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# INDUCED REPRESENTATIONS OF THE MULTIPARAMETER HOPF SUPERALGEBRAS $U_{uq}(gl(m/n))$ and $U_{uq}(sl(m/n))$

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**Abstract**

We construct induced representations of the multiparameter Hopf superalgebras $U_{u\mathbf{q}}(gl(m/n))$ and $U_{u\mathbf{q}}(sl(m/n))$. The first superalgebra we constructed earlier as the dual of the multiparameter quantum deformation of the supergroup $GL(m/n)$. The second superalgebra is a Hopf subalgebra of the first for a special choice of the parameters. The representations are labelled by $m+n$ integer numbers, respectively $m+n-1$ complex numbers, and act in the space of formal power series of $(m+n)(m+n-1)/2$ non-commuting variables, of which $mn$ are odd and the rest are even. These variables generate a $q$-deformation of a flag supermanifold of the supergroup $GL(m/n)$, respectively $SL(m/n)$.

## 1. Introduction

The extension of the activity on quantum groups to the field of supersymmetry was started with the paper of Manin [1], where the standard multiparametric quantum deformation of the supergroup $GL(m/n)$ was introduced. These deformations of $GL(m/n)$ were further studied in, e.g., [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]. (For the non-standard two-parameter deformations of $GL(1/1)$ we refer to [12], [13], [14].) In the case of one-parametric deformation the superalgebra $U_q(gl(m/n))$ in duality with $GL_q(m/n)$ and its quantum subsuperalgebra $U_q(sl(m/n))$ were studied in, e.g., [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35]. However, there was not much study of the multiparameter deformations of $U(gl(m/n))$ and $U(sl(m/n))$ and their interrelations, namely, two-parameter deformations were obtained for $m = n = 1$ in [36], [5], [8], and multiparameter deformations of $U(sl(m/n))$ were obtained in [37], and of $U(sl(m/1))$ in [38]. However, until recently the superalgebra in duality with the standard multiparameter deformation $GL_{u\mathbf{q}}(m/n)$ was not known. This dual Hopf superalgebra, which we denote as $\mathcal{U} \equiv \mathcal{U}_{u\mathbf{q}}(gl(m/n))$, was found in [39]. There were also found the conditions on the parameters for which $\mathcal{U}$ has as Hopf subsuperalgebra the multiparameter deformation $\mathcal{U}' \equiv \mathcal{U}_{u\mathbf{q}}(sl(m/n))$. (For $m = n = 1$ the latter holds always.)

In the present paper we construct the induced holomorphic representations of $\mathcal{U}$ and $\mathcal{U}'$. The representations are labelled by $m+n$ integer numbers, respectively $m+n-1$ complex numbers and act in the space of formal power series of $(m+n)(m+n-1)/2$ non-commuting variables, of which $mn$ are odd and the rest are even. These variables generate a $q$-deformation of a flag supermanifold of the supergroup $GL(m/n)$, respectively $SL(m/n)$. The construction is achieved by using the Gauss decomposition of the generators of the multiparameter matrix quantum supergroup $\mathcal{A} \equiv GL_{u\mathbf{q}}(m/n)$. We use it to give a new basis of $\mathcal{A}$ which we use as expansion basis for our functions and which has convenient properties w.r.t. the right action of $\mathcal{U}$. Namely, we impose the conditions of right covariance [40] in order to eliminate the dependence of our functions on the strictly upper diagonal generators in the Gauss decomposition, while the dependence on the diagonal generators in the Gauss decomposition is fixed for all functions. These fixed powers of the diagonal generators are the integer numbers which parametrize our representations. For $u = \mathbf{q} = 1$ our representations coincide with the holomorphic representations induced from the upper diagonal Borel subsupergroup $B$ of $G \equiv GL(m/n)$ and acting on the coset $G/G^+$, where $G^+$ is the strictly upper diagonal supergroup of $G$. That is why we call our representations induced. Further, we enforce the conditions under which $\mathcal{U}'$ is a Hopf superalgebra. Then we can set the superdeterminant to unity and consider the representations of $\mathcal{U}'$. Finally, we eliminate also the dependence on the diagonal generators of the Gauss decomposition. This is done invariantly, so that the representation parameters remain in the matrix elements. For $u = \mathbf{q} = 1$ these latter representations coincide with the standard holomorphic representations induced from $B$ and acting on the coset $G/B$. These representations can be extended to arbitrary complex values of the $m + n - 1$ representations parameters.

The paper is organized as follows. In Section 2 we recall the multiparameter matrix quantum supergroup $\mathcal{A}$ and the dual multiparameter Hopf superalgebra $\mathcal{U}$. In Section 3 we give the left and right actions of $\mathcal{U}$ on $\mathcal{A}$. In Section 4 we give the Gauss decomposition of the generators of $\mathcal{A}$ and a new basis of $\mathcal{A}$. In Section 5 we give the explicit construction of the induced representations of $\mathcal{U}$ and $\mathcal{U}'$. Section 6 contains an Outlook. There are also three Appendices.

## 2. Multiparameter deformation of GL(m/n) and the dual superalgebra

The multiparameter quantum deformation $\mathcal{A} = GL_{u\mathbf{q}}(m/n)$ of the supergroup $GL(m/n)$ was introduced first in [1], and later, in a slightly different form, in [11]. It is generated by the elements of a quantum supermatrix $M$ :

$$M = (T_{IJ}) = \begin{pmatrix} A_{ij} & B_{i\alpha} \\ C_{\beta j} & D_{\beta\alpha} \end{pmatrix} \tag{2.1}$$

where $I, J = 1, ..., m+n;\ \ i,j = 1,...,m$ and $\alpha, \beta = m+1, ..., m+n$, which obey the following commutaion relations:

$$\begin{aligned}
T_{IN}T_{IL} &= (-1)^{\widehat{I}\widehat{N}+\widehat{I}\widehat{L}+\widehat{N}\widehat{L}}(-u^2)^{\widehat{I}} p\ T_{IL}T_{IN}\ , \quad \text{for}\ \ N<L\ , \\
T_{IN}T_{JN} &= (-1)^{\widehat{N}\widehat{I}+\widehat{N}\widehat{J}+\widehat{I}\widehat{J}}(-u^2)^{\widehat{N}} q\ T_{JN}T_{IN}\ , \quad \text{for}\ \ I<J\ , \\
p\ T_{IL}T_{JN} &= (-1)^{(\widehat{I}+\widehat{L})(\widehat{J}+\widehat{N})} q\ T_{JN}T_{IL}\ , \quad \text{for}\ \ I<J\ ,\ N<L\ , \\
(-1)^{(\widehat{I}+\widehat{N})(\widehat{J}+\widehat{L})}\ uq\ T_{JL}T_{IN} - (up)^{-1}\ T_{IN}T_{JL} &= (-1)^{\widehat{J}\widehat{N}+\widehat{J}\widehat{L}+\widehat{N}\widehat{L}}(u-u^{-1})\ T_{IL}T_{JN} \\
&\text{for}\ \ I<J\ ,\ N<L \\
(T_{IN})^2 &= 0\ , \quad \text{for}\ \ \widehat{I}+\widehat{N}=1 \\
p = \frac{q_{NL}}{u^2}, &\quad q = \frac{1}{q_{IJ}}
\end{aligned} \tag{2.2}$$

where $\widehat{}$ denotes the parity, which for the indices is defined by: $\widehat{I} = 0$ if $I = i = 1, ..., m$ and $\widehat{I} = 1$ if $I = \alpha = m+1, ..., m+n$. Further, we define the parity $\widehat{T_{IJ}}$ of the generators $T_{IJ}$ through the parity of the indices, namely we set: $\widehat{T_{IN}} = (\widehat{I}+\widehat{N})$ (mod 2). Thus, the supermatrix $M$ is in the so-called standard form, so that the elements of $A$ and $D$ are even and those of $B$ and $C$ are odd. We shall not need explicitly the basis of $\mathcal{A}$ which was introduced in [39], but we shall use the fact that it is homogeneous, i.e., each element of the basis has a definite parity.

Considered as a superbialgebra, $\mathcal{A}$ has the following comultiplication $\delta_{\mathcal{A}}$ and counit $\varepsilon_{\mathcal{A}}$ [1]:

$$\delta_{\mathcal{A}}(T_{IJ}) = \sum_{N=1}^{m+n} T_{IN} \otimes T_{NJ} = (T_{IJ})_{(1)} \otimes (T_{IJ})_{(2)} \tag{2.3$a$}$$

$$\varepsilon_{\mathcal{A}}(T_{IJ}) = \delta_{IJ} \tag{2.3$b$}$$

where in (2.3$a$) we have used Sweedler's notation for the co-product of an element $a$ : $\delta_{\mathcal{A}}(a) = a_{(1)} \otimes a_{(2)}$. We also recall that for a superbialgebra the coproduct preserves the parity, (cf., e.g., [39]). In particular, $\widehat{a} = (\widehat{a}_{(1)} + \widehat{a}_{(2)})$ (mod 2).

The Hopf superalgebra $\mathcal{U} \equiv \mathcal{U}_{u\mathbf{q}}(gl(m/n))$ which is in duality with $GL_{u\mathbf{q}}(m/n)$ was found in [39]. Naturally $\mathcal{U}$ is a multiparameter deformation of the superalgebra $U(gl(m/n))$. We have shown that as a commutation algebra we have the classical structure, namely, a splitting in two subalgebras: $\mathcal{U} \cong \mathcal{U}' \otimes \mathcal{Z}$, where $\mathcal{U}'$ is isomorphic to the standard one-parametric deformation $U_u(sl(m/n))$, and $\mathcal{Z}$ is central in $\mathcal{U}$ for $m \neq n$. However, as a coalgebra $\mathcal{U}$ can not be split in this way, as only $\mathcal{Z}$ is a Hopf subalgebra, while $\mathcal{U}'$ is not a Hopf subalgebra unless $m = n = 1$ or some special relations between the parameters exist. These special relations were established in [39] and used to obtain explicit multiparameter Hopf superalgebra deformations of $U(sl(m/n))$ which we use here.

Let us denote the Chevalley generators of $sl(m/n)$ by $H_I$, $X_I^{\pm}$ $I = 1, \ldots, m + n - 1$. Then we take for the 'Chevalley' generators of $\mathcal{U}'$ : $K_I = u^{d_I H_I/2}$, $K_I^{-1} = u^{-d_I H_I/2}$, $X_I^{\pm}$ , $I = 1, \ldots,\ m+n-1$, $d_1 = \ldots = d_m = -d_{m+1} = \ldots = -d_{m+n} = 1$, with the following algebra relations

$$K_I K_J \;=\; K_J K_I, \quad K_I K_I^{-1} \;=\; K_I^{-1} K_I \;=\; 1_{\mathcal{U}} \tag{2.4a}$$
$$K_I X_J^{\pm} \;=\; u^{\pm c_{IJ}} X_J^{\pm} K_I \tag{2.4b}$$
$$[X_I^+, X_J^-] \;=\; \delta_{IJ} \frac{K_I^2 \;-\; K_I^{-2}}{\lambda_I} \tag{2.4c}$$
$$X_I^{\pm} X_J^{\pm} \;=\; X_J^{\pm} X_I^{\pm} \quad |I - J| > 1 \tag{2.4d}$$
$$(ad_{u^{\kappa}} X^{\pm})^2 X_J^{\pm} \;=\; 1 \quad |I - J| = 1 \tag{2.4e}$$
$$[\,[\,X_m^{\pm}\,,\,X_{m-1}^{\pm}]_{u^{\kappa}}\,,\,[\,X_m^{\pm}\,,\,X_{m+1}^{\pm}]_{u^{\kappa}}\,]_{u^{\kappa}} \;=\; 0\,, \quad \kappa = \pm \tag{2.4f}$$

where $c_{IJ}$ is the Cartan matrix of $sl(m/n)$ and $\lambda_I = d_I \lambda$, $(\lambda = u - u^{-1})$.

Further $\mathcal{Z}$ is generated by $\mathcal{K} = u^{K'/2}$ with $K' = K$ $(m \neq n)$, $K' = \tilde{K}$ if $m = n$. Here $K$ is the standard central generator of $gl(m/n)$, being given in the defining matrix representation by $1_{m+n}$. The generator $K$ is not used for $m = n$ since then it belongs also to the Cartan subalgebra of $sl(m/m)$), (being a linear combination of the $H_I$). For $m = n$ we introduce the generator $\tilde{K}$ which belongs to the Cartan subalgebra of $gl(m/m)$, but not to the subsuperalgebra $sl(m/m)$. In the defining matrix representation $\tilde{K}_{IJ} = d_I \delta_{IJ}$.

The Hopf structure of $\mathcal{U}$ is given by [39]:

$$\delta_{\mathcal{U}}(K_I^{\pm}) \;= K_I^{\pm} \otimes K_I^{\pm} \tag{2.5a}$$
$$\delta_{\mathcal{U}}(X_I^+) \;= X_I^+ \otimes \mathcal{P}_I^{1/2} \;+\; \mathcal{P}_I^{-1/2} \otimes X_I^+ \tag{2.5b}$$
$$\delta_{\mathcal{U}}(X_I^-) \;= X_I^- \otimes \mathcal{Q}_I^{1/2} \;+\; \mathcal{Q}_I^{-1/2} \otimes X_I^- \tag{2.5c}$$
$$\varepsilon_{\mathcal{U}}(K_I^{\pm}) \;= 1_{\mathcal{U}} \qquad \varepsilon_{\mathcal{U}}(X_I^{\pm}) \;=\; 0 \tag{2.5d}$$
$$\gamma_{\mathcal{U}}(K_I) \;= K_I^{-1}\,, \qquad \gamma_{\mathcal{U}}(X_I^{\pm}) \;=\; -u^{\pm(d_I + d_{I+1})/2} X_I^{\pm} \tag{2.5e}$$
$$\delta_{\mathcal{U}}(\mathcal{K}) \;= \mathcal{K} \otimes \mathcal{K}, \quad \varepsilon_{\mathcal{U}}(\mathcal{K}) \;=\; 1, \quad \gamma_{\mathcal{U}}(\mathcal{K}) \;=\; \mathcal{K}^{-1} \tag{2.6}$$

where

$$\mathcal{P}_I \;= (\tilde{q}_I)^{\hat{K}'} \prod_{S=1}^{m+n} Q_{IS}^{d_S \widehat{H}_S}\,, \qquad \mathcal{Q}_I^{1/2} = K_I^2 \mathcal{P}_I^{-1/2} \tag{2.7a}$$

$$Q_{II} \;=\; \begin{cases} \frac{u^2}{q_{i,i+1}}\,, & i \leq m \\ \frac{u'^2}{q'_{\alpha,\alpha+1}}\,, & I = \alpha > m \end{cases} \tag{2.7b}$$
$$u' \;\equiv\; 1/u\,, \qquad q'_{IJ} \;\equiv\; q_{IJ}/u^2$$

$$Q_{I,I+1} \;=\; \begin{cases} \frac{1}{q_{i,i+1}} \,, & i < m \\ \frac{1}{q'_{m,m+1}} \,, & I = m \\ \frac{1}{q'_{\alpha,\alpha+1}} \,, & I = \alpha > m \end{cases} \tag{2.7c}$$

$$Q_{IS} \;=\; \begin{cases} \frac{q_{SI}}{q_{S,I+1}} \,, & S \leq I-1 \\ \frac{q_{I+1,S}}{q_{IS}} \,, & I+2 \leq S \end{cases} \tag{2.7d}$$

$$\tilde{q}_I \;=\; \left( \prod_{s=1}^{m} Q_{Is} \right) \prod_{\alpha=m+1}^{m+n} Q_{I\alpha}^{-1} \tag{2.7e}$$

$$\widehat{H}_S \;\equiv\; \sum_{J=S}^{m+n-1} d_J H_J \,, \qquad \widehat{H}_{m+n} \;=0 \tag{2.7f}$$

and for $m \neq n$ we have:

$$\hat{K}' \;\equiv\; \frac{1}{m-n} \,(K \;-\; K_0) \tag{2.8}$$

$$K_0 \;\equiv\; \sum_{j=1}^{m} j H_j \;+\; \sum_{\beta=m+1}^{m+n-1} (\beta - 2m) H_\beta$$

while for $m = n$ we have:

$$\hat{K}' \;\equiv\; \frac{1}{2m} \,(\tilde{K} \;-\; \tilde{K}_0) \tag{2.9}$$

$$\tilde{K}_0 \;\equiv\; \sum_{I=1}^{2m-1} I d_I H_I \;.$$

We have also:

$$\delta_{\mathcal{U}}(\mathcal{P}_I) \;= \mathcal{P}_I \otimes \mathcal{P}_I \qquad \delta_{\mathcal{U}}(\mathcal{Q}_I) \;=\; \mathcal{Q}_I \otimes \mathcal{Q}_I \tag{2.10a}$$

$$\varepsilon_{\mathcal{U}}(\mathcal{P}_I) \;= 1_{\mathcal{U}} \qquad \varepsilon_{\mathcal{U}}(\mathcal{Q}_I) \;=\; 1_{\mathcal{U}} \tag{2.10b}$$

$$\gamma_{\mathcal{U}}(\mathcal{P}_I) \;= \mathcal{P}_I^{-1} \,, \qquad \gamma_{\mathcal{U}}(\mathcal{Q}_I) \;=\; \mathcal{Q}_I^{-1} \tag{2.10c}$$

Note that from the generators $X_I^\pm$, $K_I$, $\mathcal{K}$, only $X_m^\pm$ are odd, while the rest are even.

As we said we shall also use the conditions on the deformation parameters that decouple $K'$ from $\mathcal{P}$ and $\mathcal{Q}$, namely

$$\tilde{q}_I \;=\; 1 \;. \tag{2.11}$$

If (2.11) holds then $\mathcal{U}'$ is a Hopf subalgebra of $\mathcal{U}$ [39]. Note that for $m = n = 1$ (2.11) holds always.

The bilinear form giving the duality between $\mathcal{U}$ and $\mathcal{A}$ is given by [39]:

$$\langle K_I, T_{JL} \rangle \;=\; u^{d_I(\delta_{IJ} \;-\; \frac{d_I}{d_{I+1}} \delta_{I+1,J})/2} \delta_{JL} \tag{2.12a}$$

$$\langle X_I^+, T_{JL} \rangle \;=\; u^{\tilde{I}/2} Q_{I,I+1}^{-1/2} \delta_{IJ} \delta_{J+1,L} \tag{2.12b}$$

$$\langle X_I^-, T_{JL} \rangle \;=\; (-1)^{\tilde{I}} u^{(\tilde{I}-2d_I)/2} Q_{II}^{1/2} \delta_{IL} \delta_{J-1,L} \tag{2.12c}$$

from which follows:

$$\begin{aligned} \langle \mathcal{P}_I^{1/2}, T_{JL} \rangle &= Q_{IJ}^{1/2} \delta_{JL} && (2.12d) \\ \langle \mathcal{Q}_I^{1/2}, T_{JL} \rangle &= u^{d_I(\delta_{IJ} - \frac{d_I}{d_{I+1}}\delta_{I+1,J})} Q_{IJ}^{-1/2} \delta_{JL} && (2.12e) \end{aligned}$$

Finally:

$$\begin{aligned} \langle \mathcal{K}, T_{JL} \rangle &= u^{1/2} \delta_{JL} \qquad m \neq n && (2.13a) \\ \langle \mathcal{K}, T_{JL} \rangle &= u^{d_J/2} \delta_{JL} \qquad m = n && (2.13b) \end{aligned}$$

where $\tilde{I} = 1$ if $I = m$ and 0 otherwise.

The pairing between arbitrary elements of $\mathcal{U}$ and $\mathcal{A}$ follows from the properties of the duality pairing. The pairing (2.12) is standardly supplemented with

$$\langle y, 1_{\mathcal{A}} \rangle = \varepsilon_{\mathcal{U}}(y) \ . \qquad (2.14)$$

## 3. Left and right actions of $\mathcal{U}$ and $\mathcal{U}'$

We begin by defining two actions of the dual algebra $\mathcal{U}$ on $\mathcal{A}$. First we introduce (as in [41]) the left regular representation of $\mathcal{U}$ by:

$$\begin{aligned} \pi(y)\, T_{IL} &= \sum_{N=1}^{m+n} \langle \gamma_{\mathcal{U}}(y), T_{IN} \rangle \ T_{NL} \ = && (3.1a) \\ &= \langle \gamma_{\mathcal{U}}(y), (T_{IL})_{(1)} \rangle \, (T_{IL})_{(2)} && (3.1b) \end{aligned}$$

where in the second line we have used (2.3$a$). From (3.1) we find the explicit action of the generators of $\mathcal{U}$:

$$\begin{aligned} \pi(K_I) T_{JL} &= u^{d_I(\frac{d_I}{d_{I+1}}\delta_{I+1,J} - \delta_{IJ})/2} T_{JL} && (3.2a) \\ \pi(X_I^+) T_{JL} &= -\, u^{(\tilde{I}+d_I+d_{I+1})/2} Q_{I,I+1}^{-1/2} \delta_{IJ} T_{J+1,L} && (3.2b) \\ \pi(X_I^-) T_{JL} &= -\ (-1)^{\tilde{I}} u^{(\tilde{I}-3d_I-d_{I+1})/2} Q_{II}^{1/2} \delta_{I+1,J} T_{J-1,L} && (3.2c) \end{aligned}$$

from which follows:

$$\begin{aligned} \pi(\mathcal{P}_I^{1/2}) T_{JL} &= Q_{IJ}^{-1/2} T_{JL} && (3.2d) \\ \pi(\mathcal{Q}_I^{1/2}) T_{JL} &= u^{d_I(\frac{d_I}{d_{I+1}}\delta_{I+1,J} - \delta_{IJ})} Q_{IJ}^{1/2} T_{JL} && (3.2e) \end{aligned}$$

Finally:

$$\begin{aligned} \pi(\mathcal{K}) T_{JL} &= u^{-1/2} T_{JL} \qquad m \neq n && (3.3a) \\ \pi(\mathcal{K}) T_{JL} &= u^{-d_J/2} T_{JL} \qquad m = n && (3.3b) \end{aligned}$$

The above is supplemented with the following action on the unit element of $\mathcal{A}$:

$$\pi(K_I) 1_{\mathcal{A}} = 1_{\mathcal{A}}, \quad \pi(X_I^{\pm}) 1_{\mathcal{A}} = 0, \quad \pi(\mathcal{K}) 1_{\mathcal{A}} = 1_{\mathcal{A}} \ . \qquad (3.4)$$

In order to derive the action of $\pi(y)$ on $\mathcal{A}$ we shall use the general form [42], which is the same as (3.1$b$) but for an arbitrary element $\psi$ of $\mathcal{A}$ :

$$\pi(y)\,\psi \;=\; \langle \gamma\mathcal{U}(y), \psi_{(1)}\rangle \psi_{(2)} \tag{3.5}$$

So the action on the product of two homogeneous elements may be calculated using the properties of pairing, the graded tensor product, coproduct and antipode, namely,

$$\langle y_1 \otimes y_2, \psi_1 \otimes \psi_2\rangle \;= (-1)^{\widehat{y_2}\widehat{\psi_1}}\; \langle y_1, \psi_1\rangle\; \langle y_2, \psi_2\rangle \tag{3.6$a$}$$

$$(a \otimes b)(c \otimes d) \;= (-1)^{\widehat{bc}} ac \otimes bd \tag{3.6$b$}$$

$$\delta(\phi\psi) \;= (-1)^{\widehat{\phi_{(2)}}\;\widehat{\psi_{(1)}}}\; \phi_{(1)}\,\psi_{(1)} \otimes \phi_{(2)}\,\psi_{(2)} \tag{3.6$c$}$$

$$\gamma(ab) \;= (-1)^{\widehat{ab}}\gamma(b)\gamma(a) \tag{3.6$d$}$$

We find using (3.5) and (3.6):

$$\pi(y)\phi\psi \;=\; (-1)^{\widehat{y}_{(1)}\,(\widehat{\phi}+\widehat{y}_{(2)})}\; (\pi(y_{(2)})\phi)\; (\pi(y_{(1)})\psi) \tag{3.7}$$

Thus we have for the generating elements $y = K_I, X_I^{\pm}$, (note that in all cases we have $\widehat{y}_{(1)}\,\widehat{y}_{(2)} = 0$) :

$$\pi(K_I)\phi\psi \;= (\pi(K_I)\phi)\; \pi(K_I)\psi \tag{3.8$a$}$$

$$\begin{aligned}\pi(X_I^+)\phi\psi \;&= (-1)^{\widehat{\phi}\tilde{I}}\; (\pi(\mathcal{P}_I^{1/2})\phi)\; \pi(X_I^+)\psi \;+\\ &+\; (\pi(X_I^+)\phi)\; \pi(\mathcal{P}_I^{-1/2})\psi\end{aligned} \tag{3.8$b$}$$

$$\begin{aligned}\pi(X_I^-)\phi\psi \;&= (-1)^{\widehat{\phi}\tilde{I}}\; (\pi(\mathcal{Q}_I^{1/2})\phi)\; \pi(X_I^-)\psi \;+\\ &+\; (\pi(X_I^-)\phi)\; \pi(\mathcal{Q}_I^{-1/2})\psi\end{aligned} \tag{3.8$c$}$$

From (3.8$a$) follows:

$$\pi(\mathcal{P}_I^{1/2})\phi\psi \;= (\pi(\mathcal{P}_I^{1/2})\phi)\; \pi(\mathcal{P}_I^{1/2})\psi \tag{3.8$d$}$$

$$\pi(\mathcal{Q}_I^{1/2})\phi\psi \;= (\pi(\mathcal{Q}_I^{1/2})\phi)\; \pi(\mathcal{Q}_I^{1/2})\psi \tag{3.8$e$}$$

For $\mathcal{K}$ we have:

$$\pi(\mathcal{K})\phi\psi \;=\; (\pi(\mathcal{K})\phi)\; \pi(\mathcal{K})\psi \tag{3.9}$$

Applying the above rules one obtains:

$$\pi(K_I)(T_{JL})^n \;= u^{nd_I(\frac{d_I}{d_{I+1}}\delta_{I+1,J}\;-\;\delta_{IJ})/2}(T_{JL})^n \tag{3.10$a$}$$

$$\pi(\mathcal{P}_I^{1/2})(T_{JL})^n \;= Q_{IJ}^{-n/2}(T_{JL})^n \tag{3.10$b$}$$

$$\pi(\mathcal{Q}_I^{1/2})(T_{JL})^n \;= u^{nd_I(\frac{d_I}{d_{I+1}}\delta_{I+1,J}\;-\;\delta_{IJ})}Q_{IJ}^{n/2}(T_{JL})^n \tag{3.10$c$}$$

$$\pi(X_I^+)(a_{jl})^n \;=\; -\,u^{-3\tilde{I}/2}uq^{-n/2}[n]_u\delta_{Ij}(a_{jl})^{n-1}T_{j+1,l} \tag{3.11$a$}$$

$$\pi(X_I^+)(d_{\alpha\beta})^n \;=\; -\,u'q'^{-n/2}[n]_u\delta_{I\alpha}(d_{\alpha\beta})^{n-1}d_{\alpha+1,\beta} \tag{3.11$b$}$$

$$\pi(X_I^-)(a_{jl})^n = - \ u^{-1}q^{-(n-2)/2}[n]_u\delta_{I+1,j}a_{j-1,l}(a_{jl})^{n-1} \quad (3.12a)$$
$$\pi(X_I^-)(d_{\alpha\beta})^n = - \ (-1)^{\tilde I}u^{-3\tilde I/2}u'^{-1}q'^{-(n-2)/2}[n]_u\delta_{I+1,\alpha}T_{\alpha-1,\beta}(d_{\alpha\beta})^{n-1} \quad (3.12b)$$

where $u' = u^{-1}$, $q' = u^2q = u^2/q_{I,I+1}$ and $[n]_u = (u^n - u^{-n})/\lambda$. For $\mathcal{K}$ we have

$$\pi(\mathcal{K})(T_{JL})^n = u^{-n/2}(T_{JL})^n \ , \quad m \neq n \quad (3.13a)$$
$$\pi(\mathcal{K})(T_{JL})^n = u^{-nd_J/2}(T_{JL})^n \ , \quad m = n \quad (3.13b)$$

Next we introduce the right action of $\mathcal{U}$ following [41] (cf. also [43], where it is called left action and denoted by $\pi_l$), but taking into account the graded structure:

$$\pi_R(y)T_{IL} = \sum_{N=1}^{m+n}(-1)^{\widehat{y}\,\widehat{T_{IN}}}\ T_{IN}\ \langle y, T_{NL}\rangle \quad (3.14a)$$
$$= (-1)^{\widehat{y}\,\widehat{T_{IL}}_{(1)}}\ (T_{IL})_{(1)}\ \langle y, (T_{IL})_{(2)}\rangle \quad (3.14b)$$

where $y \in \mathcal{U}$.

From (3.14) we find the explicit right action of the generators of $\mathcal{U}$:

$$\pi_R(K_I)T_{JL} = u^{d_I(\delta_{IL} \ - \ \frac{d_I}{d_{I+1}}\delta_{I+1,L})/2}T_{JL} \quad (3.15a)$$
$$\pi_R(X_I^+)T_{JL} = \delta_{I+1,L}\ (-1)^{\tilde I\ \widehat{T_{J,L-1}}}\ u^{\tilde I/2}Q_{I,I+1}^{-1/2}\ T_{J,L-1} \quad (3.15b)$$
$$\pi_R(X_I^-)T_{JL} = \delta_{IL}\,(-1)^{\tilde I(1+\widehat{T_{J,L+1}})}\ u^{(\tilde I-2d_I)/2}Q_{II}^{1/2}\ T_{J,L+1} \quad (3.15c)$$

From $(3.15a)$ follows:

$$\pi_R(\mathcal{P}_I^{1/2})T_{JL} = Q_{IL}^{1/2}T_{JL} \quad (3.15d)$$
$$\pi_R(\mathcal{Q}_I^{1/2})T_{JL} = u^{d_I(\delta_{IL} \ - \ \frac{d_I}{d_{I+1}}\delta_{I+1,L})}Q_{IL}^{-1/2}T_{JL} \quad (3.15e)$$

Finally:

$$\pi_R(\mathcal{K})T_{JL} = u^{1/2}T_{JL} \quad m \neq n \quad (3.16a)$$
$$\pi_R(\mathcal{K})T_{JL} = u^{d_L/2}T_{JL} \quad m = n \quad (3.16b)$$

The above are supplemented with the following action on the unit element of $\mathcal{A}$:

$$\pi_R(K_I)1_{\mathcal{A}} = 1_{\mathcal{A}}, \quad \pi_R(X_I^{\pm})1_{\mathcal{A}} = 0, \quad \pi_R(\mathcal{K})1_{\mathcal{A}} = 1_{\mathcal{A}}. \quad (3.17)$$

In order to derive the action $\pi_R(y)$ on $\mathcal{A}$ we shall use the general form [42], which is the same as $(3.14b)$ but for an arbitrary homogeneous element $\psi$ of $\mathcal{A}$ :

$$\pi_R(y)\psi = (-1)^{\widehat{y}\widehat{\psi}_{(1)}}\ \psi_{(1)}\ \langle y, \psi_{(2)}\rangle \quad (3.18)$$

So the action of an arbitrary homogeneous element $y \in \mathcal{U}$ on the product of two homogeneous elements of $\mathcal{A}$ is given by:

$$\pi_R(y)\phi\psi = (-1)^{\widehat{y\phi_{(1)}}+\widehat{y}_{(2)}(\widehat{\phi}_{(2)}+\widehat{\psi}_{(1)})}\ \phi_{(1)}\langle y_{(1)},\phi_{(2)}\rangle\ \psi_{(1)}\langle y_{(2)},\psi_{(2)}\rangle = \tag{3.19a}$$
$$= (-1)^{\widehat{\phi y_{(2)}}}\ (\pi_R(y_{(1)})\phi)\,\pi_R(y_{(2)})\psi \tag{3.19b}$$

Thus we have for the generating elements $y = K_I, X_I^{\pm}$,

$$\pi_R(K_I)\,\phi\psi = (\pi_R(K_I)\phi)\,\pi_R(K_I)\psi \tag{3.20a}$$
$$\pi_R(X_I^+)\phi\psi = (\pi_R(X_I^+)\phi)\,\pi_R(\mathcal{P}_I^{1/2})\psi + + (-1)^{\widehat{\phi}\tilde{I}}(\pi_R(\mathcal{P}_I^{-1/2})\phi)\,\pi_R(X_I^+)\psi \tag{3.20b}$$
$$\pi_R(X_I^-)\phi\psi = (\pi_R(X_I^-)\phi)\,\pi_R(\mathcal{Q}_I^{1/2})\psi + + (-1)^{\widehat{\phi}\tilde{I}}(\pi_R(\mathcal{Q}_I^{-1/2})\phi)\,\pi_R(X_I^-)\psi \tag{3.20c}$$

From $(3.20a)$ follows:

$$\pi_R(\mathcal{P}_I^{1/2})\phi\psi = (\pi_R(\mathcal{P}_I^{1/2})\phi)\,\pi_R(\mathcal{P}_I^{1/2})\psi \tag{3.20d}$$
$$\pi_R(\mathcal{Q}_I^{1/2})\phi\psi = (\pi_R(\mathcal{Q}_I^{1/2})\phi)\,\pi_R(\mathcal{Q}_I^{1/2})\psi \tag{3.20e}$$

For $\mathcal{K}$ we have

$$\pi_R(\mathcal{K})\,\phi\psi = (\pi_R(\mathcal{K})\phi)\,\pi_R(\mathcal{K})\psi \tag{3.21}$$

Using this we find:

$$\pi_R(K_I)(T_{JL})^n = u^{nd_I(\delta_{IL} - \frac{d_I}{d_{I+1}}\delta_{I+1,L})/2}(T_{JL})^n \tag{3.22a}$$
$$\pi_R(\mathcal{P}_I^{1/2})(T_{JL})^n = Q_{IL}^{n/2}(T_{JL})^n \tag{3.22b}$$
$$\pi_R(\mathcal{Q}_I^{1/2})(T_{JL})^n = u^{nd_I(\delta_{IL} - \frac{d_I}{d_{I+1}}\delta_{I+1,L})}Q_{IL}^{-n/2}(T_{JL})^n \tag{3.22c}$$

$$\pi_R(X_I^+)(a_{jl})^n = u^{n-1}q^{(n-2)/2}[n]_u\delta_{I+1,l}a_{j,l-1}(a_{jl})^{n-1} \tag{3.23a}$$
$$\pi_R(X_I^+)(d_{\alpha\beta})^n = u^{\tilde{I}/2}u'^{(n-1)}q'^{(n-2)/2}[n]_u\delta_{I+1,\beta}T_{\alpha,\beta-1}(d_{\alpha\beta})^{n-1} \tag{3.23b}$$

$$\pi_R(X_I^-)(a_{jl})^n = (-1)^{\tilde{I}}u^{\tilde{I}/2}u^{n-1}q^{n/2}[n]_u\delta_{I,l}(a_{jl})^{n-1}T_{j,l+1} \tag{3.24a}$$
$$\pi_R(X_I^-)(d_{\alpha\beta})^n = u'^{(n-1)}q'^{n/2}[n]_u\delta_{I,\beta}(d_{\alpha\beta})^{n-1}d_{\alpha,\beta+1} \tag{3.24b}$$

For $\mathcal{K}$ we have

$$\pi_R(\mathcal{K})(T_{JL})^n = u^{n/2}(T_{JL})^n \qquad m \neq n \tag{3.25a}$$
$$\pi_R(\mathcal{K})(T_{JL})^n = u^{nd_L/2}(T_{JL})^n \qquad m = n \tag{3.25b}$$

## 4. Basis via Gauss decomposition

Until here we have used implicitly the basis for $\mathcal{A}$ given in [39], however, it is not suitable for the construction of the induced representations following [40], [41]. From the latter references we know that the suitable basis is via the use of a Gauss decomposition. The point is that we shall use right covariance [40] to reduce the number of variables on which our functions depend. Right covariance with respect to the raising generators $X_I^+$ means that their right action will annihilate our functions. It so happens that this right action will annihilate automatically the 'lower triangular' and 'diagonal' entries of the Gauss decomposition. Thus, right covariance eliminates dependence on the 'upper triangular' entries of the Gauss decomposition. Right covariance with respect to the Cartan generators means that their right action will be scalar on our functions. For this it is sufficient that the right action of the Cartan generators will be scalar on the 'lower triangular' and 'diagonal' entries of the Gauss decomposition.

We give the simple cases $m = n = 1$ and $m = 2, n = 1$ in Appendix A and Appendix B, resp. Below we treat the general case.

The matrix $T$ in (2.1) may be written as

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ F & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} 1 & E \\ 0 & 1 \end{pmatrix} \tag{4.1}$$

where

$$H = D - CA^{-1}B \tag{4.2a}$$
$$E = A^{-1}B , \qquad F = CA^{-1} \tag{4.2b}$$

and $A^{-1}$ is the inverse of the quantum matrix $A$. Furthermore, the quantum matrices $A$ and $H$ may be decomposed as follows

$$A = A_L A_D A_U \tag{4.3a}$$
$$H = H_L H_D H_U \tag{4.3b}$$

where the index $L$ indicates the strictly lower triangular matrix (with units on the main diagonal), $D$ for the diagonal matrix and $U$ for the strictly upper triangular matrix (with units at the main diagonal). Then, the quantum supermatrix $T$ may be decomposed as follows

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_L & 0 \\ \Gamma & H_L \end{pmatrix} \begin{pmatrix} A_D & 0 \\ 0 & H_D \end{pmatrix} \begin{pmatrix} A_U & \Lambda \\ 0 & H_U \end{pmatrix} \tag{4.4}$$

where

$$\Lambda = A_U E = A_U A^{-1} B \tag{4.5a}$$
$$\Gamma = F A_L = C A^{-1} A_L \tag{4.5b}$$

In fact, the elements of the quantum matrix $A$ are even and their commutation relations are that of $GL_{u\mathbf{q}}(m)$, so we can get its Gauss decomposition directly from [41]. For this we have to suppose that the principal minor determinants of A:

$$\begin{aligned} D_r &= \sum_{\rho\in S_r} \epsilon(\rho) a_{1\rho(1)}\dots a_{r\rho(r)} \\ &= \sum_{\rho\in S_r} \epsilon'(\rho) a_{\rho(1)1}\dots a_{\rho(r)r}\ , \quad r\leq m \end{aligned} \tag{4.6}$$

$$\epsilon(\rho) \;=\; \prod_{\substack{j<k\\ \rho(j)>\rho(k)}} \left(\frac{-q_{\rho(k)\rho(j)}}{u^2}\right) \tag{4.7a}$$

$$\epsilon'(\rho) \;=\; \prod_{\substack{j<k\\ \rho(j)>\rho(k)}} \left(\frac{-1}{q_{\rho(k)\rho(j)}}\right) \tag{4.7b}$$

are invertible; note that $D_m$ is just the quantum determinant of $A$ (we will denote it by $\mathcal{D}_A$). Further, for the ordered set $I=\{i_1<\dots<i_r\}$ and $J=\{j_1<\dots<j_r\}$, let $\xi^I_J$ be the $r$-minor determinant with respect to rows $I$ and columns $J$ such that:

$$\xi^I_J \;=\; \sum_{\rho\in S_r} \epsilon'(\rho) a_{i_{\rho(1)}j_1}\dots a_{i_{\rho(r)}j_r} \tag{4.8}$$

Note that $\xi^{1\dots i}_{1\dots i}=D_i$. Then one has as in [41] $(1\leq i,k,l\leq m)$

$$a_{il} \;=\; Y_{ik}D_{kk}U_{kl} \tag{4.9}$$

where $Y_{ik}$ are elements of $A_L$, $D_{kk}$ are elements of $A_D$ and $U_{kl}$ are those of $A_U$. They are given explicitly by:

$$Y_{ik} \;= \prod_{s=1}^{k-1} \frac{q_{si}}{q_{sk}}\ \xi^{1\dots k-1\,i}_{1\dots k} D_k^{-1} \tag{4.10a}$$

$$D_{kk} \;= D_k D_{k-1}^{-1} \quad (D_0=1) \tag{4.10b}$$

$$U_{kl} \;= D_k^{-1}\xi^{1\dots k}_{1\dots k-1\,l} \tag{4.10c}$$

Now let us calculate the right action of $X^+_I$ on $Y_{il}$ and $D_{ll}$. From $(3.20b)$ we deduce that

$$\pi_R(X^+_I)\phi\psi \;=\; (\pi_R(X^+_I)\,\phi)\ \pi_R(\mathcal{P}_I^{1/2})\,\psi \;+\; (\pi_R(\mathcal{P}_I^{-1/2})\,\phi)\ \pi_R(X^+_I)\,\psi \tag{4.11}$$

where $\phi$ is an arbitrary product of $a_{jl}$ with $1\leq j,l\leq m$. Then, using $(3.15b,d)$ one can prove by a direct calculus that:

$$\pi_R(X^+_I)\xi^N_L \;=\; 0, for L=\{1,\dots l\}, \forall N \tag{4.12}$$

and in particular case we have

$$\pi_R(X^+_I)D_j \;=\; 0. \tag{4.13}$$

Then using (4.11) we get

$$\pi_R(X^+_I)Y_{jl} \;=\; 0, \quad \pi_R(X^+_I)D_{ll} \;=\; 0 \tag{4.14}$$

To calculate the right action of $X_I^+$ on $\Gamma$, we first introduce the left and right quantum cofactor matrices $A_{ij}$ and $A'_{ij}$ associated to $A$:

$$A_{ij} = \sum_{\rho(i)=j} \frac{\epsilon(\rho\sigma_i)}{\epsilon(\sigma_i)} a_{1\rho(1)} \ldots \check{a}_{ij} \ldots a_{m\rho(m)} \tag{4.15a}$$

$$A'_{ij} = \sum_{\rho(j)=i} \frac{\epsilon'(\rho\sigma'_j)}{\epsilon'(\sigma'_j)} a_{\rho(1)1} \ldots \check{a}_{ij} \ldots a_{\rho(m)m} \tag{4.15b}$$

where $\sigma_i$ and $\sigma'_j$ denote the cyclic permutations:

$$\sigma_i = \{i, \ldots 1\}, \quad \sigma'_j = \{j, \ldots m\} \tag{4.16}$$

and the notation $\check{x}$ in (4.15) indicates that $x$ is to be omitted. Then one can show that

$$\sum a_{ij} A_{kj} = \sum A'_{ji} a_{jk} = \delta_{ik} \mathcal{D}_A \tag{4.17}$$

and obtain the left and right inverse of $A$ as

$$M_{ij} = \mathcal{D}_A^{-1} A'_{ji} = A_{ji} \mathcal{D}_A^{-1}. \tag{4.18}$$

One can calculate the following

$$\pi_R(\mathcal{P}_I^{1/2}) \mathcal{D}_A = \prod_{s=1}^{m} Q_{Is}^{1/2} \mathcal{D}_A \tag{4.19a}$$

$$\pi_R(\mathcal{P}_I^{1/2}) \mathcal{D}_A^{-1} = \prod_{s=1}^{m} Q_{Is}^{-1/2} \mathcal{D}_A^{-1} \tag{4.19b}$$

$$\pi_R(\mathcal{P}_I^{1/2}) M_{ij} = Q_{Ii}^{-1/2} M_{ij} \tag{4.19c}$$

Now we have to calculate the right action on $F_{\alpha l}$. First, using (4.11), (4.18) and (4.19$b$), we note:

$$\pi_R(X_I^+) M_{jl} = -Q_{II}^{1/2} Q_{I,I+1}^{-1} \delta_{Ij} M_{j+1,l} \tag{4.20}$$

and then we get:

$$\begin{aligned} \pi_R(X_I^+)\, F_{\alpha l} &= \pi_R(X_I^+)\, C_{\alpha j} M_{jl} = \\ &= (\pi_R(X_I^+)\, C_{\alpha j})\ \pi_R(\mathcal{P}_I^{1/2})\, M_{jl} + \\ &\quad + (-1)^{\tilde{I}\widehat{C_{\alpha j}}}\ (\pi_R(\mathcal{P}_I^{-1/2})\, C_{\alpha j})\ \pi_R(X_I^+)\, M_{jl} = 0 \end{aligned} \tag{4.21}$$

It remains now to calculate the right action of $X_I^+$ on the lower triangular matrix $H_L$ and the diagonal one $H_D$. Note that the defining commutation relations of $GL_{u\mathbf{q}}(m/n)$ in (2.2) are in fact the explicit of the following super-RTT equation:

$$(-1)^{\widehat{N}(\widehat{N}+\widehat{L})} R^{IJ}{}_{MN} T_{MN} T_{NL} = (-1)^{\widehat{M}(\widehat{J}+\widehat{N})} T_{IM} T_{JN} R^{MN}{}_{NL} \tag{4.22}$$

where the finite-dimensional $R$-matrix is given by

$$R^{IJ}{}_{NL} = \delta^I_L \delta^J_N \{(-u^2)^{\widehat{I}} \delta^{IJ} + \theta^{IJ}(-1)^{\widehat{I}\widehat{J}} q_{JI} + \theta^{JI}(-1)^{\widehat{I}\widehat{J}} \frac{u^2}{q_{IJ}}\} + \delta^I_N \delta^J_L \theta^{JI}(1-u^2) \quad (4.23)$$

where $\theta^{IJ} = 1$ if $I > J$ and 0 otherwise. (For $n = 0$ and $q_i = u$, $\forall i$, the above relations will reduce to the RTT relations for $GL_u(m)$, [44].) On the other hand, starting from (4.22) one can prove that the matrix $H$ satisfies the same super-RTT equation with all indices are odd. This is proved in [45]. So the elements of $H$ satisfy the defining commutation relations of $GL_{u'\mathbf{q}'}(n)$. Further one can prove that the right action on $H$ is as follows:

$$\pi_R(K_I)h_{\alpha\beta} = u^{d_I(\delta_{I\beta} - \frac{d_I}{d_{I+1}}\delta_{I+1,\beta})/2} h_{\alpha\beta} \quad (4.24a)$$

$$\pi_R(X^+_I)h_{\alpha\beta} = \begin{cases} u^{\tilde{I}/2} Q^{-1/2}_{I,I+1} \delta_{I+1,\beta} h_{\alpha,\beta-1} \,, & \beta > m+1 \\ 0 \,, & \beta = m+1 \end{cases} \quad (4.24b)$$

$$\pi_R(X^-_I)h_{\alpha\beta} = \delta_{I\beta}\, u\, Q^{1/2}_{\beta\beta} h_{\alpha\beta+1} - \delta_{Im}\, u^{-1}\, h_{\alpha,m+1}\, E_{m\beta}, \quad (4.24c)$$

$$\pi_R(\mathcal{K})h_{\alpha\beta} = \begin{cases} u^{1/2} h_{\alpha\beta} \,, & \text{for} \quad m \neq n \\ u'^{1/2} h_{\alpha\beta} \,, & \text{for} \quad m = n \end{cases} \quad (4.24d)$$

$$\pi_R(\mathcal{P}^{1/2}_I)h_{\alpha\beta} = Q^{1/2}_{I\beta} h_{\alpha\beta} \quad (4.24e)$$

$$\pi_R(\mathcal{Q}^{1/2}_I)h_{\alpha\beta} = u^{d_I(\delta_{I\beta} - \frac{d_I}{d_{I+1}}\delta_{I+1,\beta})} Q^{-1/2}_{I\beta} h_{\alpha\beta} \quad (4.24f)$$

Now, one can get the Gauss decomposition of $H$ in the same way as it was done for the quantum matrix $A$. For this we have to suppose that the principal minor determinant of $H$:

$$\begin{aligned} G_\alpha &= \sum_{\rho \in S_{\alpha-m}} \tilde{\epsilon}(\rho)\, h_{m+1\rho(m+1)} \ldots h_{\alpha\rho(\alpha)} \\ &= \sum_{\rho \in S_{\alpha-m}} \tilde{\epsilon}'(\rho)\, h_{\rho(m+1)m+1} \ldots h_{\rho(\alpha)\alpha} \,, \quad m+1 \leq \alpha \leq m+n \end{aligned} \quad (4.25)$$

$$\tilde{\epsilon}(\rho) = \prod_{\substack{\alpha<\beta \\ \rho(\alpha)>\rho(\beta)}} \left( \frac{-q'_{\rho(\beta)\rho(\alpha)}}{u'^2} \right) \quad (4.26a)$$

$$\tilde{\epsilon}'(\rho) = \prod_{\substack{\alpha<\beta \\ \rho(\alpha)>\rho(\beta)}} \left( \frac{-1}{q'_{\rho(\beta)\rho(\alpha)}} \right) \quad (4.26b)$$

are invertible; note that $G_{m+n}$ is just the quantum determinant of $H$ (we will denote it by $\mathcal{D}_H$). Further, for the ordered set $I = \{\alpha_1 < \ldots < \alpha_r\}$ and $J = \{\beta_1 < \ldots < \beta_r\}$, let ${\xi'}^I_J$ be the $r$-minor determinant with respect to rows $I$ and columns $J$ such that:

$${\xi'}^I_J = \sum_{\rho \in S_r} \tilde{\epsilon}'(\rho)\, h_{\alpha_{\rho(1)}\beta_1} \ldots h_{\alpha_{\rho(r)}\beta_r} \quad (4.27)$$

Note that ${\xi'}^{m+1\ldots\alpha}_{m+1\ldots\alpha} = G_\alpha$. Then one has $(m+1 \leq \alpha, \beta, \gamma \leq m+n)$:

$$h_{\alpha\gamma} = Z_{\alpha\beta} G_{\beta\beta} V_{\beta\gamma} \quad (4.28)$$

where $Z_{\alpha\beta}$ are elements of $H_L$, $G_{\beta\beta}$ are elements of $H_D$ and $V_{\beta\gamma}$ are elements of $H_U$. They are given explicitly by

$$\begin{aligned} Z_{\alpha\beta} &= \prod_{\gamma=m+1}^{\beta-1} \frac{q_{\gamma\alpha}}{q_{\gamma\beta}} \xi'^{m+1\ldots\beta-1\alpha}_{m+1\ldots\beta} G_\beta^{-1} && (4.29a) \\ G_{\beta\beta} &= G_\beta G_{\beta-1}^{-1} \quad (G_m = 1) && (4.29b) \\ V_{\beta\gamma} &= G_\beta^{-1} \xi'^{m+1\ldots\beta}_{m+1\ldots\beta-1\gamma} && (4.29c) \end{aligned}$$

Now let us calculate the right action of $X_I^+$ on $Z_{\alpha\beta}$ and $G_{\alpha\alpha}$. Using (3.20$b$) and $\widehat{h_{\alpha\beta}} = 0 \ (mod 2)$ we get:

$$\pi_R(X_I^+) h_{\alpha\beta}\psi \ = \ (\pi_R(X_I^+) h_{\alpha\beta})\, \pi_R(\mathcal{P}_I^{1/2})\psi \ + \ (\pi_R(\mathcal{P}_I^{-1/2}) h_{\alpha\beta})\, \pi_R(X_I^+)\psi \qquad (4.30)$$

from which we deduce

$$\pi_R(X_I^+) \phi\psi \ = \ (\pi_R(X_I^+) \phi)\, \pi_R(\mathcal{P}_I^{1/2})\psi \ + \ (\pi_R(\mathcal{P}_I^{-1/2}) \phi)\, \pi_R(X_I^+)\psi \qquad (4.31)$$

where $\phi$ is an arbitrary product of $h_{\alpha\beta}$ with $m+1 \leq \alpha, \beta \leq m+n$. Then, one can prove in the same way as for $A$ that:

$$\pi_R(X_I^+) \xi'^N_L \ = \ 0, \quad for \quad L = \{m+1, \ldots \alpha\}, \qquad \forall N \qquad (4.32)$$

and in particular case we have

$$\pi_R(X_I^+) G_\alpha \ = \ 0 \ . \qquad (4.33)$$

Then using (4.31) we get

$$\pi_R(X_I^+) Z_{\alpha\beta} \ = \ 0, \quad \pi_R(X_I^+) G_{\beta\beta} \ = \ 0 \ . \qquad (4.34)$$

Finally, we write down the superdeterminant:

$$\mathcal{F} \ = \ \prod_{s=1}^{m} D_{ss} \prod_{\alpha=m+1}^{m+n} G_{\alpha\alpha}^{-1} \ = \ D_m G_{m+n}^{-1} \qquad (4.35)$$

for which we also obtain:

$$\pi_R(X_I^+)\, \mathcal{F} \ = \ 0 \ . \qquad (4.36)$$

Thus, we have proved that the right action of $X_I^+$ on the strictly lower and diagonal matrices in the Gauss decomposition of $T$ is zero. On the other hand the right action of of $X_I^+$ on the strictly upper diagonal matrices in the Gauss decomposition of $T$ is nontrivial.

We have now for the right action of the Cartan generators:

$$\begin{aligned} \pi_R(K_I) \xi^N_L &= u^{\delta_{Il}/2} \xi^N_L, \quad L = \{1, \ldots l\} \ \forall N && (4.37a) \\ \pi_R(K_I) \xi'^N_L &= u^{\delta_{Im}/2} u'^{\delta_{I\alpha}/2} \xi'^N_L, \quad L = \{m+1, \ldots \alpha\} \ \forall N && (4.37b) \end{aligned}$$

from which follows

$$\pi_R(K_I)D_j \ = u^{\delta_{Ij}/2}D_j, \quad \pi_R(K_I)G_\beta \ = \ u^{\delta_{Im}/2}u'^{\delta_{I\beta}/2}G_\beta \tag{4.38a}$$

$$\pi_R(K_I)Y_{jl} \ = Y_{jl}, \quad \pi_R(K_I)Z_{\alpha\beta} \ = \ Z_{\alpha\beta} \tag{4.38b}$$

we have also

$$\pi_R(K_I)\Gamma_{\alpha l} \ = \ \Gamma_{\alpha l}. \tag{4.39}$$

Now we give the action of $\mathcal{K}$ in both cases $m \neq n$ and $m = n$.

For $m \neq n$ we have

$$\pi_R(\mathcal{K})\mathcal{D}_\mathcal{A} \ = u^{m/2}\mathcal{D}_\mathcal{A}, \quad \pi_R(\mathcal{K})M_{jl} \ = \ u^{-1/2}M_{jl} \tag{4.40a}$$

$$\pi_R(\mathcal{K})\xi^N_L \ = u^{l/2}\xi^N_L, \quad L = \{1, \ldots l\} \ \forall N \tag{4.40b}$$

$$\pi_R(\mathcal{K}){\xi'}^N_L \ = u^{(\beta-m)/2}{\xi'}^N_L, \quad L = \{m+1, \ldots \beta\} \ \forall N \tag{4.40c}$$

from which it follows

$$\pi_R(\mathcal{K})D_j \ = u^{j/2}D_j, \quad \pi_R(\mathcal{K})G_\beta \ = \ u^{(\beta-m)/2}G_\beta \tag{4.41a}$$

$$\pi_R(\mathcal{K})Y_{jl} \ = Y_{jl}, \quad \pi_R(\mathcal{K})\Gamma_{\alpha l} = \Gamma_{\alpha l} \quad \pi_R(\mathcal{K})Z_{\alpha\beta} \ = \ Z_{\alpha\beta} \tag{4.41b}$$

For $m = n$ we have

$$\pi_R(\mathcal{K})\mathcal{D}_\mathcal{A} \ = u^{m/2}\mathcal{D}_\mathcal{A}, \quad \pi_R(\mathcal{K})M_{jl} \ = \ u^{-1/2}M_{jl} \tag{4.42a}$$

$$\pi_R(\mathcal{K})\xi^N_L \ = u^{l/2}\xi^N_L, \quad L = \{1, \ldots l\} \ \forall N \tag{4.42b}$$

$$\pi_R(\mathcal{K}){\xi'}^N_L \ = u'^{(\beta-m)/2}{\xi'}^N_L, \quad L = \{m+1, \ldots \beta\} \ \forall N \tag{4.42c}$$

from which it follows

$$\pi_R(\mathcal{K})D_j \ = u^{j/2}D_j, \quad \pi_R(\mathcal{K})G_\beta \ = \ u'^{(\beta-m)/2}G_\beta \tag{4.41a}$$

$$\pi_R(\mathcal{K})Y_{jl} \ = Y_{jl}, \quad \pi_R(\mathcal{K})\Gamma_{\alpha l} = \Gamma_{\alpha l} \quad \pi_R(\mathcal{K})Z_{\alpha\beta} \ = \ Z_{\alpha\beta} \tag{4.41b}$$

Thus, we have shown that right action of the Cartan generators is scalar on all entries of the Gauss decomposition.

The generators $Y_{lj}, \Gamma_{\alpha l}, Z_{\beta\alpha}$ are the $q$-analogues of the strictly lower triangular supermatrices of $GL(m/n)$, while the generators $U_{jl}, \Lambda_{i\alpha}, V_{\alpha\beta}$ are the $q$-analogues of the strictly upper triangular supermatrices of $GL(m/n)$. The generators $D_{jj}, G_{\alpha\alpha}, \mathcal{F}$ are the $q$-analogues of the diagonal supermatrices of $GL(m/n)$. In the following we shall need their commutation relations. Since these are rather lengthy they are given in Appendix C.

Clearly one can replace the basis of $\mathcal{A}$ in terms of $T_{JL}$ with a basis in terms of $X_{LJ} = (Y_{lj}, \Gamma_{\alpha j}, Z_{\beta\alpha})$ with $(L > J)$, $D_i$, $G_\alpha$, $(\alpha \leq m+n-1)$, $\mathcal{F}$, and $W_{JL} = (U_{jl}, \Lambda_{j\alpha}, V_{\alpha\beta})$. More precisely, the basis will be given as follows:

$$\begin{aligned} f_{\bar{v},\bar{k},\bar{w}} \ \doteq \ & (Y_{21})^{v_{21}} \ldots (Y_{m,m-1})^{v_{m,m-1}} \ (\Gamma_{m+1,1})^{v_{m+1,1}} \ldots (\Gamma_{m+n,m})^{v_{m+n,m}} \ \times \\ & \times \ (Z_{m+2,m+1})^{v_{m+2,m+1}} \ldots (Z_{m+n,m+n-1})^{v_{m+n,m+n-1}} \ \times \\ & \times \ (D_1)^{k_1} \ldots (D_m)^{k_m} \ (G_{m+1})^{k_{m+1}} \ldots (G_{m+n-1})^{k_{m+n-1}} \ (\mathcal{F})^{k_{m+n}} \ \times \\ & \times \ (V_{m+n-1,m+n})^{w_{m+n-1,m+n}} \ldots (V_{m+1,m+2})^{w_{m+1,m+2}} \ \times \\ & \times \ (\Lambda_{m,m+n})^{w_{m,m+n}} \ldots (\Lambda_{1,m+1})^{w_{1,m+1}} \ (U_{m-1,m})^{w_{m-1,m}} \ldots (U_{12})^{w_{12}} \end{aligned} \tag{4.44}$$

$$\begin{aligned} \bar{v} \ &\doteq \ \{v_{IJ} \mid 1 \leq J < I \leq m+n\} \ , \quad v_{IJ} \in \mathbb{Z}_+ \ , \quad v_{\alpha i} \leq 1 \\ \bar{k} \ &\doteq \ \{k_I \mid 1 \leq I \leq m+n\} \ , \quad k_I \in \mathbb{Z} \\ \bar{w} \ &\doteq \ \{w_{IJ} \mid 1 \leq I < J \leq m+n\} \ , \quad w_{IJ} \in \mathbb{Z}_+ \ , \quad w_{i\alpha} \leq 1 \end{aligned}$$

and we are using the normal ordering similar to [41], namely, we first put the elements $Y_{ij}$ in lexicographic order, (i.e., if $i < k$ then $Y_{ij}$ is before $Y_{k\ell}$ and $Y_{ti}$ is before $Y_{tk}$), then the elements $\Gamma_{\alpha i}$ in lexicographic order, then the elements $Z_{\alpha\beta}$ in lexicographic order, then the elements $D_I$ and $\mathcal{F}$, then the elements $V_{\alpha\beta}$ in antilexicographic order, (i.e., if $\alpha > \gamma$ then $V_{\alpha\beta}$ is before $V_{\gamma\delta}$ and $V_{\tau\alpha}$ is before $V_{\tau\gamma}$), then the elements $\Lambda_{i\alpha}$ in antilexicographic order, finally, the elements $U_{ij}$ in antilexicographic order. Note that the basis includes the unit element of $\mathcal{A}$ :

$$f_{0,0,0} \;=\; 1_{\mathcal{A}} \tag{4.45}$$

Finally, we should note that the commutation relations in Appendix C are given in anticipation of this basis.

## 5. Representations of $\mathcal{U}$ and $\mathcal{U}'$

We have already seen that the basis introduced in (4.44) has the necessary right covariance properties we mentioned earlier. Thus, we consider a candidates for our representation spaces the formal power series:

$$\varphi \;=\; \sum_{\substack{k_i\in\mathbb{Z}, v_{\alpha i}, w_{i\alpha}\in\{0,1\} \\ v_{ji}, v_{\beta\alpha}, w_{ij}, w_{\alpha\beta}\in\mathbb{Z}_+}} \mu_{\bar{v},\bar{k},\bar{w}}\; f_{\bar{v},\bar{k},\bar{w}}\;, \qquad \mu_{\bar{v},\bar{k},\bar{w}} \in \mathbb{C} \tag{5.1}$$

We impose now right covariance with respect to $X_I^+$; i.e., we require:

$$\pi_R(X_I^+)\varphi \;=\; 0. \tag{5.2}$$

This means that our functions $\varphi$ do not depend on $W_{IJ}$, since (5.2) is fulfilled automatically for the other elements of the basis, as we saw in the previous Section. Thus, the function obeying (5.2) are:

$$\varphi \;=\; \sum_{\substack{k_i\in\mathbb{Z},\; v_{\alpha i}\in\{0,1\} \\ v_{ji}, v_{\beta\alpha}\in\mathbb{Z}_+}} \mu_{\bar{v},\bar{k}}\; f_{\bar{v},\bar{k}}\;, \qquad \mu_{\bar{v},\bar{k}} \;\doteq\; \mu_{\bar{v},\bar{k},0}\;, \quad f_{\bar{v},\bar{k}} \;\doteq\; f_{\bar{v},\bar{k},0} \tag{5.3}$$

Next we impose right covariance with respect to $K_I$ and $\mathcal{K}$:

$$\pi_R(K_I)\varphi \;= u^{d_I r_I/2}\varphi \tag{5.4a}$$
$$\pi_R(\mathcal{K})\varphi \;= u^{\hat{r}/2}\varphi \quad \text{if} \quad m \neq n \tag{5.4b}$$
$$\pi_R(\mathcal{K})\varphi \;= u^{\tilde{r}/2}\varphi \quad \text{if} \quad m = n \tag{5.4c}$$

where $r_I$ and $\hat{r}, \tilde{r}$ are parameters to be specified below. Using the following:

$$\pi_R(K_I)\mathcal{F} \;= \mathcal{F} \tag{5.5a}$$
$$\pi_R(\mathcal{K})\mathcal{F} \;= u^{(m-n)/2}\mathcal{F} \quad \text{if} \quad m \neq n \tag{5.5b}$$
$$\pi_R(\mathcal{K})\mathcal{F} \;= u^{m}\mathcal{F} \quad \text{if} \quad m = n \tag{5.5c}$$

and the actions of $K_I$ and $\mathcal{K}$ on the new generators and their products we find:

$$\pi_R(K_I)\varphi \ = u^{d_I k_I/2}\varphi, \quad for \quad I \leq m+n-1, I \neq m \tag{5.6a}$$

$$\pi_R(K_m)\varphi \ = u^{\frac{1}{2}\left(k_m+\sum_{\beta=m+1}^{m+n-1} k_\beta\right)}\varphi \tag{5.6b}$$

$$\pi_R(\mathcal{K})\varphi \ = u^{\frac{1}{2}\left(\sum_{j=1}^{m} jk_j+\sum_{\beta=m+1}^{m+n-1}(\beta-m)k_\beta+(m-n)k_{m+n}\right)}\varphi \quad \text{if} \quad m \neq n \tag{5.6c}$$

$$\pi_R(\mathcal{K})\varphi \ = u^{\frac{1}{2}\left(\sum_{j=1}^{m} jk_j-\sum_{\beta=m+1}^{m+n-1}(\beta-m)k_\beta+2mk_{m+n}\right)}\varphi \quad \text{if} \quad m = n \tag{5.6d}$$

Comparing the right covariance (5.4) the direct calculations (5.6) we obtain:

$$k_I \ = r_I, \quad for \quad I \leq m+n-1, I \neq m \tag{5.7a}$$

$$k_m \ = r_m - \sum_{\beta=m+1}^{m+n-1} r_\beta \tag{5.7b}$$

$$\begin{aligned}\hat{r} \ &= \sum_{j=1}^{m} jk_j + \sum_{\beta=m+1}^{m+n-1} (\beta-m)k_\beta + (m-n)k_{m+n} \ = \\ &= \sum_{j=1}^{m} jr_j + \sum_{\beta=m+1}^{m+n-1} (\beta-2m)r_\beta + (m-n)k_{m+n} \ , \quad \text{if} \quad m \neq n\end{aligned} \tag{5.7c}$$

$$\begin{aligned}\tilde{r} \ &= \sum_{j=1}^{m} jk_j - \sum_{\beta=m+1}^{2m-1} (\beta-m)k_\beta + 2mk_{2m} \ = \\ &= \sum_{J=1}^{2m-1} Jd_J r_J + 2mk_{2m} \ , \quad \text{if} \quad m = n\end{aligned} \tag{5.7d}$$

This means that $r_I, \hat{r}, \tilde{r} \in \mathbb{Z}$ and there is no summation in $k_I$ ; also we have:

$$k_{m+n} \ = \frac{1}{m-n}(\hat{r} - \sum_{j=1}^{m} jr_j - \sum_{\beta=m+1}^{m+n-1} (\beta-2m)r_\beta) \quad \text{if} \quad m \neq n \tag{5.8a}$$

$$k_{2m} \ = \frac{1}{2m}(\tilde{r} - \sum_{J=1}^{2m-1} Jd_J r_J) \quad \text{if} \quad m = n \tag{5.8b}$$

Thus, the reduced functions obeying (5.2) and (5.4) are

$$\varphi \ = \sum_{\substack{v_{\alpha i}\in\{0,1\} \\ v_{ji}, v_{\beta\alpha}\in\mathbb{Z}_+}} \mu_{\bar{v}} \ f_{\bar{v}} \ \Xi_{\bar{r}} \ , \qquad \mu_{\bar{v}} \ \doteq \ \mu_{\bar{v},0} \ , \qquad f_{\bar{v}} \ \doteq \ f_{\bar{v},0} \tag{5.9a}$$

$$\begin{aligned}\Xi_{\bar{r}} \ &\doteq (D_1)^{r_1} \ldots (D_{m-1})^{r_{m-1}} (D_m)^{\hat{s}} (G_{m+1})^{r_{m+1}} \ldots (G_{m+n-1})^{r_{m+n-1}} (\mathcal{F})^{\hat{t}} \\ \bar{r} \ &= \{r_1, \ldots, r_{m+n-1}, \hat{r}(\text{or } \tilde{r})\}\end{aligned} \tag{5.9b}$$

where

$$\hat{s} = r_m - \sum_{\beta=m+1}^{m+n-1} r_\beta \tag{5.10a}$$

$$\hat{t} \ = \begin{cases} \frac{1}{m-n}(\hat{r} - \sum_{j=1}^{m} jr_j - \sum_{\beta=m+1}^{m+n-1}(\beta-2m)r_\beta) & \text{if} \quad m \neq n \\ \frac{1}{2m}(\tilde{r} - \sum_{J=1}^{2m-1} Jd_J r_J) & \text{if} \quad m = n \end{cases} \tag{5.10b}$$

Next we shall give the $\mathcal{U}$ representation (left) action $\pi$ on $\varphi$. Besides the action of the 'Chevalley' generators $K_I, X_I^\pm, \mathcal{K}$ we shall give for the readers convenience also the action of $\mathcal{P}_I, \mathcal{Q}_I$ though it follows from that of $K_I$. We have:

$$
\begin{aligned}
\pi(K_I)Y_{lj} &= u^{(\delta_{I+1,l}-\delta_{I+1,j}-\delta_{Il}+\delta_{Ij})/2}Y_{lj} && (5.11a)\\
\pi(X_I^+)Y_{lj} &= -\,u^{(\tilde{I}+d_I+d_{I+1})/2}Q_{I,I+1}^{-1/2}Q_{Ij}^{-1/2}\delta_{Il}(\delta_{lm}\Gamma_{m+1,m}+(1-\delta_{lm})Y_{l+1,j})\; + \\
&+\; uQ_{I,I+1}^{-1/2}Q_{Il}^{-1/2}\left(\frac{q_{j,j+1}q_{j+1,l}}{q_{jl}}\right)^{(1-\delta_{l,j+1})}\delta_{Ij}Y_{j+1,j}Y_{lj}\; + \\
&+\; uQ_{I,I+1}^{-1/2}Q_{Il}^{1/2}Q_{I,j-1}^{-1/2}Q_{Ij}^{-1/2}\delta_{I+1,j}\;\times \\
&\times\; \{\frac{q_{j-1,l}}{q_{j-1,j}q_{jl}}Y_{l,j-1}-Y_{j,j-1}Y_{lj}\} && (5.11b)\\
\pi(X_I^-)Y_{lj} &= -\,u^{-2}Q_{II}^{1/2}Q_{Ij}^{1/2}u^{-\delta_{Ij}}\delta_{I+1,l}Y_{l-1,j} && (5.11c)\\
\pi(\mathcal{K})Y_{lj} &= Y_{lj} && (5.11d)\\
\pi(\mathcal{P}_I^{1/2})Y_{lj} &= Q_{Il}^{-1/2}Q_{Ij}^{1/2}Y_{lj} && (5.11e)\\
\pi(\mathcal{Q}_I^{1/2})Y_{lj} &= u^{(\delta_{I+1,l}-\delta_{I+1,j}-\delta_{Il}+\delta_{Ij})}Q_{Il}^{1/2}Q_{Ij}^{-1/2}Y_{lj} && (5.11f)
\end{aligned}
$$

$$
\begin{aligned}
\pi(K_I)\Gamma_{\alpha j} &= u^{d_I(\frac{d_I}{d_{I+1}}(\delta_{I+1,\alpha}-\delta_{I+1,j})-\delta_{I\alpha}+\delta_{Ij})/2}\Gamma_{\alpha j} && (5.12a)\\
\pi(X_I^+)\Gamma_{\alpha j} &= -\,u^{-1}Q_{I,I+1}^{-1/2}Q_{Ij}^{-1/2}\delta_{I\alpha}\Gamma_{\alpha+1,j}+ \\
&+\; u^{(\tilde{I}+d_I+d_{I+1})/2}Q_{I,I+1}^{-1/2}Q_{I\alpha}^{-1/2}(\frac{q_{j,j+1}q_{j+1,\alpha}}{q_{j\alpha}})^{(1-\delta_{\alpha,j+1})}\delta_{Ij}\;\times \\
&\times\; ((1-\delta_{jm})Y_{j+1,j}-\delta_{jm}\Gamma_{m+1,m})\Gamma_{\alpha j}\; + \\
&+\; uQ_{I,I+1}^{-1/2}Q_{I\alpha}^{1/2}Q_{I,j-1}^{-1/2}Q_{Ij}^{-1/2}\delta_{I+1,j}\;\times \\
&\times\; \{\frac{q_{j-1,\alpha}}{q_{j-1,j}q_{j\alpha}}\Gamma_{\alpha,j-1}-Y_{j,j-1}\Gamma_{\alpha j}\} && (5.12b)\\
\pi(X_I^-)\Gamma_{\alpha j} &= -\,(-1)^{\tilde{I}}u^{(\tilde{I}-3d_I-d_{I+1})1/2}Q_{Ij}^{1/2}u^{-\delta_{Ij}}\delta_{I+1,\alpha}\;\times \\
&\times\; \{\delta_{\alpha,m+1}Y_{mj}+(1-\delta_{\alpha,m+1})\Gamma_{\alpha-1,j}\} && (5.12c)
\end{aligned}
$$

$$
\begin{aligned}
\pi(\mathcal{K})\Gamma_{\alpha j} &= \begin{cases}\Gamma_{\alpha j} & \text{if } m\neq n\\ u\Gamma_{\alpha j} & \text{if } m=n\end{cases} && (5.12d)\\
\pi(\mathcal{P}_I^{1/2})\Gamma_{\alpha j} &= Q_{I\alpha}^{-1/2}Q_{Ij}^{1/2}\Gamma_{\alpha j} && (5.12e)\\
\pi(\mathcal{Q}_I^{1/2})\Gamma_{\alpha j} &= u^{d_I(\frac{d_I}{d_{I+1}}(\delta_{I+1,\alpha}-\delta_{I+1,j})-\delta_{I\alpha}+\delta_{Ij})}Q_{I\alpha}^{1/2}Q_{Ij}^{-1/2}\Gamma_{\alpha j} && (5.12f)
\end{aligned}
$$

$$
\begin{aligned}
\pi(K_I)Z_{\beta\alpha} &= u^{d_I(\frac{d_I}{d_{I+1}}(\delta_{I+1,\beta}-\delta_{I+1,\alpha})-\delta_{I\beta}+\delta_{I\alpha})/2}Z_{\beta\alpha} && (5.13a)\\
\pi(X_I^+)Z_{\beta\alpha} &= -\,u'Q_{I,I+1}^{-1/2}Q_{I\alpha}^{-1/2}\delta_{I\beta}Z_{\beta+1,\alpha}\; + \\
&+\; u'Q_{I,I+1}^{-1/2}Q_{I\beta}^{-1/2}(\frac{q'_{\alpha,\alpha+1}q'_{\alpha+1,\beta}}{q'_{\alpha\beta}})^{(1-\delta_{\beta,\alpha+1})}\delta_{I\alpha}Z_{\alpha+1,\alpha}Z_{\beta\alpha}\; +
\end{aligned}
$$

$$+(-1)^{\widehat{I}+\widehat{I+1}}u^{2\tilde{I}}\ u^{(\tilde{I}+d_I+d_{I+1})/2}Q_{I,I+1}^{-1/2}Q_{I\beta}^{1/2}Q_{I,\alpha-1}^{-1/2}Q_{I\alpha}^{-1/2}\delta_{I+1,\alpha}\times$$

$$\times\ \{\frac{q'_{\alpha-1,\beta}}{q'_{\alpha-1,\alpha}q'_{\alpha\beta}}(\delta_{Im}\Gamma_{\beta m}+(1-\delta_{Im})Z_{\beta,\alpha-1})-$$

$$-\ \delta_{Im}\Gamma_{m+1,m}Z_{\beta\alpha}-(1-\delta_{Im})Z_{\alpha,\alpha-1}Z_{\beta\alpha}\} \quad (5.13b)$$

$$\pi(X_I^-)Z_{\beta\alpha} = -u'^{-2}Q_{II}^{1/2}Q_{I\alpha}^{1/2}u'^{-\delta_{I\alpha}}\delta_{I+1,\beta}Z_{\beta-1,\alpha} \quad (5.13c)$$

$$\pi(\mathcal{K})Z_{\beta\alpha} = Z_{\beta\alpha} \quad (5.13d)$$

$$\pi(\mathcal{P}_I^{1/2})Z_{\beta\alpha} = Q_{I\beta}^{-1/2}Q_{I\alpha}^{1/2}Z_{\beta\alpha} \quad (5.13e)$$

$$\pi(\mathcal{Q}_I^{1/2})Z_{\beta\alpha} = u^{d_I(\frac{d_I}{d_{I+1}}(\delta_{I+1,\beta}-\delta_{I+1,\alpha})-\delta_{I\beta}+\delta_{I\alpha})}Q_{I\beta}^{1/2}Q_{I\alpha}^{-1/2}Z_{\beta\alpha} \quad (5.13f)$$

$$\pi(K_I)D_j = u^{-\delta_{Ij}/2}D_j \quad (5.14a)$$

$$\pi(X_I^+)D_j = -u^{(\tilde{I}+d_I+d_{I+1})/2}Q_{I,I+1}^{-1/2}\prod_{s=1}^{j-1}Q_{Is}^{1/2}\delta_{Ij}\times$$

$$\times\ (\delta_{jm}\Gamma_{m+1,m}+(1-\delta_{jm})Y_{j+1,j})D_j \quad (5.14b)$$

$$\pi(X_I^-)D_j = 0 \quad (5.14c)$$

$$\pi(\mathcal{K})D_j = u^{-j/2}D_j \quad (5.14d)$$

$$\pi(\mathcal{P}_I^{1/2})D_j = \prod_{s=1}^{j}Q_{Is}^{-1/2}D_j \quad (5.14e)$$

$$\pi(\mathcal{Q}_I^{1/2})D_j = u^{-\delta_{Ij}}\prod_{s=1}^{j}Q_{Is}^{1/2}D_j \quad (5.14f)$$

$$\pi(K_I)G_\beta = u^{(-\delta_{Im}+\delta_{I\beta})/2}G_\beta \quad (5.15a)$$

$$\pi(X_I^+)G_\beta = -u^{(\tilde{I}+d_I+d_{I+1})/2}Q_{I,I+1}^{-1/2}\{\prod_{\alpha=m+1}^{\beta-1}Q_{I\alpha}^{1/2}\delta_{I\beta}Z_{\beta+1,\beta}+$$

$$+\prod_{\alpha=m+2}^{\beta}Q_{m\alpha}^{1/2}\delta_{Im}\Gamma_{m+1,m}\}G_\beta \quad (5.15b)$$

$$\pi(X_I^-)G_\beta = 0 \quad (5.15c)$$

$$\pi(\mathcal{K})G_\beta = \begin{cases} u^{-(\beta-m)/2}G_\beta & \text{if } m\neq n \\ u'^{-(\beta-m)/2}G_\beta & \text{if } m=n \end{cases} \quad (5.15d)$$

$$\pi(\mathcal{P}_I^{1/2})G_\beta = \prod_{\alpha=m+1}^{\beta}Q_{I\alpha}^{-1/2}G_\beta \quad (5.15e)$$

$$\pi(\mathcal{Q}_I^{1/2})G_\beta = u^{(-\delta_{Im}+\delta_{I\beta})}\prod_{\alpha=m+1}^{\beta}Q_{I\alpha}^{1/2}G_\beta \quad (5.15f)$$

$$\pi(K_I)\mathcal{F} = \mathcal{F} \tag{5.16a}$$
$$\pi(X_I^+)\mathcal{F} = 0 \tag{5.16b}$$

$$\pi(X_I^-)\mathcal{F} = 0 \tag{5.16c}$$

$$\pi(\mathcal{K})\mathcal{F} = \begin{cases} u^{(n-m)/2}\mathcal{F} & \text{if } m \neq n \\ u^{-m}\mathcal{F} & \text{if } m = n \end{cases} \tag{5.16d}$$
$$\pi(\mathcal{P}_I^{1/2})\mathcal{F} = \mathcal{F} \tag{5.16e}$$
$$\pi(\mathcal{Q}_I^{1/2})\mathcal{F} = \mathcal{F} \tag{5.16f}$$

Now we note that from (5.14), (5.15), (5.16) we have the important consequence that the degrees of variables $D_j$, $G_\beta$, $\mathcal{F}$ are not changed by the action of $\mathcal{U}$. Thus, the parameters $r_I$ and $\hat{r}$ (or $\tilde{r}$) indeed characterize the action of $\mathcal{U}$, i.e., we have obtained representations of $\mathcal{U}$.

• Thus, by formulae (5.11), (5.12), (5.13), (5.14), (5.15), (5.16), we have given the induced representations of $\mathcal{U}$ labelled by the $m+n$ integer numbers $r_I$ and $\hat{r}$ (or $\tilde{r}$) and acting in the space of formal power series of $(m+n)(m+n-1)/2$ non-commuting variables, of which the $mn$ variables $\Gamma_{\alpha i}$ are odd and the variables $Y_{ij}$ and $Z_{\alpha\beta}$ are even.

*Remark:* For $u = \mathbf{q} = 1$ our representations coincide with the holomorphic representations induced from the upper diagonal Borel subsupergroup $B$ of $G \equiv GL(m/n)$ and acting on the coset $G/G^+$, where $G^+$ is the strictly upper diagonal supergroup of $G$. That is why we call our representations induced. ♢

To obtain our representation more explicitly one is using these formulae together with the rules (3.8) and (3.9). In particular, we see that:

$$\pi(\mathcal{K})\varphi = \begin{cases} u^{-\hat{r}/2}\varphi, & \text{if } m \neq n \\ u^{-\tilde{r}/2}\varphi', & \text{if } m = n \end{cases} \tag{5.17a}$$

$$\varphi' = \sum_{\substack{v_{\alpha i}\in\{0,1\} \\ v_{ji}, v_{\beta\alpha}\in\mathbb{Z}_+}} \mu_{\bar{v}}\, u^{\sum_{\alpha,i} v_{\alpha i}}\, f_{\bar{v}}\, \Xi_{\bar{r}} \tag{5.17b}$$

We notice from (5.16) that $\mathcal{U}'$ acts trivially on $\mathcal{F}$. Thus, the action of $\mathcal{U}'$ involves only the parameters $r_I$, $I \leq m+n-1$. On the other hand by (5.17) we see that the action of $\mathcal{K}$ involves only the parameter $\tilde{r}'$ ($\tilde{r}' = \hat{r}$ if $m \neq n$, $\tilde{r}' = \tilde{r}$ if $m = n$). Thus we can consistently also from the representation theory point of view restrict to $SL_{u\mathbf{q}}(m/n)$, i.e., we set

$$\mathcal{F} = \mathcal{F}^{-1} = 1_{\mathcal{A}} . \tag{5.18}$$

Note that in order to enforce this condition it is also necessary that $\mathcal{F}$ commutes with all generators, and the conditions for this which follow from the explicit commutation relation in Appendix C are just conditions (2.11).

With (5.18) enforced the dual algebra is $\mathcal{U}' \equiv \mathcal{U}_{u\mathbf{q}}(sl(m/n))$. Thus, the reduced functions for the $\mathcal{U}'$ action are:

$$\begin{aligned} \varphi \ &= \sum_{\substack{v_{\alpha i}\in\{0,1\} \\ v_{ji},v_{\beta\alpha}\in\mathbb{Z}_+}} \mu_{\bar v}\ f_{\bar v}\ \Xi^0_{\bar r} && (5.19a) \\ \Xi^0_{\bar r} \ &\doteq D_1^{r_1}\ldots D_{m-1}^{r_{m-1}}\, D_m^{\hat s} G_{m+1}^{r_{m+1}}\ldots G_{m+n-1}^{r_{m+n-1}} && (5.19b) \end{aligned}$$

• Thus, by formulae (5.11), (5.12), (5.13), (5.14), (5.15), we have given the induced representations of $\mathcal{U}'$ labelled by the $m+n-1$ integer numbers $r_I$. For $u=\mathbf{q}=1$ our representations coincide with the standard holomorphic representations induced from $B$ and acting on the coset $G/B$.

To obtain the representations more explicitly one is using these formulae together with the rules (3.8). In particular, we have:

$$\begin{aligned} \pi(K_I)(Y_{lj})^k \ &= u^{k(\delta_{I+1,l}-\delta_{I+1,j}-\delta_{Il}+\delta_{Ij})/2}(Y_{lj})^k && (5.20a) \\ \pi(X_I^+)(Y_{lj})^k \ &= -\,u^{(\tilde I+d_I+d_{I+1})/2}Q_{I,I+1}^{-1/2}Q_{Ij}^{(k-2)/2}c_l\delta_{Il}(Y_{lj})^{k-1}\ \times \\ &\times\ (\delta_{lm}\Gamma_{m+1,m}+(1-\delta_{lm})Y_{l+1,j})\ + \\ &+\ uQ_{I,I+1}^{-1/2}Q_{Il}^{(k-2)/2}c_j\left(\frac{q_{j,j+1}q_{j+1,l}}{q_{jl}}\right)^{(1-\delta_{l,j+1})}\delta_{Ij}Y_{j+1,j}(Y_{lj})^k\ + \\ &+\ uQ_{I,I+1}^{-1/2}Q_{Il}^{k/2}(\frac{q_{j-1,j}}{u})^k\tilde c_{j-1}\delta_{I+1,j}\ \times \\ &\times\ \{\frac{q_{j-1,l}}{q_{j-1,j}q_{jl}}Y_{l,j-1}(Y_{lj})^{k-1}-Y_{j,j-1}(Y_{lj})^k\} && (5.20b) \\ \pi(X_I^-)(Y_{lj})^k \ &= -\,u^{-2}Q_{II}^{1/2}Q_{Ij}^{k/2}u^{-k\delta_{Ij}}c_{l-1}\delta_{I+1,l}Y_{l-1,j}(Y_{lj})^{k-1} && (5.20c) \\ \pi(\mathcal{P}_I^{1/2})(Y_{lj})^k \ &= Q_{Il}^{-k/2}Q_{Ij}^{k/2}(Y_{lj})^k && (5.20d) \\ \pi(\mathcal{Q}_I^{1/2})(Y_{lj})^k \ &= u^{k(\delta_{I+1,l}-\delta_{I+1,j}-\delta_{Il}+\delta_{Ij})}Q_{Il}^{k/2}Q_{Ij}^{-k/2}(Y_{lj})^k && (5.20e) \end{aligned}$$

$$\begin{aligned} \pi(K_I)(Z_{\beta\alpha})^k \ &= u^{kd_I(\frac{d_I}{d_{I+1}}(\delta_{I+1,\beta}-\delta_{I+1,\alpha})-\delta_{I\beta}+\delta_{I\alpha})/2}(Z_{\beta\alpha})^k && (5.21a) \\ \pi(X_I^+)(Z_{\beta\alpha})^k \ &= -\,u'Q_{I,I+1}^{-1/2}Q_{I\alpha}^{(k-2)/2}c_\beta\delta_{I\beta}(Z_{\beta\alpha})^{k-1}Z_{\beta+1,\alpha}\ + \\ &+\ u'Q_{I,I+1}^{-1/2}Q_{I\beta}^{(k-2)/2}c_\alpha(\frac{q'_{\alpha,\alpha+1}q'_{\alpha+1,\beta}}{q'_{\alpha\beta}})^{(1-\delta_{\beta,\alpha+1})}\delta_{I\alpha}Z_{\alpha+1,\alpha}(Z_{\beta\alpha})^k\ + \\ &+(-1)^{\widehat{I}+\widehat{I+1}}u^{(\tilde I+d_I+d_{I+1})/2}Q_{I,I+1}^{-1/2}Q_{I\beta}^{k/2}\tilde c_I\delta_{I+1,\alpha}\ \times \\ &\times\ \{\frac{q'_{\alpha-1,\beta}}{q'_{\alpha-1,\alpha}q'_{\alpha\beta}}(\delta_{Im}(q_{m,m+1})^k\Gamma_{\beta m}\ + \\ &+\ (1-\delta_{Im})(\frac{q'_{\alpha-1,\alpha}}{u'})^kZ_{\beta,\alpha-1})(Z_{\beta\alpha})^{k-1}\ - \\ &-\ \delta_{Im}(q_{m,m+1})^k\Gamma_{m+1,m}(Z_{\beta\alpha})^k\ - \\ &-\ (1-\delta_{Im})(\frac{q'_{\alpha-1,\alpha}}{u'})^kZ_{\alpha,\alpha-1}(Z_{\beta\alpha})^k\} && (5.21b) \end{aligned}$$

$$\pi(X_I^-)(Z_{\beta\alpha})^k = -u'^{-2}Q_{II}^{1/2}Q_{I\alpha}^{k/2}u'^{-k\delta_{I\alpha}}c_{\beta-1}\delta_{I+1,\beta}Z_{\beta-1,\alpha}(Z_{\beta\alpha})^{k-1} \quad (5.21c)$$

$$\pi(\mathcal{P}_I^{1/2})(Z_{\beta\alpha})^k = Q_{I\beta}^{-k/2}Q_{I\alpha}^{k/2}(Z_{\beta\alpha})^k \quad (5.21d)$$

$$\pi(\mathcal{Q}_I^{1/2})(Z_{\beta\alpha})^k = u^{kd_I(\frac{d_I}{d_{I+1}}(\delta_{I+1,\beta}-\delta_{I+1,\alpha})-\delta_{I\beta}+\delta_{I\alpha})}Q_{I\beta}^{k/2}Q_{I\alpha}^{-k/2}(Z_{\beta\alpha})^k \quad (5.21e)$$

where

$$c_I = \begin{cases} (q_{i,i+1})^{(k-1)/2}[k]_u, & \text{if } I=i\le m \\ (q'_{\alpha,\alpha+1})^{(k-1)/2}[k]_u, & \text{if } I=\alpha>m \end{cases} \quad (5.22a)$$

$$\tilde{c}_I = \begin{cases} (q_{i,i+1})^{(1-k)/2}[k]_u, & \text{if } I=i\le m \\ (q'_{\alpha,\alpha+1})^{(1-k)/2}[k]_u, & \text{if } I=\alpha>m \end{cases} \quad (5.22b)$$

$$\pi(K_I)(D_j)^k = u^{-k\delta_{Ij}/2}(D_j)^k \quad (5.23a)$$

$$\pi(X_I^+)(D_j)^k = -u^{(\tilde{I}+d_I+d_{I+1})/2}Q_{I,I+1}^{-1/2}\prod_{s=1}^{j-1}Q_{Is}^{k/2}\tilde{c}_j\delta_{Ij}\times$$

$$\times\ (\delta_{jm}\Gamma_{m+1,m}+(1-\delta_{jm})Y_{j+1,j})(D_j)^k \quad (5.23b)$$

$$\pi(X_I^-)(D_j)^k = 0 \quad (5.23c)$$

$$\pi(\mathcal{P}_I^{1/2})(D_j)^k = \prod_{s=1}^{j}Q_{Is}^{-k/2}(D_j)^k \quad (5.23d)$$

$$\pi(\mathcal{Q}_I^{1/2})(D_j)^k = u^{-k\delta_{Ij}}\prod_{s=1}^{j}Q_{Is}^{k/2}(D_j)^k \quad (5.23e)$$

$$\pi(K_I)(G_\beta)^k = u^{k(-\delta_{Im}+\delta_{I\beta})/2}(G_\beta)^k \quad (5.24a)$$

$$\pi(X_I^+)(G_\beta)^k = -u^{(\tilde{I}+d_I+d_{I+1})/2}Q_{I,I+1}^{-1/2}\tilde{c}_I\{\prod_{\alpha=m+1}^{\beta-1}Q_{I\alpha}^{k/2}\delta_{I\beta}Z_{\beta+1,\beta}+$$

$$+\prod_{\alpha=m+2}^{\beta}Q_{m\alpha}^{k/2}\delta_{Im}\Gamma_{m+1,m}\}(G_\beta)^k \quad (5.24b)$$

$$\pi(X_I^-)(G_\beta)^k = 0 \quad (5.24c)$$

$$\pi(\mathcal{P}_I^{1/2})(G_\beta)^k = \prod_{\alpha=m+1}^{\beta}Q_{I\alpha}^{-k/2}(G_\beta)^k \quad (5.24d)$$

$$\pi(\mathcal{Q}_I^{1/2})(G_\beta)^k = u^{k(-\delta_{Im}+\delta_{I\beta})}\prod_{\alpha=m+1}^{\beta}Q_{I\alpha}^{k/2}(G_\beta)^k \quad (5.24e)$$

As a consequence we have, e.g.,

$$\begin{aligned}\pi(K_I)\,\varphi \;&=\; u^{-\frac{1}{2}d_I r_I} \sum_{\substack{v_{\gamma k}\in\{0,1\}\\ v_{ji},v_{\beta\alpha}\in\mathbb{Z}_+}} u^{\frac{1}{2}v_{ji}\,(\delta_{I+1,j}-\delta_{I+1,i}-\delta_{Ij}+\delta_{Ii})}\;\times \\ &\times\; u^{\frac{1}{2}v_{\gamma k}\,d_I\left(\frac{d_I}{d_{I+1}}(\delta_{I+1,\beta}-\delta_{I+1,\alpha})-\delta_{I\beta}+\delta_{I\alpha}\right)}\; u^{\frac{1}{2}v_{\beta\alpha}\,d_I\left(\frac{d_I}{d_{I+1}}(\delta_{I+1,\beta}-\delta_{I+1,\alpha})-\delta_{I\beta}+\delta_{I\alpha}\right)}\;\times \\ &\times\; \mu_{\bar v}\; f_{\bar v}\; \Xi^0_{\bar r}\end{aligned} \tag{5.25}$$

Finally, since the action of $\mathcal{U}'$ is not affecting the degrees of $D_j$ and $G_\beta$, we may introduce (as in [40], [41]) the restricted functions:

$$\tilde\varphi \;=\; \sum_{\substack{v_{\alpha i}\in\{0,1\}\\ v_{ji},v_{\beta\alpha}\in\mathbb{Z}_+}} \mu_{\bar v}\; f_{\bar v} \tag{5.26}$$

using the intertwining operator:

$$\tilde\varphi \;\equiv\; \mathcal{I}\,\varphi \;\doteq\; \varphi|_{D_i=G_\alpha=1_{\mathcal{A}}} \tag{5.27}$$

We denote the representation space of $\varphi$ by $\mathcal{C}_{\bar r}$, the representation space of $\tilde\varphi$ by $\tilde{\mathcal{C}}_{\bar r}$, and the representation acting on $\tilde\varphi$ by $\tilde\pi$. Thus, the operator $\mathcal{I}$ acts from $\mathcal{C}_{\bar r}$ to $\tilde{\mathcal{C}}_{\bar r}$. The properties of $\tilde{\mathcal{C}}_{\bar r}$ follow from the intertwining requirement for $\mathcal{I}$ [40]:

$$\tilde\pi\,\mathcal{I} \;=\; \mathcal{I}\,\pi\,. \tag{5.28}$$

In particular, we have:

$$\begin{aligned}\tilde\pi(K_I)\,\tilde\varphi \;&=\; u^{-\frac{1}{2}d_I r_I} \sum_{\substack{v_{\gamma k}\in\{0,1\}\\ v_{ji},v_{\beta\alpha}\in\mathbb{Z}_+}} u^{\frac{1}{2}v_{ji}\,(\delta_{I+1,j}-\delta_{I+1,i}-\delta_{Ij}+\delta_{Ii})}\;\times \\ &\times\; u^{\frac{1}{2}v_{\gamma k}\,d_I\left(\frac{d_I}{d_{I+1}}(\delta_{I+1,\beta}-\delta_{I+1,\alpha})-\delta_{I\beta}+\delta_{I\alpha}\right)}\; u^{\frac{1}{2}v_{\beta\alpha}\,d_I\left(\frac{d_I}{d_{I+1}}(\delta_{I+1,\beta}-\delta_{I+1,\alpha})-\delta_{I\beta}+\delta_{I\alpha}\right)}\;\times \\ &\times\; \mu_{\bar v}\; f_{\bar v}\end{aligned} \tag{5.29}$$

• We finish by noting that the functions $\tilde\varphi$ have the important advantage that the representation action $\tilde\pi$ can be extended to arbitrary complex $r_I$. This is seen, e.g., from (5.29).

## 6. Outlook

The representations constructed in this paper will have many applications. The most interesting ones seem to be connected with the case of the multiparameter quantum conformal supergroup which is a real form of $\mathcal{U}'$ for $m=4$, i.e., $U_{u\mathbf{q}}(sl(4/N))$. In this case the non-commuting variables $Y_{ij}$ contain a deformation of Minkowski space (as in [46]) which together with the variables $\Gamma_{\alpha i}$ will give a deformation of $N$-extended Minkowski superspace. Following [47] we shall analyze the reducibility of our representations and construct intertwining differential operators on them.

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## Appendix A. Basis for the case m=n=1

Here we give separately the simplest case $m = n = 1$, i.e., $GL_{uq}(1/1)$. We have:

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \Gamma & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & \Lambda \\ 0 & 1 \end{pmatrix} \tag{A.1}$$

where we suppose now that there exists an element $a^{-1}$ :

$$A = a \,, \qquad D = d - ca^{-1}b \tag{A.2a}$$

$$\Lambda = a^{-1}b \,, \qquad \Gamma = ca^{-1} \tag{A.2b}$$

The commutation relation between the old generators are

$$\begin{array}{ll} ab = pba, & db = pbd \\ ac = qca, & dc = qcd \\ pbc = -qcb, & b^2 = c^2 = 0 \\ ad - da = (q^{-1} - p)bc & \end{array} \tag{A.3}$$

The superdeterminant is given by:

$$\mathcal{D} = ad^{-1} - bd^{-1}cd^{-1} \tag{A.4}$$

It is central and group-like element, and we suppose that it has an inverse $(\mathcal{D})^{-1}$. The commutation relations between the new generators $\{A, D, \Lambda, \Gamma\}$ are

$$\begin{array}{ll} A\Lambda = p\Lambda A, & D\Lambda = p\Lambda D \\ A\Gamma = q\Gamma A, & D\Gamma = q\Gamma D \\ \Lambda\Gamma = -\Gamma\Lambda, & \Lambda^2 = \Gamma^2 = 0 \\ AD = DA & \end{array} \tag{A.5}$$

One extends the algebra with inverse elements $A^{-1}$ and $D^{-1}$ of $A$ and $D$, respectively. The superdeterminant is now given by

$$\mathcal{D} = AD^{-1} \tag{A.6}$$

The coalgebra structure is given by

$$\begin{aligned} \delta(A) &= A \otimes A + A\Lambda \otimes \Gamma A \\ \delta(D) &= D \otimes D + D\Lambda \otimes \Gamma D \\ \delta(\Lambda) &= 1 \otimes \Lambda + \Lambda \otimes A^{-1}D \\ \delta(\Gamma) &= \Gamma \otimes 1 + DA^{-1} \otimes \Gamma \end{aligned} \tag{A.7}$$

One can also calculate the coproduct of the inverses $A^{-1}$ and $D^{-1}$:

$$\delta(A^{-1}) = A^{-1} \otimes A^{-1} - \Lambda A^{-1} \otimes A^{-1}\Gamma \quad (A.8a)$$

$$\delta(D^{-1}) = D^{-1} \otimes D^{-1} - \Lambda D^{-1} \otimes D^{-1}\Gamma \quad (A.8b)$$

The counit and the antipode are given by:

$$\varepsilon_{\mathcal{A}}(A) = \varepsilon_{\mathcal{A}}(D) = 1 \quad (A.9a)$$

$$\varepsilon_{\mathcal{A}}(\Lambda) = \varepsilon_{\mathcal{A}}(\Gamma) = 0 \quad (A.9b)$$

$$\gamma_{\mathcal{A}}(A) = \Delta^{-1}A^{-1}, \quad \gamma_{\mathcal{A}}(D) = \Delta^{-1}D^{-1} \quad (A.9c)$$

$$\gamma_{\mathcal{A}}(\Lambda) = -\Lambda\mathcal{D}, \quad \gamma_{\mathcal{A}}(\Gamma) = -\mathcal{D}\Gamma \quad (A.9d)$$

where

$$\Delta = 1 - q^{-1}\Lambda\mathcal{D}\Gamma \quad (A.10)$$

Now let us write explicitly the right action on the old and new basis. For the basis $\{a, d, b, c\}$ we have:

$$\pi_R(K_1)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = u^{1/2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (A.11a)$$

$$\pi_R(\mathcal{P}_1^{1/2})\begin{pmatrix} a & b \\ c & d \end{pmatrix} = uq^{1/2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (A.11b)$$

$$\pi_R(\mathcal{Q}_1^{1/2})\begin{pmatrix} a & b \\ c & d \end{pmatrix} = q^{-1/2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (A.11c)$$

$$\pi_R(X_1^+)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (uq)^{-1/2}\begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} \quad (A.11d)$$

$$\pi_R(X_1^-)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -(uq)^{1/2}\begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix} \quad (A.11e)$$

$$\pi_R(\mathcal{K})\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u^{1/2}a & u^{-1/2}b \\ u^{1/2}c & u^{-1/2}d \end{pmatrix} \quad (A.11f)$$

On the new basis we have:

$$\pi_R(K_1)\begin{pmatrix} A & \Lambda \\ \Gamma & D \end{pmatrix} = \begin{pmatrix} u^{1/2}A & \Lambda \\ \Gamma & u^{1/2}D \end{pmatrix} \quad (A.12a)$$

$$\pi_R(\mathcal{P}_1^{1/2})\begin{pmatrix} A & \Lambda \\ \Gamma & D \end{pmatrix} = \begin{pmatrix} uq^{1/2}A & \Lambda \\ \Gamma & uq^{1/2}D \end{pmatrix} \quad (A.12b)$$

$$\pi_R(\mathcal{Q}_1^{1/2})\begin{pmatrix} A & \Lambda \\ \Gamma & D \end{pmatrix} = \begin{pmatrix} q^{-1/2}A & \Lambda \\ \Gamma & q^{-1/2}D \end{pmatrix} \quad (A.12c)$$

$$\pi_R(X_1^+)\begin{pmatrix} A & \Lambda \\ \Gamma & D \end{pmatrix} = \begin{pmatrix} 0 & u^{1/2} \\ 0 & 0 \end{pmatrix} \quad (A.12d)$$

$$\pi_R(X_1^-)\begin{pmatrix} A & \Lambda \\ \Gamma & D \end{pmatrix} = \begin{pmatrix} -(uq)^{1/2}A\Lambda & 0 \\ -u^{1/2}qDA^{-1} & (uq)^{1/2}D\Lambda \end{pmatrix} \quad (A.12e)$$

$$\pi_R(\mathcal{K})\begin{pmatrix} A & \Lambda \\ \Gamma & D \end{pmatrix} = \begin{pmatrix} u^{1/2}A & u^{-1}\Lambda \\ \Gamma & u^{-1/2}D \end{pmatrix} \quad (A.12f)$$

Finally the right action on $A^{-1}$ is given by

$$\pi_R(K_I)A^{-1} = u^{-1/2}A^{-1}, \quad \pi_R(\mathcal{P}_I^{1/2})A^{-1} = (uq)^{-1/2}A^{-1}, \quad \pi_R(\mathcal{Q}_I^{1/2})A^{-1} = q^{1/2}A^{-1},$$
$$\pi_R(X_I^+)A^{-1} = 0, \quad \pi_R(X_I^-)A^{-1} = (uq)^{1/2}\Lambda A^{-1}, \quad \pi_R(\mathcal{K})A^{-1} = u^{-1/2}A^{-1}. \quad (A.13)$$

## Appendix B. Basis for the case m=2,n=1

Now let us take the case of $m = 2$, $n = 1$. The quantum matrix may be decomposed as

$$T = \begin{pmatrix} a_{11} & a_{12} & b_{13} \\ a_{21} & a_{22} & b_{23} \\ c_{31} & c_{32} & d_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ z_{21} & 1 & 0 \\ \gamma_{31} & \gamma_{32} & 1 \end{pmatrix} \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & D_{33} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & \beta_{13} \\ 0 & 1 & \beta_{23} \\ 0 & 0 & 1 \end{pmatrix} \tag{B.1}$$

where we have to suppose that there exist $A_{11}^{-1}$ and $A_{22}^{-1}$ :

$$A_{11} = a_{11} \qquad A_{22} = a_{22} - a_{21}a_{11}^{-1}a_{12} \tag{B.2a}$$
$$u_{12} = a_{11}^{-1}a_{12} , \qquad z_{21} = a_{21}a_{11}^{-1} \tag{B.2b}$$

$$\beta_{13} = a_{11}^{-1}b_{13} \qquad \beta_{23} = A_{22}^{-1}(b_{23} - a_{21}a_{11}^{-1}b_{13}) \tag{B.3a}$$
$$\gamma_{31} = c_{31}a_{11}^{-1} , \qquad \gamma_{32} = (c_{32} - c_{31}a_{11}^{-1}a_{12})A_{22}^{-1} \tag{B.3b}$$

$$D_{33} = d_{33} - \gamma_{31}A_{11}\beta_{13} - \gamma_{32}A_{22}\beta_{23}. \tag{B.4}$$

The commutation relations between these generators are

$$\begin{array}{ll} A_{11}u_{12} = pu_{12}A_{11}, & A_{11}\beta_{13} = p\beta_{13}A_{11}, \\ uq\beta_{23}A_{11} - (up)^{-1}A_{11}\beta_{23} = 0, & \\ & \\ u_{12}A_{22} = qA_{22}u_{12}, & p\beta_{13}A_{22} = qA_{22}\beta_{13}, \\ A_{22}\beta_{23} = p\beta_{23}A_{22}, & \\ & \\ uqD_{33}u_{12} - (up)^{-1}u_{12}D_{33} = 0, & \\ \beta_{13}D_{33} = u^2qD_{33}\beta_{13}, & \beta_{23}D_{33} = u^2qD_{33}\beta_{23} , \end{array} \tag{B.5}$$

$$\begin{array}{ll} A_{11}z_{21} = qz_{21}A_{11}, & A_{11}\gamma_{31} = q\gamma_{31}A_{11}, \\ uq\gamma_{32}A_{11} - (up)^{-1}A_{11}\gamma_{32} = 0, & \\ & \\ z_{21}A_{22} = pA_{22}z_{21}, & pA_{22}\gamma_{31} = q\gamma_{31}A_{22}, \\ A_{22}\gamma_{32} = q\gamma_{32}A_{22}, & \\ & \\ uqD_{33}z_{21} - (up)^{-1}z_{21}D_{33} = 0, & \\ \gamma_{31}D_{33} = u^2pD_{33}\gamma_{31}, & \gamma_{32}D_{33} = u^2pD_{33}\gamma_{32} , \end{array} \tag{B.6}$$

$$[A_{11} , A_{22}] = [A_{11} , D_{33}] = [A_{22} , D_{33}] = 0, \tag{B.7}$$

$$\begin{array}{ll} u_{12}\beta_{13} = h\beta_{13}u_{12}, & \beta_{13}\beta_{23} = -g\beta_{23}\beta_{13}, \\ g\beta_{23}u_{12} - u_{12}\beta_{23} = u(u-u^{-1})\beta_{13}, & (\beta_{13})^2 = (\beta_{23})^2 = 0 \end{array} \tag{B.8}$$

$$\begin{array}{ll} z_{21}\gamma_{31} = g^{-1}\gamma_{31}z_{21}, & \gamma_{31}\gamma_{32} = -h^{-1}\gamma_{32}\gamma_{31}, \\ \gamma_{32}z_{21} - h\ z_{21}\gamma_{32} = u^{-1}(u-u^{-1})\gamma_{31}, & (\gamma_{31})^2 = (\gamma_{32})^2 = 0 \end{array} \tag{B.9}$$

$$\begin{array}{l} [u_{12} , z_{21}] = [u_{12} , \gamma_{31}] = [u_{12} , \gamma_{32}] = 0, \\ [\beta_{13} , z_{21}] = [\beta_{23} , z_{21}] = 0, \\ \beta_{13}\gamma_{31} + \gamma_{31}\beta_{13} = \beta_{13}\gamma_{23} + \gamma_{23}\beta_{13} = 0 \\ \beta_{23}\gamma_{13} + \gamma_{13}\beta_{23} = \beta_{23}\gamma_{23} + \gamma_{23}\beta_{23} = 0, \end{array} \tag{B.10}$$

where $g = q_{12}q_{23}/q_{13}$ and $h = g/u^2$. The superdeterminant is now given by

$$\mathcal{D} \;=\; A_{11}A_{22}D_{33}^{-1}. \tag{B.11}$$

It satisfies the following commutation relations with the new generators:

$$\begin{array}{lll} u_{12}\mathcal{D} \;=\; \tilde{q}_1\mathcal{D}u_{12}, & \mathcal{D}z_{21} \;=\; \tilde{q}_1 z_{21}\mathcal{D} & \text{(B.12}a) \\ \beta_{13}\mathcal{D} \;=\; \tilde{q}_1\tilde{q}_2\mathcal{D}\beta_{13}, & \mathcal{D}\gamma_{31} \;=\; \tilde{q}_1\tilde{q}_2\gamma_{31}\mathcal{D} & \text{(B.12}b) \\ \beta_{23}\mathcal{D} \;=\; \tilde{q}_2\mathcal{D}\beta_{23}, & \mathcal{D}\gamma_{32} \;=\; \tilde{q}_2\gamma_{32}\mathcal{D} & \text{(B.12}c) \\ A_{11}\mathcal{D} \;=\; \mathcal{D}A_{11}, \quad A_{22}\mathcal{D} \;=\; \mathcal{D}A_{22}, & D_{33}\mathcal{D} \;=\; \mathcal{D}D_{33} & \text{(B.12}d) \end{array}$$

The action of the right action on the new basis is as follows:

$$\pi_R(K_I)\begin{pmatrix} A_{11} & u_{12} & \beta_{13} \\ z_{21} & A_{22} & \beta_{23} \\ \gamma_{31} & \gamma_{32} & D_{33} \end{pmatrix} = \begin{pmatrix} u^{\delta_{I1}/2}A_{11} & u^{(\delta_{I2}-2\delta_{I1})/2}u_{12} & u^{(\delta_{I+1,3}-\delta_{I1})/2}\beta_{13} \\ z_{21} & u^{(\delta_{I2}-\delta_{I+1,2})/2}A_{22} & u^{(\delta_{I+1,2})/2}\beta_{23} \\ \gamma_{31} & \gamma_{32} & u^{\delta_{I+1,3}/2}D_{33} \end{pmatrix}, \tag{B.13$a$}$$

$$\pi_R(\mathcal{P}_I^{1/2})\begin{pmatrix} A_{11} & u_{12} & \beta_{13} \\ z_{21} & A_{22} & \beta_{23} \\ \gamma_{31} & \gamma_{32} & D_{33} \end{pmatrix} = \begin{pmatrix} Q_{I1}^{1/2}A_{11} & Q_{I1}^{-1/2}Q_{I2}^{1/2}u_{12} & Q_{I1}^{-1/2}Q_{I3}^{1/2}\beta_{13} \\ z_{21} & Q_{I2}^{1/2}A_{22} & Q_{I3}^{1/2}Q_{I2}^{-1/2}\beta_{23} \\ \gamma_{31} & \gamma_{32} & Q_{I3}^{1/2}D_{33} \end{pmatrix}, \tag{B.13$b$}$$

$$\pi_R(\mathcal{Q}_I^{1/2})\begin{pmatrix} A_{11} & u_{12} & \beta_{13} \\ z_{21} & A_{22} & \beta_{23} \\ \gamma_{31} & \gamma_{32} & D_{33} \end{pmatrix} =$$

$$= \begin{pmatrix} u^{\delta_{I1}}Q_{I1}^{-1/2}A_{11} & u^{(\delta_{I2}-2\delta_{I1})}Q_{I1}^{1/2}Q_{I2}^{-1/2}u_{12} & u^{(\delta_{I+1,3}-\delta_{I1})}Q_{I1}^{1/2}Q_{I3}^{-1/2}\beta_{13} \\ z_{21} & u^{(\delta_{I2}-\delta_{I+1,2})}Q_{I2}^{-1/2}A_{22} & u^{\delta_{I+1,2}}Q_{I3}^{-1/2}Q_{I2}^{1/2}\beta_{23} \\ \gamma_{31} & \gamma_{32} & u^{\delta_{I+1,3}}Q_{I3}^{-1/2}D_{33} \end{pmatrix}, \tag{B.13$c$}$$

$$\pi_R(X_I^+)\begin{pmatrix} A_{11} & u_{12} & \beta_{13} \\ z_{21} & A_{22} & \beta_{23} \\ \gamma_{31} & \gamma_{32} & D_{33} \end{pmatrix} = \begin{pmatrix} 0 & u\delta_{I+1,2} & u^{1/2}h^{1/2}\delta_{I+1,3}u_{12} \\ 0 & 0 & u^{1/2}\delta_{I+1,3} \\ 0 & 0 & 0 \end{pmatrix} \tag{B.13$d$}$$

## Appendix C. Commutation relations of the new basis

We first give the commutation relation between the generators $\{Y_{ji},\, \Gamma_{\alpha i},\, Z_{\beta\alpha},\, U_{ij},\, \Lambda_{i\alpha},\, V_{\alpha\beta}$. The indices used below obey $i < j < k < l \le m$ and $m+1 \le \alpha < \beta < \gamma < \delta$ throughout the Appendix. We also use the notation:

$$p_{IJ} \;\equiv\; \frac{q_{IJ}}{u^2}\,, \qquad p'_{IJ} \;\equiv\; \frac{q'_{IJ}}{u'^2} \tag{C.1}$$

We start with the generators $Y_{ji}, \Gamma_{\alpha i}, Z_{\beta\alpha}$ of the 'lower triangular' subsuperalgebra:

$$Y_{kj}Y_{ki} = \frac{q_{ij}q_{jk}}{q_{ik}}Y_{ki}Y_{kj} \tag{C.2a}$$

$$Y_{ki}Y_{ji} = \frac{q_{ij}q_{jk}}{q_{ik}}Y_{ji}Y_{ki} \tag{C.2b}$$

$$Y_{kj}Y_{ji} = \frac{p_{ij}p_{jk}}{p_{ik}}Y_{ji}Y_{kj} + u^{-1}(u-u^{-1})Y_{ki} \tag{C.2c}$$

$$Y_{li}Y_{kj} = \frac{q_{ik}q_{kl}}{q_{ij}q_{jl}}Y_{kj}Y_{li} \tag{C.2d}$$

$$\frac{q_{jl}}{q_{jk}q_{kl}}Y_{lj}Y_{ki} = \frac{p_{ij}p_{jl}}{p_{il}}Y_{ki}Y_{lj} + u^{-1}(u-u^{-1})Y_{kj}Y_{li} \tag{C.2e}$$

$$Y_{lk}Y_{ji} = \frac{q_{ik}q_{jl}}{q_{il}q_{jk}}Y_{ji}Y_{lk} \tag{C.2f}$$

$$\Gamma_{\alpha i}Y_{ji} = \frac{q_{ij}q_{j\alpha}}{q_{i\alpha}}Y_{ji}\Gamma_{\alpha i} \tag{C.3a}$$

$$\Gamma_{\alpha j}Y_{ji} = \frac{p_{ij}p_{j\alpha}}{p_{i\alpha}}Y_{ji}\Gamma_{\alpha j} + u^{-1}(u-u^{-1})\Gamma_{\alpha i} \tag{C.3b}$$

$$\Gamma_{\alpha i}Y_{kj} = \frac{q_{ik}q_{k\alpha}}{q_{ij}q_{j\alpha}}Y_{kj}\Gamma_{\alpha i} \tag{C.3c}$$

$$\frac{q_{j\alpha}}{q_{jk}q_{k\alpha}}\Gamma_{\alpha j}Y_{ki} = \frac{p_{ij}p_{j\alpha}}{p_{i\alpha}}Y_{ki}\Gamma_{\alpha j} + u^{-1}(u-u^{-1})Y_{kj}\Gamma_{\alpha i} \tag{C.3d}$$

$$\Gamma_{\alpha k}Y_{ji} = \frac{q_{ik}q_{j\alpha}}{q_{i\alpha}q_{jk}}Y_{ji}\Gamma_{\alpha k} \tag{C.3e}$$

$$Z_{\beta\alpha}Y_{ji} = \frac{q_{i\alpha}q_{j\beta}}{q_{i\beta}q_{j\alpha}}Y_{ji}Z_{\beta\alpha} \tag{C.4}$$

$$\Gamma_{\alpha j}\Gamma_{\alpha i} = -\frac{q'_{ij}q'_{j\alpha}}{q'_{i\alpha}}\Gamma_{\alpha i}\Gamma_{\alpha j} \tag{C.5a}$$

$$\Gamma_{\beta i}\Gamma_{\alpha i} = -\frac{q_{i\alpha}q_{\alpha\beta}}{q_{i\beta}}\Gamma_{\alpha i}\Gamma_{\beta i} \tag{C.5b}$$

$$\Gamma_{\beta i}\Gamma_{\alpha j} = -\frac{q_{i\alpha}q_{\alpha\beta}}{q_{ij}q_{j\beta}}\Gamma_{\alpha j}\Gamma_{\beta i} \tag{C.5c}$$

$$\frac{q'_{j\beta}}{q'_{j\alpha}q'_{\alpha\beta}}\Gamma_{\beta j}\Gamma_{\alpha i} = -\frac{p'_{ij}p'_{j\beta}}{p'_{i\beta}}\Gamma_{\alpha i}\Gamma_{\beta j} + u'^{-1}(u'-u'^{-1})\Gamma_{\alpha j}\Gamma_{\beta i} \tag{C.5d}$$

$$(\Gamma_{\alpha i})^2 = 0 \tag{C.5e}$$

$$Z_{\beta\alpha}\Gamma_{\beta k} = \frac{q'_{k\alpha}q'_{\alpha\beta}}{q'_{k\beta}}\Gamma_{\beta k}Z_{\beta\alpha} \tag{C.6a}$$

$$Z_{\beta\alpha}\Gamma_{\alpha k} = \frac{p'_{k\alpha}p'_{\alpha\beta}}{p'_{k\beta}}\Gamma_{\alpha k}Z_{\beta\alpha} + u'^{-1}(u'-u'^{-1})\Gamma_{\beta k} \tag{C.6b}$$

$$Z_{\beta\alpha}\Gamma_{\gamma k} \;=\; \frac{q_{k\alpha}q_{\alpha\gamma}}{q_{k\beta}q_{\beta\gamma}}\Gamma_{\gamma k}Z_{\beta\alpha} \tag{C.6c}$$

$$\frac{q'_{\alpha\gamma}}{q'_{\alpha\beta}q'_{\beta\gamma}}Z_{\gamma\alpha}\Gamma_{\beta k} \;=\; \frac{p'_{k\alpha}p'_{\alpha\gamma}}{p'_{k\gamma}}\Gamma_{\beta k}Z_{\gamma\alpha} \;+\; u'^{-1}(u'-u'^{-1})\frac{q_{k\alpha}q_{\alpha\gamma}}{q_{k\beta}q_{\beta\gamma}}\Gamma_{\gamma k}Z_{\beta\alpha} \tag{C.6d}$$

$$Z_{\gamma\beta}\Gamma_{\alpha k} \;=\; \frac{q_{k\beta}q_{\alpha\gamma}}{q_{k\gamma}q_{\alpha\beta}}\Gamma_{\alpha k}Z_{\gamma\beta} \tag{C.6e}$$

$$Z_{\gamma\beta}Z_{\gamma\alpha} \;=\; \frac{q'_{\alpha\beta}q'_{\beta\gamma}}{q'_{\alpha\gamma}}Z_{\gamma\alpha}Z_{\gamma\beta} \tag{C.7a}$$

$$Z_{\gamma\alpha}Z_{\beta\alpha} \;=\; \frac{q'_{\alpha\beta}q'_{\beta\gamma}}{q'_{\alpha\gamma}}Z_{\beta\alpha}Z_{\gamma\alpha} \tag{C.7b}$$

$$Z_{\gamma\beta}Z_{\beta\alpha} \;=\; \frac{p'_{\alpha\beta}p'_{\beta\gamma}}{p'_{\alpha\gamma}}Z_{\beta\alpha}Z_{\gamma\beta} \;+\; u'^{-1}(u'-u'^{-1})Z_{\gamma\alpha} \tag{C.7c}$$

$$Z_{\delta\alpha}Z_{\gamma\beta} \;=\; \frac{q_{\alpha\gamma}q_{\gamma\delta}}{q_{\alpha\beta}q_{\beta\delta}}Z_{\gamma\beta}Z_{\delta\alpha} \tag{C.7d}$$

$$\frac{q'_{\beta\delta}}{q'_{\beta\gamma}q_{\gamma\delta}}Z_{\delta\beta}Z_{\gamma\alpha} \;=\; \frac{p'_{\alpha\beta}p'_{\beta\delta}}{p'_{\alpha\delta}}Z_{\gamma\alpha}Z_{\delta\beta} \;+\; u'^{-1}(u'-u'^{-1})Z_{\gamma\beta}Z_{\delta\alpha} \tag{C.7e}$$

$$Z_{\delta\gamma}Z_{\beta\alpha} \;=\; \frac{q_{\alpha\gamma}q_{\beta\delta}}{q_{\alpha\delta}q_{\beta\gamma}}Z_{\beta\alpha}Z_{\delta\gamma} \tag{C.7f}$$

Next we consider the generators $U_{ij}, \Lambda_{i\alpha}, V_{\alpha\beta}$ of the 'upper triangular' subsuperalgebra:

$$U_{ij}U_{ik} \;=\; \frac{p_{ij}p_{jk}}{p_{ik}}\,U_{ik}U_{ij} \tag{C.8a}$$

$$U_{ik}U_{jk} \;=\; \frac{p_{ij}p_{jk}}{p_{ik}}\,U_{jk}U_{ik} \tag{C.8b}$$

$$U_{ij}U_{jk} \;=\; \frac{q_{ij}q_{jk}}{q_{ik}}\,U_{jk}U_{ij} \;-\; u(u-u^{-1})\,U_{ik} \tag{C.8c}$$

$$U_{ij}U_{kl} \;=\; \frac{p_{ik}p_{jl}}{p_{il}p_{jk}}\,U_{kl}U_{ij} \tag{C.8d}$$

$$\frac{p_{jl}}{p_{jk}p_{kl}}\,U_{ik}U_{jl} \;=\; \frac{q_{ij}q_{jl}}{q_{il}}\,U_{jl}U_{ik} \;-\; u(u-u^{-1})\,\frac{p_{ij}p_{jl}}{p_{ik}p_{kl}}\,U_{jk}U_{il} \tag{C.8e}$$

$$U_{il}U_{jk} \;=\; \frac{p_{ij}p_{jl}}{p_{ik}p_{kl}}\,U_{jk}U_{il} \tag{C.8f}$$

$$U_{ij}\Lambda_{i\alpha} \;=\; \frac{p_{ij}p_{j\alpha}}{p_{i\alpha}}\Lambda_{i\alpha}U_{ij} \tag{C.9a}$$

$$U_{ij}\Lambda_{j\alpha} \;=\; \frac{q_{ij}q_{j\alpha}}{q_{i\alpha}}\,\Lambda_{j\alpha}U_{ij} \;-\; u(u-u^{-1})\Lambda_{i\alpha} \tag{C.9b}$$

$$U_{ij}\Lambda_{k\alpha} \;=\; \frac{p_{ik}p_{j\alpha}}{p_{i\alpha}p_{jk}}\,\Lambda_{k\alpha}U_{ij} \tag{C.9c}$$

$$\frac{p_{j\alpha}}{p_{jk}p_{k\alpha}}\,U_{ik}\Lambda_{j\alpha} \;=\; \frac{q_{ij}q_{j\alpha}}{q_{i\alpha}}\,\Lambda_{j\alpha}U_{ik} \;-\; u(u-u^{-1})\,\Lambda_{i\alpha}U_{jk} \tag{C.9d}$$

$$U_{jk}\Lambda_{i\alpha} \;=\; \frac{p_{ik}p_{k\alpha}}{p_{ij}p_{j\alpha}}\,\Lambda_{i\alpha}U_{jk} \tag{C.9e}$$

$$U_{ij}V_{\alpha\beta} \;=\; \frac{p_{i\alpha}p_{j\beta}}{p_{i\beta}p_{j\alpha}}\,V_{\alpha\beta}U_{ij} \tag{C.10}$$

$$\Lambda_{i\alpha}\Lambda_{j\alpha} \;=\; -\frac{p'_{ij}p'_{j\alpha}}{p'_{i\alpha}}\Lambda_{j\alpha}\Lambda_{i\alpha} \tag{C.11$a$}$$

$$\Lambda_{i\alpha}\Lambda_{i\beta} \;=\; -\frac{p_{i\alpha}p_{\alpha\beta}}{p_{i\beta}}\Lambda_{i\beta}\Lambda_{i\alpha} \tag{C.11$b$}$$

$$\Lambda_{i\beta}\Lambda_{j\alpha} \;=\; -\frac{p_{ij}p_{j\beta}}{p_{i\alpha}p_{\alpha\beta}}\Lambda_{j\alpha}\Lambda_{i\beta} \tag{C.11$c$}$$

$$\frac{q_{ij}q_{j\beta}}{q_{i\beta}}\,\Lambda_{i\alpha}\Lambda_{j\beta} \;=\; -\frac{p_{j\beta}}{p_{j\alpha}p_{\alpha\beta}}\,\Lambda_{j\beta}\Lambda_{i\alpha} \;-\; u(u-u^{-1})\,\frac{p_{ij}p_{j\beta}}{p_{i\alpha}p_{\alpha\beta}}\,\Lambda_{j\alpha}\Lambda_{i\beta} \tag{C.11$d$}$$

$$(\Lambda_{i\alpha})^2 \;=\; 0 \tag{C.11$e$}$$

$$\Lambda_{k\beta}V_{\alpha\beta} \;=\; \frac{p'_{k\alpha}p'_{\alpha\beta}}{p'_{k\beta}}V_{\alpha\beta}\Lambda_{k\beta} \tag{C.12$a$}$$

$$\Lambda_{k\alpha}V_{\alpha\beta} \;=\; \frac{q'_{k\alpha}q'_{\alpha\beta}}{q'_{k\beta}}\,V_{\alpha\beta}\Lambda_{k\alpha} \;-\; u'(u'-u'^{-1})\,\Lambda_{k\beta} \tag{C.12$b$}$$

$$\Lambda_{k\alpha}V_{\beta\gamma} \;=\; \frac{p_{k\beta}p_{\alpha\gamma}}{p_{k\gamma}p_{\alpha\beta}}\,V_{\beta\gamma}\Lambda_{k\alpha} \tag{C.12$c$}$$

$$\frac{p'_{\alpha\gamma}}{p'_{\alpha\beta}p'_{\beta\gamma}}\,\Lambda_{k\beta}V_{\alpha\gamma} \;=\; \frac{q'_{k\alpha}q'_{\alpha\gamma}}{q'_{k\gamma}}\,V_{\alpha\gamma}\Lambda_{k\beta} \;-\; u'(u'-u'^{-1})\,\frac{p_{k\alpha}p_{\alpha\gamma}}{p_{k\beta}p_{\beta\gamma}}\,V_{\alpha\beta}\Lambda_{k\gamma} \tag{C.12$d$}$$

$$\Lambda_{k\gamma}V_{\alpha\beta} \;=\; \frac{p_{k\alpha}p_{\alpha\gamma}}{p_{k\beta}p_{\beta\gamma}}V_{\alpha\beta}\Lambda_{k\gamma} \tag{C.12$e$}$$

$$V_{\alpha\beta}V_{\alpha\gamma} \;=\; \frac{p'_{\alpha\beta}p'_{\beta\gamma}}{p'_{\alpha\gamma}}V_{\alpha\gamma}V_{\alpha\beta} \tag{C.13$a$}$$

$$V_{\alpha\delta}V_{\beta\gamma} \;=\; \frac{p_{\alpha\beta}p_{\beta\delta}}{p_{\alpha\gamma}p_{\gamma\delta}}V_{\beta\gamma}V_{\alpha\delta} \tag{C.13$b$}$$

$$V_{\alpha\beta}V_{\beta\gamma} \;=\; \frac{q'_{\alpha\beta}q'_{\beta\gamma}}{q'_{\alpha\gamma}}V_{\beta\gamma}V_{\alpha\beta} \;-\; u'(u'-u'^{-1})\,V_{\alpha\gamma} \tag{C.13$c$}$$

$$V_{\alpha\beta}V_{\gamma\delta} \;=\; \frac{p_{\alpha\gamma}p_{\beta\delta}}{p_{\alpha\delta}p_{\beta\gamma}}\,V_{\gamma\delta}V_{\alpha\beta} \tag{C.13$d$}$$

$$\frac{p'_{\beta\delta}}{p'_{\beta\gamma}p'_{\gamma\delta}}\,V_{\alpha\gamma}V_{\beta\delta} \;=\; \frac{q'_{\alpha\beta}q'_{\beta\delta}}{q'_{\alpha\delta}}\,V_{\beta\delta}V_{\alpha\gamma} \;-\; u'(u'-u'^{-1})\,\frac{p_{\alpha\beta}p_{\beta\delta}}{p_{\alpha\gamma}p_{\gamma\delta}}\,V_{\beta\gamma}V_{\alpha\delta} \tag{C.13$e$}$$

$$V_{\alpha\gamma}V_{\beta\gamma} \;=\; \frac{p'_{\alpha\beta}p'_{\beta\gamma}}{p'_{\alpha\gamma}}V_{\beta\gamma}V_{\alpha\gamma} \tag{C.13$f$}$$

Now we give the commutation relations of the 'diagonal' generators $D_{ii}, G_{\alpha\alpha}, \mathcal{F}$ with the 'off-diagonal' ones:

$$\begin{aligned}
D_{ii}Y_{ji} &= q_{ij}^{-1}Y_{ji}D_{ii} && \text{(C.14}a\text{)}\\
D_{jj}Y_{ji} &= \frac{u^2}{q_{ij}}Y_{ji}D_{jj} && \text{(C.14}b\text{)}\\
D_{ii}Y_{kj} &= \frac{q_{ij}}{q_{ik}}Y_{kj}D_{ii} && \text{(C.14}c\text{)}\\
D_{jj}Y_{ki} &= \frac{u^2}{q_{ij}q_{jk}}Y_{ki}D_{jj} && \text{(C.14}d\text{)}\\
D_{kk}Y_{ji} &= \frac{q_{jk}}{q_{ik}}Y_{ji}D_{kk} && \text{(C.14}e\text{)}\\
D_{ii}\Gamma_{\alpha i} &= q_{i\alpha}^{-1}\Gamma_{\alpha i}D_{ii} && \text{(C.14}f\text{)}\\
D_{ii}\Gamma_{\alpha j} &= \frac{q_{ij}}{q_{i\alpha}}\Gamma_{\alpha j}D_{ii} && \text{(C.14}g\text{)}\\
D_{jj}\Gamma_{\alpha i} &= \frac{u^2}{q_{ij}q_{j\alpha}}\Gamma_{\alpha i}D_{jj} && \text{(C.14}h\text{)}\\
D_{ii}Z_{\beta\alpha} &= \frac{q_{i\alpha}}{q_{i\beta}}Z_{\beta\alpha}D_{ii} && \text{(C.14}i\text{)}
\end{aligned}$$

$$\begin{aligned}
U_{ij}D_{ii} &= \frac{u^2}{q_{ij}}D_{ii}U_{ij} && \text{(C.15}a\text{)}\\
U_{ij}D_{jj} &= q_{ij}^{-1}D_{jj}U_{ij} && \text{(C.15}b\text{)}\\
U_{jk}D_{ii} &= \frac{q_{ij}}{q_{ik}}D_{ii}U_{jk} && \text{(C.15}c\text{)}\\
U_{ik}D_{jj} &= \frac{u^2}{q_{ij}q_{jk}}D_{jj}U_{ik} && \text{(C.15}d\text{)}\\
U_{ij}D_{kk} &= \frac{q_{jk}}{q_{ik}}D_{kk}U_{ij} && \text{(C.15}e\text{)}\\
\Lambda_{i\alpha}D_{ii} &= \frac{u^2}{q_{i\alpha}}D_{ii}\Lambda_{i\alpha} && \text{(C.15}f\text{)}\\
\Lambda_{i\alpha}D_{jj} &= \frac{u^2}{q_{ij}q_{j\alpha}}D_{jj}\Lambda_{i\alpha} && \text{(C.15}g\text{)}\\
\Lambda_{j\alpha}D_{ii} &= \frac{q_{ij}}{q_{i\alpha}}D_{ii}\Lambda_{j\alpha} && \text{(C.15}h\text{)}\\
V_{\alpha\beta}D_{ii} &= \frac{q_{i\alpha}}{q_{i\beta}}D_{ii}V_{\alpha\beta} && \text{(C.15}i\text{)}
\end{aligned}$$

$$\begin{aligned}
G_{\alpha\alpha}Y_{ji} &= \frac{q_{j\alpha}}{q_{i\alpha}}Y_{ji}G_{\alpha\alpha} && \text{(C.16}a\text{)}\\
G_{\alpha\alpha}\Gamma_{\alpha i} &= \frac{u'^2}{q'_{i\alpha}}\Gamma_{\alpha i}G_{\alpha\alpha} && \text{(C.16}b\text{)}\\
G_{\alpha\alpha}\Gamma_{\beta i} &= \frac{u^2}{q_{i\alpha}q_{\alpha\beta}}\Gamma_{\beta i}G_{\alpha\alpha} && \text{(C.16}c\text{)}
\end{aligned}$$

$$G_{\beta\beta}\Gamma_{\alpha i} = \frac{q_{\alpha\beta}}{q_{i\beta}}\Gamma_{\alpha i}G_{\beta\beta} \quad (C.16d)$$

$$G_{\alpha\alpha}Z_{\beta\alpha} = q'^{-1}_{\alpha\beta}Z_{\beta\alpha}G_{\alpha\alpha} \quad (C.16e)$$

$$G_{\beta\beta}Z_{\beta\alpha} = \frac{u'^2}{q'_{\alpha\beta}}Z_{\beta\alpha}G_{\beta\beta} \quad (C.16f)$$

$$G_{\alpha\alpha}Z_{\gamma\beta} = \frac{q_{\alpha\beta}}{q_{\alpha\gamma}}Z_{\gamma\beta}G_{\alpha\alpha} \quad (C.16g)$$

$$G_{\beta\beta}Z_{\gamma\alpha} = \frac{u^2}{q_{\alpha\beta}q_{\beta\gamma}}Z_{\gamma\alpha}G_{\beta\beta} \quad (C.16h)$$

$$G_{\gamma\gamma}Z_{\beta\alpha} = \frac{q_{\beta\gamma}}{q_{\alpha\gamma}}Z_{\beta\alpha}G_{\gamma\gamma} \quad (C.16i)$$

$$U_{ij}G_{\alpha\alpha} = \frac{q_{j\alpha}}{q_{i\alpha}}G_{\alpha\alpha}U_{ij} \quad (C.17a)$$

$$\Lambda_{i\alpha}G_{\alpha\alpha} = q'^{-1}_{i\alpha}G_{\alpha\alpha}\Lambda_{i\alpha} \quad (C.17b)$$

$$\Lambda_{i\alpha}G_{\beta\beta} = \frac{q_{\alpha\beta}}{q_{i\beta}}G_{\beta\beta}\Lambda_{i\alpha} \quad (C.17c)$$

$$\Lambda_{i\beta}G_{\alpha\alpha} = \frac{u^2}{q_{i\alpha}q_{\alpha\beta}}G_{\alpha\alpha}\Lambda_{i\beta} \quad (C.17d)$$

$$V_{\alpha\beta}G_{\alpha\alpha} = \frac{u'^2}{q'_{\alpha\beta}}G_{\alpha\alpha}V_{\alpha\beta} \quad (C.17e)$$

$$V_{\alpha\beta}G_{\beta\beta} = q'^{-1}_{\alpha\beta}G_{\beta\beta}V_{\alpha\beta} \quad (C.17f)$$

$$V_{\alpha\beta}G_{\gamma\gamma} = \frac{q_{\beta\gamma}}{q_{\alpha\gamma}}G_{\gamma\gamma}V_{\alpha\beta} \quad (C.17g)$$

$$V_{\alpha\gamma}G_{\beta\beta} = \frac{u^2}{q_{\alpha\beta}q_{\beta\gamma}}G_{\beta\beta}V_{\alpha\gamma} \quad (C.17h)$$

$$V_{\beta\gamma}G_{\alpha\alpha} = \frac{q_{\alpha\beta}}{q_{\alpha\gamma}}G_{\alpha\alpha}V_{\beta\gamma} \quad (C.17i)$$

Using (C.14), (C.15), (C.16), (C.17) we obtain the commutation relations of $D_i = \prod_{j=1}^{i} D_{jj}$ , $G_\alpha = \prod_{\beta=m+1}^{\alpha} G_{\beta\beta}$ .

$$\mathcal{F}\, Y_{ji} = \left(\prod_{s=i}^{j-1} \tilde{q}_s\right) Y_{ji}\,\mathcal{F} \quad (C.18a)$$

$$\mathcal{F}\,\Gamma_{\alpha i} = \left(\prod_{S=i}^{\alpha-1} \tilde{q}_S\right) \Gamma_{\alpha i}\,\mathcal{F} \quad (C.18b)$$

$$\mathcal{F}\, Z_{\beta\alpha} = \left(\prod_{\gamma=\alpha}^{\beta-1} \tilde{q}_\gamma\right) Z_{\beta\alpha}\,\mathcal{F} \quad (C.18c)$$

$$U_{ij}\,\mathcal{F} = \left(\prod_{s=i}^{j-1} \tilde{q}_s\right) \mathcal{F}\, U_{ij} \quad (C.18d)$$

$$\Lambda_{i\alpha}\,\mathcal{F} \;=\; \left(\prod_{S=i}^{\alpha-1}\tilde{q}_S\right)\,\mathcal{F}\,\Lambda_{i\alpha} \tag{C.18e}$$

$$V_{\alpha\beta}\,\mathcal{F} \;=\; \left(\prod_{\gamma=\alpha}^{\beta-1}\tilde{q}_\gamma\right)\,\mathcal{F}\,V_{\alpha\beta} \tag{C.18f}$$

Finally the elements of the strictly lower triangular generators $Y_{ji}, \Gamma_{\alpha i}, Z_{\beta\alpha}$ super-commute the strictly upper triangular generators $U_{ij}, \Lambda_{i\alpha}, V_{\alpha\beta}$. Analogously, the diagonal elements $D_{ii}, G_{\alpha\alpha}, \mathcal{F}$ commute with each others.

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