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A NON-ABELIAN TENSOR PRODUCT OF LEIBNIZ ALGEBRAS

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Abstract

Leibniz algebras are a non-commutative version of usual Lie algebras. We introduce a notion of (pre)crossed Leibniz algebra which is a simultaneous generalization of notions of representation and two-sided ideal of a Leibniz algebra. We construct the Leibniz algebra of biderivations on crossed Leibniz algebras and we define a non-abelian tensor product of Leibniz algebras. These two notions are adjoint to each other. A (co)homological characterization of these new algebraic objects enables us to compare the first order Milnor-type Hochschild homology of an associative algebra (non-necessarily commutative) to its classical Hochschild homology.

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Introduction. Let $\mathfrak{g}$ be a Lie algebra and let M be a representation of $\mathfrak{g}$, seen as a right $\mathfrak{g}$-module. Given a $\mathfrak{g}$-equivariant map $\mu : M \to \mathfrak{g}$, one can endow the $\mathbb{K}$-module M with a bracket ($[m, m'] := m^{\mu(m')}$) which is not skew-symmetric but satisfies the *Leibniz rule of derivations*:

$$[m, [m', m'']] = [[m, m'], m''] - [[m, m''], m'].$$

Such objects were baptized *Leibniz algebras* by Jean-Louis Loday and are studied as a non-commutative variation of Lie algebras (see [8]). One of the main examples of Lie algebras comes from the notion of *derivations*. For the Leibniz algebras, there is an analogue notion of *biderivations* (see [7]).

The aim of this article is to "integrate" the Leibniz algebra of biderivations by means of a non-abelian tensor product of Leibniz algebras as it is done for Lie algebras.

In the classical case, D. Guin (see [5]) has shown that, given crossed Lie $\mathfrak{g}$-algebras $\mathfrak{M}$ and $\mathfrak{N}$, the set of derivations $\mathrm{Der}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N})$ has a structure of pre-crossed Lie $\mathfrak{g}$-algebra. Moreover the functor $\mathrm{Der}_{\mathfrak{g}}(\mathfrak{N}, -)$ is right adjoint to the functor $- \otimes_{\mathfrak{g}} \mathfrak{N}$ where $- \otimes_{\mathfrak{g}} -$ is the non-abelian tensor product of Lie algebras defined by G. J. Ellis (see [3]). D. Guin uses these objects to construct a non-abelian (co)homology theory for Lie algebras, which enables him to compare the $\mathbb{K}$-modules $\mathrm{HC}_1(A)$ and $\mathrm{K}_2^{M\ add}(A)$ where A is an arbitrary associative algebra. We give a non-commutative version of his results, in the sense that Leibniz algebras play the role of Lie algebras, the additive Milnor K-theory $\mathrm{K}_*^{M\ add}(A)$ (resp. the cyclic homology $\mathrm{HC}_*(A)$) being replaced by the Milnor-type Hochschild homology $\mathrm{HH}_*^M(A)$ (resp. the classical Hochschild homology $\mathrm{HH}_*(A)$).

To this end, we introduce the notion of *(pre)crossed* Leibniz $\mathfrak{g}$-algebra as a simultaneous generalization of notions of representation and two-sided ideal of the Leibniz algebra $\mathfrak{g}$. Given crossed Leibniz $\mathfrak{g}$-algebras $\mathfrak{M}$ and $\mathfrak{N}$, we equip the set $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N})$ of biderivations with a structure of pre-crossed Leibniz $\mathfrak{g}$-algebra. On the other hand, we construct a *non-abelian tensor product* $\mathfrak{M} \star \mathfrak{N}$ of Leibniz algebras with mutual actions on one another. When $\mathfrak{M}$ and $\mathfrak{N}$ are crossed Leibniz $\mathfrak{g}$-algebras, this tensor product has also a structure of crossed Leibniz $\mathfrak{g}$-algebra. It turns out that the functor $- \star_{\mathfrak{g}} \mathfrak{N}$ is left adjoint to the functor $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{N}, -)$. Another characterization of this tensor product is the following. If the Leibniz algebra $\mathfrak{g}$ is perfect (and free as a $\mathbb{K}$-module), then the Leibniz algebra $\mathfrak{g} \star \mathfrak{g}$ is the universal central extension of $\mathfrak{g}$ (see [4]). We give also low-degrees (co)homological interpretations of these objects, which yield an exact sequence of $\mathbb{K}$-modules

$$A/[A,A] \otimes \mathrm{HH}_1(A) \ \oplus\ \mathrm{HH}_1(A) \otimes A/[A,A] \to \mathfrak{HL}_1(A, \mathrm{L}(A)) \to \mathfrak{HL}_1(A, [A,A]) \to$$
$$\to \mathrm{HH}_1(A) \to \mathrm{HH}_1^M(A) \to [A,A]/[A,[A,A]] \to 0$$

where $\mathrm{L}(A)$ is the $\mathbb{K}$-module $A \otimes A/\mathrm{im}(b_3)$ equipped with a suitable Leibniz bracket (see section 1.2).

Throughout this paper the symbol $\mathbb{K}$ denotes a commutative ring with a unit element and $\otimes$ stands $\otimes_{\mathbb{K}}$.

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References

1. Prerequisites on Leibniz algebras

1.1. Leibniz algebras. A *Leibniz algebra* is a $\mathbb{K}$-module $\mathfrak{g}$ equipped with a bilinear map $[-,-]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, called *bracket* and satisfying only the *Leibniz identity*

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]$$

for any $x,y,z\in\mathfrak{g}$. In the presence of the condition $[x,x]=0$, the Leibniz identity is equivalent to the so-called *Jacobi identity*. Therefore Lie algebras are examples of Leibniz algebras.

A *morphism* of Leibniz algebras is a linear map $f:\mathfrak{g}_1\to\mathfrak{g}_2$ such that

$$f([x,y]) = [f(x),f(y)]$$

for any $x,y\in\mathfrak{g}_1$. It is clear that Leibniz algebras and their morphisms form a category that we denote by (**Leib**).

A *two-sided ideal* of a Leibniz algebra $\mathfrak{g}$ is a submodule $\mathfrak{h}$ such that $[x,y]\in\mathfrak{h}$ and $[y,x]\in\mathfrak{h}$ for any $x\in\mathfrak{h}$ and any $y\in\mathfrak{g}$. For any two-sided ideal $\mathfrak{h}$ in $\mathfrak{g}$, the quotient module $\mathfrak{g}/\mathfrak{h}$ inherits a structure of Leibniz algebra induced by the bracket of $\mathfrak{g}$. In particular, let $([x,x])$ be the two-sided ideal in $\mathfrak{g}$ generated by all brackets $[x,x]$. The Leibniz algebra $\mathfrak{g}/([x,x])$ is in fact a Lie algebra, said *canonically associated* to $\mathfrak{g}$ and is denoted by $\mathfrak{g}_{Lie}$.

Let $\mathfrak{g}$ be a Leibniz algebra. Denote by $\mathfrak{g}':=[\mathfrak{g},\mathfrak{g}]$ the submodule generated by all brackets $[x,y]$. The Leibniz algebra $\mathfrak{g}$ is said to be *perfect* if $\mathfrak{g}'=\mathfrak{g}$. It is clear that any submodule of $\mathfrak{g}$ containing $\mathfrak{g}'$ is a two-sided ideal in $\mathfrak{g}$.

1.2. Examples. Let M be a representation of a Lie algebra $\mathfrak{g}$ (the action of $\mathfrak{g}$ on M being denoted by m^g for $m\in M$ and $g\in\mathfrak{g}$). For any $\mathfrak{g}$-equivariant map $\mu:M\to\mathfrak{g}$, the bracket given by $[m,m']:=m^{\mu(m')}$ induces a structure of Leibniz (non-Lie) algebra on M. Observe that any Leibniz algebra $\mathfrak{g}$ can be obtained in such a way by taking the canonical projection $\mathfrak{g}\to\mathfrak{g}_{Lie}$ (which is obviously $\mathfrak{g}_{Lie}$-equivariant).

Let A be an associative algebra and let $b_3:A^{\otimes 3}\to A^{\otimes 2}$ be the Hochschild boundary that is, the linear map defined by

$$b_3(a\otimes b\otimes c) := ab\otimes c - a\otimes bc + ca\otimes b,\ \ a,b,c\in A.$$

Then the bracket given by

$$[a\otimes b, c\otimes d] := (ab-ba)\otimes(cd-dc),\ \ a,b,c,d\in A,$$

defines a structure of Leibniz algebra on the $\mathbb{K}$-module $\mathrm{L}(A):=A^{\otimes 2}/\mathrm{im}(b_3)$. Moreover, we have an exact sequence of $\mathbb{K}$-modules

$$0\to \mathrm{HH}_1(A)\to \mathrm{L}(A)\xrightarrow{b_2} A\to \mathrm{HH}_0(A)$$

where $\mathrm{HH}_*(A)$ denotes the Hochschild homology groups and $b_2(x,y)=[x,y]:=xy-yx$ for any $x,y\in A$.

1.3. Free Leibniz algebra. Let V be a $\mathbb{K}$-module and let $\overline{\mathrm{T}}(V):=\bigoplus_{n\geq 1}V^{\otimes n}$ be the reduced tensor module. The bracket defined inductively by

$$[x,v] = x\otimes v,\ \text{ if } x\in\overline{\mathrm{T}}(V) \text{ and } v\in V$$
$$[x,y\otimes v] = [x,y]\otimes v - [x\otimes v, y],\ \text{ if } x,y\in\overline{\mathrm{T}}(V) \text{ and } v\in V,$$

satisfies the Leibniz identity. The Leibniz algebra so defined is the *free Leibniz algebra* over V and is denoted by $\mathcal{F}(V)$ (see [8]). Observe that ones has

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n = [\cdots[[v_1, v_2], v_3]\cdots], \ \forall\ v_1, \cdots, v_n \in V.$$

Moreover, the *free Lie algebra* over V is nothing but the Lie algebra $\mathcal{F}(V)_{Lie}$.

2. Crossed Leibniz algebras

2.1. Leibniz action. Let $\mathfrak{g}$ and $\mathfrak{M}$ be Leibniz algebras. A *Leibniz action* of $\mathfrak{g}$ on $\mathfrak{M}$ is a couple of bilinear maps

$$\mathfrak{g} \times \mathfrak{M} \to \mathfrak{M},\ (g, m) \mapsto {}^{g}m \quad \text{and} \quad \mathfrak{M} \times \mathfrak{g} \to \mathfrak{M},\ (m, g) \mapsto m^{g}$$

satisfying the axioms

i) $m^{[g,g']} = (m^{g})^{g'} - (m^{g'})^{g}$,
ii) ${}^{[g,g']}m = ({}^{g}m)^{g'} - {}^{g}(m^{g'})$,
iii) ${}^{g}({}^{g'}m) = -{}^{g}(m^{g'})$,
iv) ${}^{g}[m, m'] = [{}^{g}m, m'] - [{}^{g}m', m]$,
v) $[m, m']^{g} = [m^{g}, m'] + [m, m'^{g}]$,
vi) $[m, {}^{g}m'] = -[m, m'^{g}]$

for any $m, m' \in \mathfrak{M}$ and $g, g' \in \mathfrak{g}$. We say that $\mathfrak{M}$ is a *Leibniz* $\mathfrak{g}$*-algebra.* Observe that the axiom i) applied to the triples $(m; g, g')$ and $(m; g', g)$ yields the relation

$$m^{[g,g']} = -m^{[g',g]}.$$

2.2. Examples. Any two-sided ideal of a Leibniz algebra $\mathfrak{g}$ is a Leibniz $\mathfrak{g}$-algebra, the action being given by the initial bracket.

A $\mathbb{K}$-module M equipped with two operations of a Leibniz algebra $\mathfrak{g}$ satisfying the axioms i), ii) and iii) is called a *representation of* $\mathfrak{g}$ (see [8]). Therefore representations of a Leibniz algebra $\mathfrak{g}$ are abelian Leibniz $\mathfrak{g}$-algebras.

2.3. Crossed Leibniz algebras. Let $\mathfrak{g}$ be a Leibniz algebra. A *pre-crossed Leibniz* $\mathfrak{g}$*-algebra* is a Leibniz $\mathfrak{g}$-algebra $\mathfrak{M}$ equipped with a morphism of Leibniz algebras $\mu : \mathfrak{M} \to \mathfrak{g}$ such that

$$\mu({}^{g}m) = [g, \mu(m)] \quad \text{and} \quad \mu(m^{g}) = [\mu(m), g]$$

for any $g \in \mathfrak{g}$ and $m \in \mathfrak{M}$. Moreover if the relations

$${}^{\mu(m)}m' = [m, m'] \quad \text{and} \quad m^{\mu(m')} = [m, m'],\ \forall\ m, m' \in \mathfrak{M},$$

hold, then $(\mathfrak{M}, \mu)$ is called a *crossed Leibniz* $\mathfrak{g}$*-algebra.*

2.4. Examples. Any Leibniz algebra $\mathfrak{g}$, equipped with the identity map $\mathrm{id}_{\mathfrak{g}}$, is a crossed Leibniz $\mathfrak{g}$-algebra.

Any two-sided ideal $\mathfrak{h}$ of a Leibniz algebra $\mathfrak{g}$, equipped with the inclusion map $\mathfrak{h} \hookrightarrow \mathfrak{g}$, is a crossed Leibniz $\mathfrak{g}$-algebra.

Let $\alpha : \mathfrak{c} \twoheadrightarrow \mathfrak{g}$ be a central extension of Leibniz algebras (i.e., a surjective morphism whose kernel is contained in the centre of $\mathfrak{c}$, see [4]). Define operations of $\mathfrak{g}$ on $\mathfrak{c}$ by

$${}^{g}c := [\alpha^{-1}(g), c] \quad \text{and} \quad c^{g} := [c, \alpha^{-1}(g)]$$

where $\alpha^{-1}(g)$ is any pre-image of g in $\mathfrak{c}$. Then $(\mathfrak{c}, \alpha)$ is a crossed Leibniz $\mathfrak{g}$-algebra.

Proposition 2.1. *For any pre-crossed Leibniz $\mathfrak{g}$-algebra $(\mathfrak{M},\mu)$, the image* $\mathrm{im}(\mu)$ *(resp. the kernel* $\ker(\mu)$*) is a two-sided ideal in $\mathfrak{g}$ (resp. $\mathfrak{M}$). Moreover, if $(\mathfrak{M},\mu)$ is crossed, then* $\ker(\mu)$ *is contained in the centre of $\mathfrak{M}$.*

Proof. Let m be an element of $\mathfrak{M}$. For any $g \in \mathfrak{g}$, we have

$$[\mu(m), g] = \mu(m^g) \in \mathrm{im}(\mu) \quad \text{and} \quad [g, \mu(m)] = \mu({}^g m) \in \mathrm{im}(\mu).$$

Thus, $\mathrm{im}(\mu)$ is a two-sDided ideal in $\mathfrak{g}$. Assume that $m \in \ker(\mu)$; then for any $m' \in \mathfrak{M}$, we have

$$\mu([m,m']) = [\mu(m), \mu(m')] = 0 = [\mu(m'), \mu(m)] = \mu([m',m]).$$

Therefore $\ker(\mu)$ is a two-sided ideal in $\mathfrak{M}$. Moreover if the Leibniz action of $\mathfrak{g}$ on $\mathfrak{M}$ is crossed, then we have

$$[m,m'] = {}^{\mu(m)}m' = 0 = m'^{\mu(m)} = [m',m]$$

for any $m \in \ker(\mu)$ and $m' \in \mathfrak{M}$. Thus $\ker(\mu)$ is contained in the centre of $\mathfrak{M}$. □

2.5. Morphism of pre-crossed Leibniz algebras. Let $\mathfrak{g}$ be a Leibniz algebra and let $(\mathfrak{M},\mu)$ and $(\mathfrak{N},\nu)$ be pre-crossed Leibniz $\mathfrak{g}$-algebras. A *morphism* from $(\mathfrak{M},\mu)$ to $(\mathfrak{N},\nu)$ is a Leibniz algebra morphism $f : \mathfrak{M} \to \mathfrak{N}$ such that

$$f({}^g m) = {}^g(f(m)),\ f(m^g) = (f(m))^g \text{ and } \mu = \nu f$$

for any $m \in \mathfrak{M}$ and $g \in \mathfrak{g}$. A *morphism of crossed Leibniz* $\mathfrak{g}$*-algebras* is the same as a morphism of pre-crossed Leibniz $\mathfrak{g}$-algebras. It is clear that pre-crossed (resp. crossed) Leibniz $\mathfrak{g}$-algebras and their morphisms form a category that we denote by (**pc-Leib**$(\mathfrak{g})$) (resp. (**c-Leib**$(\mathfrak{g})$)).

Proposition 2.2. *Let $f : (\mathfrak{M},\mu) \to (\mathfrak{N},\nu)$ be a crossed Leibniz $\mathfrak{g}$-algebra morphism. Then $(\mathfrak{M}, f)$ is a crossed Leibniz $\mathfrak{N}$-algebra via the Leibniz action of $\mathfrak{N}$ on $\mathfrak{M}$ given by*

$${}^n m := {}^{\nu(n)}m \quad \text{and} \quad m^n := m^{\nu(n)},\ \forall\ m \in \mathfrak{M}, n \in \mathfrak{N}.$$

Proof. One easily checks that $\mathfrak{M}$ is a Leibniz $\mathfrak{N}$-algebra. For any $m, m' \in \mathfrak{M}$ and $n \in \mathfrak{N}$, we have

$$f({}^n m) = f({}^{\nu(n)}m) = {}^{\nu(n)}f(m) = [n, f(m)],$$
$$f(m^n) = f(m^{\nu(n)}) = f(m)^{\nu(n)} = [f(m), n];$$

thus $(\mathfrak{M}, f)$ is a pre-crossed Leibniz $\mathfrak{N}$-algebra. Moreover we have

$${}^{f(m)}m' = {}^{\nu(f(m))}m' = {}^{\mu(m)}m' = [m,m'],$$
$$m^{f(m')} = m^{\nu(f(m'))} = m^{\mu(m')} = [m,m'];$$

thus $(\mathfrak{M}, f)$ is a crossed Leibniz $\mathfrak{N}$-algebra. □

2.6. Exact sequences. We say that a sequence

$$(\mathfrak{L},\lambda) \overset{\alpha}{\to} (\mathfrak{M},\mu) \overset{\beta}{\to} (\mathfrak{N},\nu)$$

is *exact* in the category (**pc-Leib**$(\mathfrak{g})$) (resp. (**c-Leib**$(\mathfrak{g})$) if the sequence

$$\mathfrak{L} \overset{\alpha}{\to} \mathfrak{M} \overset{\beta}{\to} \mathfrak{N}$$

is exact as sequence of Leibniz algebras.

Proposition 2.3. *If the sequence*

$$(\mathfrak{L}, \lambda) \xrightarrow{\alpha} (\mathfrak{M}, \mu) \xrightarrow{\beta} (\mathfrak{N}, \nu)$$

is exact in the category (**pc-Leib**$(\mathfrak{g})$) *(resp.* (**c-Leib**$(\mathfrak{g})$)*), then the map λ is zero. Moreover if the Leibniz $\mathfrak{g}$-algebra $(\mathfrak{L}, \lambda)$ is crossed, then the Leibniz algebra $\mathfrak{L}$ is abelian.*

Proof. Indeed, since $\beta\alpha = 0$, we have $\lambda = \nu\beta\alpha = 0$. From whence $\ker(\lambda) = \mathfrak{L}$, and by Proposition 2.1, it is clear that the Leibniz algebra $\mathfrak{L}$ is abelian. □

3. Biderivations of Leibniz algebras

In this section, we fix a Leibniz algebra $\mathfrak{g}$.

3.1. Derivations and anti-derivations. Let $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ be pre-crossed Leibniz $\mathfrak{g}$-algebras. A *derivation* from $(\mathfrak{M}, \mu)$ to $(\mathfrak{N}, \nu)$ is a linear map $d : \mathfrak{M} \to \mathfrak{N}$ such that

$$d([m, m']) = d(m)^{\mu(m')} + {}^{\mu(m)}d(m'), \ \forall\ m, m' \in \mathfrak{M}.$$

An *anti-derivation* from $(\mathfrak{M}, \mu)$ to $(\mathfrak{N}, \nu)$ is a linear map $D : \mathfrak{M} \to \mathfrak{N}$ such that

$$D([m, m']) = D(m)^{\mu(m')} - D(m')^{\mu(m)}, \ \forall\ m, m' \in \mathfrak{M}.$$

3.2. Examples. Let $(\mathfrak{N}, \nu)$ be a crossed Leibniz $\mathfrak{g}$-algebra and let n be any element of $\mathfrak{N}$. By the axiom iii) (resp. i)) of 2.1, the linear map

$$\mathfrak{g} \to \mathfrak{N}, \ g \mapsto {}^{g}n \quad (\text{resp. } \mathfrak{g} \to \mathfrak{N}, \ g \mapsto -n^{g})$$

is a derivation (resp. an anti-derivation) from $(\mathfrak{g}, \mathrm{id}_{\mathfrak{g}})$ to $(\mathfrak{N}, \nu)$.

3.3. Biderivations. Let $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ be pre-crossed Leibniz $\mathfrak{g}$-algebras. We denote by $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N})$ the free $\mathbb{K}$-module generated by the triples (d, D, g), where d (resp. D) is a derivation (resp. an anti-derivation) from $(\mathfrak{M}, \mu)$ to $(\mathfrak{N}, \nu)$ and g is an element of $\mathfrak{g}$ such that

$$\nu(d(m)) = \mu(m^{g}), \ \nu(D(m)) = -\mu({}^{g}m),$$
$${}^{h}d(m) = {}^{h}D(m), \ D(m^{h}) = -D({}^{h}m)$$

for any $h \in \mathfrak{g}$ and $m \in \mathfrak{M}$.

Proposition 3.1. *If the Leibniz $\mathfrak{g}$-algebra $(\mathfrak{N}, \nu)$ is crossed, then there is a Leibniz algebra structure on the $\mathbb{K}$-module $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N})$ for the bracket defined by*

$$[(d, D, g), (d', D', g')] := (\delta, \Delta, [g, g'])$$

where

$$\delta(m) := d'(m^{g}) - d(m^{g'}) \quad \textit{and} \quad \Delta(m) = -D(m^{g'}) - d'({}^{g}m), \ \forall\ m \in \mathfrak{M}.$$

Proof. Let us show that the maps δ and Δ are respectively a derivation and an anti-derivation. Indeed, for any $m, m' \in \mathfrak{M}$, we have

$$\begin{aligned}
\delta([m, m']) &= d'([m, m']^{g}) - d([m, m']^{g'}) \\
&= d'([m^{g}, m']) + d'([m, m'^{g}]) - d([m^{g'}, m']) - d([m, m'^{g'}]) \\
&= d'(m^{g})^{\mu(m')} + {}^{\mu(m^{g})}d'(m') + d'(m)^{\mu(m'^{g})} + {}^{\mu(m)}d'(m'^{g}) \\
&\quad - d(m^{g'})^{\mu(m')} - {}^{\mu(m^{g'})}d(m') - d(m)^{\mu(m'^{g'})} - {}^{\mu(m)}d(m'^{g'}) \\
&= (d'(m^{g}) - d(m^{g'}))^{\mu(m')} + {}^{\mu(m)}(d'(m'^{g}) - d(m'^{g'})) + {}^{\nu(d(m))}d'(m') \\
&\quad + d'(m)^{\nu(d(m'))} - {}^{\nu(d'(m))}d(m') - d(m)^{\nu(d'(m'))} \\
&= \delta(m)^{\mu(m')} + {}^{\mu(m)}\delta(m') + [d(m), d'(m')] \\
&\quad + [d'(m), d(m')] - [d'(m), d(m')] - [d(m), d'(m')] \\
&= \delta(m)^{\mu(m')} + {}^{\mu(m)}\delta(m')
\end{aligned}$$

and

$$\begin{aligned}\Delta([m,m']) =& -D([m,m']^{g'}) - d'({}^{g}[m,m']\\ =& -D([m^{g'},m']) - D([m,m'^{g'}]) - d'([{}^{g}m,m']) + d'([{}^{g}m',m])\\ =& -D(m^{g'})^{\mu(m')} + D(m')^{\mu(m^{g'})} - D(m)^{\mu(m'^{g'})} + D(m'^{g'})^{\mu(m)}\\ & -d'({}^{g}m)^{\mu(m')} - {}^{\mu({}^{g}m)}d'(m') + d'({}^{g}m')^{\mu(m)} + {}^{\mu({}^{g}m')}d'(m)\\ =& (-D(m^{g'}) - d'({}^{g}m))^{\mu(m')} - (-D(m'^{g'}) - d'({}^{g}m'))^{\mu(m)} + D(m')^{\nu(d'(m))}\\ & -D(m)^{\nu(d'(m'))} + {}^{\nu(D(m))}d'(m') - {}^{\nu(D(m')))}d'(m)\\ =& \Delta(m)^{\mu(m')} - \Delta(m')^{\mu(m)} + [D(m'),d'(m)]\\ & -[D(m),d'(m')] + [D(m),d'(m')] - [D(m'),d'(m)]\\ =& \Delta(m)^{\mu(m')} - \Delta(m')^{\mu(m)}.\end{aligned}$$

On the other hand, we have

$$\nu(\delta(m)) = \nu(d'(m^{g})) - \nu(d(m^{g'})) = \mu((m^{g})^{g'}) - \mu((m^{g'})^{g}) = \mu(m^{[g,g']}),$$
$$\nu(\Delta(m)) = -\nu(D(m^{g'})) - \nu(d'({}^{g}m)) = \mu({}^{g}(m^{g'})) - \mu(({}^{g}m)^{g'}) = -\mu({}^{[g,g']}m),$$

$$\begin{aligned}{}^{h}\delta(m) =& {}^{h}d'(m^{g}) - {}^{h}d(m^{g'}) = {}^{h}D'(m^{g}) - {}^{h}D(m^{g'})\\ =& -{}^{h}D'({}^{g}m) - {}^{h}D(m^{g'}) = -{}^{h}d'({}^{g}m) - {}^{h}D(m^{g'})\\ =& {}^{h}\Delta(m),\end{aligned}$$

$$\begin{aligned}\Delta({}^{h}m) =& -D(({}^{h}m)^{g'}) - d'({}^{g}({}^{h}m))\\ =& -D({}^{[h,g']}m) - D({}^{h}(m^{g'})) + d'({}^{g}(m^{h}))\\ =& D((m^{h})^{g'}) + d'({}^{g}(m^{h})) = -\Delta(m^{h}).\end{aligned}$$

Therefore the triple $(\delta,\Delta,[g,g'])$ is a biderivation from $(\mathfrak{M},\mu)$ to $(\mathfrak{N},\nu)$. Moreover, let (d,D,g), (d',D',g') and (d'',D'',g'') be biderivations from $(\mathfrak{M},\mu)$ to $(\mathfrak{N},\nu)$. We set

$$\begin{aligned}(\delta,\Delta,[g',g'']) &:= [(d',D',g'),(d'',D'',g'')],\\ (\delta_0,\Delta_0,g_0) &:= [(d,D,g),(\delta,\Delta,[g',g''])],\\ (\delta',\Delta',[g,g']) &:= [(d,D,g),(d',D',g')],\\ (\delta_1,\Delta_1,g_1) &:= [(\delta',\Delta',[g,g']),(d'',D'',g'')],\\ (\delta'',\Delta'',[g,g'']) &:= [(d,D,g),(d'',D'',g'')],\\ (\delta_2,\Delta_2,g_2) &:= [(\delta'',\Delta'',[g,g'']),(d',D',g')].\end{aligned}$$

It is clear that $g_0 = g_1 - g_2$. For any $m \in \mathfrak{M}$, we have

$$\begin{aligned}(\delta_1-\delta_2)(m) =& d''(m^{[g,g']}) - \delta'(m^{g''}) - d'(m^{[g,g'']}) + \delta''(m^{g'})\\ =& d''((m^{g})^{g'}) - d''((m^{g'})^{g}) - d'((m^{g''})^{g}) + d((m^{g''})^{g'})\\ & -d'((m^{g})^{g''}) + d'((m^{g''})^{g}) + d''((m^{g'})^{g}) - d((m^{g'})^{g''})\\ =& d''((m^{g})^{g'}) - d'((m^{g})^{g''}) - d(m^{[g',g'']})\\ =& \delta(m^{g}) - d(m^{[g',g'']}) = \delta_0(m)\end{aligned}$$

and

$$\begin{aligned}(\Delta_1-\Delta_2)(m) &= -\Delta'(m^{g''})-d''({}^{[g,g']}m)+\Delta''(m^{g'})+d'({}^{[g,g'']}m)\\ &= D((m^{g''})^{g'})+d'({}^{g}(m^{g''}))-d''(({}^{g}m)^{g'})+d''({}^{g}(m^{g'}))\\ &\quad -D((m^{g'})^{g''})-d''({}^{g}(m^{g'}))+d'(({}^{g}m)^{g''})-d'({}^{g}(m^{g''}))\\ &= -D(m^{[g',g'']})-d''(({}^{g}m)^{g'})+d'(({}^{g}m)^{g''})\\ &= -D(m^{[g',g'']})-\delta({}^{g}m)=\Delta_0(m).\end{aligned}$$

Therefore the $\mathbb{K}$-module $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M},\mathfrak{N})$ is a Leibniz algebra. □

Let us equip the set $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M},\mathfrak{N})$ with a Leibniz action of $\mathfrak{g}$.

Proposition 3.2. *Let $(\mathfrak{M},\mu)$ (resp. $(\mathfrak{N},\nu)$) be a pre-crossed (resp. crossed) Leibniz $\mathfrak{g}$-algebra. The set $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M},\mathfrak{N})$ is a pre-crossed Leibniz $\mathfrak{g}$-algebra for the operations defined by*

$${}^{h}(d,D,g):=({}^{h}d,{}^{h}D,[h,g])\quad\textit{and}\quad (d,D,g)^{h}:=(d^{h},D^{h},[g,h])$$

where

$$\begin{gathered}({}^{h}d)(m)=d(m^{h})-d(m)^{h},\ \ ({}^{h}D)(m):={}^{h}d(m)-d({}^{h}m),\\ (d^{h})(m):=d(m)^{h}-d(m^{h}),\ \ (D^{h})(m):=D(m)^{h}-D(m^{h}).\end{gathered}$$

Proof. Everything can be smoothly checked and we merely give an example of these verifications. By definition we have

$$\begin{gathered}{}^{h}[(d,D,g),(d',D',g')]=({}^{h}\delta,{}^{h}\Delta,[h,[g,g']]),\\ [{}^{h}(d,D,g),(d',D',g')]=(\delta_1,\Delta_1,[[h,g],g']),\\ [{}^{h}(d',D',g'),(d,D,g)]=(\delta_2,\Delta_2,[[h,g'],g]).\end{gathered}$$

For any $m\in\mathfrak{M}$ we have

$$\begin{aligned}(\delta_1-\delta_2)(m) &= d'(m^{[h,g]})-({}^{h}d)(m^{g'})-d(m^{[h,g']})+({}^{h}d')(m^{g})\\ &= d'((m^{h})^{g})-d'((m^{g})^{h})-d((m^{g'})^{h})+d(m^{g'})^{h}\\ &\quad -d((m^{h})^{g'})+d((m^{g'})^{h})+d'((m^{g})^{h})-d'(m^{g})^{h}\\ &= (d'((m^{h})^{g})-d((m^{h})^{g'}))-(d'(m^{g})-d(m^{g'}))^{h}\\ &= \delta(m^{h})-\delta(m)^{h}=({}^{h}\delta)(m)\end{aligned}$$

and

$$\begin{aligned}(\Delta_1-\Delta_2)(m) &= -({}^{h}D)(m^{g'})-d'({}^{[h,g]}m)+({}^{h}D')(m^{g})+d({}^{[h,g']}m)\\ &= -{}^{h}D(m^{g'})+d({}^{h}(m^{g'}))-d'(({}^{h}m)^{g})+d'({}^{h}(m^{g}))\\ &\quad +{}^{h}D'(m^{g})-d'({}^{h}(m^{g}))+d(({}^{h}m)^{g'})-d({}^{h}(m^{g'}))\\ &= {}^{h}(D'(m^{g})-D(m^{g'}))-(d'(({}^{h}m)^{g})-d(({}^{h}m)^{g'}))\\ &= {}^{h}\delta(m)-\delta({}^{h}m)=({}^{h}\Delta)(m).\end{aligned}$$

Thus we get

$${}^{h}[(d,D,g),(d',D',g')]=[{}^{h}(d,D,g),(d',D',g')]-[{}^{h}(d',D',g'),(d,D,g)].$$ □

Now we can state the fundamental result which is a consequence of Propositions 3.1 and 3.2.

Theorem 3.3. *For any pre-crossed (resp. crossed) Leibniz* $\mathfrak{g}$*-algebra* $(\mathfrak{M},\mu)$ *(resp.* $(\mathfrak{N},\nu)$*), the Leibniz* $\mathfrak{g}$*-algebra* $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M},\mathfrak{N})$ *is pre-crossed for the morphism*

$$\rho : \mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M},\mathfrak{N}) \to \mathfrak{g}, \ \ (d,D,g) \mapsto g. \qquad \square$$

3.4. Remarks. For any element g of $\mathfrak{g}$, the linear map $\mathrm{ad}_g : h \mapsto [h,g]$ (resp. $\mathrm{Ad}_g : h \mapsto -[g,h]$) is a derivation (resp. an anti-derivation) of the Leibniz algebra $\mathfrak{g}$. In the classical sense (i.e., without "crossing", see [7]) the couple $(\mathrm{ad}_g, \mathrm{Ad}_g)$ is called *inner biderivation* of $\mathfrak{g}$. Therefore the pre-crossed Leibniz $\mathfrak{g}$-algebra $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M},\mathfrak{N})$ can be seen as the set of biderivations from $(\mathfrak{M},\mu)$ to $(\mathfrak{N},\nu)$ over inner biderivations of $\mathfrak{g}$.

On the other hand, given a pre-crossed Leibniz $\mathfrak{g}$-algebra $(\mathfrak{M},\mu)$, one easily checks that the map $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M},-)$ is a functor from the category of crossed Leibniz $\mathfrak{g}$-algebras to the category of pre-crossed Leibniz $\mathfrak{g}$-algebras.

4. Non-abelian tensor product of Leibniz algebras

4.1. Leibniz pairings. Let $\mathfrak{M}$ and $\mathfrak{N}$ be Leibniz algebras with mutual Leibniz actions on one another. A *Leibniz pairing* of $\mathfrak{M}$ and $\mathfrak{N}$ is a triple $(\mathfrak{P},h_1,h_2)$ where $\mathfrak{P}$ is a Leibniz algebra and $h_1 : \mathfrak{M}\times\mathfrak{N} \to \mathfrak{P}$ (resp. $h_2 : \mathfrak{N}\times\mathfrak{M} \to \mathfrak{P}$) is a bilinear map such that

$$\begin{aligned}
h_1(m,[n,n']) &= h_1(m^n,n') - h_1(m^{n'},n),\\
h_2(n,[m,m']) &= h_2(n^m,m') - h_2(n^{m'},m),\\
h_1([m,m'],n) &= h_2({}^m n,m') - h_1(m,n^{m'}),\\
h_2([n,n'],m) &= h_1({}^n m,n') - h_2(n,m^{n'}),\\
h_1(m,{}^{m'}n) = -h_1(m,n^{m'}), &\ \ h_2(n,{}^{n'}m) = -h_2(n,m^{n'}),\\
h_1(m^n,{}^{m'}n') &= [h_1(m,n),h_1(m',n')] = h_2({}^m n,m'^{n'}),\\
h_1({}^n m,n'^{m'}) &= [h_2(n,m),h_2(n',m')] = h_2(n^m,{}^{n'}m'),\\
h_1(m^n,n'^{m'}) &= [h_1(m,n),h_2(n',m')] = h_2({}^m n,{}^{n'}m'),\\
h_1({}^n m,{}^{m'}n') &= [h_2(n,m),h_1(m',n')] = h_2(n^m,m'^{n'})
\end{aligned}$$

for any $m,m' \in \mathfrak{M}$ and $n,n' \in \mathfrak{N}$.

4.2. Example. Let $\mathfrak{M}$ and $\mathfrak{N}$ be two-sided ideals of a same Leibniz algebra $\mathfrak{g}$. Take $\mathfrak{P} := \mathfrak{M}\cap\mathfrak{N}$ and define

$$h_1(m,n) := [m,n] \quad \text{and} \quad h_2(n,m) := [n,m].$$

Then the triple $(\mathfrak{P},h_1,h_2)$ is a Leibniz pairing of $\mathfrak{M}$ and $\mathfrak{N}$.

4.3. Non-abelian tensor product. A Leibniz pairing $(\mathfrak{P},h_1,h_2)$ of $\mathfrak{M}$ and $\mathfrak{N}$ is said to be *universal* if for any other Leibniz pairing $(\mathfrak{P}',h_1',h_2')$ of $\mathfrak{M}$ and $\mathfrak{N}$ there exists a unique Leibniz algebra morphism $\theta : \mathfrak{P} \to \mathfrak{P}'$ such that

$$\theta h_1 = h_1' \quad \text{and} \quad \theta h_2 = h_2'.$$

It is clear that a universal pairing, when it exists, is unique up to a unique isomorphism. Here is a construction of the universal pairing as a *non-abelian tensor product*.

Definition-Theorem 4.1. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be Leibniz algebras with mutual Leibniz actions on one another. Let V be the free $\mathbb{K}$-module generated by the symbols $m*n$ and $n*m$ where $m \in \mathfrak{M}$ and $n \in \mathfrak{N}$. Let $\mathfrak{M} \star \mathfrak{N}$ be the Leibniz algebra quotient of the free Leibniz algebra generated by V by the two-sided ideal defined by the relations*

i) $\lambda(m*n) = \lambda m * n = m * \lambda n,\ \ \lambda(n*m) = \lambda n * m = n * \lambda m,$

ii) $(m+m')*n = m*n + m'*n,\ \ (n+n')*m = n*m + n'*m,$
$m*(n+n') = m*n + m*n',\ \ n*(m+m') = n*m + n*m',$

iii) $m*[n,n'] = m^n * n' - m^{n'} * n,\ \ n*[m,m'] = n^m * m' - n^{m'} * m,$
$[m,m']*n = {}^m n * m' - m * n^{m'},\ \ [n,n']*m = {}^n m * n' - n * m^{n'},$

iv) $m * {}^{m'}n = -m * n^{m'},\ \ n * {}^{n'}m = -n * m^{n'},$

v) $m^n * {}^{m'}n' = [m*n, m'*n'] = {}^m n * m'^{n'},$
$m^n * n'^{m'} = [m*n, n'*m'] = {}^m n * {}^{n'}m',$
${}^n m * n'^{m'} = [n*m, n'*m'] = n^m * {}^{n'}m'$
${}^n m * {}^{m'}n' = [n*m, m'*n'] = n^m * m'^{n'}$

for any $\lambda \in \mathbb{K}$, $m, m' \in \mathfrak{M}$, $n, n' \in \mathfrak{N}$. Define maps

$$h_1 : \mathfrak{M} \times \mathfrak{N} \to \mathfrak{M} \star \mathfrak{N},\ \ h_1(m,n) := m*n$$

and

$$h_2 : \mathfrak{N} \times \mathfrak{M} \to \mathfrak{M} \star \mathfrak{N},\ \ h_2(n,m) := n*m.$$

Then the triple $(\mathfrak{M} \star \mathfrak{N}, h_1, h_2)$ is the universal Leibniz pairing of $\mathfrak{M}$ and $\mathfrak{N}$ and called the non-abelian tensor product (or tensor product for short) of $\mathfrak{M}$ and $\mathfrak{N}$.

Proof. It is straightforward to see that the triple $(\mathfrak{M} \star \mathfrak{N}, h_1, h_2)$ so-defined is a Leibniz pairing of $\mathfrak{M}$ and $\mathfrak{N}$. For the universality, notice that if $(\mathfrak{P}, h_1', h_2')$ is another Leibniz pairing of $\mathfrak{M}$ and $\mathfrak{N}$, then the map θ is necessarily given on generators by

$$\theta(m*n) = h_1'(m,n) \quad \text{and} \quad \theta(n*m) = h_2'(n,m)$$

for any $m \in \mathfrak{M}$ and $n \in \mathfrak{N}$. □

As an illustration of this construction, we give now a description of the non-abelian tensor product when the actions are trivial.

Proposition 4.2. *If the Leibniz algebras $\mathfrak{M}$ and $\mathfrak{N}$ act trivially on each other, then there is an isomorphism of abelian Leibniz algebras*

$$\mathfrak{M} \star \mathfrak{N} \cong \mathfrak{M}_{ab} \otimes \mathfrak{N}_{ab}\ \oplus\ \mathfrak{N}_{ab} \otimes \mathfrak{M}_{ab}$$

where $\mathfrak{M}_{ab} := \mathfrak{M}/[\mathfrak{M},\mathfrak{M}]$ and $\mathfrak{N}_{ab} := \mathfrak{N}/[\mathfrak{N},\mathfrak{N}]$.

Proof. Recall that the underlying $\mathbb{K}$-module of the free Leibniz algebra generated by V is

$$\overline{\mathrm{T}}(V) = V\ \oplus\ V^{\otimes 2}\ \oplus\ \cdots\ \oplus\ V^{\otimes n}\ \oplus \cdots$$

Since the actions are trivial, the definition of the bracket on $\overline{\mathrm{T}}(V)$ and the relations *v)* enable us to see that $\mathfrak{M}\star\mathfrak{N}$ is an abelian Leibniz algebra and that the summands $V^{\otimes n}$ (for $n \geq 2$) are killed. Relations i) and ii) of 4.1 say that the $\mathbb{K}$-module $\mathfrak{M}\star\mathfrak{N}$ is the quotient of $\mathfrak{M}\otimes\mathfrak{N}\ \oplus\ \mathfrak{N}\otimes\mathfrak{M}$ by the relations iii). These later imply that $\mathfrak{M}\star\mathfrak{N}$ is the abelian Leibniz algebra $\mathfrak{M}_{ab}\otimes\mathfrak{N}_{ab}\ \oplus\ \mathfrak{N}_{ab}\otimes\mathfrak{M}_{ab}$.
□

4.4. Compatible Leibniz actions. Let $\mathfrak{M}$ and $\mathfrak{N}$ be Leibniz algebras with mutual Leibniz actions on one another. We say that these actions are *compatible* if we have

$$
\begin{aligned}
{}^{({}^m n)}m' &= [m^n, m'], \ {}^{({}^n m)}n' = [n^m, n'],\\
{}^{(n^m)}m' &= [{}^n m, m'], \ {}^{(m^n)}n' = [{}^m n, n'],\\
m^{({}^{m'} n)} &= [m, m'^n], \ n^{({}^{n'} m)} = [n, n'^m],\\
m^{(n^{m'})} &= [m, {}^n m'], \ n^{(m^{n'})} = [n, {}^m n']
\end{aligned}
$$

for any $m, m' \in \mathfrak{M}$ and $n, n' \in \mathfrak{N}$.

4.5. Examples. If $\mathfrak{M}$ and $\mathfrak{N}$ are two-sided ideals of a same Leibniz algebra, then the actions (given by the initial bracket) are compatible.

Let $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ be pre-crossed Leibniz $\mathfrak{g}$-algebras. Then one can define a Leibniz action of $\mathfrak{M}$ on $\mathfrak{N}$ (resp. of $\mathfrak{N}$ on $\mathfrak{M}$) by setting

$$
{}^m n := {}^{\mu(m)}n \quad \text{and} \quad n^m := n^{\mu(m)}
$$
$$
(\text{resp. } {}^n m := {}^{\nu(n)}m \quad \text{and} \quad m^n := m^{\nu(n)}).
$$

If the Leibniz $\mathfrak{g}$-algebras $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ are crossed, then these Leibniz actions are compatible.

4.6. First crossed structure. Let $\mathfrak{M}$ and $\mathfrak{N}$ be Leibniz algebras with mutual compatible actions on one another. Consider the operations of $\mathfrak{M}$ on $\mathfrak{M} \star \mathfrak{N}$ given by

$$
\begin{aligned}
{}^m(m' * n') &:= [m, m'] * n' - {}^m n' * m', \ {}^m(n' * m') := {}^m n' * m' - [m, m'] * n',\\
(m * n)^{m'} &:= [m, m'] * n + m * n^{m'}, \ (n * m)^{m'} := n^{m'} * m + n * [m, m']
\end{aligned}
$$

and those of $\mathfrak{N}$ on $\mathfrak{M} \star \mathfrak{N}$ given by

$$
\begin{aligned}
{}^n(m' * n') &:= {}^n m' * n' - [n, n'] * m', \ {}^n(n' * m') := [n, n'] * m' - {}^n m' * n',\\
(m * n)^{n'} &:= m^{n'} * n + m * [n, n'], \ (n * m)^{n'} := [n, n'] * m + n * m^{n'}
\end{aligned}
$$

for any $m, m' \in \mathfrak{M}$ and $n, n' \in \mathfrak{N}$. Then we have

Proposition 4.3. *With the above operations, the map*

$$
\mu : \mathfrak{M} \star \mathfrak{N} \to \mathfrak{M}, \ m * n \mapsto m^n, \ n * m \mapsto {}^n m
$$
$$
(\textit{resp. } \nu : \mathfrak{M} \star \mathfrak{N} \to \mathfrak{N}, \ m * n \mapsto {}^m n, \ n * m \mapsto n^m)
$$

induces on $\mathfrak{M} \star \mathfrak{N}$ a structure of crossed Leibniz $\mathfrak{M}$-algebra (resp. $\mathfrak{N}$-algebra).

Proof. Once again everything can be readily checked thanks to the compatibility conditions. For example we have

$$
\begin{aligned}
{}^{\mu(m*n)}(m' * n') &= {}^{m^n}(m' * n') = [m^n, m'] * n' - {}^{(m^n)}n' * m'\\
&= {}^{(m^n)}n' * m' - m^n * n'^{m'} - {}^{(m^n)}n' * m'\\
&= m^n * {}^{m'}n = [m * n, m' * n']
\end{aligned}
$$

for any $m, m' \in \mathfrak{M}$ and $n, n' \in \mathfrak{N}$. □

4.7. Second crossed structure. Let $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ be pre-crossed Leibniz $\mathfrak{g}$-algebras, equipped with the mutual Leibniz actions given in Examples 4.5. One easily checks that the operations given by

$$
\begin{aligned}
{}^g(m * n) &:= {}^g m * n - {}^g n * m, \ {}^g(n * m) := {}^g n * m - {}^g m * n,\\
(m * n)^g &:= m^g * n + m * n^g, \ (n * m)^g := n^g * m + n * m^g,
\end{aligned}
$$

define a Leibniz action of $\mathfrak{g}$ on $\mathfrak{M} \star \mathfrak{N}$.

Proposition 4.4. *Let* $(\mathfrak{M},\mu)$ *and* $(\mathfrak{N},\nu)$ *be pre-crossed Leibniz* $\mathfrak{g}$*-algebras. Then the map* $\eta:\mathfrak{M}\star\mathfrak{N}\to\mathfrak{g}$ *defined on generators by*

$$\eta(m*n):=[\mu(m),\nu(n)] \quad \textit{and} \quad \eta(n*m):=[\nu(n),\mu(m)],$$

conferes to $\mathfrak{M}\star\mathfrak{N}$ *a structure of pre-crossed Leibniz* $\mathfrak{g}$*-algebra. Moreover, if one of the Leibniz* $\mathfrak{g}$*-algebras* $\mathfrak{M}$ *or* $\mathfrak{N}$ *is crossed, then the Leibniz* $\mathfrak{g}$*-algebra* $\mathfrak{M}\star\mathfrak{N}$ *is crossed.*

Proof. It is immediate to check that the map η passes to the quotient and defines a Leibniz algebra morphism. Moreover we have

$$\begin{aligned}
\eta({}^g(m*n)) &= [\mu({}^gm),\nu(n)]-[\nu({}^gn),\mu(m)]\\
&= [[g,\mu(m)],\nu(n)]-[[g,\nu(n)],\mu(m)]\\
&= [g,[\mu(m),\nu(n)]]=[g,\eta(m*n)];\\
\eta({}^g(n*m)) &= -\eta({}^g(m*n))=-[g,\eta(m*n)]\\
&= -[g,[\mu(m),\nu(n)]]=[g,[\nu(n),\mu(m)]]=[g,\eta(n*m)];\\
\eta((m*n)^g) &= [\mu(m^g),\nu(n)]+[\mu(m),\nu(n^g)]\\
&= [[\mu(m),g],\nu(n)]+[\mu(m),[\nu(n),g]]\\
&= [[\mu(m),\nu(n)],g]=[\eta(m*n),g];\\
\eta((n*m)^g) &= [\nu(n^g),\mu(m)]+[\nu(n),\mu(m^g)]\\
&= [[\nu(n),g],\mu(m)]+[\nu(n),[\mu(m),g]]\\
&= [[\nu(n),\mu(m)],g]=[\eta(n*m),g];
\end{aligned}$$

thus $(\mathfrak{M}\star\mathfrak{N},\eta)$ is a pre-crossed Leibniz $\mathfrak{g}$-algebra. Assume that, for instance, the Leibniz $\mathfrak{g}$-algebra $\mathfrak{M}$ is crossed. Then we have

$$\begin{aligned}
{}^{\eta(m*n)}(m'*n') &= {}^{[\mu(m),\nu(n)]}(m'*n')={}^{\mu(m^{\nu(n)})}(m'*n')\\
&= {}^{\mu(m^{\nu(n)})}m'*n'-{}^{\mu(m^{\nu(n)})}n'*m'\\
&= [m^{\nu(n)},m']*n'-{}^{\mu(m^{\nu(n)})}n'*m'\\
&= {}^{\mu(m^{\nu(n)})}n'*m'-m^{\nu(n)}*n'^{\mu(m')}-{}^{\mu(m^{\nu(n)})}n'*m'\\
&= m^{\nu(n)}*{}^{\mu(m')}n'=[m*n,m'*n']
\end{aligned}$$

and

$$\begin{aligned}
(m*n)^{\eta(m'*n')} &= (m*n)^{[\mu(m'),\nu(n')]}=(m*n)^{\mu(m'^{\nu(n')})}\\
&= m^{\mu(m'^{\nu(n')})}*n+m*n^{\mu(m'^{\nu(n')})}\\
&= [m,m'^{\nu(n')}]*n+m*n^{\mu(m'^{\nu(n')})}\\
&= {}^{\mu(m)}n*m'^{\nu(n')}-m*n^{\mu(m'^{\nu(n')})}+m*n^{\mu(m'^{\nu(n')})}\\
&= [m*n,m'*n'].
\end{aligned}$$

By the same way, one easily gets

$$\begin{aligned}
{}^{\eta(m*n)}(n'*m') &= [m*n,n'*m'],\ (m*n)^{\eta(n'*m')}=[m*n,n'*m'],\\
{}^{\eta(n*m)}(n'*m') &= [n*m,n'*m'],\ (n*m)^{\eta(n'*m')}=[n*m,n'*m'],\\
{}^{\eta(n*m)}(m'*n') &= [n*m,m'*n'],\ (n*m)^{\eta(m'*n')}=[n*m,m'*n'].
\end{aligned}$$

So we have proved that the Leibniz $\mathfrak{g}$-algebra $\mathfrak{M}\star\mathfrak{N}$ is crossed. □

4.8. Remark. It is clear that if $(\mathfrak{M},\mu)$ (resp. $(\mathfrak{N},\nu)$) is a crossed Leibniz $\mathfrak{g}$-algebra, then the map $\mathfrak{M}\star -$ (resp. $-\star\mathfrak{N}$) is a functor from the category of pre-crossed Leibniz $\mathfrak{g}$-algebras to the category of crossed Leibniz $\mathfrak{g}$-algebras.

Proposition 4.5. *Let* $(\mathfrak{N}, \nu)$ *be a crossed Leibniz* $\mathfrak{g}$*-algebra. The functor* $F(-) := - \star \mathfrak{N}$ *is a right exact functor from the category of pre-crossed Leibniz* $\mathfrak{g}$*-algebras to the category of crossed Leibniz* $\mathfrak{g}$*-algebras.*

Proof. Taking into account Proposition 2.3, let

$$0 \to (\mathfrak{P}, 0) \xrightarrow{f} (\mathfrak{Q}, \lambda) \xrightarrow{g} (\mathfrak{R}, \gamma) \to 0$$

be an exact sequence of pre-crossed Leibniz $\mathfrak{g}$-algebras. Consider the sequence of Leibniz algebras

$$F(\mathfrak{P}) \xrightarrow{F(f)} F(\mathfrak{Q}) \xrightarrow{F(g)} F(\mathfrak{R}) \to 0.$$

It is clear that the morphism $F(g)$ is surjective. Since the map $F(f)$ is a morphism of crossed Leibniz $\mathfrak{g}$-algebras, by Proposition 2.2, $(F(\mathfrak{P}), F(f))$ is a crossed Leibniz $F(\mathfrak{Q})$-algebra; and by Proposition 2.1, the image $\mathrm{im}F(f)$ is a two-sided ideal in $F(\mathfrak{Q})$. By composition we have $F(g)F(f) = F(gf) = 0$, which yields a factorisation

$$\overline{F(g)} : F(\mathfrak{Q})/\mathrm{im}F(f) \to F(\mathfrak{R}).$$

In fact, the morphism $\overline{F(g)}$ is an isomorphism. To see it, let us consider the map

$$\Gamma : F(\mathfrak{R}) \to F(\mathfrak{Q})/\mathrm{im}F(f)$$

given on generators by

$$\Gamma(r * n) := g^{-1}(r) * n \ \mathrm{mod}\ \mathrm{im}F(f) \quad \text{and} \quad \Gamma(n * r) := n * g^{-1}(r) \ \mathrm{mod}\ \mathrm{im}F(f)$$

where $g^{-1}(r)$ is any pre-image of r in $\mathfrak{Q}$. Indeed, if q and q' are two pre-images of r, then $q - q' = f(p)$ for some p in $\mathfrak{P}$. Therefore we have

$$q * m - q' * n = (q - q') * n = f(p) * n = F(f)(p * n) \in \mathrm{im}F(f),$$
$$n * q - n * q' = n * (q - q') = n * f(p) = F(f)(n * p) \in \mathrm{im}F(f);$$

thus the map Γ is well-defined. One easily checks that Γ is a morphism of Leibniz algebras and inverse to $\overline{F(g)}$. $\square$

5. Adjunction theorem

In this section we show that, for any crossed Leibniz $\mathfrak{g}$-algebra $(\mathfrak{N}, \nu)$, the functor $- \star \mathfrak{N}$ is left adjoint to the functor $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{N}, -)$. For technical reasons, we assume that the relations

iv) $$m * {}^{\mu(m')}n = -m * n^{\mu(m')}, \ n * {}^{\nu(n')}m = -n * m^{\nu(n')}$$

defining the tensor product $\mathfrak{M} \star \mathfrak{N}$ are extended to the relations

iv)' $$m * {}^{g}n = -m * n^{g}, \ n * {}^{g}m = -n * m^{g}$$

for any $m, m' \in \mathfrak{M}$, $n, n' \in \mathfrak{N}$ and $g \in \mathfrak{g}$. To avoid confusion, we denote this later tensor product by $\mathfrak{M} \star_{\mathfrak{g}} \mathfrak{N}$. For instance, the Leibniz $\mathfrak{g}$-algebras $\mathfrak{M} \star \mathfrak{N}$ and $\mathfrak{M} \star_{\mathfrak{g}} \mathfrak{N}$ coincide if the maps μ and ν are surjective.

Theorem 5.1. *Let $(\mathfrak{M},\mu)$ be a pre-crossed Leibniz $\mathfrak{g}$-algebra and let $(\mathfrak{N},\nu)$ and $(\mathfrak{P},\lambda)$ be crossed Leibniz $\mathfrak{g}$-algebras. There is an isomorphism of $\mathbb{K}$-modules*

$$\mathrm{Hom}_{(\mathbf{pc\text{-}Leib}(\mathfrak{g}))}(\mathfrak{M},\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{N},\mathfrak{P}))\cong \mathrm{Hom}_{(\mathbf{c\text{-}Leib}(\mathfrak{g}))}(\mathfrak{M}\star_{\mathfrak{g}}\mathfrak{N},\mathfrak{P}).$$

Proof. Let $\phi\in \mathrm{Hom}_{(\mathbf{pc\text{-}Leib}(\mathfrak{g}))}(\mathfrak{M},\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{N},\mathfrak{P}))$ and put $(d_m,D_m,g_m):=\phi(m)$ for $m\in\mathfrak{M}$. Notice that we have $g_m=\mu(m)$ thanks to the relation $\rho\phi=\mu$, where $\rho:\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{N},\mathfrak{P})\to\mathfrak{g}$ is the crossing morphism. We associate to ϕ the map $\Phi:\mathfrak{M}\star_{\mathfrak{g}}\mathfrak{N}\to\mathfrak{P}$ defined on generators by

$$\Phi(m*n):=-D_m(n)\quad\text{and}\quad \Phi(n*m):=d_m(n),\ \forall\ m\in\mathfrak{M}, n\in\mathfrak{N}.$$

Lemma 5.2. *The map Φ is a morphism of crossed Leibniz $\mathfrak{g}$-algebras.*

Conversely, given an element $\sigma\in\mathrm{Hom}_{(\mathbf{c\text{-}Leib}(\mathfrak{g}))}(\mathfrak{M}\star_{\mathfrak{g}}\mathfrak{N},\mathfrak{P})$, we associate the map $\Sigma:\mathfrak{M}\to \mathrm{Bider}_{\mathfrak{g}}(\mathfrak{N},\mathfrak{P})$ defined by

$$\Sigma(m):=(\delta_m,\Delta_m,\mu(m)),\ \forall\ m\in\mathfrak{M},$$

where

$$\delta_m(n):=\sigma(n*m)\quad\text{and}\quad \Delta_m(n):=-\sigma(m*n),\ \forall\ n\in\mathfrak{N}.$$

Lemma 5.3. *The map Σ is a morphism of pre-crossed Leibniz $\mathfrak{g}$-algebras.*

It is clear that the maps $\phi\mapsto\Phi$ and $\sigma\mapsto\Sigma$ are inverse to each other, which proves the adjunction theorem. $\square$

Proof of Lemma 5.2. There is a lot of things to check in order to show that the map Φ is well-defined. Let us give some examples of these verifications. For any $m,m'\in\mathfrak{M}$, $n,n'\in\mathfrak{N}$ and $h\in\mathfrak{g}$, we have

$$\begin{aligned}
\Phi({}^{n}m*n'-n*m^{n'}) &= -D_{{}^{\nu(n)}m}(n')-d_{m^{\nu(n')}}(n)\\
&= -({}^{\nu(n)}D_m)(n')-((d_m)^{\nu(n')})(n)\\
&= -{}^{\nu(n)}D_m(n')+d_m({}^{\nu(n)}n')-d_m(n)^{\nu(n')}+d_m(n^{\nu(n')})\\
&= -{}^{\nu(n)}d_m(n')+d_m([n,n'])-d_m(n)^{\nu(n')}+d_m([n,n'])\\
&= d_m([n,n'])=\Phi([n,n']*m).
\end{aligned}$$

We also compute

$$\begin{aligned}
\Phi(m*{}^{h}n)&=-D_m({}^{h}n)=D_m(n^{h})=-\Phi(m*n^{h}),\\
\Phi(n*{}^{h}m)&=d_{{}^{h}m}(n)=({}^{h}d_m)(n)=-((d_m)^{h})(n)=-d_{m^{h}}(n)=-\Phi(n*m^{h})
\end{aligned}$$

and

$$\begin{aligned}
\Phi(m^{n}*{}^{m'}n') &= -D_{m^{\nu(n)}}({}^{\mu(m')}n')=-((D_m)^{\nu(n)})({}^{\mu(m')}n')\\
&= -D_m({}^{\mu(m')}n')^{\nu(n)}+D_m(({}^{\mu(m')}n')^{\nu(n)})\\
&= -D_m({}^{\mu(m')}n')^{\nu(n)}+D_m([{}^{\mu(m')}n',n])\\
&= -D_m(n)^{\nu({}^{\mu(m')}n')}=D_m(n)^{\lambda(D_{m'}(n'))}\\
&=[D_m(n),D_{m'}(n')]=[\Phi(m*n),\Phi(m'*n')]=\Phi([m*n,m'*n']).
\end{aligned}$$

Now let $m\in\mathfrak{M}$, $n\in\mathfrak{N}$ and $g\in\mathfrak{g}$. One has successively

$$\begin{aligned}
\Phi({}^{g}(m*n))&=\Phi({}^{g}m*n)-\Phi({}^{g}n*m)=-D_{{}^{g}m}(n)-d_m({}^{g}n)\\
&=({}^{g}D_m)(n)-d_m({}^{g}n)=-{}^{g}D_m(n)={}^{g}\Phi(m*n),
\end{aligned}$$

$$\Phi({}^g(n*m)) = -\Phi({}^g(m*n)) = -{}^g\Phi(m*n) = {}^gD_m(n) = {}^gd_m(n) = {}^g\Phi(n*m),$$

$$\begin{aligned}\Phi((m*n)^g) &= \Phi(m^g*n) + \Phi(m*n^g) = -D_{m^g}(n) - D_m(n^g)\\ &= -((D_m)^g)(n) - D_m(n^g) = -D_m(n)^g = \Phi(m*n)^g,\end{aligned}$$

$$\begin{aligned}\Phi((n*m)^g) &= \Phi(n^g*m) + \Phi(n*m^g) = d_m(n^g) + d_{m^g}(n)\\ &= d_m(n^g) + ((d_m)^g)(n) = d_m(n)^g = \Phi(n*m)^g;\end{aligned}$$

$$\begin{aligned}\lambda\Phi(m*n) &= -\lambda(D_m(n)) = \nu({}^{\mu(m)}n) = [\mu(m),\nu(n)] = \eta(m*n),\\ \lambda\Phi(n*m) &= \lambda(d_m(n)) = \nu(n^{\mu(m)}) = [\nu(n),\mu(m)] = \eta(n*m).\end{aligned}$$

Therefore the map Φ is a morphism of crossed Leibniz $\mathfrak{g}$-algebras. □

Proof of Lemma 5.3. Let us first show that $\Sigma(m)$ is a well-defined biderivation. For any $n, n' \in \mathfrak{N}$, we have

$$\begin{aligned}&\delta_m(n)^{\nu(n')} + {}^{\nu(n)}\delta_m(n')\\ &= \sigma(n*m)^{\nu(n')} + {}^{\nu(n)}\sigma(n'*m) = \sigma((n*m)^{\nu(n')}) + \sigma({}^{\nu(n)}(n'*m))\\ &= \sigma(n^{\nu(n')}*m) + \sigma(n*m^{\nu(n')}) + \sigma({}^{\nu(n)}n'*m) - \sigma({}^{\nu(n')}m*n')\\ &= 2\sigma([n,n']*m) - \sigma({}^{\nu(n)}m*n' - n*m^{\nu(n')})\\ &= 2\sigma([n,n']*m) - \sigma([n,n']*m) = \sigma([n,n']*m) = \delta_m([n,n']),\end{aligned}$$

thus δ_m is a derivation. Moreover, we have

$$\begin{aligned}&\Delta_m(n)^{\nu(n')} - \Delta_m(n')^{\nu(n)}\\ &= -\sigma(m*n)^{\nu(n')} + \sigma(m*n')^{\nu(n)} = \sigma((m*n')^{\nu(n)}) - \sigma((m*n)^{\nu(n')})\\ &= \sigma(m^{\nu(n)}*n') + \sigma(m*n'^{\nu(n)}) - \sigma(m^{\nu(n')}*n) - \sigma(m*n^{\nu(n')})\\ &= \sigma(m^{\nu(n)}*n' - m^{\nu(n')}*n) - \sigma(m*{}^{\nu(n)}n') - \sigma(m*n^{\nu(n')})\\ &= \sigma(m*[n,n']) - \sigma(m*[n,n']) - \sigma(m*[n,n'])\\ &= -\sigma(m*[n,n']) = \Delta_m([n,n']),\end{aligned}$$

thus Δ_m is an anti-derivation. We have also

$$\begin{aligned}\lambda(\delta_m(n)) &= \lambda(\sigma(n*m)) = \eta(n*m) = [\nu(n),\mu(m)] = \nu(n^{\mu(m)}),\\ \lambda(\Delta_m(n)) &= -\lambda(\sigma(m*n)) = -\eta(m*n) = -[\mu(m),\nu(n)] = -\nu({}^{\mu(m)}n),\\ {}^h\delta_m(n) &= {}^h\sigma(n*m) = \sigma({}^h(n*m)) = -\sigma({}^h(m*n)) = -{}^h\sigma(m*n) = -{}^h\Delta_m(n),\\ \Delta_m({}^hn) &= -\sigma(m*{}^hn) = \sigma(m*n^h) = -\Delta_m(n^h).\end{aligned}$$

Therefore $\Sigma(m) = (\delta_m, \Delta_m, \mu(m))$ is a biderivation from $(\mathfrak{N},\nu)$ to $(\mathfrak{P},\lambda)$.

For any $h \in \mathfrak{g}$, $m \in \mathfrak{M}$ and $n \in \mathfrak{N}$, we have

$$\begin{aligned}({}^h(\delta_m))(n) &= \delta_m(n^h) - \delta_m(n)^h = \sigma(n^h*m) - \sigma(n*m)^h\\ &= -\sigma(n*m^h) = \sigma(n*{}^hm) = \delta_{{}^hm}(n),\end{aligned}$$

$$\begin{aligned}({}^h(\Delta_m))(n) &= {}^h\Delta_m(n) - \delta_m({}^hn) = {}^h\sigma(m*n) - \sigma({}^hn*m)\\ &= \sigma({}^hm*n) = \Delta_{{}^hm}(n);\end{aligned}$$

and obviously $[h, \mu(m)] = \mu({}^h m)$, thus we have $\Sigma({}^h m) = {}^h\Sigma(m)$. On the other side, we have

$$\begin{aligned}((\delta_m)^h)(n) &= \delta_m(n)^h - \delta_m(n^h) = \sigma(n * m)^h - \sigma(n^h * m)\\ &= \sigma(n * m^h) = \delta_{m^h}(n)\end{aligned}$$

and

$$\begin{aligned}((\Delta_m)^h)(n) &= \Delta_m(n)^h - \delta_m(n^h) = -\sigma(m * n)^h + \sigma(m * n^h)\\ &= -\sigma(m^h * n) = \Delta_{m^h}(n).\end{aligned}$$

Since $[\mu(m), h] = \mu(m^h)$, we get $\Sigma(m^h) = \Sigma(m)^h$. By definition of the map Σ, we have $\rho\Sigma(m) = \mu(m)$. Therefore the map Σ is a morphism of pre-crossed Leibniz $\mathfrak{g}$-algebras. □

6. Cohomological characterizations

6.1. Non-abelian Leibniz cohomology. Let $\mathfrak{g}$ be a Leibniz algebra viewed as the crossed Leibniz $\mathfrak{g}$-algebra $(\mathfrak{g}, \mathrm{id}_{\mathfrak{g}})$, and let $(\mathfrak{M}, \mu)$ be a crossed Leibniz $\mathfrak{g}$-algebra. Given an element $m \in \mathfrak{M}$, we denote by d_m (resp. D_m) the derivation (resp. anti-derivation) $g \mapsto {}^g m$ (resp. $g \mapsto -m^g$) from $(\mathfrak{g}, \mathrm{id}_{\mathfrak{g}})$ to $(\mathfrak{M}, \mu)$, and by $\overline{\mu(m)} := \mu(m) \bmod Z(\mathfrak{g})$, where $Z(\mathfrak{g})$ is the centre of $\mathfrak{g}$. One easily checks that the triple $(d_m, D_m, \overline{\mu(m)})$ is a well-defined element of $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{M})$.

Definition-Proposition 6.1. *Let $\mathfrak{J}$ be the $\mathbb{K}$-module freely generated by the biderivations $(d_m, D_m, \overline{\mu(m)})$, $m \in \mathfrak{M}$. Then $\mathfrak{J}$ is a two-sided ideal of* $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{M})$. *The Leibniz algebra* $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{M})/\mathfrak{J}$ *is denoted by* $\mathfrak{HL}^1(\mathfrak{g}, \mathfrak{M})$.

Proof. For any $m \in \mathfrak{M}$ and $(d, D, g) \in \mathrm{Bider}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{M})$, we have

$$[(d, D, g), (d_m, D_m, \overline{\mu(m)})] = (\delta_m, \Delta_m, [g, \overline{\mu(m)}])$$

with

$$\begin{aligned}\delta_m(x) &= d_m([x, g]) - d([x, \overline{\mu(m)}) = {}^{[x,g]}m - d([x, \mu(m)])\\ &= {}^{\mu(d(x))}m - d(x)^{\mu(m)} - {}^x d(\mu(m))\\ &= [d(x), m] - [d(x), m] - {}^x D(\mu(m))\\ &= d_{m_1}(x)\end{aligned}$$

where $m_1 := -D(\mu(m))$,

$$\begin{aligned}\Delta_m(x) &= -D([x, \overline{\mu(m)}]) - d_m([g, x]) = -D([x, \mu(m)]) - {}^{[g,x]}m\\ &= -D(x)^{\mu(m)} - D(\mu(m))^x + {}^{\mu(D(x))}m\\ &= -[D(x), m] + D(\mu(m))^x + [D(x), m]\\ &= D_{m_1}(x),\end{aligned}$$

$$\mu(m_1) = -\mu(D(\mu(m))) = [g, \mu(m)] = [g, \overline{\mu(m)}];$$

thus we have $[(d, D, g), (d_m, D_m, \overline{\mu(m)})] \in \mathfrak{J}$. On the other side, we have

$$[(d_m, D_m, \overline{\mu(m)}), (d, D, g)] = (\delta'_m, \Delta'_m, [\overline{\mu(m)}, g])$$

with

$$\begin{aligned}\delta'_m(x) &= d([x, \overline{\mu(m)}]) - d_m([x, g]) = d([x, \mu(m)]) - {}^{[x,g]}m\\ &= d(x)^{\mu(m)} + {}^x d(\mu(m)) - {}^{\mu(d(x))}m\\ &= [d(x), m] + {}^x d(\mu(m)) - [d(x), m]\\ &= d_{m_2}(x)\end{aligned}$$

where $m_2 := d(\mu(m))$,

$$\begin{aligned}\Delta'_m(x) &= -D_m([x,g]) - d([\overline{\mu(m)},x]) = m^{[x,g]} - d([\mu(m),x])\\ &= m^{\mu(d(x))} - d(\mu(m))^x - {}^{\mu(m)}d(x)\\ &= [m,d(x)] - d(\mu(m))^x - [m,d(x)]\\ &= D_{m_2}(x),\end{aligned}$$

$$\mu(m_2) = \mu(d(\mu(m))) = [\mu(m),g] = [\overline{\mu(m)},g];$$

thus we have $[(d_m, D_m, \overline{\mu(m)}), (d, D, g)] \in \mathfrak{J}$. Therefore the set $\mathfrak{J}$ is a two-sided ideal of $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{M})$. □

Similarly, given a crossed Leibniz $\mathfrak{g}$-algebra $(\mathfrak{M}, \mu)$, one defines

$$\mathfrak{HL}^0(\mathfrak{g}, \mathfrak{M}) := \{m \in \mathfrak{M} : {}^g m = m^g = 0,\ \forall\ g \in \mathfrak{g}\}$$

that is, the set of invariant elements of $\mathfrak{M}$. From the relations

$$[m, m'] = m^{\mu(m')} = 0 = {}^{\mu(m')}m = [m', m],\ \ m \in \mathfrak{HL}^0(\mathfrak{g}, \mathfrak{M}),\ \ m' \in \mathfrak{M},$$

it is clear that $\mathfrak{HL}^0(\mathfrak{g}, \mathfrak{M})$ is contained in the centre of the Leibniz algebra $\mathfrak{M}$.

Proposition 6.2. *For any exact sequence of crossed Leibniz* $\mathfrak{g}$*-algebras*

$$0 \to (\mathfrak{A}, 0) \xrightarrow{\alpha} (\mathfrak{B}, \lambda) \xrightarrow{\beta} (\mathfrak{C}, \mu) \to 0,$$

there exists an exact sequence of $\mathbb{K}$*-modules*

$$0 \to \mathfrak{HL}^0(\mathfrak{g}, \mathfrak{A}) \to \mathfrak{HL}^0(\mathfrak{g}, \mathfrak{B}) \to \mathfrak{HL}^0(\mathfrak{g}, \mathfrak{C}) \xrightarrow{\partial} \mathfrak{HL}^1(\mathfrak{g}, \mathfrak{A}) \to \mathfrak{HL}^1(\mathfrak{g}, \mathfrak{B}) \xrightarrow{\beta^1} \mathfrak{HL}^1(\mathfrak{g}, \mathfrak{C})$$

where β^1 *is a Leibniz algebra morphism.*

Proof. Everything goes smoothly except the definition of the connecting homomorphism ∂. Given an element $c \in \mathfrak{HL}^0(\mathfrak{g}, \mathfrak{C})$, let $b \in \mathfrak{B}$ be any pre-image of c in $\mathfrak{B}$. For any $x \in \mathfrak{g}$, we have

$$\beta({}^x b) = {}^x c = 0 = c^x = \beta(b^x).$$

Thus the element ${}^x b$ (resp. b^x) is in $\ker(\beta) = \mathrm{im}(\alpha)$. Since the morphism α is injective, the map $d^c : x \mapsto \alpha^{-1}({}^x b)$ (resp. $D^c : x \mapsto \alpha^{-1}(b^x)$) is a derivation (resp. an anti-derivation) from $(\mathfrak{g}, \mathrm{id}_{\mathfrak{g}})$ to $(\mathfrak{A}, 0)$. One easily checks that the triple $(d^c, D^c, 0)$ is a well-defined element of $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{A})$ whose class in $\mathfrak{HL}^1(\mathfrak{g}, \mathfrak{A})$ does not depend on the choice of the pre-image b. We put

$$\partial(c) := \mathrm{class}(d^c, D^c, 0).$$

□

6.2. Non-abelian Leibniz homology. Let $\mathfrak{g}$ be a Leibniz algebra viewed as the crossed Leibniz $\mathfrak{g}$-algebra $(\mathfrak{g}, \mathrm{id}_{\mathfrak{g}})$, and let $(\mathfrak{N}, \nu)$ be a crossed Leibniz $\mathfrak{g}$-algebra.

Definition-Proposition 6.3. *The map* $\Psi_{\mathfrak{N}} : \mathfrak{N} \star \mathfrak{g} \to \mathfrak{N}$ *given on generators by*

$$\Psi_{\mathfrak{N}}(n * g) := n^g \quad \textit{and} \quad \Psi_{\mathfrak{N}}(g * n) := {}^g n,\ \ g \in \mathfrak{g},\ \ n \in \mathfrak{N},$$

is a morphism of crossed Leibniz $\mathfrak{g}$*-algebras. We define the low-degrees non-abelian homology of* $\mathfrak{g}$ *with coefficients in* $\mathfrak{N}$ *to be*

$$\mathfrak{HL}_0(\mathfrak{g}, \mathfrak{N}) := \mathrm{coker}\Psi_{\mathfrak{N}} \quad \textit{and} \quad \mathfrak{HL}_1(\mathfrak{g}, \mathfrak{N}) := \ker \Psi_{\mathfrak{N}}.$$

Proof. To see that the map $\Psi_{\mathfrak{N}}$ is a Leibniz algebra morphism is equivalent to the fact that the Leibniz action of $\mathfrak{N}$ on $\mathfrak{g}$ is well-defined. The definition of the crossing homomorphism $\eta_{\mathfrak{N}} : \mathfrak{N} \star \mathfrak{g} \to \mathfrak{g}$ implies that $\Psi_{\mathfrak{N}}$ is a morphism of crossed Leibniz $\mathfrak{g}$-algebras. □

Proposition 6.4. *For any exact sequence of crossed Leibniz* $\mathfrak{g}$*-algebras*

$$0 \to (\mathfrak{A}, 0) \stackrel{\alpha}{\to} (\mathfrak{B}, \lambda) \stackrel{\beta}{\to} (\mathfrak{C}, \mu) \to 0,$$

there exists an exact sequence of $\mathbb{K}$*-modules*

$$\mathfrak{HL}_1(\mathfrak{g}, \mathfrak{A}) \to \mathfrak{HL}_1(\mathfrak{g}, \mathfrak{B}) \to \mathfrak{HL}_1(\mathfrak{g}, \mathfrak{C}) \stackrel{\partial}{\to} \mathfrak{HL}_0(\mathfrak{g}, \mathfrak{A}) \to \mathfrak{HL}_0(\mathfrak{g}, \mathfrak{B}) \to \mathfrak{HL}_0(\mathfrak{g}, \mathfrak{C}) \to 0.$$

Proof. We know that the functor $- \star \mathfrak{g}$ is right exact (Proposition 4.5). Therefore Proposition 6.4 is nothing but the "snake-lemma" applied to diagram

$$\begin{array}{ccccccccc} & & \mathfrak{A} \star \mathfrak{g} & \longrightarrow & \mathfrak{B} \star \mathfrak{g} & \longrightarrow & \mathfrak{C} \star \mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Psi_{\mathfrak{A}}} & & \downarrow{\scriptstyle \Psi_{\mathfrak{B}}} & & \downarrow{\scriptstyle \Psi_{\mathfrak{C}}} & & \\ 0 & \longrightarrow & \mathfrak{A} & \longrightarrow & \mathfrak{B} & \longrightarrow & \mathfrak{C} & \longrightarrow & 0 \end{array}$$

which is obviously commutative. □

6.3. Universal central extension. Let $\mathfrak{g}$ be a Leibniz algebra and let $\Psi := \Psi_{\mathfrak{g}}$ be the morphism defining the homolgy $\mathfrak{HL}_*(\mathfrak{g}, \mathfrak{g})$. From the relations v) of Definition-Theorem 4.1, it is clear that $\Psi : \mathfrak{g} \star \mathfrak{g} \twoheadrightarrow [\mathfrak{g}, \mathfrak{g}]$ is a central extension of Leibniz algebras (see [4]).

Theorem 6.5. *If the Leibniz algebra* $\mathfrak{g}$ *is perfect and free as a* $\mathbb{K}$*-module, then the morphism* $\Psi : \mathfrak{g} \star \mathfrak{g} \twoheadrightarrow [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ *is the universal central extension of* $\mathfrak{g}$*. Moreover, we have an isomorphism of* $\mathbb{K}$*-modules*

$$\mathfrak{HL}_1(\mathfrak{g}, \mathfrak{g}) \cong \mathrm{HL}_2(\mathfrak{g}).$$

Proof. It is enough to prove the universality of the central extension $\Psi : \mathfrak{g} \star \mathfrak{g} \twoheadrightarrow [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Let $\alpha : \mathfrak{C} \twoheadrightarrow \mathfrak{g}$ be a central extension of $\mathfrak{g}$. Since $\ker(\alpha)$ is central in $\mathfrak{C}$, the quantity $[\alpha^{-1}(x), \alpha^{-1}(y)]$ does not depend on the choice of the pre-images $\alpha^{-1}(x)$ and $\alpha^{-1}(y)$ where $x, y \in \mathfrak{g}$. One easily checks that the map $\phi : \mathfrak{g} \star \mathfrak{g} \to \mathfrak{C}$ given on generators by

$$\phi(x * y) := [\alpha^{-1}(x), \alpha^{-1}(y)]$$

is a well-defined Leibniz algebra morphism such that $\alpha\phi = \Psi$. The uniqueness of the map ϕ follows from Lemma 2.4 of [4] since the perfectness of $\mathfrak{g}$ implies that of $\mathfrak{g} \star \mathfrak{g}$:

$$x * y = (\sum_i [x_i, x_i']) * (\sum_j [y_j, y_j']) = \sum_{i,j} [x_i * x_i', y_j * y_j'].$$

By definition we have $\mathfrak{HL}_1(\mathfrak{g}, \mathfrak{g}) = \ker(\Psi)$. After [4] the kernel of the universal central extension of a Leibniz algebra $\mathfrak{g}$ is canonically isomorphic to $\mathrm{HL}_2(\mathfrak{g})$. Therefore we have

$$\mathfrak{HL}_1(\mathfrak{g}, \mathfrak{g}) \cong \mathrm{HL}_2(\mathfrak{g}).$$ □

7. The Milnor-type Hochschild homology

Let A be an associative algebra viewed as a Leibniz (in fact Lie) algebra for the bracket given by $[a, b] := ab - ba, a, b \in A$. Recall that the $\mathbb{K}$-module $\mathrm{L}(A) := A^{\otimes 2}/\mathrm{im}(b_3)$ is a Leibniz (non-Lie) algebra for the bracket defined by

$$[x \otimes y, x' \otimes y'] := (xy - yx) \otimes (x'y' - y'x'), \ \forall\ x, y, x', y' \in A.$$

Proposition 7.1. *The operations given by*

$$A \times \mathrm{L}(A) \to \mathrm{L}(A),\ \ {}^{a}(x \otimes y) := [a, x] \otimes y - [a, y] \otimes x,$$
$$\mathrm{L}(A) \times A \to \mathrm{L}(A),\ \ (x \otimes y)^{a} := [x, a] \otimes y + x \otimes [y, a]$$

confere to $\mathrm{L}(A)$ *a structure of Leibniz* A*-algebra. Moreover the map*

$$\mu_A : \mathrm{L}(A) \to A,\ \ x \otimes y \mapsto [x, y] = xy - yx$$

equips $\mathrm{L}(A)$ *with a structure of crossed Leibniz* A*-algebra.*

Proof. The operations are well-defined since we have

$$\begin{aligned} {}^{a}(b_3(x \otimes y \otimes z)) = b_3(&ax \otimes y \otimes z - a \otimes z \otimes xy - za \otimes x \otimes y \\ &+ a \otimes yz \otimes x + a \otimes zx \otimes y - a \otimes y \otimes zx) \end{aligned}$$

and

$$\begin{aligned} (b_3(x \otimes y \otimes z))^{a} = b_3(&-ax \otimes y \otimes z + xy \otimes a \otimes z + x \otimes y \otimes za \\ &- x \otimes a \otimes yz - zx \otimes a \otimes y - zx \otimes y \otimes a). \end{aligned}$$

One easily checks that the couple $(\mathrm{L}(A), \mu_A)$ is a pre-crossed Leibniz A-algebra. Moreover we have

$$\begin{aligned} {}^{\mu_A(x \otimes y)}(x' \otimes y') - [x \otimes y, x' \otimes y'] &= b_3([x, y] \otimes x' \otimes y' - [x, y] \otimes y' \otimes x') \\ (x \otimes y)^{\mu_A(x \otimes y)} - [x \otimes y, x' \otimes y'] &= b_3(x \otimes [x', y'] \otimes y - x \otimes y \otimes [x', y']). \end{aligned}$$

Thus the Leibniz A-algebra $(\mathrm{L}(A), \mu_A)$ is crossed. □

It is clear that the inclusion map $[A, A] \hookrightarrow A$ induces a structure of crossed Leibniz A-algebra on the two-sided ideal $[A, A]$, and that the map $\mu_A : \mathrm{L}(A) \to [A, A]$ is a morphism of crossed Leibniz A-algebras. Moreover we have an exact sequence of $\mathbb{K}$-modules

$$0 \to \mathrm{HH}_1(A) \to \mathrm{L}(A) \xrightarrow{\mu_A} [A, A] \to 0.$$

Lemma 7.2. *The Leibniz algebra* A *acts trivially on* $\mathrm{HH}_1(A)$*.*

Proof. One easily checks that

$${}^{a}(x \otimes y) = a \otimes [x, y] + b_3(a \otimes x \otimes y - a \otimes y \otimes x) \equiv a \otimes [x, y] \text{ in } \mathrm{L}(A)$$

and

$$(x \otimes y)^{a} = [x, y] \otimes a + b_3(x \otimes a \otimes y - x \otimes y \otimes a) \equiv [x, y] \otimes a \text{ in } \mathrm{L}(A).$$

Therefore, if $\omega = \sum \lambda_i (x_i \otimes y_i) \in \mathrm{HH}_1(A)$, that is $\sum \lambda_i [x_i, y_i] = 0$, then we have

$${}^{a}\omega = \sum \lambda_i {}^{a}(x_i \otimes y_i) \equiv \sum \lambda_i (a \otimes [x_i, y_i]) \equiv a \otimes \sum \lambda_i [x_i, y_i] = 0$$

and

$$\omega^{a} = \sum \lambda_i (x_i \otimes y_i)^{a} \equiv \sum \lambda_i ([x_i, y_i] \otimes a) \equiv (\sum \lambda_i [x_i, y_i]) \otimes a = 0$$

for any $a \in A$. □

As an immediate consequence, we get the following

Corollary 7.3. *The sequence*

$$0 \to \mathrm{HH}_1(A) \to \mathrm{L}(A) \xrightarrow{\mu_A} [A,A] \to 0$$

is an exact sequence of crossed Leibniz A-algebras. □

We deduce from Proposition 6.4 an exact sequence of $\mathbb{K}$-modules

$$\mathfrak{HL}_1(A,\mathrm{HH}_1(A)) \to \mathfrak{HL}_1(A,\mathrm{L}(A)) \to \mathfrak{HL}_1(A,[A,A]) \to$$
$$\to \mathfrak{HL}_0(A,\mathrm{HH}_1(A)) \to \mathfrak{HL}_0(A,\mathrm{L}(A)) \to \mathfrak{HL}_0(A,[A,A]) \to 0.$$

Since A and $\mathrm{HH}_1(A)$ act trivially on each other, we have

$$\mathfrak{HL}_0(A,\mathrm{HH}_1(A)) = \mathrm{HH}_1(A)$$

and

$$\mathfrak{HL}_1(A,\mathrm{HH}_1(A)) = A \star \mathrm{HH}_1(A) \cong A/[A,A] \otimes \mathrm{HH}_1(A) \;\oplus\; \mathrm{HH}_1(A) \otimes A/[A,A].$$

On the other hand, it is clear that

$$\mathfrak{HL}_1(A,[A,A]) \cong [A,A]/[A,[A,A]].$$

Therefore we can state

Theorem 7.4. *For any associative algebra A with unit, there exists an exact sequence of* $\mathbb{K}$*-modules*

$$A/[A,A] \otimes \mathrm{HH}_1(A) \;\oplus\; \mathrm{HH}_1(A) \otimes A/[A,A] \to \mathfrak{HL}_1(A,\mathrm{L}(A)) \to \mathfrak{HL}_1(A,[A,A]) \to$$
$$\to \mathrm{HH}_1(A) \to \mathrm{HH}_1^M(A) \to [A,A]/[A,[A,A]] \to 0$$

where $\mathrm{HH}_1^M(A)$ *denotes the Milnor-type Hochschild homology of* A.

Proof. Recall that $\mathrm{HH}_1^M(A)$ is defined to be the quotient of $A \otimes A$ by the relations

$$a \otimes [b,c] = 0,\;\; [a,b] \otimes c = 0,\;\; b_3(a \otimes b \otimes c) = 0$$

for any $a,b,c \in A$ (see [6, 10.6.19]). By definition $\mathrm{L}(A) = A \otimes A/\mathrm{im}(b_3)$ and from the proof of Lemma 7.2, we get

$$\Psi_{L(A)}(a * (x \otimes y)) = {}^a(x \otimes y) \equiv a \otimes [x,y]$$

and

$$\Psi_{L(A)}((x \otimes y) * a) = (x \otimes y)^a \equiv [x,y] \otimes a.$$

Therefore it is clear that $\mathfrak{HL}_0(A,\mathrm{L}(A)) = \mathrm{coker}(\Psi_{L(A)})$ is isomorphic to $\mathrm{HH}_1^M(A)$. □

Remark. The $\mathbb{K}$-modules $\mathrm{HH}_1(A)$ and $\mathrm{HH}_1^M(A)$ coincide when the associative algebra A is *superperfect* as a Leibniz algebra that is, $A = [A,A]$ and $\mathrm{HL}_2(A) = 0$. Also, if the associative algebra A is commutative, then we have

$$\mathrm{HH}_1(A) \cong \mathrm{HH}_1^M(A) \cong \Omega^1_{A|\mathbb{k}}.$$

Let us also mention that the Milnor-type Hochschild homology appears in the description of the obstruction to the stability

$$\mathrm{HL}_n(gl_{n-1}(A)) \to \mathrm{HL}_n(gl_n(A)) \to \mathrm{HH}_{n-1}^M(A) \to 0$$

where $gl_n(A)$ is the Lie algebra of matrices with entries in the associative algebra A (see [2], [6, 10.6.20]).

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