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QUANTUM GROUP $U_q(A_\ell)$ SINGULAR VECTORS IN POINCARÉ - BIRKHOFF - WITT BASIS

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Abstract

We give explicit expressions for the singular vectors of $U_q(A_\ell)$ in terms of the Poincaré-Birkhoff-Witt (PBW) basis. We relate these expressions with those in terms of the simple root vectors.

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1. Introduction

We consider the q - deformation $U_q(\mathcal{G})$ of the universal enveloping algebras $U(\mathcal{G})$ of simple Lie algebras $\mathcal{G}$ called also quantum groups [1], quantum universal enveloping algebras [2], [3], or just *quantum algebras*. The representations of $U_q(\mathcal{G})$ were first considered in [3], [4], [5], [6] for generic values of the deformation parameter. Actually all results from the representation theory of $\mathcal{G}$ carry over to the quantum group case. This is not so, however, if the deformation parameter q is a root of unity. Thus this case is very interesting from the mathematical point of view, cf. [7], [8], [9], and also [10].

The above developments use results on the embeddings of the reducible Verma modules. These embeddings are realized by the so-called *singular* vectors (or *null* or *extremal* vectors). In [10] was given a general formula for the singular vectors which however was not so explicit, except for the examples necessary for that paper. In [11] were given explicit formulae for the singular vectors of Verma modules over $U_q(\mathcal{G})$ for arbitrary $\mathcal{G}$ corresponding to a class of positive roots of $\mathcal{G}$, which were called straight roots, and some examples corresponding to arbitrary positive roots. Note that these results are complete for $\mathcal{G} = A_\ell$ since in that case all positive roots are straight. In all these cases the singular vectors were given only through the simple root vectors as in earlier work in the case $q = 1$ [12], [13], [14]. The basis in which these singular vectors were written turned out to be part of a general basis introduced later in the context of quantum groups, though for other reasons, by Lusztig [15].

The aim of the present paper is to give expressions for the singular vectors of $U_q(A_\ell)$ in terms of the Poincaré-Birkhoff-Witt (PBW) basis and to relate these expressions with those in terms of the simple root vectors of [11]. The first result may be compared for $q = 1$ with the formulae of [16], while the second result is not known also for $q = 1$, except for $\ell = 2$.

2. Definitions

Let $\mathcal{G}$ be a complex simple Lie algebra; then the quantum algebra $U_q(\mathcal{G})$ is the q-deformation of the universal enveloping algebra $U(\mathcal{G})$ defined [1], [17] as the associative algebra over $\mathbb{C}$ with Chevalley generators $X_i^\pm$, H_i , $i = 1, \ldots, \ell = \mathrm{rank}\, \mathcal{G}$ and with relations [1] :

$$[H_i\, , \, H_j] \ = \ 0, \quad [H_i\, , \, X_j^\pm] \ = \ \pm a_{ij} X_j^\pm \ , \tag{1a}$$

$$[X_i^+\, , \, X_j^-] \ = \ \delta_{ij} \frac{q_i^{H_i/2} - q_i^{-H_i/2}}{q_i^{1/2} - q_i^{-1/2}} \ = \ \delta_{ij}\ [H_i]_{q_i}\ , \quad q_i = q^{(\alpha_i\, ,\, \alpha_i)/2}\ , \tag{1b}$$

$$\sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix}_{q_i} \ (X_i^\pm)^k \ X_j^\pm \ (X_i^\pm)^{n-k} \ = \ 0\ ,\ i \neq j\ , \tag{1c}$$

where $(a_{ij}) = (2(\alpha_i\, , \, \alpha_j)/(\alpha_i\, , \alpha_i))$ is the Cartan matrix of $\mathcal{G}$, $(\cdot\, , \, \cdot)$ is the scalar product of the roots normalized so that for the short roots α we have $(\alpha, \alpha) = 2$, $n = 1 - a_{ij}$,

$$\begin{pmatrix} n \\ k \end{pmatrix}_q \ = \ \frac{[n]_q!}{[k]_q!\ [n-k]_q!}\ , \quad [m]_q! = [m]_q\ [m-1]_q \ldots [1]_q\ , \quad [m]_q = \frac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}} \tag{1d}$$

Further we shall omit the subscript q in $[m]_q$ if no confusion can arise. Note also that sometimes instead of q one uses $q' = q^2$, so that $[m]_{q'} = \frac{q^m - q^{-m}}{q - q^{-1}} \equiv [m]'_q$.

The above definition is valid also when $\mathcal{G}$ is an affine Kac-Moody algebra [1].

We use the standard decompositions into direct sums of vector subspaces $\mathcal{G} = \mathcal{H} \oplus \bigoplus_{\beta\in\Delta} \mathcal{G}_\beta = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^-$, $\mathcal{G}^\pm = \bigoplus_{\beta\in\Delta^\pm} \mathcal{G}_\beta$, where $\mathcal{H}$ is the Cartan subalgebra spanned by the elements H_i, $\Delta = \Delta^+ \cup \Delta^-$ is the root system of $\mathcal{G}$, Δ^+, Δ^-, the sets of positive, negative, roots, respectively; Δ_S will denote the set of simple roots of Δ. We recall that H_j correspond to the simple roots α_j of $\mathcal{G}$, and if $\beta^\vee = \sum_j d_j \alpha_j^\vee$, $\beta^\vee \equiv 2\beta/(\beta,\beta)$, then to β corresponds $H_\beta = \sum_j d_j H_j$. For every $\beta \in \Delta^+$ the elements of $\mathcal{G}$ which span $\mathcal{G}_{\pm\beta}$ are denoted by $X^\pm_\beta$, (dim $\mathcal{G}_{\pm\beta} = 1$). The Cartan–Weyl generators $X^\pm_\beta$ [17], [10], [18] are normalized so that

$$\begin{aligned} [X^+_\beta \ , \ X^-_\beta] \ &= \ [H_\beta]_{q_\beta} \ , \qquad q_\beta \ \equiv \ q^{(\beta,\beta)/2} \\ [H_\beta \ , \ X^\pm_{\beta'}] \ &= \ \pm(\beta^\vee, \beta')X^\pm_{\beta'} \ , \quad \beta, \beta' \in \Delta^+ \end{aligned} \tag{2}$$

We shall not use the fact that the algebra $U_q(\mathcal{G})$ is a Hopf algebra and consequently we shall not introduce the corresponding structure.

3. Highest weight modules over $U_q(\mathcal{G})$

The highest weight modules V over $U_q(\mathcal{G})$ are given by their highest weight $\Lambda \in \mathcal{H}^*$ and highest weight vector $v_0 \in V$ such that:

$$X^+_i \, v_0 \ = \ 0 \ , \quad i = 1, \ldots, \ell \ , \qquad H v_0 \ = \ \Lambda(H) v_0 \ , \quad \forall H \in \mathcal{H} \ . \tag{3}$$

We start with the *Verma* modules V^Λ such that $V^\Lambda \cong U_q(\mathcal{G}) \otimes_{U_q(\mathcal{B})} v_0 \cong U_q(\mathcal{G}^-) \otimes v_0$, where $\mathcal{B} = \mathcal{B}^+$, $\mathcal{B}^\pm = \mathcal{H} \oplus \mathcal{G}^\pm$ are Borel subalgebras of $\mathcal{G}$.

We recall several facts from [10]. The Verma module V^Λ is reducible if there exists a root $\beta \in \Delta^+$ and $m \in I\!N$ such that

$$[(\Lambda + \rho \ , \ \beta^\vee) - m]_{q_\beta} \ = \ 0 \ , \ \beta^\vee \equiv 2\beta/(\beta,\beta) \ , \tag{4}$$

holds, where $\rho = \frac{1}{2} \sum_{\alpha\in\Delta^+} \alpha$. If q is not a root of unity then (4) is also a necessary condition for reducibility and then it may be rewritten as $2(\Lambda + \rho \ , \ \beta) = m(\beta, \beta)$. (In that case it is the generalization of the (necessary and sufficient) reducibility conditions for Verma modules over finite–dimensional $\mathcal{G}$ [19] and affine Lie algebras [20].) For uniformity we shall write the reducibilty condition in the general form (4). If (4) holds then there exists a vector $v_s \in V^\Lambda$, called a *singular vector*, such that $v_s \notin I\!\!C\, v_0$, and:

$$H \ v_s \ = \ (\Lambda(H) - m\beta(H)) \ v_s \ , \ \forall \ H \in \mathcal{H} \tag{5a}$$

$$X^+_i \ v_s = 0 \ , \quad i = 1, \ldots, \ell \ . \tag{5b}$$

The space $U(\mathcal{G}^-)v_s$ is a proper submodule of V^Λ isomorphic to the Verma module $V^{\Lambda - m\beta} = U(\mathcal{G}^-) \otimes v'_0$ where v'_0 is the highest weight vector of $V^{\Lambda - m\beta}$; the isomorphism being realized by $v_s \mapsto 1 \otimes v'_0$. The singular vector is given by [13], [10]:

$$v_s \ = \ v^{\beta,m} \ = \ \mathcal{P}^\beta_m \otimes v_0 \tag{6}$$

where $\mathcal{P}^\beta_m$ is a homogeneous polynomial of weight $m\beta$. The polynomial $\mathcal{P}^\beta_m$ is unique up to a non-zero multiplicative constant. The Verma module V^Λ contains a unique proper maximal submodule I^Λ. Among the HWM with highest weight Λ there is a unique irreducible one, denoted by L_Λ, i.e., $L_\Lambda = V^\Lambda / I^\Lambda$. If V^Λ is irreducible then $L_\Lambda = V^\Lambda$. Thus further we discuss L_Λ for which V^Λ is reducible. If V^Λ reducible with respect to (w.r.t.) to every simple root (and thus w.r.t. to all positive roots), then L_Λ is a finite-dimensional highest weight module over $U_q(\mathcal{G})$ [21]. The representations of $U_q(\mathcal{G})$ are deformations of the representations of $U(\mathcal{G})$, and the latter are obtained from the former for $q \to 1$ [21].

In [11] the singular vectors were given only through the simple root vectors, namely:

$$v^{\beta,m} \ = \ \mathcal{P}^\beta_m(X_1^-, \ldots, X_\ell^-) \otimes v_0 \ , \tag{7}$$

so that $\mathcal{P}^\beta_m$ is a homogeneous polynomial in its variables of degrees mn_i, where $n_i \in \mathbb{Z}_+$ come from $\beta = \sum n_i \alpha_i$.

The aim of the present paper is to give expressions for the singular vectors in terms of the PBW basis and to relate these expressions with those of [11].

4. Singular vectors in PBW basis

In this paper we consider $U_q(\mathcal{G})$ when the deformation parameter q is not a nontrivial root of unity. This generic case is very important for two reasons. First, for $q = 1$ all formulae are valid also for the undeformed case and formulae for the relation with [11] are new also for $q = 1$. Second, the formulae for the case when q is a root of unity use the formulae for generic q as important input as is explained in [11].

Let $\mathcal{G} = A_\ell$. Let α_i, $i = 1, \ldots, \ell$ be the simple roots, so that $(\alpha_i\ ,\ \alpha_j) = -1$ for $|i-j| = 1$ and $(\alpha_i\ ,\ \alpha_j) = 2\delta_{ij}$ for $|i-j| \neq 1$. Then every root $\alpha \in \Delta^+$ is given by $\alpha = \alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$, where $1 \leq i \leq j \leq \ell$; in particular, the simple roots in this notation are: $\alpha_i = \alpha_{ii}$. We recall that for A_ℓ the highest root is given by $\tilde{\alpha} = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell$ and that every root $\alpha \in \Delta^+$ is the highest root of a subalgebra of A_ℓ; explicitly α_{ij} is the highest root of the subalgebra A_{j-i+1} with simple roots $\alpha_i, \alpha_{i+1}, \ldots, \alpha_j$. This means that it is enough to give the formula for the singular vector corresponding to the highest root.

Further we shall need the Cartan-Weyl basis of $\mathcal{G}$ and then the PBW basis of $U_q(\mathcal{G})$. First, let $X^\pm_{ij}$ be the Cartan-Weyl generators corresponding to the roots $\pm\alpha_{ij}$ with $i \leq j$; in particular, $X^\pm_{ii} = X^\pm_i$, correspond to the simple roots α_i. The CW generators corresponding to the nonsimple roots with $i < j$ are given as follows:

$$\begin{aligned} X^\pm_{ij} \ &= \ \pm\, q^{\mp 1/4} \left(q^{1/4}\ X^\pm_i X^\pm_{i+1,j} \ - \ q^{-1/4}\ X^\pm_{i+1,j} X^\pm_i \right) \ = \\ &= \ \pm\, q^{\mp 1/4} \left(q^{1/4}\ X^\pm_{i,j-1} X^\pm_j \ - \ q^{-1/4}\ X^\pm_j X^\pm_{i,j-1} \right) \end{aligned} \tag{8}$$

Now the PBW basis is given by monomials of the following kind:

$$\begin{aligned} &(X^-_\ell)^{k_\ell}\ (X^-_{\ell-1,\ell})^{k_{\ell-1,\ell}}\ \ldots\ (X^-_{1\ell})^{k_{1\ell}}\ (X^-_{\ell-1})^{k_{\ell-1}}\ \ldots\ \times \\ &\times\ \ldots\ (X^-_p)^{k_p}\ \ldots\ (X^-_{rp})^{k_{rp}}\ \ldots\ (X^-_{12})^{k_{12}}\ (X^-_1)^{k_1} \end{aligned} \tag{9}$$

This monomials are in the so-called normal order [18]. Namely, we put the simple root vectors X_j^- in the order $X_\ell^-, X_{\ell-1}^-, \ldots X_1^-$. Then we put a root vector X_α^- corresponding to the nonsimple root α between the root vectors X_β^- and X_γ^- if $\alpha = \beta+\gamma$, $\alpha,\beta,\gamma \in \Delta^+$. This order is not complete but this is not a problem, since when two roots are not ordered this means that the corresponding root vectors commute, e.g., X_i^- and $X_{i-k,i+k}^-$.

Let us have condition (4) fulfilled for $\tilde\alpha$, but not for any other positive root:

$$\begin{aligned}
[(\Lambda+\rho,\tilde\alpha^\vee) - m]_q &= [\Lambda(H_{\tilde\alpha}) + \ell - m]_q = 0 \ , \ m \in I\!N \ , && (10a)\\
[(\Lambda+\rho,\alpha_{ij}^\vee) - m']_q &= [\Lambda(H_{ij}) + j - i + 1 - m']_q \neq 0 \ , \ \forall \alpha_{ij} \neq \tilde\alpha \ , \ \forall m' \in I\!N \ . && (10b)
\end{aligned}$$

Let us denote the singular vector corresponding to $(10a)$ by $v_s^{m,\tilde\alpha}$. We take an arbitrary linear combination of the PBW basis and impose first the condition $(5a)$ with $\beta\to\tilde\alpha$ which restricts the linear combination to terms of weight $m\tilde\alpha$. Thus, we have:

$$\begin{aligned}
v_s^{m,\tilde\alpha} \ &= \sum_{t_{ij},1\leq i<j\leq\ell} C_T \ (X_\ell^-)^{m-t_{\ell-1,\ell}} (X_{\ell-1,\ell}^-)^{t_{\ell-1,\ell}-t_{\ell-2,\ell}} \ldots (X_{1\ell}^-)^{t_{1\ell}} \ \times \\
&\times (X_{\ell-1}^-)^{m-t_{\ell-1,\ell}-t_{\ell-2,\ell-1}} \ldots (X_p^-)^{m-t_{p\ell}-t_{p,\ell-1}- \ldots -t_{p,p+1}-t_{p-1,p}} \times \qquad (11)\\
&\times \ldots (X_{rp}^-)^{t_{rp}-t_{r-1,p}} \ldots (X_{12}^-)^{t_{12}} (X_1^-)^{m-t_{1\ell}-t_{1,\ell-1}- \ldots -t_{12}}
\end{aligned}$$

where the summations in the variables $t_{ij}\in Z\!\!Z_+$, $1\leq i<j\leq\ell$, are such that all powers are nonnegative. The coefficients C_T depend on the variables t_{ij}, and are to be found by imposing the annihilating condition $(5b)$. As expected they are fixed up to an arbitrary nonzero multiplicative constant, say C :

$$\begin{aligned}
C_T \ &= C \ q^{\frac12 \sum\limits_{i<j} (2m-1)t_{ij} - t_{ij}^2 \ - \ \frac12 \sum_{1\leq i<j\leq\ell-1} \sum_{p=j+1}^{\ell} t_{ij}(t_{ip}+t_{i+1,p})} \times \\
&\times \frac{q^{-\frac12 \sum\limits_{r=1}^{\ell-1} t^r (\lambda+\rho)(H^r)} \prod\limits_{r=1}^{\ell-1} [m-t^r]!}{\left(\prod\limits_{i=1}^{\ell} [m-t^i-t_{i-1,i}]!\right) \prod\limits_{1\leq i<j\leq\ell} [t_{ij}-t_{i-1,j}]!} \times \\
&\times \prod_{j=1}^{\ell-1} \frac{\Gamma_q(\lambda(H^j)+j-m+t^j)}{\Gamma_q(\lambda(H^j)+j-m)} \ , \qquad (12)\\
H^r \ &= \ H_1 + \ \ldots \ + H_r \ , \qquad t^i \ = \sum_{i+1\leq s\leq\ell} t_{is}
\end{aligned}$$

For $\ell=2$ and $t=t_{12}$ this formula is given [22], and for $q=1$ and $\ell=2,3$ in [12]. For $q=1$ our formula would coincide with the result of [16], if we correct one of the definitions there, namely, if we define $B_k^{corr} = \sum_{i\leq k<j} \alpha_{ij}$ instead of $B_k = \sum_{i\leq k\leq j} \alpha_{ij}$ in f-la (20) of [16], $\alpha_{ij} = t_{ij} - t_{i-1,j}$; note that $B_k^{corr} = t^k$.

5. Relation between the two expressions for the singular vectors

Here we would like to present the relation between the expressions for the singular vectors in the PBW basis given in (11) and in the simple root vectors basis given in [11]. The latter formula is (cf. f-la (13) from [11] for $t = \ell - 1$):

$$\begin{aligned} v^{\tilde{\alpha},m} &= \sum_{k_1=0}^{m} \cdots \sum_{k_{\ell-1}=0}^{m} a_{k_1 \ldots k_{\ell-1}} (X_1^-)^{m-k_1} \cdots (X_{\ell-1}^-)^{m-k_{\ell-1}} \times \\ &\quad \times \; (X_\ell^-)^m \; (X_{\ell-1}^-)^{k_{\ell-1}} \; \cdots \; (X_1^-)^{k_1} \; \otimes v_0 \quad , \end{aligned} \tag{13a}$$

$$\begin{aligned} a_{k_1 \ldots k_{\ell-1}} &= a \, (-1)^{k_1 + \cdots + k_{\ell-1}} \binom{m}{k_1}_q \cdots \binom{m}{k_{\ell-1}}_q \times \\ &\quad \times \; \frac{[(\lambda+\rho)(H^1)]_q}{[(\lambda+\rho)(H^1) - k_1]_q} \cdots \frac{[(\lambda+\rho)(H^{\ell-1})]_q}{[(\lambda+\rho)(H^{\ell-1}) - k_{\ell-1}]_q} = \\ &= a \, (-1)^{k_1 + \cdots + k_{\ell-1}} \binom{m}{k_1}_q \cdots \binom{m}{k_{\ell-1}}_q \times \\ &\quad \times \; \frac{[\lambda^1 + 1]_q}{[\lambda^1 + 1 - k_1]_q} \cdots \frac{[\lambda^{\ell-1} + \ell - 1]_q}{[\lambda^{\ell-1} + \ell - 1 - k_{\ell-1}]_q} \, , \quad a \neq 0 \end{aligned} \tag{13b}$$

where $\lambda^s = \lambda(H^s)$.

5.1. C-coefficients in terms of a-coefficients

The C-coefficients are given in terms of the a-coefficients by the following formula:

$$\begin{aligned} C_T &= (-1)^{\sum\limits_{i<j} t_{ij}} \; q^{\frac{1}{2} \sum\limits_{i<j} ((2m-1)t_{ij} - t_{ij}^2) \; - \frac{1}{2} \sum_{1 \le i < j \le (\ell-1)} \; \sum_{(j+1) \le p \le \ell} \; t_{ij}(t_{i+1,p} + t_{i,p})} \times \\ &\times \; \frac{q^{\frac{1}{2}(1-\ell)m^2} \prod\limits_{r=1}^{\ell-1} [m - t^r]!}{\left(\prod\limits_{s=1}^{\ell} [m - t^s - t_{s-1,s}]! \right) \prod\limits_{1 \le i < j \le \ell} [t_{ij} - t_{i-1,j}]!} \times \\ &\times \sum_{k_1, k_2, \ldots, k_{\ell-1}} a_{k_1, k_2, \ldots, k_{\ell-1}} \; q^{\frac{1}{2} \sum\limits_{i=1}^{\ell-1} k_i (m - t^i)} \prod_{r=1}^{\ell-1} \frac{[m - k_r]!}{[m - t^r - k_r]!} \end{aligned} \tag{14}$$

where $0 \le k_r \le m - t^r$.

To prove the above formula one can use the following lemmas:

Lemma 1:

$$\sum_{j=0}^{m} \frac{(-1)^j q^{\frac{mj}{2}}}{[a-j][m-j]![j]!} = \frac{(-1)^m q^{\frac{ma}{2}}}{\prod\limits_{k=0}^{m} [a-k]} \tag{15}$$

Lemma 2:

$$\frac{(X_r^-)^m (X_s^-)^n}{[m]![n]!} \;=\; \sum_{0\le p\le min(m,n)} (-1)^p q^{\frac{1}{2}((p-m)(n-p)-p)} \frac{(X_s^-)^{n-p}(X_{rs}^-)^p (X_r^-)^{m-p}}{[n-p]![p]![m-p]!} \qquad (16)$$

Lemma 1 follows from formula (60) of [23] which is:

$$\begin{aligned} {}_2F_1{}^q(-\nu,\ b;\ c;\ q^{\pm\frac{1}{2}(b-c+1-\nu)}) \;&=\; \sum_{s=0}^{\nu} (-1)^s \begin{pmatrix} \nu \\ s \end{pmatrix}_q \frac{(b)_s^q}{(c)_s^q}\ q^{\pm\frac{1}{2}s(b-c+1-\nu)} \;= \\ &=\; \frac{(c-b)_\nu^q}{(c)_\nu^q}\ q^{\pm\frac{1}{2}b\nu} \end{aligned} \qquad (17)$$

in which here we set $b=-a,\ \ c=1-a,\ \ \nu=m$, and use sign minus. Lemma 2 follows just from $(1c)$ and (8).

5.2. a-coefficients in terms of C-coefficients

The a-coefficients are given in terms of the C-coefficients by the following formula:

$$\begin{aligned} a_{j_1,\ldots,j_{\ell-1}} \;&=(-1)^{j_1+\cdots+j_{\ell-1}+l(m-1)} \sum_{\substack{m-t^1\le j_1\\ \vdots \\ m-t^{\ell-1}\le j_{\ell-1}}} C_T \prod_{i=1}^{\ell-1} \frac{q^{\frac{1}{2}((m-t^i)(1-j_i)-j_i)}}{[m-t^i]![m-j_i]![t^i+j_i-m]!}\ \times \\ &\times \left(\prod_{i=1}^{\ell}[m-t^i-t_{i-1,i}]!\right) \prod_{1\le i<j\le \ell} [t_{ij}-t_{i-1,j}]!\ \times \\ &\times\ q^{\frac{1}{2}\sum_{i<j}((1-2m)t_{ij}+t_{ij}^2)\ +\ \frac{1}{2}\sum_{1\le i<j\le \ell-1}\sum_{p=j+1}^{\ell} t_{ij}(t_{i+1,p}+t_{ip})+\frac{1}{2}(\ell-1)m^2} \end{aligned} \qquad (18)$$

To prove (18) one can use the relation:

$$\begin{aligned} &\sum_{t=0}^{p}(-1)^{p+t} q^{\frac{t(k-p+1)-p}{2}} \begin{pmatrix} m-k \\ m-t \end{pmatrix}_q \begin{pmatrix} m-t \\ m-p \end{pmatrix}_q \;= \\ &= (-1)^{p-k} q^{\frac{1}{2}(k+1)(k-p)}\ \frac{[m-k]!}{[m-p]!} \sum_{s=0}^{p-k} \frac{(-1)^s q^{\frac{1}{2}s(1-p+k)}}{[s]![p-k-s]!} \;=\; \delta_{p,k} \end{aligned} \qquad (19)$$

which also follows from (17) setting $b=c,\ \nu=p-k$, and using the fact that $(0)_\nu^q = \Gamma_q(\nu)/\Gamma_q(0) = \delta_{\nu,0}$.

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