ_k are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{57}$$

It is easy to see that $\Phi_{(\pm)}$ are expressed through our functions (for $q = 1$) as:

$$\Phi_{(+)} \ = \ \tfrac{1}{2} \begin{pmatrix} \hat{\varphi}_0 \\ -\hat{\varphi}_1 \end{pmatrix} , \qquad \Phi_{(-)} \ = \ \tfrac{1}{2} \begin{pmatrix} \hat{\varphi}'_1 \\ \hat{\varphi}'_0 \end{pmatrix} . \tag{58}$$

Thus our field $\hat{\varphi}$ corresponds to the antineutrino, while $\hat{\varphi}'$ corresponds to the neutrino.

We start first with the q-deformation of the neutrino equations (53) and (54). We shall look for solutions in terms of the q-plane wave (17):

$$\hat{\varphi}' \ = \ \sum_{s=0}^{\infty} \frac{1}{[s]_q!}\ \psi'_s \tag{59}$$

where ψ'_s are the analogues of f_s , so we shall solve:

$$([1 - N_{\bar{z}}]_q \hat{\mathcal{D}}_+ T_{\bar{v}} T_v^{-1} - q^2 \hat{\mathcal{D}}_v \hat{\mathcal{D}}_{\bar{z}} T_-)\ T_- T_{\bar{v}}\ \psi'_s \ = \ 0 \tag{60}$$

$$([1 - N_{\bar{z}}]_q \hat{\mathcal{D}}_{\bar{v}} - q^2 \hat{\mathcal{D}}_- \hat{\mathcal{D}}_{\bar{z}} T_v^2)\ T_{\bar{v}}\ \psi'_s \ = \ 0 \tag{61}$$

Furthermore we shall make the following Ansatz:

$$\psi'_s \ = \ (\alpha k_+ + \beta k_- + \gamma k_v + \delta k_{\bar{v}} + \bar{z}(\alpha' k_+ + \beta' k_- + \gamma' k_v + \delta' k_{\bar{v}}))\ f_s \tag{62}$$

where $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'$ are constants to be determined. We substitute (62) in (60) and (61) for commutative Minkowski coordinates and noncommutative momenta on the q-light cone. Solving we find that:

$$\beta = 0; \quad \gamma = 0; \quad \alpha' = 0; \quad \delta' = 0; \qquad \beta' = -\delta; \qquad \gamma' = -\alpha \tag{63}$$

Thus, the general solution of (60) and (61) is:

$$\psi_s'^{\alpha,\delta} \ = \ (\alpha k_+ + \delta k_{\bar{v}} - (\delta k_- + \alpha k_v)\bar{z})\ f_s \tag{64}$$

and so the two independent solutions are given in terms of the q-plane wave:

$$\psi'^{(1)} \ = (k_+ - k_v \bar{z})\ \exp_q(k \cdot x) \tag{65a}$$

$$\psi'^{(2)} \ = (k_{\bar{v}} - k_- \bar{z})\ \exp_q(k \cdot x) \tag{65b}$$

Let us stress that the prefactors do not depend on q, i.e., they coincide with the classical ones (which, of course, are obtained by a much shorter calculation).

Now we pass to the antineutrino field:

$$\hat{\varphi} \ = \ \sum_{s=0}^{\infty} \frac{1}{[s]_q!} \ \psi_s \tag{66}$$

where we shall solve the q-deformed equations:

$$\{q^2[1-N_z]_q \hat{\mathcal{D}}_v T_- - \hat{\mathcal{D}}_- \hat{\mathcal{D}}_z\} \ T_z T_v T_+ \ \psi_s \ = \ 0 \tag{67}$$

$$\{q^2[1-N_z]_q \hat{\mathcal{D}}_+ T_- T_{\bar{v}} T_v - \hat{\mathcal{D}}_{\bar{v}} \hat{\mathcal{D}}_z\} \ T_z T_v T_+ \ \psi_s \ = \ 0 \tag{68}$$

Analogously to the above we shall write:

$$\psi_s \ = \ (\alpha k_+ + \beta k_- + \gamma k_v + \delta k_{\bar{v}} + z(\alpha' k_+ + \beta' k_- + \gamma' k_v + \delta' k_{\bar{v}})) \ \tilde{f}_s \tag{69}$$

where $\tilde{f}_s$ is from the conjugated q-plane wave. [If we use the other basis the prefactors will depend on s and we would not be able to express the solutions in terms of a q-plane wave.] Now the general solution of (67) and (68) is:

$$\psi_s^{\alpha,\gamma} \ = \ (\alpha k_+ + \gamma k_v - q^4 z(\alpha k_{\bar{v}} + \gamma k_-)) \ \tilde{f}_s \tag{70}$$

and so the two independent solutions of (51) and (52) are:

$$\tilde{\psi}^{(1)} \ = (k_+ - q^4 k_{\bar{v}} z) \ \widetilde{\exp}_q(k \cdot x) \tag{71a}$$

$$\tilde{\psi}^{(2)} \ = (k_v - q^4 k_- z) \ \widetilde{\exp}_q(k \cdot x) \tag{71b}$$

Note that unlike the neutrino case, the antineutrino prefactors are q-deformed.

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