Available at: http://www.ictp.trieste.it/~pub_off
IC/98/158

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

OBSERVABILITY OF GENERAL LINEAR PAIRS

Ayse Kara Hacibekiroglu[1]
Department of Mathematics, Faculty of Arts and Science, Yildiz Technical University, 80270 Sisli, Istanbul, Turkey
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

and

Victor Ayala Bravo[2]
Department of Mathematics, Universidad Catolica del Norte, Casilla 1280, Antofagasta, Chile.

Abstract

Let G be a connected Lie group with Lie algebra $\mathbf{g}$. In this work, we deal with the observability of a general linear pair (X, π_K) on G. By definition the vector field X belongs to the normalizer of $\mathbf{g}$ related to the Lie algebra of all smooth vector fields on G. K is a closed Lie subgroup of G and π_K is the canonical projection from G onto the homogeneous space G/K. We compute the Lie algebra of the equivalence class of the identity element and characterize local and global observability of (X, π_k). We extend the well known observability rank condition for linear control systems on $\mathbb{R}^n$ and also the work about observability of linear pairs appear in [2].

MIRAMARE – TRIESTE

September 1998

[1] E-mail: kara@yildiz.edu.tr
[2] E-mail: vayala@socompa.ucn.cl

ϕ1 **Preliminaires**

Let G be a connected Lie group of dimension n with Lie algebra $\mathbf{g}$. Here we consider $\mathbf{g}$ as the set of left invariant vector fields on G. Denote by $X(G)$ the Lie algebra of all smooth vector fields on G and by $\mathrm{norm}_{X(G)}(\mathbf{g})$ the normalizer of $\mathbf{g}$ related to the Lie algebra $X(G)$. In other words,

$$\mathrm{norm}_{X(G)}(\mathbf{g}) = \{X \in X(G) \mid ad(X)(Y) = [X, Y] \in \mathbf{g}, \text{ for all } Y \in \mathbf{g}\}$$

In [1], the authors generalize the notion of Linear Control Systems from $\mathbb{R}^n$ to an arbitrary connected Lie group G. Related to the observability property of this class of systems, the authors in [2] introduce the notion of linear pair. Our interest in this work is to generalize this notion in a natural way and to obtain more general results for general linear pairs where the dynamic is given by a vector field in the normalizer. In fact, we extend all the results appear in [2].

So, let us start with the definition of this notion:

Definition 1.1 A general linear pair (X, π_K) on G is determined by $X \in \mathrm{norm}_{X(G)}(\mathbf{g})$ and by a closed Lie subgroup K of G.

Remarks 1.2

1. Just observe that K induces a well defined homogeneous space K/G and also a canonical projection output map $\pi_K : G \to K/G$.

2. Definition 1.1 extends:

a) The classical pair (A, C) induced by a linear control system Σ on $\mathbb{R}^n$

In fact, this class of control systems on $\mathbb{R}^n$ is defined by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad and \quad h(x(t)) = Cx(t) \in \mathbb{R}^s$$

where $x(t) \in \mathbb{R}^n$ for every $t \in \mathbb{R}$. And A, B and C are matrices of appropriate orders, [4]. It is well known that in order to study the observability property of Σ the pair (A, C) contains all the information, [3]. We identify the matrix C with the canonical projection

$$\pi_{Ker(C)} : \mathbb{R}^n \to \mathbb{R}^n / Ker(C).$$

Of course, $K = Ker(C)$ is a closed subspace of $\mathbb{R}^n$. And, the matrix $A \in M_n(\mathbb{R})$ belongs to $\mathrm{norm}_{X(\mathbb{R}^n)}(\mathbb{R}^n)$. In fact, the Lie algebra of $\mathbb{R}^n$ is the own $\mathbb{R}^n$and a simple computation shows that $[Ax, b] = -Ab$, for each invariant vector field b on $\mathbb{R}^n$. Actually, $\mathrm{norm}_{X(\mathbb{R}^n)}(\mathbb{R}^n)$ is isomorphic to the semidirect product of Lie algebras $\mathbb{R}^n \otimes M_n(\mathbb{R})$, (see Theorem 1.3 in the following). In particular, (A, C) is a general linear pair defined on the simply conneted Abelian Lie group $\mathbb{R}^n$. We also appoint that for every admissible constant control u the associated vector field X^u of Σ defined by $X^u(x) = Bu + Ax$ belongs to the $\mathrm{norm}_{X(\mathbb{R}^n)}(\mathbb{R}^n)$.

b) The notion of linear pairs (X, h) on a connected Lie group G.

In [2], the authors introduce the notion of linear pairs. By definition, (X, h) is given by the infinitesimal automorphism X on G, i.e., the flow $(X_t)_{t \in \mathbb{R}}$ induced by the vector field X is a one parameter subgroup of $Aut(G)$. And the output map h is a Lie group homomorphism from G to any Lie group V. It follows that $X \in \mathrm{norm}_{X(G)}(\mathbf{g})$. A simple proof is given as follows. Let us denote by e the identity element of G and by L_x and R_x the left and right translations by x on G, respectively. Pick any left invariant vector field $Y \in \mathbf{g}$. Since $X_t(e) = e$ for each $t \in \mathbb{R}$, we have:

$$[X, Y](e) = -(\frac{d}{ds})_{s=0} X_{exp(sY)}.$$

On the other hand,

$$[X,Y](x) = (\frac{d}{dt})_{t=0} d(X_{-t})(Y_{X_t(x)}) = (\frac{d}{dt})_{t=0}(\frac{d}{ds})_{s=0} X_{-t} \circ Y_s \circ X_t(x)$$

$$= (\frac{d}{dt})_{t=0}(\frac{d}{ds})_{s=0} L_x \circ X_{-t}(expsY) = -(\frac{d}{ds})_{s=0} d(L_x) X_{exp(sY)}$$

So, $[X,Y](x) = d(L_x)[X,Y](e)$, and $ad(X)(Y) \in \mathbf{g}$, for each $Y \in \mathbf{g}$. Therefore, $X \in \mathrm{norm}_{X(G)}(\mathbf{g})$. If we denote by K the kernel of h and consider the canonical map $\pi_K : G \to G/K \cong Im(h) \subset V$, we get that (X, π_K) is also a general linear pair.

Denote by $Aut(G)$ the Lie group of all automorphisms of G and by $aut(G)$ its Lie algebra and by $\partial\mathbf{g}$ the Lie algebra of all $\mathbf{g}-$ derivations, i.e., the elements D of $End(\mathbf{g})$ such that,

$$D(\left[Y^1, Y^2\right]) = \left[D(Y^1), Y^2\right] + \left[Y^1, D(Y^2)\right], \forall Y^1, Y^2 \in \mathbf{g}.$$

We conclude this Section with a characterization of $\mathrm{norm}_{X(G)}(\mathbf{g})$ which will be used to define our dynamic. In [1], the authors prove the following result :

Theorem 1.3 Let G be a connected Lie group. Then,

$$\mathrm{norm}_{X(G)}(\mathbf{g}) \cong \mathbf{g} \otimes aut(G).$$

If G is also simply connected, then $\mathrm{norm}_{X(G)}(\mathbf{g}) \cong \mathbf{g} \otimes \partial\mathbf{g}$.

Just observe that $aut(G) \subset \partial\mathbf{g}$. So, in the simply connected case the isomorphim is onto $\partial\mathbf{g}$. We shall consider general linear pairs of the form (X, π_K) where the dynamic is determined by the vector field X such that:

$$X = X^1 + X^2 \in \mathbf{g} \otimes aut(G), i.e., X^1 \in \mathbf{g} \quad and \quad ad(X^2) \in aut(G).$$

Finally, let us establish the solution of X for any arbitrary initial condition $x \in G$. The authors had been proved in [1] the following :

Theorem 1.4 Each vector field $X \in \mathrm{norm}_{X(G)}(\mathbf{g})$ is complete and its flow is given by

$$X_t(x) = X_t^2(x) exp\zeta(t) \tag{1}$$

where $\zeta(t)$ is a differentiable curve in $\mathbf{g}$. Actually, Jacobi identity yields that $ad(X^2) \in \partial\mathbf{g}$, and they show that :

$$\zeta(t) = \Sigma_{k \geq 1}(-1)^{k+1} t^k d_k(X^1, ad(X^2)), \tag{2}$$

where $d_1(X) = X^1$, $d_2(X) = \frac{1}{2}\left[X^2, X^1\right]$,

$$d_3(X) = \frac{1}{12}[X^1, \left[X^2, X^1\right]] + \frac{1}{6}\left[X^2, \left[X^2, X^1\right]\right].$$

In general, for each $k \geq 1$, d_k is a homogeneous polynomial map of degree k from the semidirect product $\mathbf{g} \otimes \partial\mathbf{g}$ into $\mathbf{g}$.

In order to characterize local and global observability properties of general linear pairs we shall use the global form of the solution established in (1), for any X in the normalizer.

ϕ2.**Observability**

First of all we recall the notion of observability. So, let us start with the following one:

Definition 2.1 The general linear pair (X, π_K) is said to be :

i) *observable* at x_1, if for all $x_2 \in G$, $x_1 \neq x_2$ there exist $t \geq 0$ such that

$$\pi_K(X_t(x_1)) \neq \pi_K(X_t(x_2))$$

ii) *locally observable* at x_1, if there exists a neighborhood of x_1 such that the condition (**i**) is satisfied for each x_2 in the neigborhood.

iii) *observable (locally observable)* if it is observable (locally observable) at every $x \in G$.

We note that $X_t^2 \in Aut(G), \forall t \in \mathbb{R}$. For any $x_1, x_2 \in G$, let us define $\sim$ by :

$$x_1 \sim x_2 \Leftrightarrow \pi_K(X_t(x_1)) = \pi_K(X_t(x_2)), \forall t \geq 0.$$

Then, $\sim$ is an equivalence relation. From (1) we get,

$$x_1 \sim x_2 \Leftrightarrow X_t^2(x_1) exp\zeta(t) K = X_t^2(x_2) exp\zeta(t) K, \forall t \geq 0.$$

So, for any $x_1, x_2 \in G$ we obtain :

$$x_1 \sim x_2 \Leftrightarrow i_{exp\zeta(t)}(X_t^2(x_2^{-1}x_1)) \in K, \forall t \geq 0$$

$$\Leftrightarrow i_{(exp\zeta(t))^{-1}}(X_t^2(x_1^{-1}x_2)) \in K, \forall t \geq 0$$

where $i_x : G \to G$ is the usual inner automorphism given by conjugation.

Fix $t \in \mathrm{IR}$ and denote $\varphi_t = i_{exp\zeta(t)} \circ X_t^2$, $\tilde{\varphi}_t = i_{(exp\zeta(t))^{-1}} \circ X_t^2$ and by I the equivalence class of e. It follows that :

Proposition 2.2 Let (X, π_K) be a general linear pair. Then, I is the largest $(\varphi_t)_{t \in \mathrm{IR}}$−invariant closed Lie subgroup of G contained in K. Furthermore, for any $x \in G$ the equivalence class $\widetilde{x}$ of x is given by left translation.

Proof. From Definition 2.1, it is clear that

$$I = \{x \in G \mid \varphi_t(x) \in K, \forall t \geq 0\}.$$

For each $t \in \mathrm{IR}$, $\varphi_t \in Aut(G)$. It follows that I is a subgroup of K. Since K is a closed subgroup of G standard continuity arguments shows that I is also a closed set. In particular, I is Lie subgroup of G, [5] . On the other hand, I is φ_t−invariant for every non negative t. From standard analytical arguments we get the $(\varphi_t)_{t \in \mathrm{IR}}$−invariance of I. In particular,

$$I = \{x \in G \mid \varphi_t(x) \in K, \forall t \in \mathbb{R}\}.$$

Moreover, for each $x \in G$, $\widetilde{x} = xI$. Indeed, $x_2 \sim x_1 \Leftrightarrow x^{-1}x_2 \in I \Leftrightarrow x_2 \in x_1 I$.

In order to be able to compute the Lie algebra of I, we need the following one:

Lemma 2.3 Let (X, π_K) be a general linear pair on G. Then,

1. There exists $Z \in aut(G)$ and a right invariant vector field Y on G with $X = Z + Y$.

2. The linear transformations $ad(Z)$ and $ad(X)$ defined on $X(G)$ agree on **g**.

3. $ad(Z) \in \partial\mathbf{g}$ is a derivation such that for every $U \in \mathbf{g}$ and $t \in \mathrm{IR}$.

$$\tilde{\varphi}_t(\exp U) = \exp(e^{tad(Z)}U)$$

Proof. 1. As we know the flow $(\varphi_t)_{t\in\mathrm{IR}} \subset Aut(G)$. On the other hand,

$$\tilde{\varphi}_t = L_{(\exp\zeta(t))^{-1}} \circ X_t, \forall t \in \mathrm{IR}.$$

For each $x \in G$,

$$(\frac{d}{dt})_t\, \tilde{\varphi}_t\,(x) = d(L_{\exp^{-1}\zeta(t)})_{X_t(x)}(X_{X_t(x)}) + d(R_{X_t(x)})_{\exp^{-1}\zeta(t)}((\frac{d}{dt})_t(\exp^{-1}(\zeta(t)).$$

Thus, the one parameter group $\tilde{\varphi}_t \in Aut(G)$ induced a well defined vector field $Z \in aut(G)$, such that , for every $x \in G$, $Z_x = X_x - (dR_x)_e(X^1)$. In fact, from (2), we get

$$(\frac{d}{dt})_{t=0}(\exp^{-1}(\zeta(t)) = -X^1.$$

So, $Y = (dR_x)_e(X^1)$ is a right invariant vector field with $X = Z + Y$.

2. The map $X \longmapsto ad(X)_{\mathrm{g}}$ is a Lie algebra homomorphism of $\mathrm{norm}_{X(G)}(\mathbf{g})$ into the derivation algebra $\partial\mathbf{g}$ of $\mathbf{g}$. The kernel of this homomorphism is the centralizer $Z(\mathbf{g})$ of $\mathbf{g}$ in $X(G)$, i.e., the set of all vector fields on G commuting with each element of $\mathbf{g}$. In order to prove the assertion we compute the bracket $[Y, U]$ for each $U \in \mathbf{g}$. By definition, [5],

$$[Y, U]_x = (\frac{d}{dt})_{t=0^+}\gamma(\sqrt{t}), \quad where, \quad \gamma(s) = Y_{-s^\circ} \circ U_{-s} \circ Y_{s^\circ} \circ U_s(x).$$

Since, we consider U as a left invariant vector field we get $\gamma(s) = x$, for each $s \in \mathrm{IR}$. Consequently, we have seen that $Y \in Z(\mathbf{g})$. Thus, the proof is complete, because $[Y, U] = 0$.

3. Since $\tilde{\varphi}_t \in Aut(G)$ and $\tilde{\varphi}_t\,(e) = e, \forall t \in \mathrm{IR}$, the flow $(d(\tilde{\varphi}_t)_e)_{t\in\mathrm{IR}} \subset Aut(\mathbf{g})$ is a linear flow. From the standard Lie series expansion we have

$$d(\tilde{\varphi}_t)_e = \Sigma_{i=0}^{\infty}\frac{t^i}{k!}ad^i(Z) = e^{tad(Z)}.$$

By standard commutative diagrams envolving the exponential map a homomorphim and its derivative the proof of the lemma is complete.

Let us denote by $\mathcal{I}$ the Lie algebra of the equivalent class I of the identity element and by $\mathcal{K}$ the Lie algebra of K. The following theorem establishes an algebraic charaterization of $\mathcal{I}$.

Theorem 2.4 Let (X, π_K) be a general linear pair on G. Then, $\mathcal{I}$ is the largest $ad(X)-$invariant subalgebra of $\mathbf{g}$ contained in $\mathcal{K}$.

Proof. From Prosition 2.2, the Lie subgroup I is $(\varphi_t)_{t\in\mathrm{IR}}-$invariant. It follows that I is also $(\tilde{\varphi}_t)_{t\in\mathrm{IR}}-$invariant. Since the one parameter subgroup $(\tilde{\varphi}_t)_{t\in\mathrm{IR}}$ defines the vector field Z, it is clear that $\mathcal{I}$ is $ad(Z)-$ invariant. From Lemma 2.3 we obtain that $\mathcal{I}$ is also $ad(X)-$ invariant, i.e., the map $ad(X) : \mathcal{I} \to I$ is well defined. So, for each $i \geq 0$, $ad^i(X)(\mathcal{I}) \subset \mathcal{K}$, where $ad^0(X) = Id$. Actually, we are able to prove that

$$\mathcal{I} = \cap_{i\geq 0} ad^{-i}(X)(\mathcal{K}).$$

In fact, fix an element $U \in \cap_{i \geq 0} ad^{-i}(X)(\mathcal{K})$, then $ad^i(X)(U) \in \mathcal{K}$, for each $i \geq 0$. Then, for every $t,\ s \in$ IR we have:

$$d(\tilde{\varphi}_t)(sU) = \sum_{i=0}^{\infty} \frac{t^i}{i!} ad^i(X)(sU).$$

By hypotesis, $d(\tilde{\varphi}_t)(sU) \in \mathcal{K}$. Therefore, $(\frac{d}{dt})_{t=0}\tilde{\varphi}_t(\exp(sU)) \in \mathcal{K}$. Thus, for $t,\ s \in$ IR,$\tilde{\varphi}_t(\exp(sU))$ and $\varphi_t(\exp(sU)) \in K$. As a matter of fact, Proposition 2.2 shows that $\exp(sU) \in I$. In particular, $U \in I$ as we want to prove. Finally, if $K = \{e\}$ we get $\mathcal{I} = \{0\}$. On the other hand, if $K \neq \{e\}$ we obtain that at most in $n-1$ $ad(X)-$steps we should reach the Lie algebra $\mathcal{I}$. This yields that the real face of $\mathcal{I}$ is:

$$\mathcal{I} = \bigcap_{i=0}^{n-1} ad^{-i}(X)(\mathcal{K}).$$

Remarks 2.5 Let (X, π_K) a general linear pair on G. Proposition 2.2 shows that:

1. (X, π_K) is observable if and only if I is trivial.

2. For any $x \in G$, the tangent space at any point y of the equivalence class $\widetilde{x}$ is also given by left translation, i.e., $T_y \widetilde{x} = dL_y(\mathcal{I})$.

An immediate consequence of it is the following :

Corollary 2.6 Let (X, π_K) a general linear pair on G. Therefore,

(X, π_K) is locally observable if and only if $\mathcal{I}$ is trivial.

Proof. $\mathcal{I} = \{0\} \Leftrightarrow I$ is discrete.

Remark 2.7

1. Let (A, C) be a linear pair induced by a linear control system Σ on $\mathbb{R}^n$. For this set, Corollary 2.6 also determines global observability. In fact, in this case $I = \mathcal{I}$. In particular, Theorem 2.4 give us the well known formula

$$\widetilde{0} = \bigcap_{i=0}^{n-1} Ker(CA^i)$$

In fact, as we know $\widetilde{0}$ is the largest $A-$invariant subspace of $\mathbb{R}^n$ contained in $Ker(C)$.

2. As showed in [2] , local and global observability are independent notions for linear pairs (X, h) where $X = 0 + X^2 \in \mathbf{g} \otimes\ aut(G)$ and h is a Lie group homomorphism output map. So, for a general linear pair given by $X = X^1 + X^2 \in \mathbf{g} \otimes\ aut(G)$ we must expect the same.

Lemma 2.3 yields the existence of a vector field $Z \in aut(G)$ and a right invariant vector field Y on G such that $X = Z + Y$ and $ad(Z)$ and $ad(X)$ agree on $\mathbf{g}$. Let us denote by $S(Z)$ the set of the singularities of Z, *i.e.*, $S(Z) = \{x \in G : Z_x = 0\}$.

The following result illustrates necessary and sufficient conditions for global observability.

Theorem 2.8 A general linear pair (X, π_K) on G is observable if and only if

i) (X, π_K) is locally observable

ii) $S(Z) \cap K = \{e\}$

Proof. Of course, the locally observable property is a necessary condition to the global one. If $x \in S(Z) \cap K$ it follows immediately that $\widetilde{\varphi}_t\ (x) \in K$ for every $t \in$ IR. Thus, $x \in I$. Conversely, we shall show that the property of being observable is a necessary condition. Assume

I is discrete and fix $x \in I \subset K$. As we proved, I is $(\tilde{\varphi}_t)_{t \in \mathrm{IR}}$−invariant thus by continuity arguments, $\tilde{\varphi}_t\ (x) = x$, for each $t \in$ IR. So, $x \in S(Z)$. By hypothesis, we get $x = e$. It follows that (X, π_K) is observable.

Remark 2.9 Let G be a connected Lie group such that the exponential map $\exp : \mathbf{g} \to G$ is a global diffeomorphism. For instance, it happens if G is a simply connected nilpotent Lie group. By Lemma 2.3, $\tilde{\varphi}_t(x) = \exp(e^{tad(Z)} \log x)$, for every $x \in G$ and $t \in$ IR. Here, log denotes the inverse map of exp . So,

$$Z_x = 0 \Leftrightarrow \log x \in Ker(ad(Z))$$

Therefore, in this situation Theorem 2.8 implies that locally and globally observability properties are equivalent. In fact, the only one discrete vector subspace of **g** is the trivial one. As a matter of fact, in this case the observability property of a general linear pair (X, π_K) on G reduces to a test at the algebra level.

ϕ3.**Algorithm and Examples**

In order to compute the Lie algebra $\mathcal{I}$ it is also suitable to use a general algorithm proved by Isidori in [3]. In fact, from this result and starting with the Lie algebra $\mathcal{K}$ it is possible to construct a finite sequence of left invariant subspaces of the dual space $\mathbf{g}^*$ convergent to $\mathcal{I}^*$, (see [2] for details).

Algorithm

Consider a general linear pair (X, π_K) on G and the following steps:

1. Choose a basis $\mathcal{B} = \{Y^1, ..., Y^l\}$ to the Lie subalgebra $\mathcal{K}$,

2. Find the $\mathcal{B}$-dual basis $\mathcal{B}^* = \{w_1, ..., w_{n-l}\}$,

3. Find the $ad(X)(\mathcal{B}^*)$-associated basis to $\mathcal{I}^*$, i.e.,

$$ad(X)(\mathcal{B}^*) = \{ad^i(X)(w_j) \mid 0 \leq i, \quad 1 \leq j \leq n - l\},$$

$ad^0(X) = Id., ad(X)(w) = -\ w \circ ad(X), ad^i(X)(w) = ad(X)(ad^{i-1}(X)(w)),\ i \geq 2.$

Then, we have :

Proposition 3.1 Let (X, π_K) be a general linear pair. Therefore,

$$\mathcal{I} = (Span.ad(X)(\mathcal{B}^*))^*$$

Proof. It follows from Isidori theorem in [3] that

$$Span(ad(X)(\mathcal{B}^*) = \mathcal{I}^*.$$

In the sequel, some examples.

Examples 3.2

Let us consider the simply connected and nilpotent Heisenberg Lie group G of dimension 3, such that $G = \mathrm{IR}^3$ and $\mathbf{g} = \mathrm{IR}Y^1 + \mathrm{IR}Y^2 + \mathrm{IR}Y^3$ with the generators

$$Y^1 = \frac{\partial}{\partial x_1}, Y^2 = x_3 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \quad and \quad Y^3 = \frac{\partial}{\partial x_3}.$$

Just observe that only the Lie bracket $[Y^3, Y^2] = Y^1$ is not null. The group operation is given by

$$(x_1, x_2, x_3)(y_1, y_2, y_3) =: (x_1 + y_1 + x_3 y_2, x_2 + y_2, x_3 + y_3).$$

Consider the vector field $X \in \mathbf{g} \otimes \partial\mathbf{g}$ defined by $X = X^1 + X^2$where $X^1 = Y^2$ and X^2 is the vector field associated to the derivation

$$ad(X^2) = \begin{pmatrix} -3 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & -2 \end{pmatrix} \in \partial\mathbf{g}.$$

Observe that the vector field Z of the Lemma 2.3 has the face:

$$Z = Y^2 + X^2 - \frac{\partial}{\partial x_2} = x_3 \frac{\partial}{\partial x_1} + X^2.$$

We consider the followings general linear pairs:

i) (X, π_K) where K is the closed Lie subgroup with Lie algebra $\mathcal{K} = Span \{Y^1\}$. A simple computation shows that $ad(X)(Y^1) = -3Y^1 \in \mathcal{K}$. Thus, by Proposition 2.4, $\mathcal{I} = \mathcal{K}$. Therefore, Corollary 2.6 implies that (X, π_K) is neither locally nor globally observable.

ii) (X, π_K) where K is the closed Lie subgroup with Lie algebra $\mathcal{K} = Span \{Y^2\}$. We have, $ad(X)(Y^2) = -Y^2 + Y^3 \notin \mathcal{K}$. In this case, Corollary 2.6 (and Remark 2.9) gives us that (X, π_K) is observable.

REFERENCES

[1] **V. Ayala and J. Tirao**, "*Controllability of Linear Vector Fields on Lie Groups*". Preprint No IC/94/310, International Centre for Theoretical Physics.

[2] **V. Ayala and A.K. Hacibekiroglu, "**Observable Linear Pairs**".** Comput. Appl. Math., V. 16, n.3, pp. 205-214, 1997.

[3] **A. Isidori**, "*Nonlinear Control Systems : an introduction*". Springer Verlag,1989.

[4] **R. Kalman, Y. Ho and K. Narendra**, "*Controllability of Linear Dynamical Systems*", Contrib. to Diff. Equations 1 (2), 1962, pp. 189-213.

[5] **V. Varadarajan**, "*Lie Groups, Lie Algebras and Their Representations*", Prentice-Hall Inc. 1974.