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 k -FRACTIONAL SPIN THROUGH Q -DEFORMED (SUPER)-ALGEBRAS

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Abstract

The splitting of a Q -deformed boson, in the $Q \rightarrow q = e^{\frac{2\pi i}{k}}$ limit, is discussed. The equivalence between a Q -fermion and an ordinary one is established. The properties of quantum algebras $U_Q(sl(m))$, the quantum superalgebras $osp_Q(1/2m)$ and $U_Q(sl(m/1))$ and the deformed Virasoro algebra when their deformation parameter Q goes to a root of unity, are investigated. These properties are shown to be related to fractional supersymmetry and k -fermionic spin.

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1 Introduction

Quantum groups, (or quantum algebras) introduced by Drinfeld [1], are a subject much discussed both by physicists and mathematicians. Its essence crystallized from the intensive developments of the quantum inverse problem method [2] and from the investigation related to Yang-Baxter equation [3]. A quantum group is mathematically defined as a noncommutative and nonco-commutative Hopf algebra [4]. The applications of these new mathematical structures have grown substantially in several areas of physics, as for instance in the solvable two-dimensional systems, via inverse scattering techniques and the Yang-Baxter equation and in rational conformal field theory [5-8]. The representation theory of quantum algebras has been also an object of intensive studies. Available are the results for oscillator representations of quantum algebras and quantum superalgebras. The latter are obtained through realizations involving deformed Bose and Fermi operators [9-13].

Recently, in connection with quantum group theory, a new geometric interpretation of fractional supersymmetry has been developed in references [14, 15, 16, 17, 18]. In these works, the authors show that the one-dimensional superspace is isomorphic to the braided line when the deformation parameter goes to a root of unity. Fractional supersymmetry is identified as translational invariance along this line. In the limit $q = e^{\frac{2\pi i}{k}}$ the braided line algebra [19-21] can be separated into two parts, one described by a generalized grassmann variable and the other by an ordinary even variable. Similar techniques are used, in reference [22], to show how internal spin arises naturally in a certain limit of the Q -deformed angular momentum algebra $U_Q(sl(2))$. Indeed, using Q -Schwinger realization, it is shown that the $U_Q(sl(2))$ is nothing but a direct product of undeformed $U(sl(2))$ and $U_q(sl(2))$ which is the same version of $U_Q(sl(2))$ at $Q = q$. Since there exist Q -oscillator realizations of all deformed enveloping algebras $U_Q(g)$, it is reasonable to expect these to admit analogous decompositions (or splittings) when $Q \rightarrow q$.

The aim of this paper is to investigate the property of splitting for some deformed algebras and superalgebras in the $Q \rightarrow q$ limit. As a first step we wish to present in section (2) a number of results concerning the property of Q -boson decomposition in the $Q \rightarrow q$ limit. We shall first of all discuss the way in which one obtains two independent objects (an ordinary boson and a k -fermion) from one Q -deformed boson when Q goes to a root of unity. We also show the equivalence between a Q -deformed fermion and a conventional (ordinary or undeformed) fermion. Using the Q -Schwinger realization which presents interesting properties in the $Q \rightarrow q$ limit, we have established analogous decomposition for quantum algebra $U_Q(sl(m))$ (section 3), the orthosymplectic algebra $U_Q(osp(1/2m))$ (section 4), the quantum superalgebra $U_Q(sl(m/1))$ (section 5) and the Q -deformed Virasoro algebra (section 6). Finally we make some concluding remarks.

2 Fractional spin through Q -bosons

We start with Q -deformed bosonic oscillator algebra A_Q . The algebra A_Q is generated by an annihilation operator a^- , a creation operator a^+ and a number operator N with the following relations

$$\begin{aligned} a^- a^+ - Q a^+ a^- &= Q^{-N} \\ a^- a^+ - Q^{-1} a^+ a^- &= Q^N \\ [N, a^+] &= a^+, \quad [N, a^-] = -a^- \end{aligned} \tag{1}$$

where Q is an arbitrary complex number. From the equation (1) we obtain

$$\begin{aligned} a^- (a^+)^l &= [[l]] Q^{-N} (a^+)^{l-1} + Q^l (a^+)^l a^-, \\ (a^-)^l a^+ &= [[l]] (a^+)^{l-1} Q^{-N} + Q^l a^+ (a^-)^l \end{aligned} \tag{2}$$

where the symbol $[[\]]$ is defined by

$$[[n]] = \frac{1 - Q^{2n}}{1 - Q^2}$$

Let us define q to be k -th root of unity so that $q^k = 1$, where $k \geq 2$ is a positive integer. The cases of odd and even k have to be treated in slightly different ways and because of this it is useful to introduce a variable l defined by

$$\begin{aligned} l &= k \text{ for odd values of } k \\ l &= \frac{k}{2} \text{ for even values of } k \end{aligned} \quad (3)$$

such that for odd k we have $q^l = 1$ and for even k , $q^l = -1$.

In the particular case when $Q \rightarrow q$ equations (2) can be written as

$$\begin{aligned} a^- (a^+)^l &= \pm (a^+)^l a^-, \\ (a^-)^l a^+ &= \pm a^+ (a^-)^l \end{aligned} \quad (4)$$

In addition the equations (1) leads to:

$$\begin{aligned} N (a^+)^l &= (a^+)^l (N + l), \\ (a^-)^l N &= (N + l) (a^-)^l \end{aligned} \quad (5)$$

Equations(5)with $(Q \rightarrow q)$ and (4) are trivial if we assume

$$\begin{aligned} (a^+)^l &= 0, \\ (a^-)^l &= 0 \end{aligned} \quad (6)$$

In this paper we shall deal with a representation of the algebra A_Q such that in the limit $(Q \rightarrow q)$, the equations (6) are satisfied. We note that the algebra obtained for $k = 2$, corresponds to the ordinary fermionic algebra with $(a^+)^2 = (a^-)^2 = 0$, a relation that reflects the Pauli exclusion principle. The algebra obtained for $k \geq 2$ corresponds to k -fermions (or anyons with fractional spin in the sense of Majid [19, 20, 21]) operators [23, 24] that interpolates between fermions ($k = 2$) and bosons ($k \rightarrow \infty$).

It follows from the definition (1) that:

$$\begin{aligned} [Q^{-N} a^-, [Q^{-N} a^-, [\dots [Q^{-N} a^-, (a^+)^l]_{Q^{2l}} \dots]_{Q^4}]_{Q^2}] \\ = Q^{\frac{l(l-1)}{2}} [l]! \end{aligned} \quad (7)$$

where the Q -deformed factorial is given by

$$\begin{aligned} [l]! &= [l][l-1][l-2] \dots [1] \\ [0]! &= 1 \end{aligned} \quad (8)$$

with

$$[l] = \frac{Q^l - Q^{-l}}{Q - Q^{-1}}. \quad (9)$$

In the limit $Q \rightarrow q$, the equation (7) becomes:

$$\begin{aligned} \lim_{Q \rightarrow q} \frac{1}{[l]!} [Q^{-N} a^-, [Q^{-N} a^-, [\dots [Q^{-N} a^-, (a^+)^l]_{Q^{2l}} \dots]_{Q^4}]_{Q^2}] \\ = \lim_{Q \rightarrow q} \frac{Q^{\frac{l(l-1)}{2}}}{[l]!} [Q^{-N} (a^-)^l, (a^+)^l] \\ = q^{\frac{l(l-1)}{2}} \end{aligned} \quad (10)$$

which can be written as follows:

$$\lim_{Q \rightarrow q} \left[\frac{Q^{\frac{1N}{2}} (a^-)^l}{\sqrt{[l]!}}, \frac{(a^+)^l Q^{\frac{1N}{2}}}{\sqrt{[l]!}} \right] = 1 \quad (11)$$

We remark that since q is a root of unity, it is possible to change the sign in the exponent of $q^{\frac{1N}{2}}$ terms in the above and in the following definitions.

In the spirit of the work [22] we define

$$b^- = \lim_{Q \rightarrow q} \frac{Q^{\frac{1N}{2}} (a^-)^l}{\sqrt{[l]!}}, \quad b^+ = \lim_{Q \rightarrow q} \frac{(a^+)^l Q^{\frac{1N}{2}}}{\sqrt{[l]!}} \quad (12)$$

which satisfy the following commutation relation:

$$[b^-, b^+] = 1 \quad (13)$$

which is just the defining relation of an ordinary boson. The number operator of these new bosonic oscillators, is defined in the usual way as $N_b = b^+ b^-$.

This idea was introduced initially in [14 – 18, 22] in order to investigate the fractional supersymmetry and to show that there is an isomorphism between the braided line and the one dimensional superspace. We will now introduce the new set of generators

$$\begin{aligned} A^- &= a^- q^{-\frac{1N_b}{2}} \\ A^+ &= a^+ q^{\frac{1N_b}{2}} \\ N_A &= N - 1N_b. \end{aligned} \quad (14)$$

satisfying the following commutation relations:

$$\begin{aligned} [A^-, A^+]_{q^{-1}} &= q^{N_A} \\ [A^-, A^+]_q &= q^{-N_A} \\ [N_A, A^\pm] &= \pm A^\pm \end{aligned} \quad (15)$$

which are the defining relations of a k -fermion [23, 24]. The two algebras $\{b^+, b^-, N_b\}$ and $\{A^+, A^-, N_A\}$ are mutually commutative, i.e.

$$\begin{aligned} [A^-, b^\pm] &= 0, \quad [A^+, b^\pm] = 0 \\ [N_b, A^\pm] &= 0, \quad [N_b, N_A] = 0, \\ [N_A, b^\pm] &= 0, \end{aligned} \quad (16)$$

We conclude that in the limit $Q \rightarrow q$ the Q -deformed bosonic oscillator decomposes into two independent oscillators, an (undeformed) boson and a k -fermion.

An appealing question is to ask whether is possible to find Q -deformed fermionic operators exhibiting a similar property of splitting to Q -deformed bosons, when the deformation parameter Q reduces to a root of unity q . To answer this question, we consider the Q -fermionic oscillator algebra defined by $\{f^-, f^+, N_f\}$ and by the following equations:

$$\begin{aligned} f^- f^+ + Q f^+ f^- &= Q^{-N_f} \\ f^- f^+ + Q^{-1} f^+ f^- &= Q^{N_f} \\ [N_f, f^+] &= -f^+, \quad [N_f, f^-] = +f^- \\ (f^-)^2 &= 0, \quad (f^+)^2 = 0 \end{aligned} \quad (17)$$

If we define the new creation F^+ and annihilation F^- operators by:

$$\begin{aligned} F^- &= Q^{-\frac{N_f}{2}} f^- \\ F^+ &= f^+ Q^{-\frac{N_f}{2}} \end{aligned} \quad (18)$$

we obtain by a direct calculation the following anticommutation relation

$$\{F^+, F^-\} = 1. \quad (19)$$

Moreover, we have

$$(F^\pm)^2 = 0. \quad (20)$$

Thus we see that the Q -deformed fermion reproduce the conventional (ordinary) fermion. It should be noted that the Q -deformed fermions were used by Hayashi [25] to give Q -fermionic representation of quantum groups $U_Q(X)$ where X is a finite Lie algebra of type A_n , B_n or D_n . However the Q -fermions are nothing but the conventional ones. So, the spinor representation of these quantum groups can be given in terms of a set of fermionic oscillator algebras.

3 The quantum algebra $U_Q(sl(m))$ at Q a root of unity

The analogue of the Jordan-Schwinger representation of Lie groups exists for all quantum groups. A discussion of the Q -boson and Q -fermion representation was given by Hayashi [25], this discussion covers the quantum groups A_n , B_n , C_n and D_n . However, we concentrate here on the quantum group $U_Q(sl(m))$ and we assume that the parameter Q is generic. In the $Q \rightarrow q$ limit, the deformed bosonic oscillator algebra decomposes into a direct product of an ordinary bosonic oscillator and a k -fermionic one. Since there exists a Q -oscillator realization of the deformed enveloping algebra $U_Q(sl(m))$ and its subalgebra $U_Q(su(m))$, it is natural to expect these to exhibit analogous decompositions when $Q \rightarrow q$. For Q generic the $U_Q(sl(m))$ algebra is generated by the set of generators $\{E_i, F_i, H_i, 1 \leq i \leq m-1\}$ satisfying the following relations:

$$\begin{aligned} [E_i, F_j] &= \delta_{ij}[H_i] \\ [E_i, H_j] &= a_{ij}E_i, \\ [F_i, H_j] &= -a_{ij}F_i, \\ [H_i, H_j] &= 0, \end{aligned} \quad (21)$$

with $1 \leq i, j \leq m-1$ and the Q -deformed Serre relations

$$\begin{aligned} E_i^2 E_{i+1} - (Q + Q^{-1})E_i E_{i+1} E_i + E_{i+1} E_i^2 &= 0 \\ E_i^2 E_{i-1} - (Q + Q^{-1})E_i E_{i-1} E_i + E_{i-1} E_i^2 &= 0 \\ F_i^2 F_{i+1} - (Q + Q^{-1})F_i F_{i+1} F_i + F_{i+1} F_i^2 &= 0 \\ F_i^2 F_{i-1} - (Q + Q^{-1})F_i F_{i-1} F_i + F_{i-1} F_i^2 &= 0 \end{aligned} \quad (22)$$

where

$$[H_i] = \frac{Q^{H_i} - Q^{-H_i}}{Q - Q^{-1}}$$

In (21), a_{ij} is the ij -element of the $N \times N$ Cartan matrix

$$\begin{pmatrix} 2 & -1 & \dots & 0 \\ -1 & 2 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & -1 \\ \cdot & \cdot & \dots & -1 & 2 \end{pmatrix} \quad (23)$$

In the limit $Q \rightarrow 1$, the previous equations reduce to the standard commutation relations defining $U(sl(m))$ algebra.

The $U_Q(sl(m))$ can be realized in terms of m Q -bosonic oscillators $\{a_i^+, a_i^-, N_i, 1 \leq i \leq m\}$ by putting

$$\begin{aligned}
E_i &= a_i^+ a_{i+1}^-, \\
F_i &= a_i^- a_{i+1}^+, \\
H_i &= N_i - N_{i+1},
\end{aligned} \tag{24}$$

for $1 \leq i \leq m-1$,

where $a_i^+, a_i^-, N_i (1 \leq i \leq m)$ are operators satisfying the following relations

$$\begin{aligned}
a_i^- a_j^+ - Q^{\pm 1} a_j^+ a_i^- &= Q^{\mp N_i} \delta_{ij} \\
[N_i, a_j^\pm] &= \pm a_i^\pm \delta_{ij}, \quad [a_i^\pm, a_j^\pm] = 0, \\
N_i N_j &= N_j N_i.
\end{aligned} \tag{25}$$

The algebra generated by $a_i^+, a_i^-, N_i (1 \leq i \leq m-1)$ is nothing but the well known Q -deformed Weyl algebra. We denote it by $A_Q(m)$.

As we have discussed above, we are interested by the $Q \rightarrow q$ limit of the quantum algebra $U_Q(sl(m))$. The key tool to discuss this limit is the Q -bosonic decomposition presented in the first section when $Q \rightarrow q$. So, the m Q -bosons reproduce m ordinary bosons defined by

$$b_i^- = \lim_{Q \rightarrow q} \frac{Q^{\pm \frac{N_i}{2}} (a_i^-)^l}{\sqrt{[l]!}}, \quad b_i^+ = \lim_{Q \rightarrow q} \frac{(a_i^+)^l Q^{\pm \frac{N_i}{2}}}{\sqrt{[l]!}}. \tag{26}$$

where their number operators are given by $N_{b_i} = b_i^+ b_i^-$ for $i = 1, 2, \dots, m$.

Using operators (26), we can construct the undeformed $U(sl(m))$ algebra:

$$\begin{aligned}
e_i &= b_i^+ b_{i+1}^-, \\
f_i &= b_i^- b_{i+1}^+, \\
h_i &= N_{b_i} - N_{b_{i+1}},
\end{aligned} \tag{27}$$

(with $1 \leq i \leq m-1$), and by combining the generators $\{H_i, E_i, F_i, h_i, e_i, f_i\}$ we introduce the new generators

$$\begin{aligned}
S_i^0 &= H_i - l h_i, \\
S_i^+ &= q^{l h_i} E_i, \\
S_i^- &= F_i q^{l h_i},
\end{aligned} \tag{28}$$

(where $1 \leq i \leq m-1$) which commute with the generators given by (27) and generate the quantum algebra $U_q(sl(m))$.

As a consequence, in the $Q \rightarrow q$ limit, the quantum algebra $U_Q(sl(m))$ is a direct product of the form

$$\lim_{Q \rightarrow q} U_Q(sl(m)) = U_q(sl(m)) \otimes U(sl(m))$$

in which $U(sl(m))$ denotes the enveloping algebra of the undeformed $sl(m)$ ($Q \rightarrow 1$) and by $U_q(sl(m))$, we mean the enveloping algebra obtained by taking $Q = q$ in relations (21) and (22), rather than by taking the limit as above. This result generalizes the one obtained in reference[22].

It is important to note that we have only established the above decomposition for a particular realization, i.e. Q -bosonic realization. This remark is also valid for all decomposition which will be investigated in this work.

4 The orthosymplectic quantum supergroup $osp_Q(1/2m)$

The general definition of the orthosymplectic superalgebra quantum deformations has now been formulated and their Hopf algebras structure established [26]. The simplest example,

that is $osp_Q(1/2)$, has been examined in detail [27]. In addition, a Q -oscillator realization of $osp_Q(1/2m)$ has been given in terms of the annihilation and creation operators of m Q -deformed bosonic algebras [28, 29]. We shall be concerned in this section with the $Q \rightarrow q$ limit (q is a root of unity) of the quantum superalgebra $osp_Q(1/2m)$ (or $B(0,m)$ in Kac's notation). The symmetric Cartan matrix of $osp_Q(1/2m)$ is given by

$$\begin{pmatrix} 2 & -1 & \dots & 0 \\ -1 & 2 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & -1 \\ \cdot & \cdot & \dots & -1 & 1 \end{pmatrix} \quad (29)$$

Moreover, it is generated by $3m$ generators $\{E_i, F_i, H_i, \ 1 \leq i \leq m\}$ satisfying the following constraints

$$\begin{aligned} [E_i, F_j] &= \delta_{ij}[H_i], \\ [E_i, H_j] &= a_{ij}E_i, \\ [F_i, H_j] &= -a_{ij}F_i, \\ [H_i, H_j] &= 0, \end{aligned} \quad (30)$$

for $1 \leq i, j \leq m$, and the Q -deformed Serre relations

$$\begin{aligned} E_i^2 E_j - (Q + Q^{-1})E_i E_j E_i + E_j E_i^2 &= 0, \quad i \neq n, |i-j|=1 \\ F_i^2 F_j - (Q + Q^{-1})F_i F_j F_i + F_j F_i^2 &= 0, \quad i \neq n, |i-j|=1 \\ E_m^3 E_{m-1} - (Q + Q^{-1} - 1)(E_m^2 E_{m-1} E_m + E_m E_{m-1} E_m^2) + E_{m-1} E_m^3 &= 0 \\ F_m^3 F_{m-1} - (Q + Q^{-1} - 1)(F_m^2 F_{m-1} F_m + F_m F_{m-1} F_m^2) + F_{m-1} F_m^3 &= 0 \end{aligned} \quad (31)$$

with $\deg E_i = \deg F_i = 0$ for $i \neq m$ and $\deg E_m = \deg F_m = 1$

The symbol $[,]$, in equations (30), stands for the graded Lie product:

$$[x, y] = xy - (-1)^{\deg x \deg y} yx$$

Now we introduce the representation of $osp_Q(1/2m)$ in terms of Q -deformed bosons. Thus, we introduce the operators

$$\begin{aligned} E_i &= a_i^+ a_{i+1}^-, \\ F_i &= a_i^- a_{i+1}^+, \\ H_i &= N_i - N_{i+1}, \\ H_m &= N_m + \frac{1}{2}, \quad E_m = \frac{1}{\sqrt{Q^{\frac{1}{2}} + Q^{-\frac{1}{2}}}} a_m^+, \quad F_m = \frac{1}{\sqrt{Q^{\frac{1}{2}} + Q^{-\frac{1}{2}}}} a_m^-, \end{aligned} \quad (32)$$

where $1 \leq i \leq m-1$.

Using the Q -commutation relations of Q -bosons, one verifies that the operators given in (32) satisfy the defining relations (30) and (31) of the generators of the quantum superalgebra $osp_Q(1/2m)$. Furthermore, one obtains in this way a representation of $osp_Q(1/2m)$ on the Hilbert space formed by the state vectors of m ordinary harmonic oscillators. For a generic Q this representation will be irreducible and infinite dimensional. In the limit $Q \rightarrow 1$, the quantum orthosymplectic superalgebra reduce to the standard one. As in the case $U_Q(sl(m))$, let us now investigate the limit of $osp_Q(1/2m)$ When $Q \rightarrow q$.

In the $Q \rightarrow q$ limit, the Q -deformed bosonic algebra $\{a_i^-, a_i^+, N_i\}$ for i fixed gives one classical bosonic algebra $\{b_i^-, b_i^+, N_{b_i}\}$ (eq(12)) and one k -fermionic algebra (eq(14)) similarly to the decomposition discussed in the first section. Using the classical operators we can construct by using the Schwinger realization, the undeformed $U(osp(1/2m))$ algebra as:

$$\begin{aligned} e_i &= b_i^+ b_{i+1}^-, \\ f_i &= b_i^- b_{i+1}^+, \\ h_i &= N_{b_i} - N_{b_{i+1}}, \\ h_m &= N_m + \frac{1}{2}, \quad e_m = \frac{1}{\sqrt{2}} b_m^+, \quad f_m = \frac{1}{\sqrt{2}} b_m^-, \end{aligned} \quad (33)$$

where $1 \leq i \leq m-1$, and from the remaining generators $\{N_{A_i}, A_i^+, A_i^-\}$ we generate a q -orthosymplectic quantum superalgebra $U_q(\mathfrak{osp}(1/2m))$ which commute with the undeformed one realized above. So, we obtain

$$\lim_{Q \rightarrow q} U_Q(\mathfrak{osp}(1/2m)) = U_q(\mathfrak{osp}(1/2m)) \otimes U(\mathfrak{osp}(1/2m)).$$

Let us conclude this section by noting that the decomposition of the quantum orthosymplectic superalgebra $\mathfrak{osp}_Q(1/2m)$ is similar to the one obtained in the $U_Q(\mathfrak{sl}(m))$ case.

5 The quantum superalgebra $U_Q(\mathfrak{sl}(m/1))$

We recall that the quantum superalgebra $U_Q(\mathfrak{sl}(m/1))$ is defined as the associative algebra over C generated by the generators $\{H_i, E_i, F_i, i = 1, 2, \dots, m\}$ satisfying

$$\begin{aligned} [E_i, F_j] &= \delta_{ij}[H_i], \quad 1 \leq i, j \leq m-1 \\ \{E_m, F_m\} &= [H_m], \\ [E_i, H_j] &= a_{ij}E_i, \quad i, j = 1, 2, \dots, m \\ [F_i, H_j] &= -a_{ij}F_i, \quad i, j = 1, 2, \dots, m \\ [E_i, E_j] &= 0, \quad [F_i, F_j] = 0, \quad i, j = 1, 2, \dots, m, |i-j| \neq 1 \\ E_m^2 &= 0, \quad F_m^2 = 0. \end{aligned} \tag{34}$$

and the Q -deformed Serre relations

$$\begin{aligned} E_i^2 E_{i+1} - (Q + Q^{-1})E_i E_{i+1} E_i + E_{i+1} E_i^2 &= 0, \quad i = 1, 2, \dots, m-1, \\ E_{i+1}^2 E_i - (Q + Q^{-1})E_{i+1} E_i E_{i+1} + E_i E_{i+1}^2 &= 0, \quad i = 1, 2, \dots, m-2, \\ F_i^2 F_{i+1} - (Q + Q^{-1})F_i F_{i+1} F_i + F_{i+1} F_i^2 &= 0, \quad i = 1, 2, \dots, m-1, \\ F_{i+1}^2 F_i - (Q + Q^{-1})F_{i+1} F_i F_{i+1} + F_i F_{i+1}^2 &= 0, \quad i = 1, 2, \dots, m-2, \end{aligned} \tag{35}$$

where a_{ij} are the elements of the Cartan matrix given by

$$\begin{pmatrix} 2 & -1 & \dots & 0 \\ -1 & 2 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots \cdot 2 & -1 \\ \cdot & \cdot & \dots -1 & 0 \end{pmatrix} \tag{36}$$

The Z_2 -grading on $U_Q(\mathfrak{sl}(m/1))$ is defined by the requirement that only E_m and F_m are odd generators.

The $\mathfrak{sl}_Q(m/1)$ algebra can be realized in terms of m Q -deformed bosonic oscillators and one Q -deformed fermionic oscillator. The explicit expressions for the corresponding generators, in terms of Q -bosonic and Q -fermionic operators, are given by:

$$\begin{aligned} E_i &= a_i^+ a_{i+1}^-, \quad F_i = a_i^- a_{i+1}^+, \\ H_i &= N_i - N_{i+1}, \\ E_m &= a_m^+ f^-, \quad F_m = a_m^- f^+, \quad H_m = N_m - N_f, \end{aligned} \tag{37}$$

where $i = 1, 2, \dots, m-1$.

Let us observe that the quantum superalgebra $U_Q(\mathfrak{sl}(m/1))$ admits several representations [28, 29]. However, we shall concentrate on the Q -realization given by formulae (37). Based on the Q -boson decomposition when Q goes to the root of unity q , we introduce two set of operators $\{N_{b_i}, b_i^+, b_i^-\}$ and $\{N_{A_i}, A_i^+, A_i^-\}$ corresponding to the classical bosons and k -fermions

respectively. We have shown also that the Q -deformed fermion is equivalent to an ordinary one. Let us make use of the realization

$$\begin{aligned} e_i &= b_i^+ b_{i+1}^-, \\ f_i &= b_i^- b_{i+1}^+, \\ h_i &= N_{b_i} - N_{b_{i+1}}, \\ e_m &= b_m^+ F^-, \quad f_m = b_m^- F^+, \quad h_m = N_{b_m} - N_F, \end{aligned} \quad (38)$$

(where $i = 1, 2, \dots, m-1$). Here we have introduced the creation F^+ , annihilation F^- and number N_F operators defined by

$$\begin{aligned} F^+ &= q^{-N_f/2} f^+, \\ F^- &= f^- q^{-N_f/2}, \\ N_F &= -F^+ F^-. \end{aligned} \quad (39)$$

The generators $\{e_i, f_i, h_i, \quad i = 1, 2, \dots, m\}$ generate the undeformed superalgebra $U(sl(m/1))$. On the other hand, the remaining generators

$$\begin{aligned} A_i^- &= a_i^- q^{\frac{-N_{b_i}}{2}}, \\ A_i^+ &= a_i^+ q^{\frac{-N_{b_i}}{2}}, \\ N_{A_i} &= N_i - l N_{b_i}, \end{aligned} \quad (40)$$

for $\{i = 1, 2, \dots, m\}$ generate a k -fermionic algebra $A_q(m)$:

$$\begin{aligned} [A_i^-, A_j^+]_{q^{-1}} &= \delta_{ij} q^{N_{A_i}}, \\ [A_i^-, A_j^+]_q &= \delta_{ij} q^{-N_{A_i}}, \\ [N_{A_i}^i, A_j^+] &= \delta_{ij} A_i^+, \\ [N_{A_i}^i, A_j^-] &= -\delta_{ij} A_i^-. \end{aligned} \quad (41)$$

The two algebras $A_q(m)$ and $U(sl(m/1))$ commute mutually. So, we have the following result

$$\lim_{Q \rightarrow q} sl_Q(m/1) = A_q(m) \otimes sl(m/1)$$

which is different from the two first decompositions obtained in sections (3) and (4). We think that this type of decomposition can be extended to all quantum superalgebras .

6 The deformed centreless Virasoro algebra

The classical Virasoro algebra (Vir) is generated by the following set of generators $\{l_n, n \in Z\}$ such that

$$[l_n, l_m] = (m-n)l_{n+m}. \quad (42)$$

It is well known that the algebra(42) can be realized by considering the Schwinger construction. This realization involve one classical (undeformed) bosonic algebra $\{a^+, a^-, N\}$

$$l_n = (a^+)^{n+1} a^-. \quad n \geq -1 \quad (43)$$

Recently, a lot of attention has been paid to the Q -deformation of the centreless Virasoro algebra [30, 31, 32, 33] and its central extension[31, 34, 35]. Recall that the one parameter deformation of the centreless Virasoro algebra Vir_Q is given by

$$[L_n, L_m]_{(Q^{m-n}, Q^{n-m})} = [m-n]L_{n+m} \quad (44)$$

where

$$[A, B]_{(\alpha, \beta)} = \alpha AB - \beta BA$$

A possible realization of Q -Virasoro generators is given by

$$L_n = Q^{-N}(a^+)^{n+1}a^-. \quad (45)$$

where the operators a^+ and a^- are Q -deformed creator and annihilator respectively. At this step, we repeat the similar procedure used in the above sections to investigate the behaviour of the Q -deformed Virasoro algebra. When $Q \rightarrow q$. Using the splitting property of Q -deformed bosons, let us introduce two commuting algebras $\{j_n, n \in Z\}$ and $\{J_m, m \in Z\}$ such that

$$\begin{aligned} [j_n, j_m] &= (m - n)j_{n+m} \\ [J_n, J_m]_{(q^{m-n}, q^{n-m})} &= [m - n]J_{n+m} \\ [J_k, J_l] &= 0 \end{aligned} \quad (46)$$

where the generators j_n are defined in terms of the undeformed boson of type b (we denote by b and b^+ , the annihilation and creation operators, respectively) as:

$$j_n = (b^+)^{n+1}b^-. \quad (47)$$

The k -fermionic operators $\{A^-, A^+, N_A\}$ permit a realization of the q -deformed Virasoro algebra (Vir_q) by putting

$$J_n = q^{-N_A}(A^+)^{n+1}A^-. \quad (48)$$

Then we conclude that

$$\lim_{Q \rightarrow q} Vir_Q = Vir_q \otimes Vir$$

We remark that the q -deformed Virasoro algebra exhibit some interesting properties. Indeed, we note that when $m - n = rl$ for any $r \in Z$ the second equation in (46) reduces to:

$$[J_n, J_m] = 0 \quad (49)$$

Note also, due to the nilpotency condition $(A^+)^l = (A^-)^l = 0$, the generators J_n vanishes for any $n \geq l - 1$. This fact constitutes an interesting property of the deformed Virasoro algebra when the deformation parameter is a root of unity. In particular, for $l = 3$, the q -deformed Virasoro algebra reduces to its subalgebra $su_q(2)$ generated by $\{J_0, J_1, J_{-1}\}$.

7 Conclusion

We have presented a general method to investigate the $Q \rightarrow q = e^{\frac{2\pi i}{k}}$ limit of some Q -deformed algebras based on the decomposition of Q -bosons in this limit. We note that Q -oscillator realization is crucial in this decomposition of these algebras. We have restricted in this work our attention to $U_Q(sl(m))$, $U_Q(osp(1/2m))$, $U_Q(sl(m/1))$ and the Q -deformed Virasoro algebra. We believe that the techniques and formulae used here will be useful in extending this study to all Q -deformed Lie algebras and superalgebras. This matter will be treated in forthcoming paper[36].

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